



## OPEN Empowering decentralized identity systems for Web 3.0 in complex spherical fuzzy knowledge

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The Web 3.0 network system, the next generation of the world wide web, incorporates new technologies and algorithms to enhance accessibility, decentralization, and security, mimicking human comprehension and enabling more personalized user interactions. The key component of this environment is decentralized identity management (DIM), embracing an identity and access management strategy that empowers computing devices and individuals to manage their digital personas. Aggregation operators (AOs) are valuable techniques that facilitate combining and summarizing a finite set of imprecise data. It is imperative to employ such operators to effectively address multicriteria decision-making (MCDM) issues. Yager operators have a significant extent of adaptability in managing operational environments and exhibit excellent effectiveness in addressing decision-making (DM) uncertainties. The complex spherical fuzzy (CSF) model is more effective in capturing and reflecting the known unpredictability in a DM application. This research endeavors to enhance the DM scenario of the Web 3.0 environment using Yager aggregation operators within the CSF environment. We present two innovative aggregation operators, namely complex spherical fuzzy Yager-ordered weighted averaging (CSFYOWA) and complex spherical fuzzy Yager-ordered weighted geometric (CSFYOWG) operators. We elucidate some structural characteristics of these operators and come up with an updated score function to rectify the drawbacks of the existing score function in the CSF framework. By utilizing newly proposed operators under CSF knowledge, we develop an algorithm for MCDM problems. In addition, we adeptly employ these strategies to handle the MCDM scenario, aiming to identify the optimal approach for ensuring the privacy of digital identity or data in the evolving landscape of the Web 3.0 era. Moreover, we undertake a comparative study to highlight the veracity and proficiency of the proposed techniques compared to the previously designed approaches.

**Keywords** Complex spherical fuzzy sets, Yager ordered weighted averaging operator, Yager ordered weighted geometric operator, Multi-criteria decision-making problems

MCDM is a methodical and systematic strategy utilized to assess and compare solutions according to several criteria or objectives. MCDM aims to provide decision-makers with a structured approach to evaluate various alternatives objectively and transparently, taking into account certain factors that are important in the DM process. Traditionally, it has been interpreted that all information for accessing the alternatives ought to come in the form of crisp numbers. Real-world problems often lack precise experimental data<sup>1</sup>, however there are many instances where human people make first estimates. Human decisions are influenced by cognitive knowledge<sup>2,3</sup>. Therefore, a certain degree of imprecision is inherent in human evaluations. The first powerful tool to tackle the uncertain data was introduced by Zadeh<sup>4</sup>, which takes into account the fuzzy memberships depicting the partial truth within the range of absolute truth and absolute false. Recent research<sup>5</sup> has shown that numerous domains aggregate attributes and evaluate their differences, thus increasing their dependence on FSs. However, in some scenarios, the FS theory proves to be ineffectual. For instance, when a person is presented with knowledge, that is categorized as either true or false, the FS model is unable to deal with it.

As an expansion of fuzzy sets, Atanassov introduced the novel notion of intuitionistic fuzzy sets (IFSs) in<sup>6</sup>, wherein the membership grades of an element form a pair of values within  $[0, 1]$ , one value represents support for

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membership in the FS, while the other represents opposition to membership. The theory of IFS is strengthened and extended in real-world problems like pattern recognition and DM. Despite of its numerous advantages, Atanassov's model has some restrictions since the total of MD and NMD is confined within the range of 0 and 1. Yager<sup>7</sup> addressed this issue and proposed an eminent model of Pythagorean fuzzy sets (PFSs), in which the quadratic sum of MD and NMD is less than and equal to 1, enabling decision experts to easily surmise that the PFSs are more effective than IFSs in depicting fuzzy information. After the significant development of PFS theory, several researchers became fascinated by this concept and put forth an assortment of novel operations within the framework of PF in<sup>8–11</sup>. Moreover, Tešić and Marinković<sup>12</sup> investigated the functionality of Fermatean fuzzy weight operators and Bonferroni Mean (BM) in the context of the MCDM model. The idea of IFSs and PFSs was further extended by Yager<sup>13</sup>, defining the structure of q-rung orthopair fuzzy sets (q-ROFSs), in which sum of the qth power of MD and NMD is constrained within the range of [0, 1]. Numerous researchers have adeptly employed q-ROFS theory across diverse domains, as evidenced in seminal research works referenced from<sup>14–19</sup>.

In cases when the structure of FSs is insufficient for application, the frameworks of IFSs, PFSs, and q-ROFSs are extremely important. However, these frameworks possess inherent limits, particularly in the context of voting, when opinions cannot be simply categorized as either "satisfaction" or "dissatisfaction," but rather may involve varying degrees of resistance and abstention. Although the neutral situation is crucial for the representation of human thought, it has yet to be addressed within the above-discussed theories. To express this information, Cuong<sup>20</sup> established the idea of picture FSs (PicFSs) – which contain MD, abstinence degree (AD), and NMD – with their cumulative sum constrained inside the range of closed unit interval. Qiyas et al.<sup>21</sup> incorporated the Yager AOs within the PicF environment and explored their fundamental operational laws. However, this approach faces a constraint when the total of these degrees surpasses one. To solve this problem, Kahraman and Gundogdu<sup>22</sup> established the idea of spherical FSs (SFSs), which extended the principles of PicFSs, wherein the total of the squares of these three degrees is between 0 and 1. Chinaram et al.<sup>23</sup> defined the innovative Yager AOs in the context of SF environments and applied them to analyze MCDM framework employed in the selection of wind power facility sites. Sarfraz<sup>24</sup> introduced a powerful tool for group DM by defining the Hamy mean (HM) operators in the framework of the interval-valued T-SF weighted (IVT-SFW) Dombi HM operator, the stretch-esteemed interval-valued T-spherical dual HM, and the IVT-SFW dual Dombi HM operator. Numerous fruitful research projects based on SFS theory have been established across diverse domains of science<sup>25–28</sup>.

The above-discussed theories are concerned with solving one-dimensional problems. The intriguing situation becomes evident when two-dimensional challenges are brought in. Nevertheless, these models are inadequate for handling periodic data or two-dimensional problems. After substantial research, Ramot et al.<sup>29,30</sup> originated the theoretical framework of complex FSs (CFSs) to resolve this concern. Ramot generalized the idea of ordinary FSs by incorporating the phase term, has a crucial role within the context of DM. This characteristic distinguishes CFSs as powerful and distinct from classical FS models. CFSs have been employed by several researchers to address DM issues in<sup>31–34</sup>. Following this, Alkouri and Salleh<sup>35</sup> proposed the notion of complex IFSs (CIFs) that contain the complex-valued membership and non-membership functions in a closed unit disk, subject to the condition that the sum of real parts (as well as imaginary parts) of MD and NMD should be less than and equal to one. However, the problem arises when the decision experts choose the degrees of real and imaginary parts, whose sum exceeds a closed unit disk. Recognizing this limitation, Ullah et al.<sup>36</sup> presented a new strategy for complex PFSs (CPFSs) where the square summation of real and imaginary components of these complex numbers is bounded by a closed unit disk.

Apart from these scenarios, numerous problems and uncertain situations manifest in everyday life, necessitating the inclusion of alternatives expressing neutral behavior within the datasets of the complex plane. For instance, voters can be categorized into three distinct categories during an election: those who cast their support, those who cast their opposition, and those who abstain to vote. To address this inadequacy<sup>37</sup>, elaborated on the idea of CPFSs and came up with an innovative concept of complex PicFSs (CPicFSs), categorized by the amplitude and phase terms of MD, AD, and NMD, subject to the constraint that the sum of real parts (as well as the imaginary parts) of these three degrees bounded within a closed unit disk in a complex plane. The structure of CPicFSs is of great importance as its ability to deal with human opinion efficiently in two-dimensional space. Nevertheless, CPicFSs encounter difficulties when the quadratic sum of these three degrees surpasses 1. To tackle this restriction of CPicFSs, the novel framework of CSFSs was originated by Ali et al.<sup>38</sup>, which is defined by membership, abstinence, and non-membership function in a complex plane with the more flexible condition that square sum of the real components as well as the imaginary components of MD, AD, and NMD must lie within the range of closed unit disk. An extensive amount of information about any object is contained in a CSFS, which includes three terms, each characterized by amplitude and phase components. For instance,  $(0.2e^{i2\pi(0.7)}, 0.6e^{i2\pi(0.4)}, 0.4e^{i2\pi(0.3)})$ , the CIFs, CPFSs, and CPicFSs are unable to handle such a value. The CSF system is more advantageous and dominant than existing two-dimensional malleable approaches. The importance of CSFSs in managing imprecise and inconvenient information inspired our investigation of the theory of CSFSs. This exceptional concept has received significant attention from researchers, as evidenced by the numerous articles dedicated to its analysis.

The hybrid DM approach that incorporates the CSF information prioritized weighted averaging/geometric operators was presented by Akram et al.<sup>39</sup> and also introduces their notable properties. By utilizing the CSFS VIKOR technique, Akram et al.<sup>40</sup> designed the CSFOW aggregation operators to solve group DM problems. The operational principle of the suggested technique prioritizes the suggestion of a compromise outcome based on two key factors: group utility and the individual regret of the opposing party. Akram et al.<sup>41</sup> decided to incorporate Dombi AOs into the CSF framework by keeping the importance of t-norm and s-norm, to find the optimal solutions to the MCDM issues. Naem et al.<sup>42</sup> described the CSF power AOs and developed the CSF decision support structure, which utilizes the entropy measure and power operator. Reference<sup>43</sup> explored the

Aczel-Alsina operators in the context of the CSF settings, and explored their utilization in real world problems. The CSF TOPSIS method was developed in<sup>44</sup>, where the authors employ a sophisticated SF-soft WAO to combine the opinions of multiple experts based on the strength of the characteristics and traits of the alternatives and also introduced normalized Euclidean distances to measure the proximity of alternatives. Moreover, they propose a revised closeness index to determine the best practicable option. Akram et al.<sup>45</sup> showcased the magnificent design of a CSFS that enhances the efficiency of DM and the ranking capabilities of the ELECTRE I technique. This strategy proves to be highly beneficial and superior in the context of MADM.

Aggregation operators are essential in the DM process, particularly in the field of fuzzy logic and uncertainty modelling. These operators enable the consolidation of several pieces of information, which facilitates the collection of significant insights from different sources. The ordered weighted aggregation approaches involve sorting the input values in descending order assigning each value a weight based on its position in the ordered sequence and enhancing well-informed DM in various operational conditions and complex challenges. Under the exception of flexibility and sophistication, the Yager OWA and Yager OWG operators, an extension of the conventional OWA and OWG operators, have been designed to capture the importance of input values based on their order. It is important to recognize that existing methods cannot adjust the parameter based on the decision-maker's risk aversion dynamically. This limitation hinders the practical implementation of the MCDM solution. Nevertheless, the techniques outlined in this article are quite effective in mitigating this flaw in the given context. Inspired by this observation, we integrate these two innovative YOWA and YOWG operators with the CSF setting in this research.

This paper highlights the following principal developments:

- (i) We develop an updated score function while mitigating the deficiency of current score function in the CSF context. This enhances the precision and accuracy of the grading system.
- (ii) Based on the advantages of the Yager t-norm and s-norm, we introduce two innovative AOs, namely the CSFYOWA operator and the CSFYOWG operator. These operators are specifically created to assess a problem that involves two-dimensional spherical fuzzy data.
- (iii) We formulate fundamental Yager's operational laws for CSFNs and investigate the structural characteristics of newly defined Yager-ordered weighted aggregation operators in the CSF paradigm.
- (iv) We design a systematic mathematical approach to tackle MCDM scenarios using newly defined operators in the CSF settings.
- (v) We successfully apply recently defined techniques for MCDM problem of optimizing DID management with the framework of Web 3.0. We have also conducted an exhaustive comparative evaluation to the validity and efficacy of these techniques in comparison to the existing approaches.

The following discussion of this research article is structured in the following way: In section "[Fundamental ideas of CSF knowledge](#)", we thoroughly examined the operational rules and basic principles of CSFSs. Section "[Refinement of the existing score function for CSFNs](#)" formulates an improved score function to successfully rectify the limitation of the current score function for MCDM problem in the context of CSF settings. Section "[Structural analysis of Yager aggregation operators within CSFNs](#)" presents two innovative AOs, specifically the CSFYOWA operator and CSFYOWG operator and investigate their structural characteristics, namely idempotency, monotonicity, and boundedness. We present a structured approach in section "[Implementation of proposed strategies in MCDM problem](#)" to address the MCDM issues involving CSF data. We effectively apply newly developed approaches to solve the MCDM problem of selecting an optimal DIM technique for an inclusive Web 3.0 network system where user have more control over their digital information based on CSF knowledge. Additionally, we perform a comparative evaluation to validate the effectiveness of these strategies with the established methodologies. Finally, we culminate the research by examining possible consequences and summarize the major results in Sect. 6.

Table 1 represents the list of abbreviations and Table 2 indicates the list of symbols used in this research work.

## Fundamental ideas of CSF knowledge

This section presents core concepts and attributes that are intended to enhance comprehension and accessibility of the content offered in this article.

**Definition 1** (<sup>22</sup>). An SFS  $\Xi$  of the universal set  $E$  is defined as:

$$\Xi = \{ (x, \underline{\mu}(x), \aleph(x), \Psi(x)) \mid x \in E \},$$

where  $\underline{\mu}, \aleph, \Psi : E \rightarrow [0, 1]$ , indicate the membership, abstinence, and non-membership functions, respectively, with the constraint that  $0 \leq \underline{\mu}^2(x) + \aleph^2(x) + \Psi^2(x) \leq 1$ . The hesitance margin of SFS is stated as:

$$\kappa(x) = \sqrt{1 - (\underline{\mu}^2(x) + \aleph^2(x) + \Psi^2(x))}.$$

**Definition 2** (<sup>29</sup>). A CFS  $\mathcal{F}$  on  $E$  is defined as:

$$\mathcal{F} = \{ (x, \underline{\mu}(x)) \mid x \in E \},$$

where  $\underline{\mu}$  is a complex-valued membership function that allocates each element  $x \in E$  to the closed unit disk in a complex plane, and is written as:  $\underline{\mu}(x) = \zeta(x) \cdot e^{i2\pi\varpi(x)}$ , such that  $\zeta(x), \varpi(x) \in [0, 1]$ .

Abbreviations	Description
FSs	Fuzzy Sets
IFs	Intuitionistic Fuzzy Sets
PFSs	Pythagorean Fuzzy Sets
q-ORFSs	q-rung Orthopair Fuzzy Sets
PicFSs	Picture Fuzzy Sets
SFSs	Spherical Fuzzy Sets
CFSs	Complex Fuzzy Sets
CIFs	Complex Intuitionistic Fuzzy Sets
CPFs	Complex Pythagorean Fuzzy Sets
CPicFSs	Complex Picture Fuzzy Sets
CSFSs	Complex Spherical Fuzzy Sets
CSFNs	Complex Spherical Fuzzy Numbers
MD	Membership Degree
AD	Abstinence Degree
NMD	Non-membership Degree
DM	Decision Making
MCDM	Multi-criteria Decision Making
AOs	Aggregation Operators
PicYOWA	Picture Fuzzy Yager Ordered Weighted Averaging
PicYOWG	Picture Fuzzy Yager Ordered Weighted Geometric
SFYOWA	Spherical Fuzzy Yager Ordered Weighted Averaging
SFYOWG	Spherical Fuzzy Yager Ordered Weighted Geometric
CSFOWA	Complex Spherical Fuzzy Ordered Weighted Averaging
CSFOWG	Complex Spherical Fuzzy Ordered Weighted Geometric
CSFYOWA	Complex Spherical Fuzzy Yager Ordered Weighted Averaging
CSFYOWG	Complex Spherical Fuzzy Yager Ordered Weighted Geometric
DIM	Decentralized Identity Management

**Table 1.** List of abbreviations.

Symbol	Description
$\Xi$	SFSs
$\mathcal{F}$	CFS
$\mathcal{P}$	CPicFSs
$\mathcal{Q}$	CSFSs
$\mu, \nu, \psi$	membership, abstinence, and non-membership values
$\zeta, \varpi$	Real and imaginary components of MD
$\eta, \vartheta$	Real and imaginary components of AD
$\kappa, \Omega$	Real and imaginary components of NMD
$\kappa$	Hesitance margin
$\Theta$	Permutation function
$\beta$	Weight Vector
$(\tilde{v})$	CSFN
$(\tilde{v})_{\Theta(i)}$	Permuted CSFNs
$\Gamma$	Parametric value
$\mathcal{L}$	Collection of CSFNs
$\mathcal{C}$	Collection of criteria
$\mathcal{D}$	CSF Decision Matrix
$\mathcal{G}_{(i1)}$	Entries of CSF decision matrix
$\delta$	Preference value of an alternative

**Table 2.** List of symbols.

**Definition 3** <sup>(37)</sup>. A CPicFS  $P$  on  $E$  is defined as:

$$P = \{(x, \underline{\mu}(x), \aleph(x), \Psi(x)) : x \in E\},$$

Here,  $\underline{\mu}, \aleph,$  and  $\Psi$  are the function from  $E$  to the closed unit disk in complex plane and are called membership, abstinance, and non-membership functions, respectively. These are defined as:  $\underline{\mu}(x) = \zeta(x) \cdot e^{i2\pi\varpi(x)}, \aleph(x) = \eta(x) \cdot e^{i2\pi\vartheta(x)}$ , and  $\Psi(x) = \kappa(x) \cdot e^{i2\pi\Omega(x)}$ , such that  $0 \leq \zeta(x), \eta(x), \kappa(x), \zeta(x) + \eta(x) + \kappa(x) \leq 1$  and  $0 \leq \varpi(x), \vartheta(x), \Omega(x), \varpi(x) + \vartheta(x) + \Omega(x) \leq 1$ . The hesitance margin of CPicFS is defined as:

$$\kappa(x) = r(x) \cdot e^{i2\pi\tau(x)} = \sqrt{1 - (\zeta(x) + \eta(x) + \kappa(x))} \cdot e^{i2\pi\sqrt{(1 - (\varpi(x) + \vartheta(x) + \Omega(x)))}}.$$

**Definition 4** <sup>(38)</sup>. A CSFS  $Q$  on  $E$  is defined as follows:

$$Q = \{(x, \underline{\mu}(x), \aleph(x), \Psi(x)) | x \in E\},$$

Here,  $\underline{\mu}, \aleph,$  and  $\Psi$  are the function from  $E$  to the closed unit disk in complex plane and are called membership, abstinance, and non-membership functions, respectively. These are defined as:  $\underline{\mu}(x) = \zeta(x) \cdot e^{i2\pi\varpi(x)}, \aleph(x) = \eta(x) \cdot e^{i2\pi\vartheta(x)}$ , and  $\Psi(x) = \kappa(x) \cdot e^{i2\pi\Omega(x)}$ , such that  $0 \leq \zeta(x), \eta(x), \kappa(x), \zeta^2(x) + \eta^2(x) + \kappa^2(x) \leq 1$  and  $0 \leq \varpi(x), \vartheta(x), \Omega(x), \varpi^2(x) + \vartheta^2(x) + \Omega^2(x) \leq 1$ . The hesitancy margin of CSFSs is represented by  $\kappa(x) = r(x) \cdot e^{i2\pi\tau(x)}$  and is expressed as:

$$\kappa(x) = r(x) \cdot e^{i2\pi\tau(x)} = \sqrt{1 - (\zeta^2(x) + \eta^2(x) + \kappa^2(x))} \cdot e^{i2\pi\sqrt{(1 - (\varpi^2(x) + \vartheta^2(x) + \Omega^2(x)))}}.$$

For simplicity, we consider  $x = ((\zeta, \varpi), (\eta, \vartheta), (\kappa, \Omega))$  for  $x \in E$  in the rest of the paper. This specific representation of  $x$  is referred to as a CSFN, where  $0 \leq \zeta, \eta, \kappa, \zeta^2 + \eta^2 + \kappa^2 \leq 1$  and  $0 \leq \varpi, \vartheta, \Omega, \varpi^2 + \vartheta^2 + \Omega^2 \leq 1$

**Definition 5** <sup>(25)</sup>. Consider two SFNs:  $(\tilde{v})_1 = (\zeta_1, \eta_1, \kappa_1)$  and  $(\tilde{v})_2 = (\zeta_2, \eta_2, \kappa_2)$ . The fundamental operations on  $(\tilde{v})_1$  and  $(\tilde{v})_2$  are outlined in the subsequent way:

- (i)  $(\tilde{v})_1 \subseteq (\tilde{v})_2$  iff  $\zeta_1 \leq \zeta_2, \eta_1 \leq \eta_2, \kappa_1 \geq \kappa_2$ ,
- (ii)  $(\tilde{v})_1^c = (\kappa_1, \eta_1, \zeta_1)$ .

**Definition 6** <sup>(40)</sup>. Consider the collection of  $m$  number of CSFNs, represented by  $(\tilde{v})_i = (\zeta_i e^{i2\pi\varpi_i}, \eta_i e^{i2\pi\vartheta_i}, \kappa_i e^{i2\pi\Omega_i})$  and  $(\Theta(1), \Theta(2), \dots, \Theta(m))$  represents the permutation of  $\{1, 2, \dots, m\}$ , such that  $(\tilde{v})_{\Theta(i)} \succeq (\tilde{v})_{\Theta(i+1)} \forall i = 1, 2, \dots, m - 1$ . Moreover, the associated weight vector of  $(\tilde{v})_i$  is  $\beta = (\beta_1, \beta_2, \dots, \beta_m)^T$  with  $0 \leq \beta_i \leq 1$  and  $\sum_{i=1}^m \beta_i = 1$ . Then, the aggregated value of CSFOWA operator is mathematically expressed in the following way:

$$\text{CSFOWA}((\tilde{v})_1, (\tilde{v})_2, (\tilde{v})_3, \dots, (\tilde{v})_m) = \left( \begin{array}{c} \sqrt{1 - \prod_{i=1}^m (1 - \zeta_{\Theta(i)}^2)^{\beta_i}} e^{i2\pi\sqrt{1 - \prod_{i=1}^m (1 - \varpi_{\Theta(i)}^2)^{\beta_i}}}, \\ \prod_{i=1}^m (\eta_{\Theta(i)})^{\beta_i} e^{i2\pi\prod_{i=1}^m (\vartheta_{\Theta(i)})^{\beta_i}}, \prod_{i=1}^m (\kappa_{\Theta(i)})^{\beta_i} e^{i2\pi\prod_{i=1}^m (\Omega_{\Theta(i)})^{\beta_i}} \end{array} \right).$$

**Definition 7** <sup>(40)</sup>. Consider the collection of  $m$  number of CSFNs, denoted as  $(\tilde{v})_i = (\zeta_i e^{i2\pi\varpi_i}, \eta_i e^{i2\pi\vartheta_i}, \kappa_i e^{i2\pi\Omega_i})$  and  $(\Theta(1), \Theta(2), \dots, \Theta(m))$  represents the permutation of  $\{1, 2, \dots, m\}$ , such that  $(\tilde{v})_{\Theta(i)} \succeq (\tilde{v})_{\Theta(i+1)}$  for all  $i = 1, 2, \dots, m - 1$ . Moreover, the corresponding weight vector of  $(\tilde{v})_i$  is  $\beta = (\beta_1, \beta_2, \dots, \beta_m)^T$  with  $0 \leq \beta_i \leq 1$  and  $\sum_{i=1}^m \beta_i = 1$ . Then, the aggregated value of CSFOWG operator is mathematically represented in the following way:

$$\text{CSFOWG}((\tilde{v})_1, (\tilde{v})_2, (\tilde{v})_3, \dots, (\tilde{v})_m) = \left( \begin{array}{c} \prod_{i=1}^m (\zeta_{\Theta(i)})^{\beta_i} e^{i2\pi\prod_{i=1}^m (\varpi_{\Theta(i)})^{\beta_i}}, \prod_{i=1}^m (\eta_{\Theta(i)})^{\beta_i} e^{i2\pi\prod_{i=1}^m (\vartheta_{\Theta(i)})^{\beta_i}} \\ \left( \sqrt{1 - \prod_{i=1}^m (1 - \kappa_{\Theta(i)}^2)^{\beta_i}} e^{i2\pi\sqrt{1 - \prod_{i=1}^m (1 - \Omega_{\Theta(i)}^2)^{\beta_i}}} \right) \end{array} \right).$$

**Definition 8** <sup>(5)</sup>. For any  $(c, d) \in [0, 1]^2$  and  $\lambda \in (0, \infty)$ , the Yager t-norm and s-norm are defined as follows:

- (i)  $T(c, d) = 1 - \min(1, (1 - c)^\lambda + (1 - d)^\lambda)^{1/\lambda}$ ,
- (ii)  $S(c, d) = \min(1, (c^\lambda + d^\lambda)^{1/\lambda})$ .

**Definition 9** <sup>(39)</sup>. The score function for any CSFN  $(\tilde{v}) = ((\zeta, \varpi), (\eta, \vartheta), (\kappa, \Omega))$  is represented by  $\&((\tilde{v}))$  and is defined as follows:

$\&((\tilde{v})) = \frac{1}{3} [(2 + \zeta^2 - \eta^2 - \kappa^2) + (2 + \varpi^2 - \vartheta^2 - \Omega^2)]$ , where  $\&((\tilde{v}))$  falls within the range of  $[0.67, 2]$ . Additionally, the comparison of any two CSFNs  $(\tilde{v})_1$  and  $(\tilde{v})_2$  is governed by the following laws:

- (i) If  $\&((\tilde{v})_1) > \&((\tilde{v})_2)$ , then  $(\tilde{v})_1 > (\tilde{v})_2$ ,
- (ii) If  $\&((\tilde{v})_1) < \&((\tilde{v})_2)$ , then  $(\tilde{v})_1 < (\tilde{v})_2$ ,

(iii) If  $\&((\tilde{v})_1) = \&((\tilde{v})_2)$ , then  $(\tilde{v})_1 \sim (\tilde{v})_2$ .

### Refinement of the existing score function for CSFNs

In this section, we aim to elucidate the shortcoming of the current score function employed in CSFNs formulated in <sup>39</sup> by means of a specific illustration. Subsequently, we design a novel scoring mechanism to address and eliminate this shortcoming.

**Example 1** Consider the two CSFNs:  $(\tilde{v})_1 = ((0.6, 0.3), (0.2, 0.4), (0.5, 0.2))$  and  $(\tilde{v})_2 = ((0.8, 0.7), (0.3, 0.2), (0.1, \frac{\sqrt{5}}{10}))$ . By applying the definition 9 to CSFNs  $(\tilde{v})_1$  and  $(\tilde{v})_2$ , we deduce the following results:  $\&((\tilde{v})_1) = \&((\tilde{v})_2) = 1.65$ . According to property 3 stated in definition 9, it is prominent that the CSFNs  $(\tilde{v})_1$  and  $(\tilde{v})_2$  are incomparable.

The above example highlights the inherent inadequacy of the current scoring mechanism within the CSF knowledge. As a result, it serves as a motivation for us to enhance this score function in the subsequent way.

**Definition 10** Let  $(\tilde{v}) = ((\zeta, \varpi), (\eta, \vartheta), (\kappa, \Omega))$  be any CSFN, then the score function of  $(\tilde{v})$  is defined as follows:

$\&((\tilde{v})) = \frac{1}{3} [4 + ((1 - \kappa^2) (\zeta^2 - \eta^2) + (1 - \Omega^2) (\varpi^2 - \vartheta^2))]$ , where  $\&((\tilde{v}))$  falls within the range of  $[1.33, 2]$ . Additionally, the comparison of any two CSFNs  $(\tilde{v})_1$  and  $(\tilde{v})_2$  in the framework of the above score function is governed by the following laws:

- (i) If  $\&((\tilde{v})_1) > \&((\tilde{v})_2)$ , then  $(\tilde{v})_1 > (\tilde{v})_2$ ,
- (ii) If  $\&((\tilde{v})_1) < \&((\tilde{v})_2)$ , then  $(\tilde{v})_1 < (\tilde{v})_2$ ,
- (iii) If  $\&((\tilde{v})_1) = \&((\tilde{v})_2)$ , then  $(\tilde{v})_1 \sim (\tilde{v})_2$ .

The ensuing example illuminates the precision and legitimacy of the newly introduced score function tailored for CSFNs.

**Example 2** Let us consider the two CSFNs:  $(\tilde{v})_1 = ((0.6, 0.3), (0.2, 0.4), (0.5, 0.2))$  and  $(\tilde{v})_2 = ((0.8, 0.7), (0.3, 0.2), (0.1, \frac{\sqrt{5}}{10}))$ . Using the framework outlined in Definition 10 to these CSFNs  $(\tilde{v})_1$  and  $(\tilde{v})_2$  results in  $\&((\tilde{v})_1) = 1.39$  and  $\&((\tilde{v})_2) = 1.66$ . According to Property 2 of Definition 10, it is quite evident that  $(\tilde{v})_1$  is less than  $(\tilde{v})_2$ . The presented evidence suggests that  $(\tilde{v})_2$  is superior to  $(\tilde{v})_1$ .

The preceding discourse illuminates that the suggested score function emerges as a more apt and precise instrument, yielding refined outcomes in the realm of decision analysis.

### Structural analysis of Yager aggregation operators within CSFNs

This section outlines Yager’s operations for CSFNs. Furthermore, we define CSF Yager’s aggregating operators and conduct an investigation into their structural properties.

**Definition 11** For any two CSFNs,  $(\tilde{v})_1 = ((\zeta_1, \varpi_1), (\eta_1, \vartheta_1), (\kappa_1, \Omega_1))$  and  $(\tilde{v})_2 = ((\zeta_2, \varpi_2), (\eta_2, \vartheta_2), (\kappa_2, \Omega_2))$  with  $\geq 1$  and  $\Gamma > 0$ . The Yager’s basic operations on  $(\tilde{v})_1$  and  $(\tilde{v})_2$  are stated as follows:

(i)

$$(\tilde{v})_1 \oplus (\tilde{v})_2 = \left( \left( \sqrt{\min \left( 1, (\zeta_1^{2\Gamma} + \zeta_2^{2\Gamma})^{1/\Gamma} \right)}, \sqrt{\min \left( 1, (\varpi_1^{2\Gamma} + \varpi_2^{2\Gamma})^{1/\Gamma} \right)}, \right. \right. \\ \left. \left. \left( \sqrt{1 - \min \left( 1, ((1 - \eta_1^\Gamma) + (1 - \eta_2^\Gamma))^{1/\Gamma} \right)}, \sqrt{1 - \min \left( 1, ((1 - \vartheta_1^\Gamma) + (1 - \vartheta_2^\Gamma))^{1/\Gamma} \right)}, \right. \right. \\ \left. \left. \left( \sqrt{1 - \min \left( 1, ((1 - \kappa_1^\Gamma) + (1 - \kappa_2^\Gamma))^{1/\Gamma} \right)}, \sqrt{1 - \min \left( 1, ((1 - \Omega_1^\Gamma) + (1 - \Omega_2^\Gamma))^{1/\Gamma} \right)} \right) \right)$$

(ii)

$$(\tilde{v})_1 \otimes (\tilde{v})_2 = \left( \left( \sqrt{1 - \min \left( 1, ((1 - \zeta_1^\Gamma) + (1 - \zeta_2^\Gamma))^{1/\Gamma} \right)}, \sqrt{1 - \min \left( 1, ((1 - \varpi_1^\Gamma) + (1 - \varpi_2^\Gamma))^{1/\Gamma} \right)}, \right. \right. \\ \left. \left. \left( \sqrt{1 - \min \left( 1, ((1 - \eta_1^\Gamma) + (1 - \eta_2^\Gamma))^{1/\Gamma} \right)}, \sqrt{1 - \min \left( 1, ((1 - \vartheta_1^\Gamma) + (1 - \vartheta_2^\Gamma))^{1/\Gamma} \right)}, \right. \right. \\ \left. \left. \left( \sqrt{\min \left( 1, (\kappa_1^{2\Gamma} + \kappa_2^{2\Gamma})^{1/\Gamma} \right)}, \sqrt{\min \left( 1, (\Omega_1^{2\Gamma} + \Omega_2^{2\Gamma})^{1/\Gamma} \right)} \right) \right)$$

(iii)

$$\begin{aligned}
 \mathfrak{U}(\tilde{v})_1 &= \left( \left( \sqrt{\min \left( 1, \left( \mathfrak{U} \cdot \zeta_1^{2\Gamma} \right)^{1/\Gamma} \right)}, \sqrt{\min \left( 1, \left( \mathfrak{U} \cdot \varpi_1^{2\Gamma} \right)^{1/\Gamma} \right)} \right), \right. \\
 &\quad \left( \sqrt{1 - \min \left( 1, \left( \mathfrak{U} \cdot (1 - \eta_1^2)^\Gamma \right)^{1/\Gamma} \right)}, \sqrt{1 - \min \left( 1, \left( \mathfrak{U} \cdot (1 - \vartheta_1^2)^\Gamma \right)^{1/\Gamma} \right)} \right), \\
 &\quad \left. \left( \sqrt{1 - \min \left( 1, \left( \mathfrak{U} \cdot (1 - \kappa_1^2)^\Gamma \right)^{1/\Gamma} \right)}, \sqrt{1 - \min \left( 1, \left( \mathfrak{U} \cdot (1 - \Omega_1^2)^\Gamma \right)^{1/\Gamma} \right)} \right) \right), \\
 \text{(iv)} \\
 (\tilde{v})_1^{\mathfrak{U}} &= \left( \left( \sqrt{1 - \min \left( 1, \left( \mathfrak{U} \cdot (1 - \zeta_1^2)^\Gamma \right)^{1/\Gamma} \right)}, \sqrt{1 - \min \left( 1, \left( \mathfrak{U} \cdot (1 - \varpi_1^2)^\Gamma \right)^{1/\Gamma} \right)} \right), \right. \\
 &\quad \left( \sqrt{1 - \min \left( 1, \left( \mathfrak{U} \cdot (1 - \eta_1^2)^\Gamma \right)^{1/\Gamma} \right)}, \sqrt{1 - \min \left( 1, \left( \mathfrak{U} \cdot (1 - \vartheta_1^2)^\Gamma \right)^{1/\Gamma} \right)} \right), \\
 &\quad \left. \left( \sqrt{\min \left( 1, \left( \mathfrak{U} \cdot \kappa_1^{2\Gamma} \right)^{1/\Gamma} \right)}, \sqrt{\min \left( 1, \left( \mathfrak{U} \cdot \Omega_1^{2\Gamma} \right)^{1/\Gamma} \right)} \right) \right).
 \end{aligned}$$

**Fundamental characteristics of CSFYOWA operator**

Within this subsection, we introduce the CSFYOWA operator. This operator is designed to facilitate the aggregation of CSFNs, providing a comprehensive approach to ordered weighted averaging operator within the CSF settings. Furthermore, we prove the fundamental properties of this operator.

**Definition 12** Assume that  $\mathcal{L}$  is a collection of  $m$  number of CSFNs, represented by  $(\tilde{v})_i = ((\zeta_i, \varpi_i), (\eta_i, \vartheta_i), (\kappa_i, \Omega_i))$  and  $(\Theta(1), \Theta(2), \dots, \Theta(m))$  represents the permutation of  $\{1, 2, \dots, m\}$ , such that  $(\tilde{v})_{\Theta(i-1)} \geq (\tilde{v})_{\Theta(i)} \forall i = 1, 2, \dots, m$ . Moreover, the corresponding weight vector of  $(\tilde{v})_i$  is  $\beta = (\beta_1, \beta_2, \dots, \beta_m)^T$  with  $0 \leq \beta_i \leq 1$  and  $\sum_{i=1}^m \beta_i = 1$ . The CSF Yager ordered weighted averaging operator is a mapping CSFYOWA:  $\mathcal{L}^m \rightarrow \mathcal{L}$ , defined as follows:

$$\begin{aligned}
 \text{CSFYOWA}((\tilde{v})_1, (\tilde{v})_2, (\tilde{v})_3, \dots, (\tilde{v})_m) &= \oplus_{i=1}^m (\beta_i (\tilde{v})_{\Theta(i)}) \\
 &= \left( \left( \sqrt{\min \left( 1, \left( \sum_{i=1}^m \beta_i \cdot \zeta_{\Theta(i)}^{2\Gamma} \right)^{1/\Gamma} \right)}, \sqrt{\min \left( 1, \left( \sum_{i=1}^m \beta_i \cdot \varpi_{\Theta(i)}^{2\Gamma} \right)^{1/\Gamma} \right)} \right), \right. \\
 &\quad \left( \sqrt{1 - \min \left( 1, \left( \sum_{i=1}^m \beta_i \cdot \left( \frac{1}{(1 - \eta_{\Theta(i)}^2)^\Gamma} \right)^{1/\Gamma} \right)} \right)}, \sqrt{1 - \min \left( 1, \left( \sum_{i=1}^m \beta_i \cdot \left( \frac{1}{(1 - \vartheta_{\Theta(i)}^2)^\Gamma} \right)^{1/\Gamma} \right)} \right)} \right), \\
 &\quad \left. \left( \sqrt{1 - \min \left( 1, \left( \sum_{i=1}^m \beta_i \cdot \left( \frac{1}{(1 - \kappa_{\Theta(i)}^2)^\Gamma} \right)^{1/\Gamma} \right)} \right)}, \sqrt{1 - \min \left( 1, \left( \sum_{i=1}^m \beta_i \cdot \left( \frac{1}{(1 - \Omega_{\Theta(i)}^2)^\Gamma} \right)^{1/\Gamma} \right)} \right)} \right) \right), \tag{1}
 \end{aligned}$$

**Theorem 1** Consider the collection of CSFNs, represented by  $(\tilde{v})_i = ((\zeta_i, \varpi_i), (\eta_i, \vartheta_i), (\kappa_i, \Omega_i))$ ,  $i = 1, 2, \dots, m$  and  $(\Theta(1), \Theta(2), \dots, \Theta(m))$  denotes the permutation of  $\{1, 2, \dots, m\}$ , such that  $(\tilde{v})_{\Theta(i-1)} \geq (\tilde{v})_{\Theta(i)} \forall i = 1, 2, \dots, m$ . In addition, the corresponding weight vector of  $(\tilde{v})_i$  is  $\beta = (\beta_1, \beta_2, \beta_3, \dots, \beta_m)^T$ , where  $0 \leq \beta_i \leq 1$ ,  $\sum_{i=1}^m \beta_i = 1$  and  $\Gamma > 0$ . Then, the aggregated value in the structure of CSFYOWA operator is a CSFN and is expressed in the following way:

$$\begin{aligned}
 \text{CSFYOWA}((\tilde{v})_1, (\tilde{v})_2, (\tilde{v})_3, \dots, (\tilde{v})_m) &= \oplus_{i=1}^m (\beta_i (\tilde{v})_{\Theta(i)}) \\
 &= \left( \left( \sqrt{\min \left( 1, \left( \sum_{i=1}^m \beta_i \cdot \zeta_{\Theta(i)}^{2\Gamma} \right)^{1/\Gamma} \right)}, \sqrt{\min \left( 1, \left( \sum_{i=1}^m \beta_i \cdot \varpi_{\Theta(i)}^{2\Gamma} \right)^{1/\Gamma} \right)} \right), \right. \\
 &\quad \left( \sqrt{1 - \min \left( 1, \left( \sum_{i=1}^m \beta_i \cdot \left( \frac{1}{(1 - \eta_{\Theta(i)}^2)^\Gamma} \right)^{1/\Gamma} \right)} \right)}, \sqrt{1 - \min \left( 1, \left( \sum_{i=1}^m \beta_i \cdot \left( \frac{1}{(1 - \vartheta_{\Theta(i)}^2)^\Gamma} \right)^{1/\Gamma} \right)} \right)} \right), \\
 &\quad \left. \left( \sqrt{1 - \min \left( 1, \left( \sum_{i=1}^m \beta_i \cdot \left( \frac{1}{(1 - \kappa_{\Theta(i)}^2)^\Gamma} \right)^{1/\Gamma} \right)} \right)}, \sqrt{1 - \min \left( 1, \left( \sum_{i=1}^m \beta_i \cdot \left( \frac{1}{(1 - \Omega_{\Theta(i)}^2)^\Gamma} \right)^{1/\Gamma} \right)} \right)} \right) \right).
 \end{aligned}$$

**Proof** The verification of the validity of the theorem is accomplished by employing mathematical induction.

Let  $m = 2$ . We obtain the following equation by performing the operations developed for CSFNs:

$$\beta_1(\tilde{v})_{\Theta(1)} = \left( \begin{array}{l} \left( \sqrt{\min \left( 1, \left( \beta_1 \cdot \zeta_{\Theta(1)}^{2\Gamma} \right)^{1/\Gamma} \right)}, \sqrt{\min \left( 1, \left( \beta_1 \cdot \varpi_{\Theta(1)}^{2\Gamma} \right)^{1/\Gamma} \right)} \right), \\ \left( \sqrt{1 - \min \left( 1, \left( \beta_1 \cdot \left( 1 - \eta_{\Theta(1)}^2 \right)^\Gamma \right)^{1/\Gamma} \right)}, \sqrt{1 - \min \left( 1, \left( \beta_1 \cdot \left( 1 - \vartheta_{\Theta(1)}^2 \right)^\Gamma \right)^{1/\Gamma} \right)} \right), \\ \left( \sqrt{1 - \min \left( 1, \left( \beta_1 \cdot \left( 1 - \kappa_{\Theta(1)}^2 \right)^\Gamma \right)^{1/\Gamma} \right)}, \sqrt{1 - \min \left( 1, \left( \beta_1 \cdot \left( 1 - \Omega_{\Theta(1)}^2 \right)^\Gamma \right)^{1/\Gamma} \right)} \right) \end{array} \right)$$

and

$$\beta_2(\tilde{v})_{\Theta(2)} = \left( \begin{array}{l} \left( \sqrt{\min \left( 1, \left( \beta_2 \cdot \zeta_{\Theta(2)}^{2\Gamma} \right)^{1/\Gamma} \right)}, \sqrt{\min \left( 1, \left( \beta_2 \cdot \varpi_{\Theta(2)}^{2\Gamma} \right)^{1/\Gamma} \right)} \right), \\ \left( \sqrt{1 - \min \left( 1, \left( \beta_2 \cdot \left( 1 - \eta_{\Theta(2)}^2 \right)^\Gamma \right)^{1/\Gamma} \right)}, \sqrt{1 - \min \left( 1, \left( \beta_2 \cdot \left( 1 - \vartheta_{\Theta(2)}^2 \right)^\Gamma \right)^{1/\Gamma} \right)} \right), \\ \left( \sqrt{1 - \min \left( 1, \left( \beta_2 \cdot \left( 1 - \kappa_{\Theta(2)}^2 \right)^\Gamma \right)^{1/\Gamma} \right)}, \sqrt{1 - \min \left( 1, \left( \beta_2 \cdot \left( 1 - \Omega_{\Theta(2)}^2 \right)^\Gamma \right)^{1/\Gamma} \right)} \right) \end{array} \right)$$

According to the Definition 12, the combined values of  $(\tilde{v})_1$  and  $(\tilde{v})_2$  are determined in the following manner:

$$\begin{aligned} \text{CSFYOWA}((\tilde{v})_1, (\tilde{v})_2) &= \beta_1(\tilde{v})_{\Theta(1)} \oplus \beta_2(\tilde{v})_{\Theta(2)} \\ &= \left( \begin{array}{l} \left( \sqrt{\min \left( 1, \left( \beta_1 \cdot \zeta_{\Theta(1)}^{2\Gamma} \right)^{1/\Gamma} \right)}, \sqrt{\min \left( 1, \left( \beta_1 \cdot \varpi_{\Theta(1)}^{2\Gamma} \right)^{1/\Gamma} \right)} \right), \\ \left( \sqrt{1 - \min \left( 1, \left( \beta_1 \cdot \left( 1 - \eta_{\Theta(1)}^2 \right)^\Gamma \right)^{1/\Gamma} \right)}, \sqrt{1 - \min \left( 1, \left( \beta_1 \cdot \left( 1 - \vartheta_{\Theta(1)}^2 \right)^\Gamma \right)^{1/\Gamma} \right)} \right), \\ \left( \sqrt{1 - \min \left( 1, \left( \beta_1 \cdot \left( 1 - \kappa_{\Theta(1)}^2 \right)^\Gamma \right)^{1/\Gamma} \right)}, \sqrt{1 - \min \left( 1, \left( \beta_1 \cdot \left( 1 - \Omega_{\Theta(1)}^2 \right)^\Gamma \right)^{1/\Gamma} \right)} \right) \end{array} \right) \\ &\oplus \left( \begin{array}{l} \left( \sqrt{\min \left( 1, \left( \beta_2 \cdot \zeta_{\Theta(2)}^{2\Gamma} \right)^{1/\Gamma} \right)}, \sqrt{\min \left( 1, \left( \beta_2 \cdot \varpi_{\Theta(2)}^{2\Gamma} \right)^{1/\Gamma} \right)} \right), \\ \left( \sqrt{1 - \min \left( 1, \left( \beta_2 \cdot \left( 1 - \eta_{\Theta(2)}^2 \right)^\Gamma \right)^{1/\Gamma} \right)}, \sqrt{1 - \min \left( 1, \left( \beta_2 \cdot \left( 1 - \vartheta_{\Theta(2)}^2 \right)^\Gamma \right)^{1/\Gamma} \right)} \right), \\ \left( \sqrt{1 - \min \left( 1, \left( \beta_2 \cdot \left( 1 - \kappa_{\Theta(2)}^2 \right)^\Gamma \right)^{1/\Gamma} \right)}, \sqrt{1 - \min \left( 1, \left( \beta_2 \cdot \left( 1 - \Omega_{\Theta(2)}^2 \right)^\Gamma \right)^{1/\Gamma} \right)} \right) \end{array} \right) \end{aligned}$$

It implies that

$$\begin{aligned} \text{CSFYOWA}((\tilde{v})_1, (\tilde{v})_2) &= \left( \begin{array}{l} \left( \sqrt{\min \left( 1, \left( \sum_{i=1}^2 \beta_i \cdot \zeta_{\Theta(i)}^{2\Gamma} \right)^{1/\Gamma} \right)}, \sqrt{\min \left( 1, \left( \sum_{i=1}^2 \beta_i \cdot \varpi_{\Theta(i)}^{2\Gamma} \right)^{1/\Gamma} \right)} \right), \\ \left( \sqrt{1 - \min \left( 1, \left( \sum_{i=1}^2 \beta_i \cdot \left( 1 - \eta_{\Theta(i)}^2 \right)^\Gamma \right)^{1/\Gamma} \right)}, \sqrt{1 - \min \left( 1, \left( \sum_{i=1}^2 \beta_i \cdot \left( 1 - \vartheta_{\Theta(i)}^2 \right)^\Gamma \right)^{1/\Gamma} \right)} \right), \\ \left( \sqrt{1 - \min \left( 1, \left( \sum_{i=1}^2 \beta_i \cdot \left( 1 - \kappa_{\Theta(i)}^2 \right)^\Gamma \right)^{1/\Gamma} \right)}, \sqrt{1 - \min \left( 1, \left( \sum_{i=1}^2 \beta_i \cdot \left( 1 - \Omega_{\Theta(i)}^2 \right)^\Gamma \right)^{1/\Gamma} \right)} \right) \end{array} \right). \end{aligned}$$

Therefore, the equation holds true for  $m = 2$ .

Assuming that the outcome remains valid for  $m = k$ , we derive:

$$\begin{aligned} \text{CSFYOWA}((\tilde{v})_1, (\tilde{v})_2, (\tilde{v})_3, \dots, (\tilde{v})_k) &= \oplus_{i=1}^k (\beta_i(\tilde{v})_{\Theta(i)}) \\ &= \left( \left( \sqrt{\min \left( 1, \left( \sum_{i=1}^k \beta_i \cdot \zeta_{\Theta(i)}^{2\Gamma} \right)^{1/\Gamma} \right)}, \sqrt{\min \left( 1, \left( \sum_{i=1}^k \beta_i \cdot \varpi_{\Theta(i)}^{2\Gamma} \right)^{1/\Gamma} \right)} \right), \right. \\ &\quad \left( \sqrt{1 - \min \left( 1, \left( \frac{\sum_{i=1}^k \beta_i}{(1 - \eta_{\Theta(i)}^2)^\Gamma} \right)^{1/\Gamma} \right)}, \sqrt{1 - \min \left( 1, \left( \frac{\sum_{i=1}^k \beta_i}{(1 - \vartheta_{\Theta(i)}^2)^\Gamma} \right)^{1/\Gamma} \right)} \right), \right. \\ &\quad \left. \left( \sqrt{1 - \min \left( 1, \left( \frac{\sum_{i=1}^k \beta_i}{(1 - \kappa_{\Theta(i)}^2)^\Gamma} \right)^{1/\Gamma} \right)}, \sqrt{1 - \min \left( 1, \left( \frac{\sum_{i=1}^k \beta_i}{(1 - \Omega_{\Theta(i)}^2)^\Gamma} \right)^{1/\Gamma} \right)} \right) \right) \end{aligned}$$

Now,

$$\begin{aligned} \text{CSFYOWA}((\tilde{v})_1, (\tilde{v})_2, (\tilde{v})_3, \dots, (\tilde{v})_k, (\tilde{v})_{k+1}) &= (\beta_1(\tilde{v})_{\Theta(1)}) \oplus (\beta_2(\tilde{v})_{\Theta(2)}) \oplus (\beta_3(\tilde{v})_{\Theta(3)}) \oplus \dots \oplus \\ &(\beta_k(\tilde{v})_{\Theta(k)}) \oplus (\beta_{k+1}(\tilde{v})_{\Theta(k+1)}) \\ &= \oplus_{i=1}^k (\beta_i(\tilde{v})_{\Theta(i)}) \oplus (\beta_{k+1}(\tilde{v})_{\Theta(k+1)}) \\ &= \left( \left( \sqrt{\min \left( 1, \left( \sum_{i=1}^k \beta_i \cdot \zeta_{\Theta(i)}^{2\Gamma} \right)^{1/\Gamma} \right)}, \sqrt{\min \left( 1, \left( \sum_{i=1}^k \beta_i \cdot \varpi_{\Theta(i)}^{2\Gamma} \right)^{1/\Gamma} \right)} \right), \right. \\ &\quad \left( \sqrt{1 - \min \left( 1, \left( \frac{\sum_{i=1}^k \beta_i}{(1 - \eta_{\Theta(i)}^2)^\Gamma} \right)^{1/\Gamma} \right)}, \sqrt{1 - \min \left( 1, \left( \frac{\sum_{i=1}^k \beta_i}{(1 - \vartheta_{\Theta(i)}^2)^\Gamma} \right)^{1/\Gamma} \right)} \right), \right. \\ &\quad \left( \sqrt{1 - \min \left( 1, \left( \frac{\sum_{i=1}^k \beta_i}{(1 - \kappa_{\Theta(i)}^2)^\Gamma} \right)^{1/\Gamma} \right)}, \sqrt{1 - \min \left( 1, \left( \frac{\sum_{i=1}^k \beta_i}{(1 - \Omega_{\Theta(i)}^2)^\Gamma} \right)^{1/\Gamma} \right)} \right) \right) \\ &\oplus \left( \left( \sqrt{\min \left( 1, \left( \beta_{k+1} \cdot \zeta_{\Theta(k+1)}^{2\Gamma} \right)^{1/\Gamma} \right)}, \sqrt{\min \left( 1, \left( \beta_{k+1} \cdot \varpi_{\Theta(k+1)}^{2\Gamma} \right)^{1/\Gamma} \right)} \right), \right. \\ &\quad \left( \sqrt{1 - \min \left( 1, \left( \frac{\beta_{k+1}}{(1 - \eta_{\Theta(k+1)}^2)^\Gamma} \right)^{1/\Gamma} \right)}, \sqrt{1 - \min \left( 1, \left( \frac{\beta_{k+1}}{(1 - \vartheta_{\Theta(k+1)}^2)^\Gamma} \right)^{1/\Gamma} \right)} \right), \right. \\ &\quad \left. \left( \sqrt{1 - \min \left( 1, \left( \frac{\beta_{k+1}}{(1 - \kappa_{\Theta(k+1)}^2)^\Gamma} \right)^{1/\Gamma} \right)}, \sqrt{1 - \min \left( 1, \left( \frac{\beta_{k+1}}{(1 - \Omega_{\Theta(k+1)}^2)^\Gamma} \right)^{1/\Gamma} \right)} \right) \right) \end{aligned}$$

Consequently,

$$\begin{aligned} \text{CSFYOWA}((\tilde{v})_1, (\tilde{v})_2, (\tilde{v})_3, \dots, (\tilde{v})_{k+1}) &= \left( \left( \sqrt{\min \left( 1, \left( \sum_{i=1}^{k+1} \beta_i \cdot \zeta_{\Theta(i)}^{2\Gamma} \right)^{1/\Gamma} \right)}, \sqrt{\min \left( 1, \left( \sum_{i=1}^{k+1} \beta_i \cdot \varpi_{\Theta(i)}^{2\Gamma} \right)^{1/\Gamma} \right)} \right), \right. \\ &\quad \left( \sqrt{1 - \min \left( 1, \left( \frac{\sum_{i=1}^{k+1} \beta_i}{(1 - \eta_{\Theta(i)}^2)^\Gamma} \right)^{1/\Gamma} \right)}, \sqrt{1 - \min \left( 1, \left( \frac{\sum_{i=1}^{k+1} \beta_i}{(1 - \vartheta_{\Theta(i)}^2)^\Gamma} \right)^{1/\Gamma} \right)} \right), \right. \\ &\quad \left. \left( \sqrt{1 - \min \left( 1, \left( \frac{\sum_{i=1}^{k+1} \beta_i}{(1 - \kappa_{\Theta(i)}^2)^\Gamma} \right)^{1/\Gamma} \right)}, \sqrt{1 - \min \left( 1, \left( \frac{\sum_{i=1}^{k+1} \beta_i}{(1 - \Omega_{\Theta(i)}^2)^\Gamma} \right)^{1/\Gamma} \right)} \right) \right) \end{aligned}$$

Hence, using mathematical induction, we have shown that the result is valid for all integral values of  $m$ . A further illustration of the previous fact is provided in the subsequent example.

**Example 3** Suppose the four CSFNs,  $(\tilde{v})_1 = ((0.36, 0.7), (0.24, 0.42), (0.52, 0.15))$ ,  $(\tilde{v})_2 = ((0.45, 0.62), (0.28, 0.34), (0.51, 0.23))$ ,  $(\tilde{v})_3 = ((0.5, 0.61), (0.3, 0.18), (0.14, 0.37))$ , and  $(\tilde{v})_4 = ((0.25, 0.47), (0.11, 0.45), (0.5, 0.31))$  and their corresponding weight vector is  $\beta = (0.35, 0.22, 0.31, 0.12)^T$  with the parameter  $\Gamma = 3$ . To amalgamate these values through the CSFYOWA operator, we initially arrange these numbers based on Definition 10, yielding the subsequent information.

$$\&((\tilde{v})_1) = 1.453, \&((\tilde{v})_2) = 1.449, \&((\tilde{v})_3) = 1.483, \&((\tilde{v})_4) = 1.351.$$

In the light of Definition 12, the obtained permuted values of CSFNs are arranged in descending order as follows:  $(\tilde{v})_{\Theta(1)} = ((0.5, 0.61), (0.3, 0.18), (0.14, 0.37))$ ,  $(\tilde{v})_{\Theta(2)} = ((0.36, 0.7), (0.24, 0.42), (0.52, 0.15))$ ,  $(\tilde{v})_{\Theta(3)} = ((0.45, 0.62), (0.28, 0.34), (0.51, 0.23))$ . Then, we have

$$\left(\sum_{i=1}^4 \beta_i \cdot \zeta_{\Theta(i)}^{2\Gamma}\right)^{1/\Gamma} = \left(0.35 (0.5)^{2(3)} + 0.22 (0.36)^{2(3)} + 0.31 (0.45)^{2(3)} + 0.12 (0.25)^{2(3)}\right)^{1/3} = 0.2045.$$

Similarly,

$$\begin{aligned} \left(\sum_{i=1}^4 \beta_i \cdot \varpi_{\Theta(i)}^{2\Gamma}\right)^{1/\Gamma} &= 0.4139, & \left(\sum_{i=1}^4 \beta_i \cdot (1 - \eta_{\Theta(i)}^2)^\Gamma\right)^{1/\Gamma} &= 0.9339, \\ \left(\sum_{i=1}^4 \beta_i \cdot (1 - \vartheta_{\Theta(i)}^2)^\Gamma\right)^{1/\Gamma} &= 0.8834, & \left(\sum_{i=1}^4 \beta_i \cdot (1 - \kappa_{\Theta(i)}^2)^\Gamma\right)^{1/\Gamma} &= 0.8280, \text{ and} \\ \left(\sum_{i=1}^4 \beta_i \cdot (1 - \Omega_{\Theta(i)}^2)^\Gamma\right)^{1/\Gamma} &= 0.9289. \end{aligned}$$

The above discussion is summarized as follows:

$$\text{CSFYOWA}((\tilde{v})_1, (\tilde{v})_2, (\tilde{v})_3, (\tilde{v})_4) = \oplus_{i=1}^4 (\beta_i (\tilde{v})_{\Theta(i)}) = \left( (0.452, 0.643), (0.257, 0.341), (0.415, 0.267) \right)$$

Therefore, we conclude that the outcome of the preceding discourse is a CSFN.

**Theorem 2** (Idempotency) Consider a set of  $m$  number of CSFNs, denoted as  $(\tilde{v})_i = ((\zeta_i, \varpi_i), (\eta_i, \vartheta_i), (\kappa_i, \Omega_i))$ ,  $i = 1, 2, \dots, m$ . In addition, suppose  $(\Theta(1), \Theta(2), \dots, \Theta(m))$  denotes the permutation of  $\{1, 2, \dots, m\}$ , such that  $(\tilde{v})_{\Theta(i-1)} \geq (\tilde{v})_{\Theta(i)}$  and the corresponding weight vector of  $(\tilde{v})_i$  is  $\beta = (\beta_1, \beta_2, \beta_3, \dots, \beta_m)^T$ , where  $0 \leq \beta_i \leq 1$  such that  $\sum_{i=1}^m \beta_i = 1$  and  $\Gamma > 0$ . If  $(\tilde{v})_{\Theta(i)} = (\tilde{v})_o$  for all  $i$ . Then,

$$\text{CSFYOWA}((\tilde{v})_1, (\tilde{v})_2, (\tilde{v})_3, \dots, (\tilde{v})_m) = (\tilde{v})_o$$

**Proof** Given that  $(\tilde{v})_{\Theta(i)} = (\tilde{v})_o = ((\zeta_o, \varpi_o), (\eta_o, \vartheta_o), (\kappa_o, \Omega_o))$ , for all  $i$ . In accordance with definition 5, it follows that  $\zeta_{\Theta(i)} = \zeta_o, \varpi_{\Theta(i)} = \varpi_o, \eta_{\Theta(i)} = \eta_o, \vartheta_{\Theta(i)} = \vartheta_o, \kappa_{\Theta(i)} = \kappa_o$ , and  $\Omega_{\Theta(i)} = \Omega_o$ . The application of above relations  $\zeta_{\Theta(i)}, \varpi_{\Theta(i)}, \eta_{\Theta(i)}, \vartheta_{\Theta(i)}, \kappa_{\Theta(i)}$ , and  $\Omega_{\Theta(i)}$  in Eq. 1, gives that

$$\begin{aligned}
 \text{CSFYOWA}((\tilde{v})_1, (\tilde{v})_2, (\tilde{v})_3, \dots, (\tilde{v})_m) &= \\
 &\left( \left( \sqrt{\min\left(1, \left(\sum_{i=1}^m \beta_i \cdot \zeta_o^{2\Gamma}\right)^{1/\Gamma}\right)}, \sqrt{\min\left(1, \left(\sum_{i=1}^m \beta_i \cdot \varpi_o^{2\Gamma}\right)^{1/\Gamma}\right)} \right), \right. \\
 &\left( \sqrt{1 - \min\left(1, \left(\sum_{i=1}^m \beta_i \cdot (1 - \eta_o^2)^\Gamma\right)^{1/\Gamma}\right)}, \sqrt{1 - \min\left(1, \left(\sum_{i=1}^m \beta_i \cdot (1 - \vartheta_o^2)^\Gamma\right)^{1/\Gamma}\right)} \right), \\
 &\left( \sqrt{1 - \min\left(1, \left(\sum_{i=1}^m \beta_i \cdot (1 - \kappa_o^2)^\Gamma\right)^{1/\Gamma}\right)}, \sqrt{1 - \min\left(1, \left(\sum_{i=1}^m \beta_i \cdot (1 - \Omega_o^2)^\Gamma\right)^{1/\Gamma}\right)} \right) \\
 &= \left( \left( \sqrt{\min\left(1, (\zeta_o^2) \left(\sum_{i=1}^m \beta_i\right)^{1/\Gamma}\right)}, \sqrt{\min\left(1, (\varpi_o^2) \left(\sum_{i=1}^m \beta_i\right)^{1/\Gamma}\right)} \right), \right. \\
 &\left( \sqrt{1 - \min\left(1, (1 - \eta_o^2) \left(\sum_{i=1}^m \beta_i\right)^{1/\Gamma}\right)}, \sqrt{1 - \min\left(1, (1 - \vartheta_o^2) \left(\sum_{i=1}^m \beta_i\right)^{1/\Gamma}\right)} \right), \\
 &\left( \sqrt{1 - \min\left(1, (1 - \kappa_o^2) \left(\sum_{i=1}^m \beta_i\right)^{1/\Gamma}\right)}, \sqrt{1 - \min\left(1, (1 - \Omega_o^2) \left(\sum_{i=1}^m \beta_i\right)^{1/\Gamma}\right)} \right) \\
 &= \left( \left( \sqrt{\min(1, \zeta_o^2)}, \sqrt{\min(1, \varpi_o^2)} \right), \right. \\
 &\left. \left( \sqrt{1 - \min(1, (1 - \eta_o^2))}, \sqrt{1 - \min(1, (1 - \vartheta_o^2))} \right), \right. \\
 &\left. \left( \sqrt{1 - \min(1, (1 - \kappa_o^2))}, \sqrt{1 - \min(1, (1 - \Omega_o^2))} \right) \right) \\
 &= ((\zeta_o, \varpi_o), (\eta_o, \vartheta_o), (\kappa_o, \Omega_o)).
 \end{aligned}$$

It follows that

$$\text{CSFYOWA}((\tilde{v})_1, (\tilde{v})_2, (\tilde{v})_3, \dots, (\tilde{v})_m) = (\tilde{v})_o.$$

**Theorem 3 (Monotonicity)** Assume that  $(\tilde{v})_i = ((\zeta_i, \varpi_i), (\eta_i, \vartheta_i), (\kappa_i, \Omega_i))$  and  $(\tilde{v})_i^* = ((\zeta_i^*, \varpi_i^*), (\eta_i^*, \vartheta_i^*), (\kappa_i^*, \Omega_i^*))$  are two collections of CSFNs, where  $i$  ranges from 1 to  $m$ , and  $\beta = (\beta_1, \beta_2, \beta_3, \dots, \beta_m)^T$  is the corresponding weight vector of these CSFNs, where  $0 \leq \beta_i \leq 1$  such that  $\sum_{i=1}^m \beta_i = 1$  and  $\Gamma > 0$ . Moreover, suppose that  $(\Theta(1), \Theta(2), \dots, \Theta(m))$  denotes the permutation of  $\{1, 2, \dots, m\}$ , such that  $(\tilde{v})_{\Theta(i-1)} \geq (\tilde{v})_{\Theta(i)}$  and  $(\tilde{v})_{\Theta(i-1)}^* \geq (\tilde{v})_{\Theta(i)}^*$ . If  $\zeta_{\Theta(i)} \leq \zeta_{\Theta(i)}^*, \varpi_{\Theta(i)} \leq \varpi_{\Theta(i)}^*, \eta_{\Theta(i)} \leq \eta_{\Theta(i)}^*, \vartheta_{\Theta(i)} \leq \vartheta_{\Theta(i)}^*, \kappa_{\Theta(i)} \geq \kappa_{\Theta(i)}^*$  and  $\Omega_{\Theta(i)} \geq \Omega_{\Theta(i)}^*$ . Then,

$$\text{CSFYOWA}((\tilde{v})_1, (\tilde{v})_2, (\tilde{v})_3, \dots, (\tilde{v})_m) \leq \text{CSFYOWA}((\tilde{v})_1^*, (\tilde{v})_2^*, (\tilde{v})_3^*, \dots, (\tilde{v})_m^*).$$

**Proof** Given

that

$$\zeta_{\Theta(i)} \leq \zeta_{\Theta(i)}^* \Rightarrow \zeta_{\Theta(i)}^{2\Gamma} \leq (\zeta_{\Theta(i)}^*)^{2\Gamma} \Rightarrow \sum_{i=1}^m \beta_i \cdot \zeta_{\Theta(i)}^{2\Gamma} \leq \sum_{i=1}^m \beta_i \cdot (\zeta_{\Theta(i)}^*)^{2\Gamma} \Rightarrow \min\left(1, \left(\sum_{i=1}^m \beta_i \cdot \zeta_{\Theta(i)}^{2\Gamma}\right)^{1/\Gamma}\right) \leq \min\left(1, \left(\sum_{i=1}^m \beta_i \cdot (\zeta_{\Theta(i)}^*)^{2\Gamma}\right)^{1/\Gamma}\right)$$

This dictates that

$$\sqrt{\min\left(1, \left(\sum_{i=1}^m \beta_i \cdot \zeta_{\Theta(i)}^{2\Gamma}\right)^{1/\Gamma}\right)} \leq \sqrt{\min\left(1, \left(\sum_{i=1}^m \beta_i \cdot (\zeta_{\Theta(i)}^*)^{2\Gamma}\right)^{1/\Gamma}\right)} \tag{2}$$

In a similar fashion, we determine the following relations by adopting the above mathematical steps within the frameworks of the inequalities  $\varpi_{\rho(i)} \leq \varpi_{\rho(i)}^*, \eta_{\rho(i)} \leq \eta_{\rho(i)}^*, \vartheta_{\rho(i)} \leq \vartheta_{\rho(i)}^*$  :

$$\sqrt{\min\left(1, \left(\sum_{i=1}^m \beta_i \cdot \varpi_{\Theta(i)}^{2\Gamma}\right)^{1/\Gamma}\right)} \leq \sqrt{\min\left(1, \left(\sum_{i=1}^m \beta_i \cdot (\varpi_{\Theta(i)}^*)^{2\Gamma}\right)^{1/\Gamma}\right)} \tag{3}$$

$$\sqrt{1 - \min\left(1, \left(\sum_{i=1}^m \beta_i \cdot (1 - \eta_{\Theta(i)}^2)^\Gamma\right)^{1/\Gamma}\right)} \leq \sqrt{1 - \min\left(1, \left(\sum_{i=1}^m \beta_i \cdot (1 - (\eta_{\Theta(i)}^*)^2)^\Gamma\right)^{1/\Gamma}\right)} \tag{4}$$

$$\sqrt{1 - \min \left( 1, \left( \sum_{i=1}^m \beta_i \cdot (1 - \vartheta_{\Theta(i)}^2)^\Gamma \right)^{1/\Gamma} \right)} \leq \sqrt{1 - \min \left( 1, \left( \sum_{i=1}^m \beta_i \cdot (1 - (\vartheta_{\Theta(i)}^*)^2)^\Gamma \right)^{1/\Gamma} \right)} \quad (5)$$

Additionally, by considering  $\kappa_{\Theta(i)} \geq \kappa_{\Theta(i)}^*$ , we infer that  $1 - \kappa_{\Theta(i)}^{2\Gamma} \leq 1 - (\kappa_{\Theta(i)}^*)^{2\Gamma} \Rightarrow \min \left( 1, \left( \sum_{i=1}^m \beta_i \cdot (1 - \kappa_{\Theta(i)}^2)^\Gamma \right)^{1/\Gamma} \right) \leq \min \left( 1, \left( \sum_{i=1}^m \beta_i \cdot (1 - (\kappa_{\Theta(i)}^*)^2)^\Gamma \right)^{1/\Gamma} \right)$

Consequently, it follows that:

$$\sqrt{1 - \min \left( 1, \left( \sum_{i=1}^m \beta_i \cdot (1 - \kappa_{\Theta(i)}^2)^\Gamma \right)^{1/\Gamma} \right)} \geq \sqrt{1 - \min \left( 1, \left( \sum_{i=1}^m \beta_i \cdot (1 - (\kappa_{\Theta(i)}^*)^2)^\Gamma \right)^{1/\Gamma} \right)} \quad (6)$$

Similarly, we establish the subsequent expression by adopting the above mathematical procedures within the context of the inequality  $\Omega_{\rho(i)} \geq \Omega_{\rho(i)}^*$ :

$$\sqrt{1 - \min \left( 1, \left( \sum_{i=1}^m \beta_i \cdot (1 - \Omega_{\Theta(i)}^2)^\Gamma \right)^{1/\Gamma} \right)} \geq \sqrt{1 - \min \left( 1, \left( \sum_{i=1}^m \beta_i \cdot (1 - (\Omega_{\Theta(i)}^*)^2)^\Gamma \right)^{1/\Gamma} \right)} \quad (7)$$

Analysis of relations 2 to 7 and application of Definition 5 yields:

$$\text{CSFYOWA}((\tilde{v})_1, (\tilde{v})_2, (\tilde{v})_3, \dots, (\tilde{v})_m) \leq \text{CSFYOWA}((\tilde{v})_1^*, (\tilde{v})_2^*, (\tilde{v})_3^*, \dots, (\tilde{v})_m^*).$$

**Theorem 4 (Boundedness)** Consider a set of CSFNs,  $(\tilde{v})_i = ((\zeta_i, \varpi_i), (\eta_i, \vartheta_i), (\kappa_i, \Omega_i))$ , where  $i$  ranges from 1 to  $m$ , and corresponding weight vector of these CSFNs is  $\beta = (\beta_1, \beta_2, \beta_3, \dots, \beta_m)^T$ , where  $0 \leq \beta_i \leq 1$  such that  $\sum_{i=1}^m \beta_i = 1$  and  $\Gamma > 0$ . Moreover, suppose that  $(\Theta(1), \Theta(2), \dots, \Theta(m))$  is the permutation of  $\{1, 2, \dots, m\}$ , such that  $(\tilde{v})_{\Theta(i-1)} \geq (\tilde{v})_{\Theta(i)}$ . If  $(\tilde{v})^- = \left\{ \left( \min_i \{ \zeta_{\Theta(i)} \}, \min_i \{ \varpi_{\Theta(i)} \} \right), \left( \max_i \{ \eta_{\Theta(i)} \}, \max_i \{ \vartheta_{\Theta(i)} \} \right), \left( \max_i \{ \kappa_{\Theta(i)} \}, \max_i \{ \Omega_{\Theta(i)} \} \right) \right\}$  and  $(\tilde{v})^+ = \left\{ \left( \max_i \{ \zeta_{\Theta(i)} \}, \max_i \{ \varpi_{\Theta(i)} \} \right), \left( \min_i \{ \eta_{\Theta(i)} \}, \min_i \{ \vartheta_{\Theta(i)} \} \right), \left( \min_i \{ \kappa_{\Theta(i)} \}, \min_i \{ \Omega_{\Theta(i)} \} \right) \right\}$

Then,

$$(\tilde{v})^- \leq \text{CSFYOWA}((\tilde{v})_1, (\tilde{v})_2, (\tilde{v})_3, \dots, (\tilde{v})_m) \leq (\tilde{v})^+$$

**Proof** Consider

$$\text{CSFYOWA}((\tilde{v})_1, (\tilde{v})_2, (\tilde{v})_3, \dots, (\tilde{v})_m) = ((\zeta, \varpi), (\eta, \vartheta), (\kappa, \Omega)).$$

For each  $(\tilde{v})_i, \min_i \{ \zeta_{\Theta(i)} \} \leq \zeta_{\Theta(i)} \leq \max_i \{ \zeta_{\Theta(i)} \}$

$$\begin{aligned} &\Rightarrow \left( \min_i \{ \zeta_{\Theta(i)} \} \right)^2 \leq \zeta_{\Theta(i)}^2 \leq \left( \max_i \{ \zeta_{\Theta(i)} \} \right)^2 \Rightarrow \left( \sum_{i=1}^m \beta_i \cdot \left( \min_i \{ \zeta_{\Theta(i)} \} \right)^{2\Gamma} \right)^{1/\Gamma} \\ &\leq \left( \sum_{i=1}^m \beta_i \cdot \zeta_{\Theta(i)}^{2\Gamma} \right)^{1/\Gamma} \leq \left( \sum_{i=1}^m \beta_i \cdot \left( \max_i \{ \zeta_{\Theta(i)} \} \right)^{2\Gamma} \right)^{1/\Gamma} \\ &\Rightarrow \sqrt{1 - \min \left( 1, \left( \sum_{i=1}^m \beta_i \cdot \left( \min_i \{ \zeta_{\Theta(i)} \} \right)^{2\Gamma} \right)^{1/\Gamma} \right)} \leq \sqrt{1 - \min \left( 1, \left( \sum_{i=1}^m \beta_i \cdot \zeta_{\Theta(i)}^{2\Gamma} \right)^{1/\Gamma} \right)} \\ &\leq \sqrt{1 - \min \left( 1, \left( \sum_{i=1}^m \beta_i \cdot \left( \max_i \{ \zeta_{\Theta(i)} \} \right)^{2\Gamma} \right)^{1/\Gamma} \right)} \end{aligned}$$

Because,  $\sum_{i=1}^m \beta_i = 1$ , we get:

$$\begin{aligned} &\Rightarrow \sqrt{\min \left( 1, \left( \left( \min_i \{ \zeta_{\Theta(i)} \} \right)^{2\Gamma} \right)^{1/\Gamma} \right)} \leq \sqrt{\min \left( 1, (\zeta^{2\Gamma})^{1/\Gamma} \right)} \\ &\leq \sqrt{\min \left( 1, \left( \left( \max_i \{ \zeta_{\Theta(i)} \} \right)^{2\Gamma} \right)^{1/\Gamma} \right)} \\ &\Rightarrow \min \left( 1, \left( \left( \min_i \{ \zeta_{\Theta(i)} \} \right)^{2\Gamma} \right)^{1/\Gamma} \right) \leq \min \left( 1, (\zeta^{2\Gamma})^{1/\Gamma} \right) \leq \min \left( 1, \left( \left( \max_i \{ \zeta_{\Theta(i)} \} \right)^{2\Gamma} \right)^{1/\Gamma} \right) \\ &\Rightarrow \left( \min_i \{ \zeta_{\Theta(i)} \} \right)^2 \leq \zeta^2 \leq \left( \max_i \{ \zeta_{\Theta(i)} \} \right)^2 \end{aligned}$$

Hence,

$$\min_i \{ \zeta_{\Theta(i)} \} \leq \zeta \leq \max_i \{ \zeta_{\Theta(i)} \} \# (8)$$

Similarly, the subsequent relation is established through the application of the aforementioned mathematical procedures in the context of the inequality:

$$\min_i \{ \varpi_{\Theta(i)} \} \leq \varpi \leq \max_i \{ \varpi_{\Theta(i)} \} \# (9)$$

Furthermore,

$$\begin{aligned} &\max_i \{ \eta_{\Theta(i)} \} \leq \eta_{\Theta(i)} \leq \min_i \{ \eta_{\Theta(i)} \} \Rightarrow \left( \max_i \{ \eta_{\Theta(i)} \} \right)^2 \leq (\eta_{\Theta(i)})^2 \leq \left( \min_i \{ \eta_{\Theta(i)} \} \right)^2 \\ &\Rightarrow 1 - \left( \min_i \{ \eta_{\Theta(i)} \} \right)^2 \leq 1 - (\eta_{\Theta(i)})^2 \leq 1 - \left( \max_i \{ \eta_{\Theta(i)} \} \right)^2 \\ &\Rightarrow \left( \sum_{i=1}^m \beta_i \cdot \left( 1 - \left( \min_i \{ \eta_{\Theta(i)} \} \right)^2 \right)^\Gamma \right)^{1/\Gamma} \leq \left( \sum_{i=1}^m \beta_i \cdot \left( 1 - (\eta_{\Theta(i)})^2 \right)^\Gamma \right)^{1/\Gamma} \\ &\leq \left( \sum_{i=1}^m \beta_i \cdot \left( 1 - \left( \max_i \{ \eta_{\Theta(i)} \} \right)^2 \right)^\Gamma \right)^{1/\Gamma} \\ &\Rightarrow \min \left( 1, \left( \sum_{i=1}^m \beta_i \cdot \left( 1 - \left( \min_i \{ \eta_{\Theta(i)} \} \right)^2 \right)^\Gamma \right)^{1/\Gamma} \right) \leq \min \left( 1, \left( \sum_{i=1}^m \beta_i \cdot \left( 1 - (\eta_{\Theta(i)})^2 \right)^\Gamma \right)^{1/\Gamma} \right) \\ &\leq \min \left( 1, \left( \sum_{i=1}^m \beta_i \cdot \left( 1 - \left( \max_i \{ \eta_{\Theta(i)} \} \right)^2 \right)^\Gamma \right)^{1/\Gamma} \right) \\ &\Rightarrow \sqrt{1 - \min \left( 1, \left( \sum_{i=1}^m \beta_i \cdot \left( 1 - \left( \max_i \{ \eta_{\Theta(i)} \} \right)^2 \right)^\Gamma \right)^{1/\Gamma} \right)} \\ &\leq \sqrt{1 - \min \left( 1, \left( \sum_{i=1}^m \beta_i \cdot \left( 1 - (\eta_{\Theta(i)})^2 \right)^\Gamma \right)^{1/\Gamma} \right)} \\ &\leq \sqrt{1 - \min \left( 1, \left( \sum_{i=1}^m \beta_i \cdot \left( 1 - \left( \min_i \{ \eta_{\Theta(i)} \} \right)^2 \right)^\Gamma \right)^{1/\Gamma} \right)} \end{aligned}$$

Because  $\sum_{i=1}^m \beta_i = 1$ , we conclude that

$$\begin{aligned}
 &\Rightarrow \sqrt{1 - \min \left( 1, \left( \left( 1 - \left( \max_i \{ \eta_{\Theta(i)} \} \right)^2 \right)^\Gamma \right)^{1/\Gamma} \right)} \leq \sqrt{1 - \min \left( 1, \left( (1 - \eta^2)^\Gamma \right)^{1/\Gamma} \right)} \\
 &\leq \sqrt{1 - \min \left( 1, \left( \left( 1 - \left( \min_i \{ \eta_{\Theta(i)} \} \right)^2 \right)^\Gamma \right)^{1/\Gamma} \right)} \\
 &\Rightarrow \min \left( 1, \left( \left( 1 - \left( \min_i \{ \eta_{\Theta(i)} \} \right)^2 \right)^\Gamma \right)^{1/\Gamma} \right) \\
 &\leq \min \left( 1, \left( (1 - \eta^2)^\Gamma \right)^{1/\Gamma} \right) \leq \min \left( 1, \left( \left( 1 - \left( \max_i \{ \eta_{\Theta(i)} \} \right)^2 \right)^\Gamma \right)^{1/\Gamma} \right) \\
 &\Rightarrow \left( \left( 1 - \left( \min_i \{ \eta_{\Theta(i)} \} \right)^2 \right)^\Gamma \right)^{1/\Gamma} \leq \left( (1 - \eta^2)^\Gamma \right)^{1/\Gamma} \leq \left( \left( 1 - \left( \max_i \{ \eta_{\Theta(i)} \} \right)^2 \right)^\Gamma \right)^{1/\Gamma} \\
 &\Rightarrow \left( \max_i \{ \eta_{\Theta(i)} \} \right)^2 \leq \eta^2 \leq \left( \min_i \{ \eta_{\Theta(i)} \} \right)^2
 \end{aligned}$$

Hence,

$$\max_i \{ \eta_{\Theta(i)} \} \leq \eta \leq \min_i \{ \eta_{\Theta(i)} \} \# (10)$$

Furthermore, the application of above mathematical steps in the frameworks of the given conditions, yields the following relations

$$\max_i \{ \vartheta_{\Theta(i)} \} \leq \vartheta \leq \min_i \{ \vartheta_{\Theta(i)} \} \# (11)$$

$$\max_i \{ \kappa_{\Theta(i)} \} \leq \kappa \leq \min_i \{ \kappa_{\Theta(i)} \} \# (12)$$

and

$$\max_i \{ \Omega_{\Theta(i)} \} \leq \Omega \leq \min_i \{ \Omega_{\Theta(i)} \} \# (13)$$

Analysis of relations 8 to 13 yields:

$$(\tilde{v})^- \leq \text{CSFYOWA} ((\tilde{v})_1, (\tilde{v})_2, (\tilde{v})_3, \dots, (\tilde{v})_m) \leq (\tilde{v})^+$$

### Fundamental characteristics of CSFYOWG operator

In this subsection, we introduce the CSFYOWG operator. This operator is designed to streamline the aggregation process of CSFNs, thereby providing a comprehensive approach to the ordered weighted geometric operator within the CSF environment. We subsequently establish the fundamental properties of this operator.

**Definition 13** Assume that  $\mathcal{L}$  is a collection of  $m$  number of CSFNs, denoted as  $(\tilde{v})_i = ((\zeta_i, \varpi_i), (\eta_i, \vartheta_i), (\kappa_i, \Omega_i))$  and  $(\Theta(1), \Theta(2), \dots, \Theta(m))$  represents the permutation of  $\{1, 2, \dots, m\}$ , such that  $(\tilde{v})_{\Theta(i-1)} \geq (\tilde{v})_{\Theta(i)} \forall i = 1, 2, \dots, m$ . Moreover, the corresponding weight vector of  $(\tilde{v})_i$  is  $\beta = (\beta_1, \beta_2, \dots, \beta_m)^T$  with  $0 \leq \beta_i \leq 1$  and  $\sum_{i=1}^m \beta_i = 1$ . The CSF Yager ordered weighted geometric operator is a mapping CSFYOWG:  $\mathcal{L}^m \rightarrow \mathcal{L}$ , defined by the following principle:

$$\begin{aligned}
 &\text{CSFYOWG}((\tilde{v})_1, (\tilde{v})_2, (\tilde{v})_3, \dots, (\tilde{v})_m) = \otimes_{i=1}^m (\tilde{v}_{\Theta(i)})^{\beta_i} \\
 &= \left( \left( \left( \sqrt{1 - \min \left( 1, \left( \left( \sum_{i=1}^m \beta_i \cdot \left( 1 - \zeta_{\Theta(i)}^2 \right)^\Gamma \right)^{1/\Gamma} \right)} \right), \sqrt{1 - \min \left( 1, \left( \left( \sum_{i=1}^m \beta_i \cdot \left( 1 - \varpi_{\Theta(i)}^2 \right)^\Gamma \right)^{1/\Gamma} \right)} \right) \right), \right. \\
 &\left. \left( \left( \sqrt{1 - \min \left( 1, \left( \left( \sum_{i=1}^m \beta_i \cdot \left( 1 - \eta_{\Theta(i)}^2 \right)^\Gamma \right)^{1/\Gamma} \right)} \right), \sqrt{1 - \min \left( 1, \left( \left( \sum_{i=1}^m \beta_i \cdot \left( 1 - \vartheta_{\Theta(i)}^2 \right)^\Gamma \right)^{1/\Gamma} \right)} \right) \right), \right. \\
 &\left. \left( \sqrt{\min \left( 1, \left( \sum_{i=1}^m \beta_i \cdot \kappa_{\Theta(i)}^{2\Gamma} \right)^{1/\Gamma} \right)}, \sqrt{\min \left( 1, \left( \sum_{i=1}^m \beta_i \cdot \Omega_{\Theta(i)}^{2\Gamma} \right)^{1/\Gamma} \right)} \right) \right)
 \end{aligned} \# (14)$$

**Theorem 5** Consider the collection of CSFNs, represented by  $(\tilde{v})_i = ((\zeta_i, \varpi_i), (\eta_i, \vartheta_i), (\kappa_i, \Omega_i)), i = 1, 2, \dots, m$  and  $(\Theta(1), \Theta(2), \dots, \Theta(m))$  denotes the permutation of  $\{1, 2, \dots, m\}$ , such that  $(\tilde{v})_{\Theta(i-1)} \geq (\tilde{v})_{\Theta(i)} \forall i = 1, 2, \dots, m$ . In addition, the corresponding weight vector of  $(\tilde{v})_i$  is  $\beta = (\beta_1, \beta_2, \beta_3, \dots, \beta_m)^T$  where  $0 \leq \beta_i \leq 1$  such that  $\sum_{i=1}^m \beta_i = 1$  and  $\Gamma > 0$ . Then, the aggregated value in the structure of CSFYOWG operator is a CSFN and is expressed in the following way:

$$\begin{aligned} \text{CSFYOWG}((\tilde{v})_1, (\tilde{v})_2, (\tilde{v})_3, \dots, (\tilde{v})_m) &= \otimes_{i=1}^m ((\tilde{v})_{\Theta(i)})^{\beta_i} \\ &= \left( \left( \sqrt{\frac{1 - \min \left( 1, \left( \frac{\sum_{i=1}^m \beta_i}{(1 - \zeta_{\Theta(i)}^2)^\Gamma} \right)^{1/\Gamma} \right)}{1 - \min \left( 1, \left( \frac{\sum_{i=1}^m \beta_i}{(1 - \varpi_{\Theta(i)}^2)^\Gamma} \right)^{1/\Gamma} \right)}}, \sqrt{\frac{1 - \min \left( 1, \left( \frac{\sum_{i=1}^m \beta_i}{(1 - \eta_{\Theta(i)}^2)^\Gamma} \right)^{1/\Gamma} \right)}{1 - \min \left( 1, \left( \frac{\sum_{i=1}^m \beta_i}{(1 - \vartheta_{\Theta(i)}^2)^\Gamma} \right)^{1/\Gamma} \right)}}, \right. \\ &\quad \left. \left( \sqrt{\min \left( 1, \left( \frac{\sum_{i=1}^m \beta_i \cdot \kappa_{\Theta(i)}^{2\Gamma}} \right)^{1/\Gamma} \right)}, \sqrt{\min \left( 1, \left( \frac{\sum_{i=1}^m \beta_i \cdot \Omega_{\Theta(i)}^{2\Gamma}} \right)^{1/\Gamma} \right)} \right) \right) \end{aligned}$$

**Proof** The verification of the validity of the theorem is accomplished by employing mathematical induction.

Let  $m = 2$ . We obtain the following equation by performing the operations developed for CSFNs:

$$((\tilde{v})_{\Theta(1)})^{\beta_1} = \left( \left( \sqrt{\frac{1 - \min \left( 1, \left( \beta_1 \cdot (1 - \zeta_{\Theta(1)}^2)^\Gamma \right)^{1/\Gamma} \right)}{1 - \min \left( 1, \left( \beta_1 \cdot (1 - \varpi_{\Theta(1)}^2)^\Gamma \right)^{1/\Gamma} \right)}}, \sqrt{\frac{1 - \min \left( 1, \left( \beta_1 \cdot (1 - \eta_{\Theta(1)}^2)^\Gamma \right)^{1/\Gamma} \right)}{1 - \min \left( 1, \left( \beta_1 \cdot (1 - \vartheta_{\Theta(1)}^2)^\Gamma \right)^{1/\Gamma} \right)}}, \right. \\ \left. \left( \sqrt{\min \left( 1, \left( \beta_1 \cdot \kappa_{\Theta(1)}^{2\Gamma} \right)^{1/\Gamma} \right)}, \sqrt{\min \left( 1, \left( \beta_1 \cdot \Omega_{\Theta(1)}^{2\Gamma} \right)^{1/\Gamma} \right)} \right) \right)$$

and

$$((\tilde{v})_{\Theta(2)})^{\beta_2} = \left( \left( \sqrt{\frac{1 - \min \left( 1, \left( \beta_2 \cdot (1 - \zeta_{\Theta(2)}^2)^\Gamma \right)^{1/\Gamma} \right)}{1 - \min \left( 1, \left( \beta_2 \cdot (1 - \varpi_{\Theta(2)}^2)^\Gamma \right)^{1/\Gamma} \right)}}, \sqrt{\frac{1 - \min \left( 1, \left( \beta_2 \cdot (1 - \eta_{\Theta(2)}^2)^\Gamma \right)^{1/\Gamma} \right)}{1 - \min \left( 1, \left( \beta_2 \cdot (1 - \vartheta_{\Theta(2)}^2)^\Gamma \right)^{1/\Gamma} \right)}}, \right. \\ \left. \left( \sqrt{\min \left( 1, \left( \beta_2 \cdot \kappa_{\Theta(2)}^{2\Gamma} \right)^{1/\Gamma} \right)}, \sqrt{\min \left( 1, \left( \beta_2 \cdot \Omega_{\Theta(2)}^{2\Gamma} \right)^{1/\Gamma} \right)} \right) \right)$$

According to the Definition 13, the combined values of  $(\tilde{v})_1$  and  $(\tilde{v})_2$  are determined in the following manner:

$$\begin{aligned}
 \text{CSFYOWG}((\tilde{v})_1, (\tilde{v})_2) &= ((\tilde{v})_{\Theta(1)})^{\uparrow\text{beta}_1} \otimes ((\tilde{v})_{\Theta(2)})^{\uparrow\text{beta}_2} \\
 &= \left( \left( \sqrt{1 - \min\left(1, \left(\beta_1 \cdot (1 - \zeta_{\Theta(1)}^2)^\Gamma\right)^{1/\Gamma}\right)}, \sqrt{1 - \min\left(1, \left(\beta_1 \cdot (1 - \varpi_{\Theta(1)}^2)^\Gamma\right)^{1/\Gamma}\right)} \right), \right. \\
 &\quad \left( \sqrt{1 - \min\left(1, \left(\beta_1 \cdot (1 - \eta_{\Theta(1)}^2)^\Gamma\right)^{1/\Gamma}\right)}, \sqrt{1 - \min\left(1, \left(\beta_1 \cdot (1 - \vartheta_{\Theta(1)}^2)^\Gamma\right)^{1/\Gamma}\right)} \right), \right. \\
 &\quad \left. \left( \sqrt{\min\left(1, \left(\beta_1 \cdot \kappa_{\Theta(1)}^{2\Gamma}\right)^{1/\Gamma}\right)}, \sqrt{\min\left(1, \left(\beta_1 \cdot \Omega_{\Theta(1)}^{2\Gamma}\right)^{1/\Gamma}\right)} \right) \right) \\
 &\otimes \left( \left( \sqrt{1 - \min\left(1, \left(\beta_2 \cdot (1 - \zeta_{\Theta(2)}^2)^\Gamma\right)^{1/\Gamma}\right)}, \sqrt{1 - \min\left(1, \left(\beta_2 \cdot (1 - \varpi_{\Theta(2)}^2)^\Gamma\right)^{1/\Gamma}\right)} \right), \right. \\
 &\quad \left( \sqrt{1 - \min\left(1, \left(\beta_2 \cdot (1 - \eta_{\Theta(2)}^2)^\Gamma\right)^{1/\Gamma}\right)}, \sqrt{1 - \min\left(1, \left(\beta_2 \cdot (1 - \vartheta_{\Theta(2)}^2)^\Gamma\right)^{1/\Gamma}\right)} \right), \right. \\
 &\quad \left. \left( \sqrt{\min\left(1, \left(\beta_2 \cdot \kappa_{\Theta(2)}^{2\Gamma}\right)^{1/\Gamma}\right)}, \sqrt{\min\left(1, \left(\beta_2 \cdot \Omega_{\Theta(2)}^{2\Gamma}\right)^{1/\Gamma}\right)} \right) \right)
 \end{aligned}$$

It infers that

$$\begin{aligned}
 \text{CSFYOWG}((\tilde{v})_1, (\tilde{v})_2) \\
 &= \left( \left( \sqrt{1 - \min\left(1, \left(\sum_{i=1}^2 \beta_i \cdot (1 - \zeta_{\Theta(i)}^2)^\Gamma\right)^{1/\Gamma}\right)}, \sqrt{1 - \min\left(1, \left(\sum_{i=1}^2 \beta_i \cdot (1 - \varpi_{\Theta(i)}^2)^\Gamma\right)^{1/\Gamma}\right)} \right), \right. \\
 &\quad \left( \sqrt{1 - \min\left(1, \left(\sum_{i=1}^2 \beta_i \cdot (1 - \eta_{\Theta(i)}^2)^\Gamma\right)^{1/\Gamma}\right)}, \sqrt{1 - \min\left(1, \left(\sum_{i=1}^2 \beta_i \cdot (1 - \vartheta_{\Theta(i)}^2)^\Gamma\right)^{1/\Gamma}\right)} \right), \right. \\
 &\quad \left. \left( \sqrt{\min\left(1, \left(\sum_{i=1}^2 \beta_i \cdot \kappa_{\Theta(i)}^{2\Gamma}\right)^{1/\Gamma}\right)}, \sqrt{\min\left(1, \left(\sum_{i=1}^2 \beta_i \cdot \Omega_{\Theta(i)}^{2\Gamma}\right)^{1/\Gamma}\right)} \right) \right)
 \end{aligned}$$

Therefore, the equation holds true for  $m = 2$ .

Assuming that the outcome remains valid for  $m = k$ , we derive:

$$\begin{aligned}
 \text{CSFYOWG}((\tilde{v})_1, (\tilde{v})_2, (\tilde{v})_3, \dots, (\tilde{v})_k) &= \otimes_{i=1}^k ((\tilde{v})_{\Theta(i)})^{\beta_i} \\
 &= \left( \left( \sqrt{1 - \min\left(1, \left(\sum_{i=1}^k \beta_i \cdot (1 - \zeta_{\Theta(i)}^2)^\Gamma\right)^{1/\Gamma}\right)}, \sqrt{1 - \min\left(1, \left(\sum_{i=1}^k \beta_i \cdot (1 - \varpi_{\Theta(i)}^2)^\Gamma\right)^{1/\Gamma}\right)} \right), \right. \\
 &\quad \left( \sqrt{1 - \min\left(1, \left(\sum_{i=1}^k \beta_i \cdot (1 - \eta_{\Theta(i)}^2)^\Gamma\right)^{1/\Gamma}\right)}, \sqrt{1 - \min\left(1, \left(\sum_{i=1}^k \beta_i \cdot (1 - \vartheta_{\Theta(i)}^2)^\Gamma\right)^{1/\Gamma}\right)} \right), \right. \\
 &\quad \left. \left( \sqrt{\min\left(1, \left(\sum_{i=1}^k \beta_i \cdot \kappa_{\Theta(i)}^{2\Gamma}\right)^{1/\Gamma}\right)}, \sqrt{\min\left(1, \left(\sum_{i=1}^k \beta_i \cdot \Omega_{\Theta(i)}^{2\Gamma}\right)^{1/\Gamma}\right)} \right) \right)
 \end{aligned}$$

Moreover, consider

$$\begin{aligned}
 \text{CSFYOWG}((\tilde{v})_1, (\tilde{v})_2, (\tilde{v})_3, \dots, (\tilde{v})_k, (\tilde{v})_{k+1}) &= ((\tilde{v})_{\Theta(1)})^{\uparrow\text{beta}_1} \otimes ((\tilde{v})_{\Theta(2)})^{\uparrow\text{beta}_2} \otimes ((\tilde{v})_{\Theta(3)})^{\uparrow\text{beta}_3} \otimes \dots \otimes \\
 &\quad ((\tilde{v})_{\Theta(k)})^{\uparrow\text{beta}_k} \otimes ((\tilde{v})_{\Theta(k+1)})^{\uparrow\text{beta}_{k+1}} \\
 &= \otimes_{i=1}^k ((\tilde{v})_{\Theta(i)})^{\uparrow\text{beta}_i} \otimes ((\tilde{v})_{\Theta(k+1)})^{\uparrow\text{beta}_{k+1}}
 \end{aligned}$$

$$\begin{aligned}
 &= \left( \left( \sqrt[1/\Gamma]{1 - \min \left( 1, \left( \frac{\sum_{i=1}^k \beta_i}{(1 - \zeta_{\Theta(i)}^2)^\Gamma} \right)^{1/\Gamma} \right)}, \sqrt[1/\Gamma]{1 - \min \left( 1, \left( \frac{\sum_{i=1}^k \beta_i}{(1 - \varpi_{\Theta(i)}^2)^\Gamma} \right)^{1/\Gamma} \right)} \right), \right. \\
 &\quad \left( \sqrt[1/\Gamma]{1 - \min \left( 1, \left( \frac{\sum_{i=1}^k \beta_i}{(1 - \eta_{\Theta(i)}^2)^\Gamma} \right)^{1/\Gamma} \right)}, \sqrt[1/\Gamma]{1 - \min \left( 1, \left( \frac{\sum_{i=1}^k \beta_i}{(1 - \vartheta_{\Theta(i)}^2)^\Gamma} \right)^{1/\Gamma} \right)} \right), \\
 &\quad \left( \sqrt{\min \left( 1, \left( \sum_{i=1}^k \beta_i \cdot \kappa_{\Theta(i)}^{2\Gamma} \right)^{1/\Gamma} \right)}, \sqrt{\min \left( 1, \left( \sum_{i=1}^k \beta_i \cdot \Omega_{\Theta(i)}^{2\Gamma} \right)^{1/\Gamma} \right)} \right) \Bigg) \\
 \otimes &\left( \left( \sqrt[1/\Gamma]{1 - \min \left( 1, \left( \frac{\beta_{k+1}}{(1 - \zeta_{\Theta(k+1)}^2)^\Gamma} \right)^{1/\Gamma} \right)}, \sqrt[1/\Gamma]{1 - \min \left( 1, \left( \frac{\beta_{k+1}}{(1 - \varpi_{\Theta(k+1)}^2)^\Gamma} \right)^{1/\Gamma} \right)} \right), \right. \\
 &\quad \left( \sqrt[1/\Gamma]{1 - \min \left( 1, \left( \frac{\beta_{k+1}}{(1 - \eta_{\Theta(k+1)}^2)^\Gamma} \right)^{1/\Gamma} \right)}, \sqrt[1/\Gamma]{1 - \min \left( 1, \left( \frac{\beta_{k+1}}{(1 - \vartheta_{\Theta(k+1)}^2)^\Gamma} \right)^{1/\Gamma} \right)} \right), \\
 &\quad \left( \sqrt{\min \left( 1, \left( \beta_{k+1} \cdot \kappa_{\Theta(k+1)}^{2\Gamma} \right)^{1/\Gamma} \right)}, \sqrt{\min \left( 1, \left( \beta_{k+1} \cdot \Omega_{\Theta(k+1)}^{2\Gamma} \right)^{1/\Gamma} \right)} \right) \Bigg)
 \end{aligned}$$

It follows that

$$\begin{aligned}
 &\text{CSFYOWG}((\tilde{v})_1, (\tilde{v})_2, (\tilde{v})_3, \dots, (\tilde{v})_{k+1}) \\
 &= \left( \left( \sqrt[1/\Gamma]{1 - \min \left( 1, \left( \frac{\sum_{i=1}^{k+1} \beta_i}{(1 - \zeta_{\Theta(i)}^2)^\Gamma} \right)^{1/\Gamma} \right)}, \sqrt[1/\Gamma]{1 - \min \left( 1, \left( \frac{\sum_{i=1}^{k+1} \beta_i}{(1 - \varpi_{\Theta(i)}^2)^\Gamma} \right)^{1/\Gamma} \right)} \right), \right. \\
 &\quad \left( \sqrt[1/\Gamma]{1 - \min \left( 1, \left( \frac{\sum_{i=1}^{k+1} \beta_i}{(1 - \eta_{\Theta(i)}^2)^\Gamma} \right)^{1/\Gamma} \right)}, \sqrt[1/\Gamma]{1 - \min \left( 1, \left( \frac{\sum_{i=1}^{k+1} \beta_i}{(1 - \vartheta_{\Theta(i)}^2)^\Gamma} \right)^{1/\Gamma} \right)} \right), \\
 &\quad \left( \sqrt{\min \left( 1, \left( \sum_{i=1}^{k+1} \beta_i \cdot \kappa_{\Theta(i)}^{2\Gamma} \right)^{1/\Gamma} \right)}, \sqrt{\min \left( 1, \left( \sum_{i=1}^{k+1} \beta_i \cdot \Omega_{\Theta(i)}^{2\Gamma} \right)^{1/\Gamma} \right)} \right) \Bigg)
 \end{aligned}$$

Therefore, it can be concluded from the preceding that the result is valid for all integral values of  $m$ .

The following illustrated example signifies the previously stated fact.

**Example 4** Consider  $(\tilde{v})_1 = ((0.3, 0.4), (0.37, 0.16), (0.62, 0.22))$ ,  $(\tilde{v})_2 = ((0.53, 0.7), (0.28, 0.31), (0.14, 0.23))$ ,  $(\tilde{v})_3 = ((0.13, 0.25), (0.39, 0.08), (0.54, 0.67))$ , and  $(\tilde{v})_4 = ((0.45, 0.67), (0.31, 0.25), (0.1, 0.51))$  are CSFNs and their corresponding weight vector is  $\beta = (0.35, 0.22, 0.31, 0.12)^T$  with the parameter  $\Gamma = 4$ . To integrate all the above information using the CSFYOWG operator, we first sort these numbers in descending order by means of Definition 10, yielding the subsequent information.

$$\&((\tilde{v})_1) = 1.366, \&((\tilde{v})_2) = 1.524, \&((\tilde{v})_3) = 1.312, \&((\tilde{v})_4) = 1.464.$$

In the view of Definition 13, the obtained permuted values of CSFNs are arranged in descending order as follows:  $(\tilde{v})_{\Theta(1)} = ((0.53, 0.7), (0.28, 0.31), (0.14, 0.23))$ ,  $(\tilde{v})_{\Theta(2)} = ((0.45, 0.67), (0.31, 0.25), (0.1, 0.51))$ ,  $(\tilde{v})_{\Theta(3)} = ((0.3, 0.4), (0.37, 0.16), (0.62, 0.22))$ , and  $(\tilde{v})_{\Theta(4)} = ((0.13, 0.25), (0.39, 0.08), (0.54, 0.67))$ . Consider

$$\begin{aligned}
 \left( \sum_{i=1}^4 \beta_i \cdot (1 - \zeta_{\Theta(i)}^2)^\Gamma \right)^{1/\Gamma} &= \left( 0.35 (1 - (0.53)^2)^4 + 0.22 (1 - (0.45)^2)^4 + 0.31 (1 - (0.3)^2)^4 + 0.12 (1 - (0.13)^2)^4 \right)^{1/4} \\
 &= 0.8439,
 \end{aligned}$$

Similarly,

$$\begin{aligned} \left(\sum_{i=1}^4 \beta_i \cdot (1 - \varpi_{\Theta(i)}^2)^{\Gamma}\right)^{1/\Gamma} &= 0.6627, & \left(\sum_{i=1}^4 \beta_i \cdot (1 - \eta_{\Theta(i)}^2)^{\Gamma}\right)^{1/\Gamma} &= 0.8588, \\ \left(\sum_{i=1}^4 \beta_i \cdot (1 - \vartheta_{\Theta(i)}^2)^{\Gamma}\right)^{1/\Gamma} &= 0.9283, & \left(\sum_{i=1}^4 \beta_i \cdot \kappa_{\Theta(i)}^{2\Gamma}\right)^{1/\Gamma} &= 0.8478, \text{ and} \\ \left(\sum_{i=1}^4 \beta_i \cdot \Omega_{\Theta(i)}^{2\Gamma}\right)^{1/\Gamma} &= 0.8494. \end{aligned}$$

This shows that,

$$\text{CSFYOWG}((\tilde{v})_1, (\tilde{v})_2, (\tilde{v})_3, (\tilde{v})_4) = \otimes_{i=1}^4 ((\tilde{v})_{\Theta(i)})^{\beta_i} = \left( (0.395, 0.581), (0.376, 0.268), (0.921, 0.922) \right).$$

Therefore, we deduce that the outcome of the above computation is a CSFN.

**Theorem 6 (Idempotency)** Consider a set of  $m$  number of CSFNs, denoted as  $(\tilde{v})_i = ((\zeta_i, \varpi_i), (\eta_i, \vartheta_i), (\kappa_i, \Omega_i))$ ,  $i = 1, 2, \dots, m$ , and the corresponding weight vector of  $(\tilde{v})_i$  is  $\beta = (\beta_1, \beta_2, \beta_3, \dots, \beta_m)^T$  where  $0 \leq \beta_i \leq 1$  such that  $\sum_{i=1}^m \beta_i = 1$  and  $\Gamma > 0$ . In addition, suppose  $(\Theta(1), \Theta(2), \dots, \Theta(m))$  is the permutation of  $\{1, 2, \dots, m\}$  such that  $(\tilde{v})_{\Theta(i-1)} \geq (\tilde{v})_{\Theta(i)}$ . If  $(\tilde{v})_{\Theta(i)} = (\tilde{v})_o$  for all  $i$ . Then,

$$\text{CSFYOWG}((\tilde{v})_1, (\tilde{v})_2, (\tilde{v})_3, \dots, (\tilde{v})_m) = (\tilde{v})_o$$

**Proof** This theorem can be proved using the same arguments as Theorem 2.

**Theorem 7 (Monotonicity)** Assume that  $(\tilde{v})_i = ((\zeta_i, \varpi_i), (\eta_i, \vartheta_i), (\kappa_i, \Omega_i))$  and  $(\tilde{v})_i^* = ((\zeta_i^*, \varpi_i^*), (\eta_i^*, \vartheta_i^*), (\kappa_i^*, \Omega_i^*))$  are two collections of CSFNs, where  $i$  ranges from 1 to  $m$  and  $\beta = (\beta_1, \beta_2, \beta_3, \dots, \beta_m)^T$  is the corresponding weight vector of these CSFNs, where  $0 \leq \beta_i \leq 1$  such that  $\sum_{i=1}^m \beta_i = 1$  and  $\Gamma > 0$ . Moreover, suppose  $(\Theta(1), \Theta(2), \dots, \Theta(m))$  denotes the permutation of  $\{1, 2, \dots, m\}$ , such that  $(\tilde{v})_{\Theta(i-1)} \geq (\tilde{v})_{\Theta(i)}$  and  $(\tilde{v})_{\Theta(i-1)}^* \geq (\tilde{v})_{\Theta(i)}^*$ . If  $\zeta_{\Theta(i)} \leq \zeta_{\Theta(i)}^*, \varpi_{\Theta(i)} \leq \varpi_{\Theta(i)}^*, \eta_{\Theta(i)} \leq \eta_{\Theta(i)}^*, \vartheta_{\Theta(i)} \leq \vartheta_{\Theta(i)}^*, \kappa_{\Theta(i)} \geq \kappa_{\Theta(i)}^*$ , and  $\Omega_{\Theta(i)} \geq \Omega_{\Theta(i)}^*$ . Then,

$$\text{CSFYOWG}((\tilde{v})_1, (\tilde{v})_2, (\tilde{v})_3, \dots, (\tilde{v})_m) \leq \text{CSFYOWG}((\tilde{v})_1^*, (\tilde{v})_2^*, (\tilde{v})_3^*, \dots, (\tilde{v})_m^*)$$

**Proof** This theorem can be proved using the same arguments as Theorem 3.

**Theorem 8 (Boundedness)** Consider a set of  $m$  number of CSFNs,  $(\tilde{v})_i = ((\zeta_i, \varpi_i), (\eta_i, \vartheta_i), (\kappa_i, \Omega_i))$ , and the corresponding weight vector of these CSFNs is  $\beta = (\beta_1, \beta_2, \beta_3, \dots, \beta_m)^T$ , where  $0 \leq \beta_i \leq 1$  such that  $\sum_{i=1}^m \beta_i = 1$  and  $\Gamma > 0$ . In addition, suppose  $(\Theta(1), \Theta(2), \dots, \Theta(m))$  is the permutation of  $\{1, 2, \dots, m\}$ , where  $(\tilde{v})_{\Theta(i-1)} \geq (\tilde{v})_{\Theta(i)}$ .

If  $(\tilde{v})^- = \left\{ \left( \min_i \{ \zeta_{\Theta(i)} \}, \min_i \{ \varpi_{\Theta(i)} \} \right), \left( \min_i \{ \eta_{\Theta(i)} \}, \min_i \{ \vartheta_{\Theta(i)} \} \right), \left( \max_i \{ \kappa_{\Theta(i)} \}, \max_i \{ \Omega_{\Theta(i)} \} \right) \right\}$  and  $(\tilde{v})^+ = \left\{ \left( \max_i \{ \zeta_{\Theta(i)} \}, \max_i \{ \varpi_{\Theta(i)} \} \right), \left( \max_i \{ \eta_{\Theta(i)} \}, \max_i \{ \vartheta_{\Theta(i)} \} \right), \left( \min_i \{ \kappa_{\Theta(i)} \}, \min_i \{ \Omega_{\Theta(i)} \} \right) \right\}$ . Then,

$$(\tilde{v})^- \leq \text{CSFYOWG}((\tilde{v})_1, (\tilde{v})_2, (\tilde{v})_3, \dots, (\tilde{v})_m) \leq (\tilde{v})^+.$$

**Proof** This theorem can be proved using the same arguments as Theorem 4.

### Implementation of proposed strategies in MCDM problem

Within this section, we design a systematic methodology to address MCDM issues utilizing the complex spherical fuzzy Yager ordered weighted AOs. We also articulate a sequential procedure to adeptly navigate the MCDM problem.

- For this, suppose  $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \dots, \mathcal{Q}_m)$  is a discrete collection of alternatives and  $(\Theta(1), \Theta(2), \dots, \Theta(m))$  represents the permutation of  $(1, 2, \dots, m)$ , such that  $\mathcal{Q}_{\Theta(i-1)} \geq \mathcal{Q}_{\Theta(i)} \forall i = 1, 2, \dots, m$ .
- Let  $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \dots, \mathcal{C}_n)$  be a set of attributes, each augmented by its corresponding weight vector denoted by  $\beta = (\beta_1, \beta_2, \beta_3, \dots, \beta_n)^T$ , where  $\beta_j \in [0, 1]$  and  $\sum_{j=1}^n \beta_j = 1$ .
- Consider the CSF decision matrix  $\mathcal{D} = (\mathcal{G}_{ij})_{m \times n} = ((\zeta_{ij}, \varpi_{ij}), (\eta_{ij}, \vartheta_{ij}), (\kappa_{ij}, \Omega_{ij}))_{m \times n}$ , where it encapsulates the complex spherical fuzzy data obtained from expert assessments about the finite number of alternatives  $\mathcal{Q}_i$  based on the criteria  $\mathcal{C}_j$ .

The essential steps to address the MCDM problem using newly designed CSFYOWA and CSFYOWG operators are outlined below:

*Step 1.* Construct the CSF decision matrix in light of the knowledge furnished by the decision-makers subsequently:

$$D = \begin{pmatrix} \left( \begin{matrix} (\zeta_{11}, \varpi_{11}), (\eta_{11}, \vartheta_{11}), \\ (\kappa_{11}, \Omega_{11}) \end{matrix} \right) & \left( \begin{matrix} (\zeta_{12}, \varpi_{12}), (\eta_{12}, \vartheta_{12}), \\ (\kappa_{12}, \Omega_{12}) \end{matrix} \right) & \dots & \left( \begin{matrix} (\zeta_{1n}, \varpi_{1n}), (\eta_{1n}, \vartheta_{1n}), \\ (\kappa_{1n}, \Omega_{1n}) \end{matrix} \right) \\ \left( \begin{matrix} (\zeta_{21}, \varpi_{21}), (\eta_{21}, \vartheta_{21}), \\ (\kappa_{21}, \Omega_{21}) \end{matrix} \right) & \left( \begin{matrix} (\zeta_{22}, \varpi_{22}), (\eta_{22}, \vartheta_{22}), \\ (\kappa_{22}, \Omega_{22}) \end{matrix} \right) & \dots & \left( \begin{matrix} (\zeta_{2n}, \varpi_{2n}), (\eta_{2n}, \vartheta_{2n}), \\ (\kappa_{2n}, \Omega_{2n}) \end{matrix} \right) \\ \vdots & \vdots & \ddots & \vdots \\ \left( \begin{matrix} (\zeta_{m1}, \varpi_{m1}), (\eta_{m1}, \vartheta_{m1}), \\ (\kappa_{m1}, \Omega_{m1}) \end{matrix} \right) & \left( \begin{matrix} (\zeta_{m2}, \varpi_{m2}), (\eta_{m2}, \vartheta_{m2}), \\ (\kappa_{m2}, \Omega_{m2}) \end{matrix} \right) & \dots & \left( \begin{matrix} (\zeta_{mn}, \varpi_{mn}), (\eta_{mn}, \vartheta_{mn}), \\ (\kappa_{mn}, \Omega_{mn}) \end{matrix} \right) \end{pmatrix},$$

where the rows of matrix represent the attributes and columns of matrix represent the alternatives. Moreover, each entry shows CSFNs in the matrix.

**Step 2.** In order to attain the CSF permuted decision matrix  $D = (\mathcal{G}_{\Theta(ij)})_{m \times n} = ((\zeta_{\Theta(ij)}, \varpi_{\Theta(ij)}), (\eta_{\Theta(ij)}, \vartheta_{\Theta(ij)}), (\kappa_{\Theta(ij)}, \Omega_{\Theta(ij)}))_{m \times n}$ , we implement the following two stages:

Compute the score values  $\Lambda_i$  for all criteria  $C_j$  associated with each alternative  $Q_i$  using the principles, outlined in Definition 10.

For instance, calculate the score value  $\Lambda_1$  of criterion  $C_1$  corresponding to alternative  $Q_1 = ((\zeta_1, \varpi_1), (\eta_1, \vartheta_1), (\kappa_1, \Omega_1))$  using Definition 10 as follows:

$$\Lambda_1 = \frac{1}{3} [4 + ((1 - \kappa_1^2) (\zeta_1^2 - \eta_1^2) + (1 - \Omega_1^2) (\varpi_1^2 - \vartheta_1^2))],$$

Similarly, we compute the score values  $\Lambda_i$  corresponding to remaining criteria  $C_j (j > 1)$  associated to the alternative  $Q_1$  by using Definition 10. The outcomes of this mathematical procedure are represented as  $\Lambda_1, \Lambda_2, \dots, \Lambda_m$ .

Likewise, by adopting the above mathematical procedure, we can compute the score values of each criterion  $C_j$  corresponding to the remaining alternatives  $Q_i$  where  $i > 1$ .

Arrange the obtained values from the above stage of all criterion  $C_j$ , corresponding to each alternative  $Q_i$  in descending order to formulate a CSF permuted decision matrix  $D = (\mathcal{G}_{\Theta(ij)})_{m \times n} = ((\zeta_{\Theta(ij)}, \varpi_{\Theta(ij)}), (\eta_{\Theta(ij)}, \vartheta_{\Theta(ij)}), (\kappa_{\Theta(ij)}, \Omega_{\Theta(ij)}))_{m \times n}$  based on this information.

**Step 3. (a)** Aggregate all the preference values  $\delta_i$  of the alternatives  $Q_i$  by means of CSFYOWA operator in the following way:

$$\begin{aligned} \delta_i &= \text{CSFYOWA}(\mathcal{G}_{\Theta(i1)}, \mathcal{G}_{\Theta(i2)}, \dots, \mathcal{G}_{\Theta(in)}) \\ &= \left( \left( \sqrt{\min \left( 1, \left( \sum_{j=1}^n \beta_j \cdot \zeta_{\Theta(ij)}^{2\Gamma} \right)^{1/\Gamma} \right)}, \sqrt{\min \left( 1, \left( \sum_{j=1}^n \beta_j \cdot \varpi_{\Theta(ij)}^{2\Gamma} \right)^{1/\Gamma} \right)} \right), \right. \\ &\quad \left( \left( \sqrt{1 - \min \left( 1, \left( \frac{\sum_{j=1}^n \beta_j \cdot \eta_{\Theta(ij)}^2}{(1 - \eta_{\Theta(ij)}^2)^\Gamma} \right)^{1/\Gamma} \right)}, \sqrt{1 - \min \left( 1, \left( \frac{\sum_{j=1}^n \beta_j \cdot \vartheta_{\Theta(ij)}^2}{(1 - \vartheta_{\Theta(ij)}^2)^\Gamma} \right)^{1/\Gamma} \right)} \right), \right. \\ &\quad \left. \left( \left( \sqrt{1 - \min \left( 1, \left( \frac{\sum_{j=1}^n \beta_j \cdot \kappa_{\Theta(ij)}^2}{(1 - \kappa_{\Theta(ij)}^2)^\Gamma} \right)^{1/\Gamma} \right)}, \sqrt{1 - \min \left( 1, \left( \frac{\sum_{j=1}^n \beta_j \cdot \Omega_{\Theta(ij)}^2}{(1 - \Omega_{\Theta(ij)}^2)^\Gamma} \right)^{1/\Gamma} \right)} \right) \right) \end{aligned}.$$

For instance, if  $i = 1$ , then we obtain the preference value  $\delta_1$  corresponding to the alternative  $Q_1$  as follows:

$$\begin{aligned} \delta_1 &= \text{CSFYOWA}(\mathcal{G}_{\Theta(11)}, \mathcal{G}_{\Theta(12)}, \dots, \mathcal{G}_{\Theta(1n)}) \\ &= \left( \left( \sqrt{\min \left( 1, \left( \sum_{j=1}^n \beta_j \cdot \zeta_{\Theta(1j)}^{2\Gamma} \right)^{1/\Gamma} \right)}, \sqrt{\min \left( 1, \left( \sum_{j=1}^n \beta_j \cdot \varpi_{\Theta(1j)}^{2\Gamma} \right)^{1/\Gamma} \right)} \right), \right. \\ &\quad \left( \left( \sqrt{1 - \min \left( 1, \left( \frac{\sum_{j=1}^n \beta_j \cdot \eta_{\Theta(1j)}^2}{(1 - \eta_{\Theta(1j)}^2)^\Gamma} \right)^{1/\Gamma} \right)}, \sqrt{1 - \min \left( 1, \left( \frac{\sum_{j=1}^n \beta_j \cdot \vartheta_{\Theta(1j)}^2}{(1 - \vartheta_{\Theta(1j)}^2)^\Gamma} \right)^{1/\Gamma} \right)} \right), \right. \\ &\quad \left. \left( \left( \sqrt{1 - \min \left( 1, \left( \frac{\sum_{j=1}^n \beta_j \cdot \kappa_{\Theta(1j)}^2}{(1 - \kappa_{\Theta(1j)}^2)^\Gamma} \right)^{1/\Gamma} \right)}, \sqrt{1 - \min \left( 1, \left( \frac{\sum_{j=1}^n \beta_j \cdot \Omega_{\Theta(1j)}^2}{(1 - \Omega_{\Theta(1j)}^2)^\Gamma} \right)^{1/\Gamma} \right)} \right) \right) \end{aligned}.$$

Similarly, by adopting the above mathematical procedure, we can compute the preference values  $\delta_i$  corresponding to each alternative  $Q_i$ , where  $i > 1$ .

(b) Find the aggregated preference values  $\delta_i$  of all alternatives  $Q_i$  employing CSFYOWG operator as follows:

$$\delta_i = \text{CSFYOWG}(\mathcal{G}_{\Theta(i1)}, \mathcal{G}_{\Theta(i2)}, \dots, \mathcal{G}_{\Theta(im)})$$

$$= \left( \left( \sqrt[1/\Gamma]{1 - \min \left( 1, \left( \frac{\sum_{j=1}^n \beta_j}{(1 - \zeta_{\Theta(ij)}^2)^\Gamma} \right)^{1/\Gamma} \right)}, \sqrt[1/\Gamma]{1 - \min \left( 1, \left( \frac{\sum_{j=1}^n \beta_j}{(1 - \varpi_{\Theta(ij)}^2)^\Gamma} \right)^{1/\Gamma} \right)} \right), \left( \sqrt[1/\Gamma]{1 - \min \left( 1, \left( \frac{\sum_{j=1}^n \beta_j}{(1 - \eta_{\Theta(ij)}^2)^\Gamma} \right)^{1/\Gamma} \right)}, \sqrt[1/\Gamma]{1 - \min \left( 1, \left( \frac{\sum_{j=1}^n \beta_j}{(1 - \vartheta_{\Theta(ij)}^2)^\Gamma} \right)^{1/\Gamma} \right)} \right), \left( \sqrt[1/\Gamma]{\min \left( 1, \left( \frac{\sum_{j=1}^n \beta_j \cdot \kappa_{\Theta(ij)}^{2\Gamma}}{\Omega_{\Theta(ij)}^{2\Gamma}} \right)^{1/\Gamma} \right)}, \sqrt[1/\Gamma]{\min \left( 1, \left( \frac{\sum_{j=1}^n \beta_j \cdot \Omega_{\Theta(ij)}^{2\Gamma}}{\Omega_{\Theta(ij)}^{2\Gamma}} \right)^{1/\Gamma} \right)} \right) \right)$$

For instance, if  $i = 1$ , then we obtain the preference value  $\delta_1$  corresponding to the alternative  $Q_1$  as follows:

$$\delta_1 = \text{CSFYOWG}(\mathcal{G}_{\Theta(11)}, \mathcal{G}_{\Theta(12)}, \dots, \mathcal{G}_{\Theta(1n)})$$

$$= \left( \left( \sqrt[1/\Gamma]{1 - \min \left( 1, \left( \frac{\sum_{j=1}^n \beta_j}{(1 - \zeta_{\Theta(1j)}^2)^\Gamma} \right)^{1/\Gamma} \right)}, \sqrt[1/\Gamma]{1 - \min \left( 1, \left( \frac{\sum_{j=1}^n \beta_j}{(1 - \varpi_{\Theta(1j)}^2)^\Gamma} \right)^{1/\Gamma} \right)} \right), \left( \sqrt[1/\Gamma]{1 - \min \left( 1, \left( \frac{\sum_{j=1}^n \beta_j}{(1 - \eta_{\Theta(1j)}^2)^\Gamma} \right)^{1/\Gamma} \right)}, \sqrt[1/\Gamma]{1 - \min \left( 1, \left( \frac{\sum_{j=1}^n \beta_j}{(1 - \vartheta_{\Theta(1j)}^2)^\Gamma} \right)^{1/\Gamma} \right)} \right), \left( \sqrt[1/\Gamma]{\min \left( 1, \left( \frac{\sum_{j=1}^n \beta_j \cdot \kappa_{\Theta(1j)}^{2\Gamma}}{\Omega_{\Theta(1j)}^{2\Gamma}} \right)^{1/\Gamma} \right)}, \sqrt[1/\Gamma]{\min \left( 1, \left( \frac{\sum_{j=1}^n \beta_j \cdot \Omega_{\Theta(1j)}^{2\Gamma}}{\Omega_{\Theta(1j)}^{2\Gamma}} \right)^{1/\Gamma} \right)} \right) \right)$$

Likewise, by employing the above mathematical procedure, we can compute the preference values  $\delta_i$  corresponding to each alternative  $Q_i$ , where  $i > 1$ .

**Step 4.** Compute the score values  $\&(\delta_i)$  corresponding to each alternative  $Q_i$  by means of Definition 10.

**Step 5.** Arrange the alternatives  $Q_i$  in light of the obtained values of  $\&(\delta_i)$  and identify the optimal one.

The pseudocode for the proposed algorithms is shown in Table 3.

### Case study: Decentralized identity management for inclusive Web 3.0 network system

Web 3.0, the envisioned future of the internet, promises a decentralized internet and user empowerment where users have greater control, ownership, and privacy over their data, transactions, and digital interactions<sup>46,47</sup>. The essential aspect of this transformation is the reconceptualization of identity management systems to foster inclusivity, privacy, and security<sup>48</sup>. Conventional centralized identification systems face challenges related to data breaches, identity theft, and user privacy concerns. Decentralized identity management promises to address these issues by shifting the control of personal data back to the users, enabling a more inclusive and secure digital environment. The phenomenon of determining the best DIM technique to enable an inclusive and vibrant Web 3.0 ecosystem in a diverse environment such as Pakistan, where large segments of the population reside in the physical world without official documentation or digital access, which in turn means that there is no verifiable identity, is rooted in major security and privacy issues<sup>49,50</sup>. The implementation of Web 3.0 network system benefits the society in the following way:

- This phenomenon enables individuals who do not possess legal documentation or banking capabilities to participate in the digital economy and gain access to a wide range of services. As a result, they receive a specific level of authority. The provision of comparable economic opportunities thus facilitates the establishment of an atmosphere that fosters improved self-governance
- In Web 3.0 ecosystems, the delegation of legislative authority via DIM grants users the ability to select democratic administrations that prioritize security and usability. Consequently, the community would have a responsibility to supervise the DM process.
- Increases in credibility and transparency: The integration of verifiable credentials into DIM diminishes the probability of fraudulent activities, bolsters user trust in electronic transactions, and ultimately elevates accountability.

To ensure successful implementation of a DIM system in a Web 3.0 network, the following obstacles must be addressed:

Algorithm: MCDM Using CSFYOWA and CSFYONG Operators

Input:

- Decision matrix  $A = \{(\zeta_{ij}, \omega_{ij}), (\eta_{ij}, \phi_{ij}), (\kappa_{ij}, \rho_{ij})\}$  for all alternatives  $Q_i$  and criteria  $C_j$
- $m$ : Number of alternatives ( $Q_1, Q_2, \dots, Q_m$ )
- $n$ : Number of criteria ( $C_1, C_2, \dots, C_n$ )

Output:

- Optimal alternative  $Q^*$

Step 1: Construct the CSF decision matrix

```

for each alternative  $Q_i$  ( $i = 1$  to  $m$ ) do
  for each criterion  $C_j$  ( $j = 1$  to  $n$ ) do
     $A[i][j] = \{(\zeta_{ij}, \omega_{ij}), (\eta_{ij}, \phi_{ij}), (\kappa_{ij}, \rho_{ij})\}$ 
  end for
end for

```

Step 2: Compute the CSF permuted decision matrix  $D$

```

for each alternative  $Q_i$  ( $i = 1$  to  $m$ ) do
  for each criterion  $C_j$  ( $j = 1$  to  $n$ ) do
    Compute score value  $\lambda_{ij}$  using Definition 10
  end for
  Arrange  $\lambda_{ij}$  for all  $j$  in descending order
  Formulate permuted decision matrix  $D[i]$  based on the arranged values
end for

```

Step 3: Aggregate preference values using CSFYOWA and CSFYONG operators

```

for each alternative  $Q_i$  ( $i = 1$  to  $m$ ) do
  Compute  $\delta_i$  using CSFYOWA operator:
   $\delta_i = \text{CSFYOWA}(G_\theta(i1), G_\theta(i2), \dots, G_\theta(in))$ 
  Compute  $\delta_i$  using CSFYONG operator:
   $\delta_i = \text{CSFYONG}(G_\theta(i1), G_\theta(i2), \dots, G_\theta(in))$ 
end for

```

Step 4: Compute the score values for each alternative

```

for each alternative  $Q_i$  ( $i = 1$  to  $m$ ) do
  Compute the score value  $S(\delta_i)$  using Definition 10
end for

```

Step 5: Rank the alternatives

```

Sort alternatives  $Q_i$  based on the score values  $S(\delta_i)$  in descending order
Select the optimal alternative  $Q^*$  with the highest score value  $S(\delta_i)$ 

```

End Algorithm

**Table 3.** Pseudocode for the developed mechanisms.

- 
- (i) Consideration of both technical and non-technical factors such as user behavior, regulatory frameworks, and legacy systems is crucial. Trust anchors are vital for transitioning from obsolete systems.
  - (ii) The inherent technological complexity of Web 3.0 networks may limit user participation.
  - (iii) Decentralized identity systems are vulnerable to cyber intrusions, posing risks of illicit access and financial harm.
  - (iv) Web 3.0's early stage may delay widespread adoption of DIM solutions.

- (v) Legal safeguards are lacking against unauthorized use or monitoring of user information, highlighting the need for individual control over personal data.
- (vi) Interoperability challenges among various organizations and DIM protocols may hinder user participation and innovation discovery.
- (vii) Thorough resource allocation and cost analysis are essential for DIM system establishment and management.
- (viii) Ensuring resource efficiency and scalability is crucial for large-scale DIM implementation, requiring attention to technological concerns and user adoption.

The challenge is to provide feasible recommendations on strategy selection, considering strict criteria that resolve weaknesses and enhance resilience in the Web 3.0 network system. Web 3.0 is a revolutionary technology that has the potential to shape the digital future and make the online space autonomous. It is anticipated that the decentralized system will provide greater security. It integrates advanced technologies including blockchain and artificial intelligence<sup>51</sup>. Web 3.0 network systems grant users increased autonomy. They promote enhanced collaboration and stimulate innovative thought. As Web 3.0 continues to evolve, a more equitable digital environment is imminent. Individuals will retain the authority to govern their data and digital identities<sup>52–55</sup>. In essence, each strategy should provide individuals with the power to control and oversee their confidential information and online personas. It ought to be an effortless incorporation into their daily regimen.

The following discussion presents a solution to the MCDM problem through a numerical example, showcasing the efficacy of the innovative approaches suggested.

XYZ Corporation, a prominent fintech company that specializes in digital payments, is studying Web 3.0 decentralized identity management approaches to boost transaction security across all of its digital platforms. XYZ Firm's diverse clientele—enterprises, institutional partners, and individual investors—drives daily transaction volume. The organization knows the need to keep these transactions confidential and legal. Single points of failure, hackers, and lengthy verification procedures have plagued XYZ Firm's traditional centralized identity management solutions in recent years. The startup is trying to employ Web 3.0 technologies like blockchain and decentralized authentication protocols to increase transaction security and efficiency. By decentralized identity management approach, the organization envisions, is to enable customers to safely verify their identities and approve transactions without the need for middlemen or the exposure of private data to possible security risks. In order to accomplish this goal, the firm is assessing many decentralized identity management frameworks, each of which has special features and capabilities catered to the needs of the company. As such, it understands the necessity for a thorough investigation to determine the best strategy from the set of potential alternatives  $\{Q_1 = \text{Self-Sovereign identity}, Q_2 = \text{Zero-knowledge proof}, Q_3 = \text{Hyperledger Indy}, Q_4 = \text{OAuth-based identity management}\}$  that strikes a balance between security, scalability, and usability, where.

- Self-Sovereign identity: It signifies a digital identity framework that empowers individuals to control the information they provide to establish their identity on various websites, services, and applications accessible over the internet.
- Zero-knowledge proof: It is a type of proof that establishes the validity of a statement without revealing any further information beyond the statement's validity. A proof system comprising a prover, a verifier, and a challenge is employed to enable users to publicly disseminate proof of knowledge or ownership while maintaining the confidentiality of its details.
- Hyperledger Indy: It provides a comprehensive suite of tools, libraries, and reusable components that facilitate the establishment of digital identities based on blockchains or other distributed ledgers. It is specifically designed for the DIM system. This enables seamless interoperability between various administrative domains, applications, and other technological silos.
- OAuth-based identity management: An OAuth-based identity management system is a system that makes use of the Open Authorization (OAuth) protocol to manage user identities and control access to resources across a variety of platforms or services. OAuth is commonly utilized for delegated authorization, which enables third-party apps to access a user's resources without revealing the user's credentials.

In an increasingly digital and interlinked world, the firm seeks to enhance its reputation as a reliable provider of financial services by using the best-decentralized identity management method. They set up a structure of essential criteria based on the weight vector  $(0.35, 0.22, 0.31, 0.12)^T$  for building an inclusive Web 3.0 system, including

- $C_1$ : Privacy protection
- $C_2$ : Authenticity and Scarcity
- $C_3$ : User-friendly experience
- $C_4$ : Transparency,

which remain hurdles to overcome on this path to a truly secure and private digital existence.

- Privacy protection: This refers to the strategies implemented to ensure the security and confidentiality of an individual's personal information and digital resources, thereby preventing illegal access, utilization, or theft.
- Authenticity and Scarcity: Scarcity can increase the value or security of identities, credentials, or access rights, while authenticity ensures their legitimacy and trustworthiness.
- User-friendly experience: This system is characterized by its intuitive nature, efficiency, and accessibility, enabling users to effectively complete activities or achieve goals with minimal exertion and dissatisfaction.

	$C_1$	$C_2$	$C_3$	$C_4$
$Q_1$	$\left( \begin{array}{l} (0.71, 0.52), (0.23, 0.40), \\ (0.14, 0.30) \end{array} \right)$	$\left( \begin{array}{l} (0.63, 0.80), (0.50, 0.25), \\ (0.41, 0.30) \end{array} \right)$	$\left( \begin{array}{l} (0.58, 0.33), (0.31, 0.25), \\ (0.60, 0.56) \end{array} \right)$	$\left( \begin{array}{l} (0.47, 0.66), (0.12, 0.21), \\ (0.27, 0.39) \end{array} \right)$
$Q_2$	$\left( \begin{array}{l} (0.14, 0.50), (0.39, 0.24), \\ (0.32, 0.46) \end{array} \right)$	$\left( \begin{array}{l} (0.41, 0.17), (0.31, 0.42), \\ (0.65, 0.51) \end{array} \right)$	$\left( \begin{array}{l} (0.16, 0.31), (0.26, 0.55), \\ (0.23, 0.40) \end{array} \right)$	$\left( \begin{array}{l} (0.41, 0.53), (0.60, 0.24), \\ (0.13, 0.30) \end{array} \right)$
$Q_3$	$\left( \begin{array}{l} (0.33, 0.41), (0.19, 0.20), \\ (0.60, 0.56) \end{array} \right)$	$\left( \begin{array}{l} (0.70, 0.90), (0.50, 0.37), \\ (0.24, 0.18) \end{array} \right)$	$\left( \begin{array}{l} (0.60, 0.50), (0.27, 0.45), \\ (0.16, 0.24) \end{array} \right)$	$\left( \begin{array}{l} (0.10, 0.25), (0.30, 0.60), \\ (0.44, 0.52) \end{array} \right)$
$Q_4$	$\left( \begin{array}{l} (0.52, 0.60), (0.20, 0.70), \\ (0.17, 0.35) \end{array} \right)$	$\left( \begin{array}{l} (0.53, 0.39), (0.10, 0.40), \\ (0.72, 0.61) \end{array} \right)$	$\left( \begin{array}{l} (0.65, 0.60), (0.40, 0.30), \\ (0.28, 0.54) \end{array} \right)$	$\left( \begin{array}{l} (0.35, 0.72), (0.13, 0.54), \\ (0.46, 0.21) \end{array} \right)$

**Table 4.** CSF Decision Matrix.

	$C_1$	$C_2$	$C_3$	$C_4$
$Q_1$	$\left( \begin{array}{l} (0.63, 0.80), (0.50, 0.25), \\ (0.41, 0.30) \end{array} \right)$	$\left( \begin{array}{l} (0.71, 0.52), (0.23, 0.40), \\ (0.14, 0.30) \end{array} \right)$	$\left( \begin{array}{l} (0.47, 0.66), (0.12, 0.21), \\ (0.27, 0.39) \end{array} \right)$	$\left( \begin{array}{l} (0.58, 0.33), (0.31, 0.25), \\ (0.60, 0.56) \end{array} \right)$
$Q_2$	$\left( \begin{array}{l} (0.14, 0.50), (0.39, 0.24), \\ (0.32, 0.46) \end{array} \right)$	$\left( \begin{array}{l} (0.41, 0.53), (0.60, 0.24), \\ (0.13, 0.30) \end{array} \right)$	$\left( \begin{array}{l} (0.41, 0.17), (0.31, 0.42), \\ (0.65, 0.51) \end{array} \right)$	$\left( \begin{array}{l} (0.16, 0.31), (0.26, 0.55), \\ (0.23, 0.40) \end{array} \right)$
$Q_3$	$\left( \begin{array}{l} (0.70, 0.90), (0.50, 0.37), \\ (0.24, 0.18) \end{array} \right)$	$\left( \begin{array}{l} (0.60, 0.50), (0.27, 0.45), \\ (0.16, 0.24) \end{array} \right)$	$\left( \begin{array}{l} (0.33, 0.41), (0.19, 0.20), \\ (0.60, 0.56) \end{array} \right)$	$\left( \begin{array}{l} (0.10, 0.25), (0.30, 0.60), \\ (0.44, 0.52) \end{array} \right)$
$Q_4$	$\left( \begin{array}{l} (0.52, 0.60), (0.20, 0.70), \\ (0.17, 0.35) \end{array} \right)$	$\left( \begin{array}{l} (0.53, 0.39), (0.10, 0.40), \\ (0.72, 0.61) \end{array} \right)$	$\left( \begin{array}{l} (0.35, 0.72), (0.13, 0.54), \\ (0.46, 0.21) \end{array} \right)$	$\left( \begin{array}{l} (0.65, 0.60), (0.40, 0.30), \\ (0.28, 0.54) \end{array} \right)$

**Table 5.** CSF permuted decision matrix.

- **Transparency:** Digital identity verification transparency involves disclosing the methods, algorithms, and data sources utilized to verify an individual’s identification online. It involves informing people about how their personal data is gathered, maintained, and utilized for identity verification.

To construct a CSFN, these elements are separated into two independent features as described below:

- Privacy protection consists of the selective disclosure and decentralized identity.
- Authenticity and scarcity consist of provenance verification and limited issuance.
- User-friendly experience consists of the intuitive user interface and accessible documentation.
- Transparency consists of the public ledger and real-time updates.

The essential stages for solving this MCDM problem within the scope of CSFYOWA and CSFYOWG operators are described below:

*Step 1.* Table 4 presents the decision-maker’s feedback on  $Q_i$  for each  $C_j$ , expressed as CSFN.

*Step 2.* Compute the score values  $\Lambda_1, \Lambda_2, \Lambda_3$ , and  $\Lambda_4$  relative to each criterion  $C_1, C_2, C_3$ , and  $C_4$ , corresponding to each alternative  $Q_1, Q_2, Q_3$  and  $Q_4$  using Definition 10. For instance, the score values  $\Lambda_1, \Lambda_2, \Lambda_3$ , and  $\Lambda_4$  relative to each criterion  $C_1, C_2, C_3$ , and  $C_4$  corresponding to alternative  $Q_1$  are calculated in the framework of Definition 10 as follows:

$$\Lambda_1 = \frac{1}{3} [4 + ((1 - 0.14^2) (0.31^2 - 0.23^2) + (1 - 0.30^2) (0.52^2 - 0.40^2))] = 1.5142$$

Similarly,

$$\Lambda_2 = 1.5492, \Lambda_3 = 1.3952, \text{ and } \Lambda_4 = 1.5078.$$

Likewise, by adopting the above mathematical procedure, we compute the respective score values of all criteria corresponding to the remaining three alternatives.

Moreover, the above-computed score values are arranged in descending order to formulate the CSF permuted decision matrix in Table 5 as follows:

*Step 3.* (a) Compute the overall preference values

$\delta_i$  of each alternative  $Q_i$  using the CSFYOWA operator for the particular value  $\Gamma = 3$  with the corresponding weight vector  $(0.35, 0.22, 0.31, 0.12)^T$ . For instance, the preference value

$\delta_1$  of alternative  $Q_1$  in the framework of CSFYOWA operator is calculated as follows:

$$\begin{aligned} & \sqrt{\min \left( 1, \left( \sum_{j=1}^4 \beta_j \cdot \zeta_{\Theta(1j)}^6 \right)^{1/3} \right)} \\ &= \sqrt{\min \left( 1, \left( (0.35 (0.63)^6) + (0.22 (0.71)^6) + (0.31 (0.47)^6) + (0.12 (0.58)^6) \right)^{1/3} \right)} \\ &= 0.622, \end{aligned}$$

Similarly,

$$\begin{aligned} & \sqrt{\min \left( 1, \left( \sum_{j=1}^4 \beta_j \cdot \varpi_{\Theta(1j)}^6 \right)^{1/3} \right)} = 0.704, \\ & \sqrt{1 - \min \left( 1, \left( \sum_{j=1}^4 \beta_j \left( 1 - \eta_{\Theta(1j)}^2 \right)^3 \right)^{1/3} \right)} = 0.322, \quad \sqrt{1 - \min \left( 1, \left( \sum_{j=1}^4 \beta_j \left( 1 - \vartheta_{\Theta(1j)}^2 \right)^3 \right)^{1/3} \right)} = 0.276, \\ & \sqrt{1 - \min \left( 1, \left( \sum_{j=1}^4 \beta_j \left( 1 - \kappa_{\Theta(1j)}^2 \right)^3 \right)^{1/3} \right)} = 0.343, \quad \sqrt{1 - \min \left( 1, \left( \sum_{j=1}^4 \beta_j \left( 1 - \Omega_{\Theta(1j)}^2 \right)^3 \right)^{1/3} \right)} = 0.361 \\ \text{Thus, } \delta_1 &= \left( \begin{matrix} (0.622, 0.704), (0.322, 0.276), \\ (0.343, 0.361) \end{matrix} \right). \end{aligned}$$

Likewise, by adopting the above mathematical procedure, we compute the respective preference values corresponding to the remaining three alternatives.

The outcomes of the above mathematical procedure are tabulated in Table 6.

**Step 3. (b)** Compute the overall preference values  $\delta_i$  of each alternative  $Q_i$  using the CSFYOWG operator for the particular value  $\Gamma = 3$  with the corresponding weight vector  $(0.35, 0.22, 0.31, 0.12)^T$ . For instance, the preference value  $\delta_1$  of alternative  $Q_1$  in the framework of CSFYOWG operator is calculated as follows:

$$\begin{aligned} & \sqrt{1 - \min \left( 1, \left( \sum_{j=1}^4 \beta_j \left( 1 - \zeta_{\Theta(1j)}^2 \right)^3 \right)^{1/3} \right)} \\ &= \sqrt{\min \left( 1, \left( (0.35 (1 - 0.63^2)^3) + (0.22 (1 - 0.71^2)^3) + (0.31 (1 - 0.47^2)^3) + (0.12 (1 - 0.58^2)^3) \right)^{1/3} \right)} \\ &= 0.585, \end{aligned}$$

Similarly,

Alternatives	$\delta_i$
$Q_1$	$\left( \begin{matrix} (0.622, 0.704), (0.322, 0.276), \\ (0.343, 0.361) \end{matrix} \right)$
$Q_2$	$\left( \begin{matrix} (0.369, 0.468), (0.397, 0.340), \\ (0.382, 0.434) \end{matrix} \right)$
$Q_3$	$\left( \begin{matrix} (0.611, 0.759), (0.340, 0.370), \\ (0.371, 0.367) \end{matrix} \right)$
$Q_4$	$\left( \begin{matrix} (0.527, 0.636), (0.196, 0.527), \\ (0.401, 0.393) \end{matrix} \right)$

**Table 6.** Preference values of alternatives using CSFYOWA operator.

$$\sqrt[3]{1 - \min \left( 1, \left( \sum_{j=1}^4 \beta_j (1 - \varpi_{\Theta(1,j)}^2)^3 \right)^{1/3} \right)} = 0.322, \quad \sqrt[3]{1 - \min \left( 1, \left( \sum_{j=1}^4 \beta_j (1 - \eta_{\Theta(1,j)}^2)^3 \right)^{1/3} \right)} = 0.276,$$

$$\sqrt[3]{1 - \min \left( 1, \left( \sum_{j=1}^4 \beta_j (1 - \vartheta_{\Theta(1,j)}^2)^3 \right)^{1/3} \right)} = 0.343, \quad \sqrt[3]{\min \left( 1, \left( \sum_{j=1}^4 \beta_j \cdot \kappa_{\Theta(1,j)}^6 \right)^{1/3} \right)} = 0.704,$$

$$\sqrt[3]{\min \left( 1, \left( \sum_{j=1}^4 \beta_j \cdot \Omega_{\Theta(1,j)}^6 \right)^{1/3} \right)} = 0.704,$$

Thus,  $\delta_1 = ((0.585, 0.614), (0.322, 0.276), (0.441, 0.416))$ .

Likewise, by adopting the above mathematical procedure, we compute the respective preference values corresponding to the remaining three alternatives.

The outcomes of the above mathematical procedure are tabulated in Table 7.

*Step 4. (a)* In the light of Table 6, we compute score values  $\&(\delta_i)$  corresponding to each alternative  $Q_i$  using Definition 10. For instance, the score value  $\&(\delta_i)$  of the alternative  $Q_1$  in the framework of Definition 10 is calculated as follows:

$$\begin{aligned} \&(\delta_1) &= \frac{1}{3} [4 + ((1 - 0.34^2) (0.62^2 - 0.32^2) + (1 - 0.36^2) (0.70^2 - 0.27^2))] \\ &= 1.5383, \end{aligned}$$

Similarly,

$$\&(\delta_2) = 1.3549, \&(\delta_3) = 1.5337, \text{ and } \&(\delta_4) = 1.4359.$$

*Step 4. (b)* In the light of Table 7, we compute score values  $\&(\delta_i)$  corresponding to each alternative  $Q_i$  using Definition 10. For instance, the score value  $\&(\delta_i)$  of the alternative  $Q_1$  in the framework of Definition 10 is calculated as follows:

$$\begin{aligned} \&(\delta_1) &= \frac{1}{3} [4 + ((1 - 0.44^2) (0.58^2 - 0.32^2) + (1 - 0.41^2) (0.61^2 - 0.28^2))] \\ &= 1.4800, \end{aligned}$$

Similarly,

$$\&(\delta_2) = 1.3284, \&(\delta_3) = 1.4011, \text{ and } \&(\delta_4) = 1.3922.$$

*Step 5.* In light of the information determined from 4(a) and 4(b), we observe that:

$\&(\delta_1) > \&(\delta_3) > \&(\delta_4) > \&(\delta_2)$ , it follows that the alternatives are ranked in the following order:  $Q_1 > Q_3 > Q_4 > Q_2$ .

Therefore, self-sovereign identity is the most suitable DIM technique for an inclusive Web 3.0 network system where the user has great control over their digital data.

### Comparative analysis

This subsection aims to highlight and illustrate the strength and viability of our proposed methodologies by conducting a thorough examination of various established approaches, specifically PicFYOWA<sup>21</sup>, PicFYOWG<sup>21</sup>, SFYOWA<sup>23</sup>, SFYOWG<sup>23</sup>, CSFOWA<sup>40</sup>, and CSFOWG<sup>40</sup>. The summarized results obtained by these techniques are succinctly presented in Table 8, whereas Table 9 depicts the ranking of alternatives.

The above information reveals that the recently established strategies provide more extensive approaches compared to existing methods due to their superior preferences. Notably, the approaches presented in<sup>21</sup> cannot be used in situations such as those in our investigation where the sum of MD, AD, and NDM surpasses 1 and unable to possess the two-dimensional data, whereas CSF Yager aggregation operators can address these situations more efficiently. In addition, it is also important to notice that the frameworks described in<sup>23</sup> exhibit limitations due to their uni-dimensional processing. Some data can be lost as a consequence of this particular limitation of

Alternatives	$\delta_i$
$Q_1$	$((0.585, 0.614), (0.322, 0.276), (0.441, 0.416))$
$Q_2$	$((0.305, 0.395), (0.397, 0.340), (0.536, 0.459))$
$Q_3$	$((0.489, 0.529), (0.340, 0.370), (0.499, 0.478))$
$Q_4$	$((0.483, 0.580), (0.196, 0.527), (0.568, 0.496))$

**Table 7.** Preference Values of Alternatives Using the CSFYOWG operator.

	PicFYOWA <sup>21</sup>	PicFYOWG <sup>21</sup>	SFYOWA <sup>23</sup>	SFYOWG <sup>23</sup>
$Q_1$	$\left( \begin{matrix} 0.605, 0.784, \\ 0.698 \end{matrix} \right)$	$\left( \begin{matrix} 0.437, 0.784, \\ 0.384 \end{matrix} \right)$	$\left( \begin{matrix} 0.566, 0.174, \\ 0.318 \end{matrix} \right)$	$\left( \begin{matrix} 0.516, 0.174, \\ 0.472 \end{matrix} \right)$
$Q_2$	$\left( \begin{matrix} 0.336, 0.625, \\ 0.686 \end{matrix} \right)$	$\left( \begin{matrix} 0.738, 0.625, \\ 0.462 \end{matrix} \right)$	$\left( \begin{matrix} 0.476, 0.269, \\ 0.179 \end{matrix} \right)$	$\left( \begin{matrix} 0.458, 0.269, \\ 0.189 \end{matrix} \right)$
$Q_3$	$\left( \begin{matrix} 0.563, 0.695, \\ 0.687 \end{matrix} \right)$	$\left( \begin{matrix} 0.589, 0.695, \\ 0.436 \end{matrix} \right)$	$\left( \begin{matrix} 0.453, 0.275, \\ 0.436 \end{matrix} \right)$	$\left( \begin{matrix} 0.419, 0.275, \\ 0.471 \end{matrix} \right)$
$Q_4$	$\left( \begin{matrix} 0.504, 0.829, \\ 0.647 \end{matrix} \right)$	$\left( \begin{matrix} 0.533, 0.829, \\ 0.489 \end{matrix} \right)$	$\left( \begin{matrix} 0.683, 0.338, \\ 0.419 \end{matrix} \right)$	$\left( \begin{matrix} 0.546, 0.338, \\ 0.481 \end{matrix} \right)$
	CSFOWA <sup>40</sup>	CSFOWG <sup>40</sup>		
$Q_1$	$\left( \begin{matrix} (0.606, 0.679), (0.256, 0.0263), \\ (0.298, 0.351) \end{matrix} \right)$	$\left( \begin{matrix} (0.0263, 0.374), (0.249, 0.205) \end{matrix} \right)$	$(0.159, 0.163),$	
$Q_2$	$\left( \begin{matrix} (0.319, 0.420), (0.380, 0.4258), \\ (0.314, 0.425) \end{matrix} \right)$	$\left( \begin{matrix} (0.4258, 0.391), (0.180, 0.192) \end{matrix} \right)$	$(0.264, 0.239),$	
$Q_3$	$\left( \begin{matrix} (0.557, 0.712), (0.404, 0.0338), \\ (0.313, 0.309) \end{matrix} \right)$	$\left( \begin{matrix} (0.496), (0.449, 0.461) \end{matrix} \right)$	$(0.256, 0.395),$	
$Q_4$	$\left( \begin{matrix} (0.501, 0.614), (0.163, 0.251), \\ (0.338, 0.256) \end{matrix} \right)$	$\left( \begin{matrix} (0.413), (0.442, 0.381) \end{matrix} \right)$	$(0.323, 0.344),$	

**Table 8.** The aggregated Values of the alternatives using Existing Operators.

	$Q_1$	$Q_2$	$Q_3$	$Q_4$	Ranking
PicFYOWA <sup>21</sup>	1.3203	1.3031	1.3196	1.2951	$Q_1 > Q_3 > Q_2 > Q_4$
PicFYOWG <sup>21</sup>	1.2723	1.3877	1.3732	1.2830	$Q_2 > Q_3 > Q_4 > Q_1$
SFYOWA <sup>23</sup>	1.4167	1.3271	1.4072	1.4002	$Q_1 > Q_3 > Q_4 > Q_2$
SFYOWG <sup>23</sup>	1.3973	1.3079	1.3643	1.3773	$Q_1 > Q_4 > Q_3 > Q_2$
CSFOWA <sup>40</sup>	1.5396	1.3416	1.5173	1.4318	$Q_1 > Q_3 > Q_4 > Q_2$
CSFOWG <sup>40</sup>	1.5022	1.3163	1.4043	1.4029	$Q_1 > Q_3 > Q_4 > Q_2$
CSFYOWA	1.5383	1.3549	1.5337	1.4359	$Q_1 > Q_3 > Q_4 > Q_2$
CSFYOWG	1.4800	1.3284	1.4011	1.3922	$Q_1 > Q_3 > Q_4 > Q_2$

**Table 9.** Ranking of alternatives.

SF Yager aggregation operators, whereas CSF Yager aggregation operators are able to handle this circumstance successfully without losing any important data. Moreover, the frameworks suggested by<sup>40</sup> do not possess the ability to dynamically adjust their parameters based on the risk tolerance of the decision-experts, therefore reducing their adaptability and inability to integrate individual preferences. To overcome this deficiency, the newly introduced operators showcase greater adaptability through the incorporation of parametric values. As a result, the methodologies formulated in this study more versatile and effective in addressing MCDM difficulties.

### Advantages of the current study

The proposed system is more advantageous and dominant than existing malleable approaches because it modifies parameter values dynamically based on the decision-maker's risk aversion and has a unique ability to process two-dimensional data simultaneously, which other existing methods such as SF Yager ordered weighted aggregation operators and CSF ordered weighted aggregation operators lack. Consequently, these enhancements make our operators more versatile, enabling them to tackle complex DM problems with greater precision and adaptability.

### Sensitivity analysis

In this subsection, we establish the sensitivity analysis of the recently proposed strategies by changing the weight coefficients of the criteria. The outcomes of this mathematical procedure in the framework of CSFYOWA and CSFYOWG are tabulated in Tables 10 and 11, respectively.

After modifying the coefficient of the weights of criterion, it is clear that the rankings of the alternatives stay constant. Hence, the approach demonstrates a considerable degree of robustness and offers a more feasible resolution for the MCDM problem being examined.

Weight Coefficients	$\delta_1$	$\delta_2$	$\delta_3$	$\delta_4$	Ranking
(0.35, 0.22, 0.31, 0.12)	1.5383	1.3549	1.5337	1.4359	$Q_1 > Q_3 > Q_4 > Q_2$
(0.29, 0.24, 0.26, 0.21)	1.5287	1.3503	1.5138	1.4443	$Q_1 > Q_3 > Q_4 > Q_2$
(0.33, 0.23, 0.28, 0.16)	1.5405	1.3531	1.5266	1.4394	$Q_1 > Q_3 > Q_4 > Q_2$
(0.36, 0.21, 0.30, 0.13)	1.5334	1.3538	1.5272	1.4345	$Q_1 > Q_3 > Q_4 > Q_2$

**Table 10.** Ranking of alternatives by changing weight coefficients of criteria using CSFYOWA operator.

Weight Coefficients	$\delta_1$	$\delta_2$	$\delta_3$	$\delta_4$	Ranking
(0.35, 0.22, 0.31, 0.12)	1.4800	1.3284	1.4011	1.3922	$Q_1 > Q_3 > Q_4 > Q_2$
(0.29, 0.24, 0.26, 0.21)	1.4643	1.3240	1.3821	1.3688	$Q_1 > Q_3 > Q_4 > Q_2$
(0.33, 0.23, 0.28, 0.16)	1.4725	1.3266	1.3931	1.3850	$Q_1 > Q_3 > Q_4 > Q_2$
(0.36, 0.21, 0.30, 0.13)	1.4826	1.3279	1.4004	1.3930	$Q_1 > Q_3 > Q_4 > Q_2$

**Table 11.** Ranking of alternatives by changing weight coefficients of criteria using CSFYOWG operator.

## Conclusions

In this article, two novel aggregation operators, namely CSFYOWA and CSFYOWG operators, have been introduced within the CSF environment. We have designed an improved score function to alleviate the limitations of the existing score function, enabling CSFNs to effectively rank and choose the optimal alternatives. Additionally, the operational laws based on the newly defined CSF Yager order-weighted aggregation operators have been formulated, and the fundamental structural characteristics of these operators have also been explored. Expanding our contributions, a robust MCDM step-by-step mathematical mechanism has been devised in light of our proposed aggregation operators and score function. In addition, the recently defined techniques have also effectively been utilized to address the MCDM problem of optimizing decentralized identity management within the framework of the Web 3.0 network system. Finally, a comprehensive comparative study has been established to showcase the reliability and superiority of suggested approaches to current techniques.

## Limitations

Despite the several advantages of the approaches developed in this article, they are nevertheless bound by specific limitations:

- (i) These strategies are not optimal for addressing problems in which the square summation of membership, abstinence and non-membership degrees surpasses the unit interval.
- (ii) The lack of an inherent dynamic modification system to address MCDM challenges makes this study are not suitable for situations that involve data collection at varying specific time intervals.

## Anticipated future research avenues

We intend to broaden the range of Yager aggregation operators to encompass more generalized contexts, particularly complex T-spherical fuzzy settings and CSF dynamic knowledge. In addition, through these advancements, we will enhance the flexibility and usefulness of proposed models in many DM situations, including decision support systems in cybersecurity, healthcare, natural disaster response planning, IOT network optimization, and assessment of environmental impacts. Moreover, we will also explore the significance of various DM scenarios, including selecting emergency medical suppliers through an interactive and feedback mechanism<sup>56</sup>, choosing a cloud storage provider<sup>57</sup>, selecting unmanned aerial vehicles for precision agriculture<sup>58</sup>, and analyzing the combinative distance-based assessment approach<sup>59</sup> using the proposed methodologies.

## Data availability

All data generated or analyzed during this study are included in this article.

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## Author contributions

All authors contributed equally to this research.

## Declaration

## Competing interests

The authors declare no conflict of interest.

## Additional information

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