reference. The area under the extrapolated part of the curve represents the unknown volume fraction below detectable size. In Fig. 1, two different extrapolations are suggested. The first one, which in simplified form is frequently used in our laboratory, for example in Coulter-counter analysis of cement, is given by the straight line between the central point, *a*, of the top of the finest histogram column, and the origin, *e*. The other is given by the 'best' straight line through the three finest histogram columns, a somewhat inaccurate formulation which allows different investigators to arrive at different results, although the differences would be rather small. In general, it is advisable to choose a standardized, well-defined extrapolation procedure to eliminate the individual variations among different investigators.

In Fig. 2, the cumulative volume percentages above stated sizo are plotted in a log-probability graph with three different 100 per cent references, one of them being identical with the result of Harris and Jowett and equal to 20,600. The other two are based on the above-described procedures, and the unknown percentages below detectable size are given by the areas of the triangles *ecb* and *dcb'* in Fig. 1.

Finally, attention is drawn to the dotted horizontal line through the point *f* on the ordinate axis in Fig. 1. This line represents the top of the histogram column from zero to 5·86μ based on the 100 per cent value 20,600 of Harris and Jowett. It seems to be reasonable to expect the decreasing pattern of the three finest histogram columns to continue towards the origin in some way, and, if that is true, the 100 per cent value of Harris and Jowett is too large. I therefore suggest that a 100 per cent estimate of either 19,621 or 19,259 should bo used instead of 20,600.

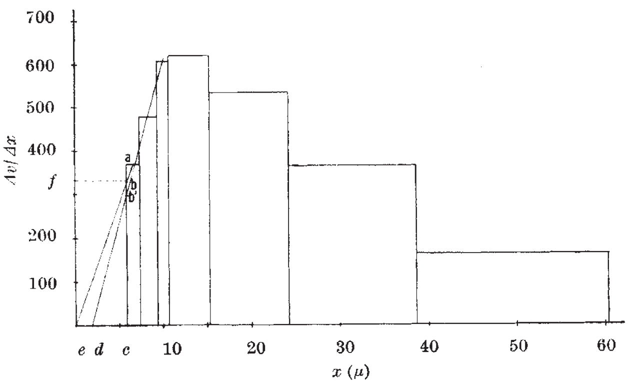


Fig. 1. Differential histogram of the Coulter-counter data of Harris and Jowett (ref. 1)

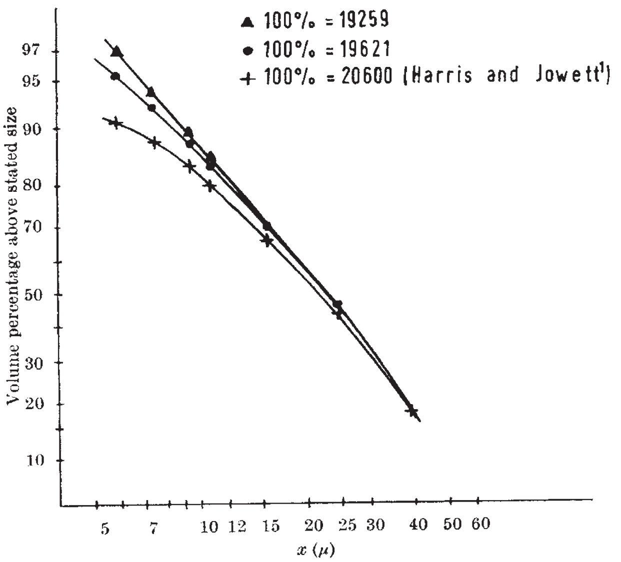


Fig. 2. Cumulative log-probability plots of volume percentage above stated size based on the data of Harris and Jowett (ref. 1)

Table 1. COULTER-COUNTER DATA OF HARRIS AND JOWETT (REF. 1)

| $x$ (μ) | $\Delta x$ | Volume $v$ above stated size (arbitrary units) | $\Delta v$ | $\Delta v/\Delta x$ | Volume percentage above stated size 100% = 20,600 | 100% = 19,621 | 100% = 19,259 |
|---|---|---|---|---|---|---|---|
| 60·40 | | 0 | | | | | |
| | 21·72 | | 3,528 | 162·4 | | | |
| 38·68 | | 3,528 | | | 17·13 | 17·98 | 18·32 |
| | 14·49 | | 5,264 | 363·3 | | | |
| 24·19 | | 8,792 | | | 42·68 | 44·81 | 45·65 |
| | 8·91 | | 4,731 | 531·6 | | | |
| 15·28 | | 13,523 | | | 65·65 | 68·92 | 70·22 |
| | 4·60 | | 2,842 | 617·8 | | | |
| 10·68 | | 16,365 | | | 79·44 | 83·41 | 84·97 |
| | 1·35 | | 818 | 605·9 | | | |
| 9·33 | | 17,183 | | | 83·41 | 87·57 | 89·22 |
| | 1·93 | | 920 | 476·7 | | | |
| 7·40 | | 18,103 | | | 87·88 | 92·26 | 94·00 |
| | 1·54 | | 566 | 367·5 | | | |
| 5·86 | | 18,669 | | | 90·63 | 95·15 | 96·94 |
| | | Volume below detectable size: | | | 1,931 | 952 | 590 |

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## Distribution of the Derivative of the Likelihood Function

It is common practice to approximate the distribution of the derivative of the log-likelihood, $\partial L/\partial\theta$, by a normal distribution with mean zero and variance $V = -[E(\partial^2 L/\partial\theta^2)]$. Calculations show that the three-moment $\chi^2$-approximation:

$$(\partial L/\partial\theta) = a + b\chi^2_{(\nu)} \quad (1)$$

can be a considerable improvement over the normal approximation, where $a$, $b$ and $\nu$ (the degrees of freedom of the $\chi^2$) are determined so that the right-hand side of (1) has its first three moments in common with $(\partial L/\partial\theta)$, that is:

$$b = \tfrac{1}{4}\mu_3/V,\ \nu = \tfrac{1}{2}V/b^2 \text{ and } a = -b\nu \quad (2)$$

where $\mu_3 = 3(\partial V/\partial\theta) + 2E(\partial^3 L/\partial\theta^3)$.

Table 1. ERROR IN THE LIMITS OF θ: (APPROX. − TRUE)$10^2$

| | Lower 0·01 | | Lower 0·05 | | Upper 0·05 | | Upper 0·01 | |
|---|---|---|---|---|---|---|---|---|
| $x$ | Normal | $\chi^2$ | Normal | $\chi^2$ | Normal | $\chi^2$ | Normal | $\chi^2$ |
| 1 | 3 | −1 | 2 | 0 | 54 | 6 | 150 | 21 |
| 2 | 13 | −3 | 7 | −3 | 45 | 4 | 136 | 17 |
| 3 | 20 | −5 | 10 | −2 | 47 | 4 | 128 | 14 |
| 5 | 30 | −6 | 14 | −2 | 43 | 4 | 119 | 10 |
| 10 | 41 | −6 | 18 | −2 | 39 | 3 | 107 | 8 |

To illustrate this, we consider a Poisson variate, $x$, with mean $\theta$ so that $\partial L/\partial\theta = (x - \theta)/\theta$. Substituting (2) in (1), we obtain in this case $\chi^2_{(\nu)} = 4(x + \theta)$ and $\nu = 8\theta$. Confidence limits for $\theta$ were obtained from Pearson and Hartley's[1] and Harter's[2] tables after introducing the usual continuity corrections, that is, using $x - \frac{1}{2}$ and $x + \frac{1}{2}$ when obtained the lower and upper limits for $\theta$, respectively. The values are given in Table 1 and compared with the normal values. The true values are given by Bartlett[3]. For $\mu_3$ small, that is, $\nu$ large, the square or cube root normal approximation of $\chi^2$ can be used.

For several other applications of the $\chi^2$-approximation (2), see Pearson[4] and Tiku[5-7].

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[6] Tiku, M. L., *Biometrika*, **53**, 630 (1965).
[7] Tiku, M. L., *Austral. J. Statist.*, **7** (in the press).