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Sampled-data stabilization of delayed Boolean control networks with state inequality constraints

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This paper studies the set stabilization of delayed Boolean control networks (DBCNs) with state inequality constraints via time-variant nonuniform sampled-data control. State inequality constraints are introduced into DBCNs. Firstly, the equivalent algebraic forms of DBCNs and the solution set of state inequality constraints are constructed via the algebraic state space representation approach, based on which an inequality constrained controllability matrix is constructed. Then, by the inequality constrained controllability matrix, new criteria are proposed for the nonuniform sampled-data inequality constrained reachability of DBCNs. Finally, a time-variant nonuniform sampled-data stabilizers are designed for DBCNs by utilizing the nonuniform sampled-data inequality constrained reachability. The effectiveness of the obtained results is verified through the cell apoptosis network.

Keywords Delayed Boolean control networks, State inequality constraints, Stabilization, Nonuniform sampled-data control, Algebraic state space representation

Boolean networks (BNs) are a special kind of nonlinear systems with logical operations[1,2]. The model of BNs was established by Kauffman to study gene regulatory networks (GRNs)[3]. Furthermore, BNs with inputs are referred to Boolean control networks (BCNs), which is a correct method to describe the dynamics of GRNs[4]. For some basic theory and applications of BNs, please refer to[5,6]. However, due to the lack of mathematical methods for handling logical processes, it becomes very inconvenient to study the control issues of BNs[7].

Recently, a new matrix product, called semi-tensor product of matrices, has provided great convenience for the study of BNs[8]. Based on the characteristics of matrix product, an algebraic state space representation (ASSR) framework has been constructed for the research of BNs. Therefore, researchers can use the ASSR to investigate BNs through the traditional control theory[9]. In the past twenty years, many excellent results have been achieved for BNs via the ASSR framework[10,11]. Observability and controllability for BNs were investigated in[12,13]. Stability and stabilization of BCNs were investigated in[14–16].

Control design is always a core issue in control theory[17,18]. Recently, many control design methods have been introduced to address the control problems of BCNs through the ASSR[19]. Furthermore, how to design appropriate control techniques to reduce control costs is an important topic in modern control theory. As we all know, sampled-data control (SDC) is an effective technique, which reduces the update frequency of the controller and significantly reduces the computational burden[20,21]. In[22], a self-triggered implementation of the proposed event-triggered sampling scheme was presented. Based on the ASSR framework, SDC method is introduced into the control of BCNs[19,23], and some basic results are proposed for the sampled-data stabilization and controllability of BCNs[24]. There are two types of SDC: uniform SDC and nonuniform SDC (NSDC). Since the sampling length of NSDC is time-varying, it can more effectively utilize information resources than uniform SDC[25], which also makes the controller design more challenging.

In GRNs, time delay is generally used to represent slow biochemical reactions[26]. In addition, external environmental factors, such as nutrient concentration and temperature may also lead to time delays in GRNs. For example, the coupled oscillatory biochemical network in cell cycle is simplified as the delayed BN[27]:

$$\begin{cases} C_1(t+1) = \neg\Big(C_1(t-2) \wedge C_2(t-1)\Big), \\ C_2(t+1) = \neg\Big(C_1(t-1) \wedge C_2(t-2)\Big), \end{cases} \tag{1}$$

where $C_1(t)$ and $C_2(t)$ represent the state of two cells at time t, and time delays are caused by the delayed translocation between cells. In the past ten or more years, delayed BNs have attracted the research interest

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of many scholars. Delayed BNs were firstly investigated through the ASSR in[8]. Subsequently, delayed BCNs (DBCNs) were also well investigated in some works[12,28]. In[29], fault detectability of asynchronous DBCNs with sampled-data control was investigated.

As is well known, constraints play a very important role in nonlinear systems[30]. In GRNs, it is necessary to impose constraints on certain gene states that may lead to diseases and treatment options that may lead to serious complications[31]. For example, in WNT5A gene networks, states of the WNT5A=1 are undesirable as they accelerate the opportunity for transfer[32]. State inequality constraints, as an important form of constraints, often appear in the control logic design of dynamic systems[33]. They are usually physical constraints of the system and have wide practical applications[34]. For example, in a semi-batch reactor, due to the volume constraint of the reactor, only a limited amount of substrate can be fed. Similarly, for safety reasons, one may not want the reactor to operate above a certain maximum temperature. In a feed batch bioreactor, there may be constraints on cell mass concentration (beyond which oxygen transfer is restricted) or substrate concentration (beyond which unexpected side reactions may occur). However, there are no relevant results on the investigation of DBCNs with state inequality constraints via NSDC.

In this paper, based on the ASSR framework, we analyze the inequality constrained set reachability and inequality constrained set stabilization of DBCNs with state inequality constraints via NSDC. The main contributions can be summarized in two aspects: (i) State inequality constraints are introduced into DBCNs, and an inequality constrained controllability matrix is constructed for studying the inequality constrained set stabilization of DBCNs. (ii) NSDC provides us with more control schemes to achieve control objectives and reduce control costs. The traditional control and uniform SDC can be considered as a special case of NSDC.

The remainder of this paper is organized as follows: Section "Problem formulation" presents the equivalent system of DBCNs with state inequality constraints through the ASSR framework. The inequality constrained controllability matrix and nonuniform sampled-data inequality constrained reachability are considered in section "Inequality constrained set reachability". The time-variant state feedback inequality constrained set stabilization of DBCNs is studied in section "Inequality constrained set stabilization". Section "Illustrative example" and section "Conclusions" provide an illustrative example and a brief summary, respectively.

Notations: $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{Z}_+$ represent the set of real numbers, integers, nonnegative integers and positive integers, respectively. $0_s := (\underbrace{0, \cdots, 0}_{s})^\top$. $1_s := (\underbrace{1, \cdots, 1}_{s})^\top$. I_s denotes the s-order identity matrix. $(A)_{s,m}$ denotes the (s, m)-th element of matrix A. $Col_i(A)$ denotes the i-th column of matrix A. $\mathscr{D} := \{0, 1\}$, $\mathscr{D}^m := \underbrace{\mathscr{D} \times \cdots \times \mathscr{D}}_{m}$. $\Delta_s := \{\delta_s^i : i = 1, \cdots, s\}$, where $\delta_s^i = Col_i(I_s)$. $A_{s\times m}$ is called a (s, m) logical matrix, if $Col_i(A_{s\times m}) \in \Delta_s, i = 1, \cdots, m$. $\mathscr{L}_{s\times m}$ denotes the set of $s \times m$ logical matrices. $W_{[s,m]}$ and $M_{r,n}$ are swap matrix and power-reducing matrix, respectively[8]. $[a, b)|_{\mathbb{Z}} = \{a, \cdots, b-1\} \subseteq \mathbb{Z},\ a, b \in \mathbb{Z}, a < b$. $\lfloor c \rfloor$ represents the maximum integer not greater than c. Denote $a_1 \bigwedge a_2 = \min\{a_1, a_2\}$. In this paper, the default matrix product is semi-tensor product ($\ltimes$)[8].

Problem formulation

Consider the following DBCN:

$$\begin{cases} x_1(t+1) = \kappa_1(X(t-\varsigma+1), \cdots, X(t), U(t)), \\ \quad \vdots \\ x_n(t+1) = \kappa_n(X(t-\varsigma+1), \cdots, X(t), U(t)), \end{cases} \tag{2}$$

where $\varsigma \in \mathbb{Z}_+$ is the time delay, $X(i) := (x_1(i), \cdots, x_n(i)) \in \mathscr{D}^n, i = -\varsigma+1, \cdots, 0, 1, \cdots$ denote the state, here $U(t) := (v_1(t), \cdots, v_m(t)) \in \mathscr{D}^m$ denotes the control input, and $\kappa_i : \mathscr{D}^{n\varsigma+m} \to \mathscr{D}, i = 1, \cdots, n$ are Boolean functions. Assume that $Y_0 := (X(-\varsigma+1), \cdots, X(0)) \in \mathscr{D}^{n\varsigma}$ denotes the initial state trajectory.

Furthermore, consider state inequality constraints for binary variables $x_j(i) \in \mathscr{D}, j = 1, \cdots, n$:

$$a \le f(x_1(i), \cdots, x_n(i)) \le b, \tag{3}$$

where $f : \mathscr{D}^n \to \mathbb{R}$ is the inequality constrained function, and $a, b \in \mathbb{R}$ are inequality constrained boundaries.

For DBCN (2), we assume that $f(x_1(i), \cdots, x_n(i))$ satisfies the following linear form:

$$f(x_1(i), \cdots, x_n(i)) = (2^0, 2^1, \cdots, 2^{n-1})(x_1(i), \cdots, x_n(i))^\top. \tag{4}$$

It is worth pointing out that all possible states in $\mathscr{D}^n$ correspond one-to-one to indicator set $[0, 2^n)|_{\mathbb{Z}}$, and some other types of inequalities (nonlinear) can be transformed into linear constraint forms of (4) through column expansion[35].

Example 1 Consider the cell apoptosis network[36]

$$\begin{cases} IAP(t+1) = \neg C3a(t) \wedge TNF(t), \\ C3a(t+1) = \neg IAP(t) \wedge C8a(t), \\ C8a(t+1) = C3a(t) \vee TNF(t), \end{cases} \tag{5}$$

where IAP, C3a, and C8a represent concentration levels (low or high) of apoptosis inhibitor protein, active cystatin 3, and active cystatin8, respectively; the concentration level of tumor necrosis factor (TNF) is considered as a control input. The network graph of (5) is shown in Fig. 1.
Set $x_1(t)=IAP(t), x_2(t)=C3a(t), x_3(t)=C8a(t), v(t)=TNF(t)$, and

$$f(X(t))=(2^0,2^1,2^2)(X(t))^\top=2^0x_1(t)+2^1x_2(t)+2^2x_3(t).$$

From $\mathscr{D}^3=\{(0,0,0),(1,0,0),(0,1,0),(1,1,0),(0,0,1),(1,0,1),(0,1,1),(1,1,1)\}$, the corresponding indicator can be obtained as $f(0,0,0)=0, f(1,0,0)=1, f(0,1,0)=2, f(1,1,0)=3, f(0,0,1)=4, f(1,0,1)=5, f(0,1,1)=6, f(1,1,1)=7$.

In practice, the concentration ratio of IAP, C3a, and C8a can only be effective within an indicator range, so only the data within the range needs to be considered. For example, the indicator range requires $2\le f(X(t))\le 5$, then the corresponding solution set is $C_x=\{(0,1,0),(1,1,0),(0,0,1),(1,0,1)\}$, that is, the states of system (5) are constrained to C_x.

Firstly, we give the definitions of NSDC and nonuniform sampled-data inequality constrained set stabilization[5].

Definition 1 Given a set of sampling points $\{t_h : h\in\mathbb{N}\}$ with $t_0=0$. $\{U(t):t\in\mathbb{N}\}\subseteq\mathscr{D}^m$ is said to be an NSDC, if

$$U(t)=U(t_h), t\in[t_h,t_{h+1})|_{\mathbb{Z}}, t_{h+1}-t_h=\tau_h, \tag{6}$$

where the interval length $\tau_h\in\mathbb{Z}_+$ between sampling points are time-variant.

Especially, when $\tau_h=\tau$ holds for any $h\in\mathbb{N}$, the definition of uniform SDC can be given, where $\tau\in\mathbb{Z}_+$ is called the sampling period[37].

Definition 2 Let a state inequality constraint (3) and a nonempty state set $\mathscr{E}_e$ that satisfies (3) be given. DBCN (2) is said to be nonuniform sampled-data inequality constrained set stabilizable to $\mathscr{E}_e$, if for any Y_0 that satisfies (3), there exist a time-variant state feedback NSDC

$$v_i(t)=g_i(t_h, X(t_h-\varsigma+1),\cdots,X(t_h)), t\in[t_h,t_{h+1})|_{\mathbb{Z}} \tag{7}$$

with $g_i:\{t_h:h\in\mathbb{N}\}\times\mathscr{D}^{n\varsigma}\to\mathscr{D},\ i=1,\cdots,m$ being time-variant logical functions, and a positive integer T such that $X(t)\in\mathscr{E}_e, \forall\, t\ge T$, and $a\le f(X(t))\le b, \forall\, t\ge 1$.
Secondly, based on the ASSR[8], we provide the equivalent algebraic form of DBCN (2).

Identifying $1\sim\delta_2^1$, $0\sim\delta_2^2$. Setting $x(t)=\ltimes_{i=1}^n x_i(t)\in\Delta_{2^n}$, $v(t)=\ltimes_{i=1}^m v_i(t)\in\Delta_{2^m}$, $y(t)=\ltimes_{i=t-\varsigma+1}^t x(i)\in\Delta_{2^{n\varsigma}}$, we can convert DBCN (2) into the algebraic form

$$\begin{cases} x_1(t+1)=K_1v(t)y(t),\\ \quad\vdots\\ x_n(t+1)=K_nv(t)y(t),\end{cases} \tag{8}$$

where $K_i\in\mathscr{L}_{2\times 2^{n\varsigma+m}}$ is the structural matrix of κ_i, $i=1,\cdots,n$. Multiplying the *n* equations in (8), we can obtain the following form of (8):

$$x(t+1)=Kv(t)y(t), \tag{9}$$

where $K\in\mathscr{L}_{2^n\times 2^{n\varsigma+m}}$ satisfies $Col_j(K)=\ltimes_{i=1}^n Col_j(K_i), j=1,\cdots,2^{n\varsigma+m}$. For detailed instructions on how to use the ASSR to represent logical functions, please refer to[8].

In addition, from (4) and the construction of $x(t)=\ltimes_{i=1}^n x_i(t)\in\Delta_{2^n}$, we can easily obtain the following result.

Proposition 1 *If* $f(p_1,\cdots,p_n)=p,\ (p_1,\cdots,p_n)\in\mathscr{D}^n$, *then* $\delta_{2^n}^{p+1}=\delta_2^{p_n+1}\ltimes\cdots\ltimes\delta_2^{p_1+1}\in\Delta_{2^n}$ *holds.*

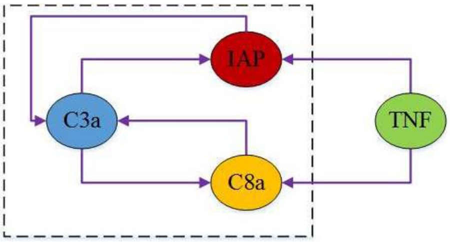

Fig. 1. Network of the cell apoptosis network (5).

Based on Proposition 1, we present the following algorithm to determine the solution set for the state inequality constraint (3).

Input: a, b
Output: C_x
1: $C_x \leftarrow \emptyset$
2: for all $a \leq s \leq b$ **do**
3: Decompose $\delta_{2^n}^{s+1}$ **as** $\delta_{2^n}^{s+1} = \delta_2^{s_n+1} \ltimes \cdots \ltimes \delta_2^{s_1+1}$
4: $C_x \leftarrow C_x \cup \{\ltimes_{i=1}^{n} \delta_2^{2-s_i}\}$
5: end for

Algorithm 1. Construction of the solution set for the state inequality constraint (3).

Similar to DBCN (2), we can convert the NSDC (7) into the algebraic form:

$$v(t) = G(t_h)y(t_h), \ t \in [t_h, t_{h+1})|_{\mathbb{Z}}, \tag{10}$$

where $G(t_h) \in \mathscr{L}_{2^m \times 2^{n\varsigma}}$ is referred to as the time-variant state feedback sampled-data gain matrix.

In order to unify the dimensions of states in system (9), we convert system (9) into the augmented form

$$\begin{aligned} y(t+1) = & \ltimes_{i=t-\varsigma+2}^{t+1} x(i) \\ &= (1_{2^n}^{\top} \otimes I_{2^n})y(t)x(t+1) \\ &= (1_{2^n}^{\top} \otimes I_{2^n})y(t)Kv(t)y(t) \\ &= (1_{2^n}^{\top} \otimes I_{2^n})(I_{2^{n\varsigma}} \otimes K)W_{[2^m,2^{n\varsigma}]}v(t)y^2(t) \\ &= (1_{2^n}^{\top} \otimes I_{2^n})(I_{2^{n\varsigma}} \otimes K)W_{[2^m,2^{n\varsigma}]}(I_{2^m} \otimes M_{r,2^{n\varsigma}})v(t)y(t) \\ &:= \overline{K}v(t)y(t), \end{aligned} \tag{11}$$

where $\overline{K} = (1_{2^n}^{\top} \otimes I_{2^n})(I_{2^{n\varsigma}} \otimes K)W_{[2^m,2^{n\varsigma}]}(I_{2^m} \otimes M_{r,2^{n\varsigma}}) \in \mathscr{L}_{2^{n\varsigma} \times 2^{n\varsigma+m}}$.

Assume $C_x = \{\delta_{2^n}^{\psi_1}, \cdots, \delta_{2^n}^{\psi_\beta}\} \subseteq \Delta_{2^n}$, where $\psi_1 < \cdots < \psi_\beta$. Correspondingly, the states of (11) are constrained to

$$C_y = \{y = \ltimes_{i=1}^{\varsigma} x^i : x^i \in C_x\} := \{\delta_{2^{n\varsigma}}^{\mu_1}, \cdots, \delta_{2^{n\varsigma}}^{\mu_{\beta^\varsigma}}\}, \tag{12}$$

where $\mu_1 < \cdots < \mu_{\beta^\varsigma}$. For the nonempty set $\mathscr{E}_e \subseteq C_x$ in Definition 2, we denote

$$\widehat{\mathscr{E}_e} = \{y = \ltimes_{i=1}^{\varsigma} x^i : x^i \in \mathscr{E}_e\} \subseteq C_y. \tag{13}$$

Finally, we study the relation between nonuniform sampled-data inequality constrained set stabilizable to $\mathscr{E}_e$ of DBCN (2) and nonuniform sampled-data inequality constrained set stabilizable to $\widehat{\mathscr{E}_e}$ of system (11). Before this, we first provide the definition of nonuniform sampled-data inequality constrained set stabilizable to $\widehat{\mathscr{E}_e}$ for (11).

Definition 3 For the nonempty set $\widehat{\mathscr{E}_e} \subseteq C_y$, system (11) is said to be nonuniform sampled-data inequality constrained set stabilizable to $\widehat{\mathscr{E}_e}$, if $\forall\ y_0 \in C_y$, there exist an NSDC (10) and an integer $\widehat{T} > 0$ such that $y(t) \in \widehat{\mathscr{E}_e}, \forall\ t \geq \widehat{T}$, and $y(t) \in C_y, \forall\ t \geq 0$.

Then, we have the following result.

Proposition 2 *DBCN* (2) *is nonuniform sampled-data inequality constrained set stabilizable to* $\mathscr{E}_e$, *if and only if system* (11) *is nonuniform sampled-data inequality constrained set stabilizable to* $\widehat{\mathscr{E}_e}$.

Proof (Necessity) From Definition 2, for any $y_0 = \ltimes_{i=-\varsigma+1}^{0} x(i) \in C_y$, there exist a time-variant state feedback NSDC sequence and a positive integer T such that $x(t) \in \mathscr{E}_e,\ \forall\ t \geq T$, and $x(t) \in C_x,\ \forall\ t \geq 1$, which implies $y(t) = \ltimes_{i=t-\varsigma+1}^{t} x(i) \in \widehat{\mathscr{E}_e},\ \forall\ t \geq T + \varsigma - 1$, and $y(t) = \ltimes_{i=t-\varsigma+1}^{t} x(i) \in C_y,\ \forall\ t \geq 1$. From Definition 3, system (11) is nonuniform sampled-data inequality constrained set stabilizable to $\widehat{\mathscr{E}_e}$.

(Sufficiency) From Definition 3, for any $y_0 \in C_y$, there exist a time-variant state feedback NSDC sequence and a positive integer $\widehat{T}$ such that $y(t) \in \widehat{\mathscr{E}_e},\ \forall\ t \geq \widehat{T}$, and $y(t) \in C_y, \forall\ t \geq 1$. From the construction of $\widehat{\mathscr{E}_e}$ and the unique factorization of $y(t) = \ltimes_{i=t-\varsigma+1}^{t} x(i)$, we have $x(t) \in \mathscr{E}_e, \forall\ t \geq \widehat{T}$, and $x(t) \in C_x, \forall\ t \geq 1$. From Definition 2, DBCN (2) is nonuniform sampled-data inequality constrained set stabilizable to $\mathscr{E}_e$. □

Inequality constrained set reachability

In this section, by constructing an inequality constrained controllability matrix, we investigate the nonuniform sampled-data inequality constrained set reachability of the augmented system (11).

Firstly, we give the concept of nonuniform sampled-data inequality constrained set reachability of system (11).

Definition 4 Given a nonempty set $\widehat{\mathscr{E}}_d \subseteq C_y$ and an initial state $y_0 \in C_y$. $\widehat{\mathscr{E}}_d$ is called nonuniform sampled-data inequality constrained set reachable from y_0 at sampling point t_h under NSDC, if one can find an NSDC sequence $\{v(0), v(1), \cdots, v(t_h-1)\} \subseteq \Delta_{2^m}$ such that $y(t_h) \in \widehat{\mathscr{E}}_d$ and $y(t) \in C_y, \forall\ 1 \le t \le t_h$.

Secondly, we construct the inequality constrained controllability matrix.

For system (11), split $\overline{K}$ into 2^m equal blocks as

$$\overline{K} = [Blk_1(\overline{K}) \ \cdots \ Blk_{2^m}(\overline{K})]. \tag{14}$$

Then, $Blk_i(\overline{K})$ corresponds to the control $\delta_{2^m}^i, i = 1, \cdots, 2^m$. In order to reduce the computational complexity caused by state inequality constraints, define $E \in \mathbb{R}^{2^{n_\varsigma} \times 2^{n_\varsigma}}$ with

$$Row_i(E) = \begin{cases} (\delta_{2^{n_\varsigma}}^i)^\top, & i \in \{\mu_1, \cdots, \mu_{\beta^\varsigma}\}, \\ 0_{2^{n_\varsigma}}^\top, & \text{otherwise}. \end{cases} \tag{15}$$

Let

$$Blk_i(\widehat{\overline{K}}) = E\big(Blk_i(\overline{K})\big)E^\top, i = 1, \cdots, 2^m. \tag{16}$$

Intuitively, $Blk_i(\widehat{\overline{K}})$ is obtained from $Blk_i(\overline{K})$ by substituting zeros in the corresponding rows and columns with indices $\{1, \cdots, 2^{n_\varsigma}\} \setminus \{\mu_1, \cdots, \mu_{\beta^\varsigma}\}$. Then, the inequality constrained controllability matrix is constructed as follows:

$$Q_{\tau_h} = \sum_{i=1}^{2^m} \left(Blk_i(\widehat{\overline{K}})\right)^{\tau_h}, \tag{17}$$

where $\tau_h = t_{h+1} - t_h, h \in \mathbb{N}$.

Finally, we present a criterion for the nonuniform sampled-data inequality constrained set reachability by (17).

Theorem 1 *Given a nonempty set $\widehat{\mathscr{E}}_d \subseteq C_y$ and an initial state $y(0) = \delta_{2^{n_\varsigma}}^{\mu_\theta} \in C_y$. $\widehat{\mathscr{E}}_d$ is nonuniform sampled-data inequality constrained set reachable from y(0) at sampling point t_h under NSDC, if and only if*

$$\sum_{\delta_{2^{n_\varsigma}}^{\mu_{h_i}} \in \widehat{\mathscr{E}}_d} (Q_{\tau_{h-1}} \cdots Q_{\tau_0})_{\mu_{h_i}, \mu_\theta} \ge 1. \tag{18}$$

Proof (Necessity) Assuming that $\widehat{\mathscr{E}}_d$ is inequality constrained set reachable from $y(0) = \delta_{2^{n_\varsigma}}^{\mu_\theta}$ at t_h under NSDC, we prove (18) by induction.

For $h = 1$, from Definition 4, there exist $v(0) = \delta_{2^m}^{\xi_0}, \cdots, v(t_1 - 1) = \delta_{2^m}^{\xi_0}$ and $y(t_1) = \delta_{2^{n_\varsigma}}^{\mu_{\bar{1}_i}} \in \widehat{\mathscr{E}}_d$ such that $\delta_{2^{n_\varsigma}}^{\mu_{\bar{1}_i}} = \left(Blk_{\xi_0}(\widehat{\overline{K}})\right)^{\tau_0} \delta_{2^{n_\varsigma}}^{\mu_\theta}$ and $y(t) \in C_y, \forall\ 1 \le t \le t_1$. Thus, $\left(\left(Blk_{\xi_0}(\widehat{\overline{K}})\right)^{\tau_0}\right)_{\mu_{\bar{1}_i}, \mu_\theta} = 1$, which shows that

$$\sum_{\delta_{2^{n_\varsigma}}^{\mu_{1_i}} \in \widehat{\mathscr{E}}_d} (Q_{\tau_0})_{\mu_{1_i}, \mu_\theta} \ge \left(\sum_{j=1}^{2^m} \left(Blk_j(\widehat{\overline{K}})\right)^{\tau_0}\right)_{\mu_{\bar{1}_i}, \mu_\theta} \ge \left(\left(Blk_{\xi_0}(\widehat{\overline{K}})\right)^{\tau_0}\right)_{\mu_{\bar{1}_i}, \mu_\theta} = 1, \tag{19}$$

that is, (18) holds for $h = 1$.

Assume that (18) holds for some $h = \lambda > 1$, that is

$$\sum_{\delta_{2^{n_\varsigma}}^{\mu_{\lambda_i}} \in \widehat{\mathscr{E}}_d} (Q_{\tau_{\lambda-1}} \cdots Q_{\tau_0})_{\mu_{\lambda_i}, \mu_\theta} \ge 1. \tag{20}$$

Then, there exist $v(t)|_{t=t_0}^{t_1-1} = \delta_{2^m}^{\xi_0}, \cdots, v(t)|_{t=t_{\lambda-1}}^{t_\lambda - 1} = \delta_{2^m}^{\xi_{\lambda-1}}$ and $y(t_\lambda) = \delta_{2^{n_\varsigma}}^{\mu_{\bar{\lambda}_i}} \in \widehat{\mathscr{E}}_d$ such that

$$\left(\left(Blk_{\xi_{\lambda-1}}(\widehat{\widehat{K}})\right)^{\tau_{\lambda-1}}\cdots\left(Blk_{\xi_0}(\widehat{\widehat{K}})\right)^{\tau_0}\right)_{\mu_{\tilde{\lambda}_i},\mu_\theta}=1, \tag{21}$$

where $v(t)|_{t=t_h}^{t_{h+1}-1}=\delta_{2^m}^{\xi_h}$ denotes an NSDC sequence $\{v(t_h)=\delta_{2^m}^{\xi_h},\cdots,v(t_{h+1}-1)=\delta_{2^m}^{\xi_h}\}\subseteq\Delta_{2^m}$, $h\in\mathbb{N}$.

We prove that (18) holds for $h=\lambda+1$. By (21) and Definition 4, there exist $v(t)|_{t=t_0}^{t_1-1}=\delta_{2^m}^{\xi_0},\ \cdots,\ v(t)|_{t=t_{\lambda-1}}^{t_\lambda-1}=\delta_{2^m}^{\xi_{\lambda-1}},\ v(t)|_{t=t_\lambda}^{t_{\lambda+1}-1}=\delta_{2^m}^{\xi_\lambda}$ and $y(t_{\lambda+1})=\delta_{2^{n_\varsigma}}^{\mu_{\widetilde{(\lambda+1)}_i}}\in\widehat{\mathscr{E}}_d$ such that the trajectory from $\delta_{2^{n_\varsigma}}^{\mu_\theta}$ to $\delta_{2^{n_\varsigma}}^{\mu_{\widetilde{(\lambda+1)}_i}}$ can be decomposed to the trajectory from $\delta_{2^{n_\varsigma}}^{\mu_\theta}$ to

$$y(t_\lambda)=\left(\left(Blk_{\xi_{\lambda-1}}(\widehat{\widehat{K}})\right)^{\tau_{\lambda-1}}\cdots\left(Blk_{\xi_0}(\widehat{\widehat{K}})\right)^{\tau_0}\right)\delta_{2^{n_\varsigma}}^{\mu_\theta}=\delta_{2^{n_\varsigma}}^{\mu_{\tilde{\lambda}_i}}\in C_y \tag{22}$$

at sampling point t_λ and the trajectory from $\delta_{2^{n_\varsigma}}^{\mu_{\tilde{\lambda}_i}}$ to $\delta_{2^{n_\varsigma}}^{\mu_{\widetilde{(\lambda+1)}_i}}$ in τ_λ steps. Then, from (19) and (22), we have

$$\left(\left(Blk_{\xi_\lambda}(\widehat{\widehat{K}})\right)^{\tau_\lambda}\right)_{\mu_{\widetilde{(\lambda+1)}_i},\mu_{\tilde{\lambda}_i}}=1\quad\text{and}\quad\left(\left(Blk_{\xi_{\lambda-1}}(\widehat{\widehat{K}})\right)^{\tau_{\lambda-1}}\cdots\left(Blk_{\xi_0}(\widehat{\widehat{K}})\right)^{\tau_0}\right)_{\mu_{\tilde{\lambda}_i},\mu_\theta}=1,$$

which show that

$$\begin{aligned}\sum_{\delta_{2^{n_\varsigma}}^{\mu_{(\lambda+1)_i}}\in\widehat{\mathscr{E}}_d}(Q_{\tau_\lambda}Q_{\tau_{\lambda-1}}\cdots Q_{\tau_0})_{\mu_{(\lambda+1)_i},\mu_\theta}&\geq\sum_{j=1}^{2^{n_\varsigma}}(Q_{\tau_\lambda})_{\mu_{\widetilde{(\lambda+1)}_i},j}(Q_{\tau_{\lambda-1}}\cdots Q_{\tau_0})_{j,\mu_\theta}\\&\geq\left(\left(Blk_{\xi_\lambda}(\widehat{\widehat{K}})\right)^{\tau_\lambda}\right)_{\mu_{\widetilde{(\lambda+1)}_i},\mu_{\tilde{\lambda}_i}}\left(\left(Blk_{\xi_{\lambda-1}}(\widehat{\widehat{K}})\right)^{\tau_{\lambda-1}}\cdots\left(Blk_{\xi_0}(\widehat{\widehat{K}})\right)^{\tau_0}\right)_{\mu_{\tilde{\lambda}_i},\mu_\theta}=1.\end{aligned}$$

Thus, (18) holds for $h=\lambda+1$. By induction, the necessity is proved.

(Sufficiency) Assume that (18) holds, that is,

$$\sum_{\delta_{2^{n_\varsigma}}^{\mu_{h_i}}\in\widehat{\mathscr{E}}_d}(Q_{\tau_{h-1}}\cdots Q_{\tau_0})_{\mu_{h_i},\mu_\theta}=\sum_{\delta_{2^{n_\varsigma}}^{\mu_{h_i}}\in\widehat{\mathscr{E}}_d}\left(\sum_{j=1}^{2^m}\left(Blk_j(\widehat{\widehat{K}})\right)^{\tau_{h-1}}\cdots\sum_{j=1}^{2^m}\left(Blk_j(\widehat{\widehat{K}})\right)^{\tau_0}\right)_{\mu_{h_i},\mu_\theta}\geq 1.$$

Then, there exist $v(t)|_{t=t_0}^{t_1-1}=\delta_{2^m}^{\xi_0},\cdots,v(t)|_{t=t_{h-1}}^{t_h-1}=\delta_{2^m}^{\xi_{h-1}}$ and $y(t_h)=\delta_{2^{n_\varsigma}}^{\mu_{\tilde{h}_i}}\in\widehat{\mathscr{E}}_d$ such that

$$\left(\left(Blk_{\xi_{h-1}}(\widehat{\widehat{K}})\right)^{\tau_{h-1}}\cdots\left(Blk_{\xi_0}(\widehat{\widehat{K}})\right)^{\tau_0}\right)_{\mu_{\tilde{h}_i},\mu_\theta}=1.$$

Thus, $y(t_h)=\delta_{2^{n_\varsigma}}^{\mu_{\tilde{h}_i}}=\left(\left(Blk_{\xi_{h-1}}(\widehat{\widehat{K}})\right)^{\tau_{h-1}}\cdots\left(Blk_{\xi_0}(\widehat{\widehat{K}})\right)^{\tau_0}\right)\delta_{2^{n_\varsigma}}^{\mu_\theta}\in\widehat{\mathscr{E}}_d$.

Finally, we prove $y(t)\in C_y,\ \forall\ 1\leq t\leq t_h$ by reduction to absurdity. In fact, if there exists $1\leq t'<t_h$ satisfying $y(t')\notin C_y$, then by (16), $y(t)=0_{2^{n_\varsigma}}$ for any $t'<t\leq t_h$, which is a contradiction to $y(t_h)\in\widehat{\mathscr{E}}_d$.

Thus, from Definition 4, $\widehat{\mathscr{E}}_d$ is inequality constrained set reachable from y(0) at t_h under NSDC. □

Example 2 Consider DBCN (2) with equivalent algebraic form (11), where $\varsigma=1$, $C_x=\Delta_{2^3}$, and $\overline{K}=\delta_8[1\ 7\ 8\ 3\ 4\ 5\ 7\ 6\ 3\ 6\ 3\ 2\ 4\ 6\ 7\ 8]$. Assume $\widehat{\mathscr{E}}_d=\{\delta_8^7\}, y(0)=\delta_8^1$.

(i) Suppose $\tau_h=1,\forall\ h\in\mathbb{N}$. According to Theorem 1, $\widehat{\mathscr{E}}_d$ is inequality constrained set reachable from δ_8^1 at time $t=7$ under the traditional state feedback control sequence $\{v(0)=\delta_2^2,v(1)=\delta_2^1,v(2)=\delta_2^1,v(3)=\delta_2^1,v(4)=\delta_2^2,v(5)=\delta_2^2,v(6)=\delta_2^1\}$;
(ii) Suppose $\tau_h=\tau=2,\forall\ h\in\mathbb{N}$, where $\tau=2$ is the sampling period. According to Theorem 1, $\widehat{\mathscr{E}}_d$ is unreachable from δ_8^1 under any state feedback uniform SDC sequence;
(iii) Suppose $\tau_0=2,\tau_1=4,\tau_2=1,\tau_3=2,\cdots$. According to Theorem 1, $\widehat{\mathscr{E}}_d$ is inequality constrained set reachable from δ_8^1 at sampling point t_4 under the state feedback NSDC sequence $\{v(t)|_{t=t_0}^{t_1-1}=\delta_2^2,v(t)|_{t=t_1}^{t_2-1}=\delta_2^1,v(t)|_{t=t_2}^{t_3-1}=\delta_2^2,v(t)|_{t=t_3}^{t_4-1}=\delta_2^1\}$.

Remark 1 Traditional state feedback control[15] and state feedback uniform SDC[37] can be viewed as a special case of NSDC. NSDC can provide us with more control schemes to achieve control objectives and reduce control costs (see Fig. 2).

Inequality constrained set stabilization

Based on the inequality constrained set reachability, we investigate the nonuniform sampled-data inequality constrained set stabilization.

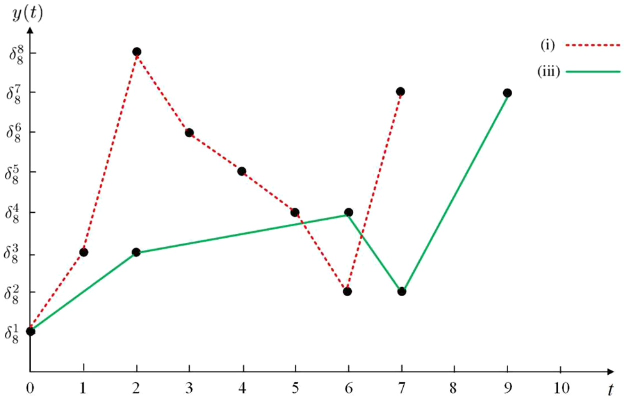

Fig. 2. State trajectories in (i) and (iii).

Firstly, we introduce the concept of the largest nonuniform sampled-data inequality constrained invariant subset.

Definition 5 A nonempty subset $\mathscr{E}' \subseteq C_y$ is referred to as a nonuniform sampled-data inequality constrained invariant subset, if for any $y(0) \in \mathscr{E}'$, one can find a state feedback NSDC $v(t_0) \in \Delta_{2^m}$ such that $y(t) \in \mathscr{E}'$ holds for any $t \leq \overline{\tau}, \overline{\tau} = \max\{\tau_h, h \in \mathbb{N}\}$.

From Definition 5, the union of inequality constrained invariant subsets is still an inequality constrained invariant subset.

Definition 6 Given a nonempty set $\mathscr{E} \subseteq C_y$, $I(\mathscr{E})$ is said to be the largest nonuniform sampled-data inequality constrained invariant subset of $\mathscr{E}$, if $I(\mathscr{E})$ is the union of all nonuniform sampled-data inequality constrained invariant subsets contained in $\mathscr{E}$.

From Definition 6, the construction method of the largest nonuniform sampled-data inequality constrained invariant subset $I(\mathscr{E})$ is given as fellow:

(i) Set $\Gamma_0 := \{\mu : \delta_{2^{n_\varsigma}}^{\mu} \in \mathscr{E}\}$;

(ii) Set $\Gamma_j := \Big\{\mu \in \Gamma_{j-1} :$ there exists an integer $0 < \xi_\mu \leq 2^m,$ such that $\widetilde{\bigwedge}_{l=1}^{\overline{\tau}}\Big(\sum_{i\in\Gamma_{j-1}}\big((Blk_{\xi_\mu}(\widehat{\overline{K}}))^l\big)_{i,\mu}\Big) = 1\Big\}, \ j \in \mathbb{Z}_+;$

(iii) Find the smallest positive integer $q \leq |\mathscr{E}|$ such that $\Gamma_q = \Gamma_{q+1}$;

(iv) $I(\mathscr{E}) = \{\delta_{2^{n_\varsigma}}^{\mu} : \mu \in \Gamma_q\}$.

Secondly, given $\chi \in \mathbb{Z}_+$, we define the nonuniform sampled-data inequality constrained reachable sets as

$$E_{\tau_{\chi-1}}(I(\widehat{\mathscr{E}}_e)) = \Big\{\delta_{2^{n_\varsigma}}^{\alpha} \in C_y : \text{there exists an integer } 1 \leq \xi_\alpha \leq 2^m, \text{such that} \sum_{\delta_{2^{n_\varsigma}}^{\mu_{\chi_i}} \in I(\widehat{\mathscr{E}}_e)} \big((Blk_{\xi_\alpha}(\widehat{\overline{K}}))^{\tau_{\chi-1}}\big)_{\mu_{\chi_i},\alpha} = 1\Big\},$$

and

$$E_{\tau_{\chi-2}+\tau_{\chi-1}}(I(\widehat{\mathscr{E}}_e)) = \Big\{\delta_{2^{n_\varsigma}}^{\alpha} \in C_y : \text{there exists an integer} 1 \leq \xi_\alpha \leq 2^m, \text{ such that} \sum_{\delta_{2^{n_\varsigma}}^{\mu_{(\chi-1)_i}} \in E_{\tau_{\chi-1}}(I(\widehat{\mathscr{E}}_e))} \big((Blk_{\xi_\alpha}(\widehat{\overline{K}}))^{\tau_{\chi-2}}\big)_{\mu_{(\chi-1)_i},\alpha} = 1\Big\}.$$

Keeping this procedure going, we define

$$E_{\Sigma_{i=0}^{\chi-1}\tau_i}(I(\widehat{\mathscr{E}}_e)) = \Big\{\delta_{2^{n\varsigma}}^{\alpha} \in C_y : \text{there exists an integer} 1 \leq \xi_\alpha \leq 2^m, \text{ such that } \sum_{\delta_{2^{n\varsigma}}^{\mu_{1_i}} \in E_{\Sigma_{i=1}^{\chi-1}\tau_i}(I(\widehat{\mathscr{E}}_e))} \Big(\big(Blk_{\xi_\alpha}(\widehat{\overline{K}})\big)^{\tau_0}\Big)_{\mu_{1_i},\alpha} = 1\Big\}.$$

Thus, by the construction process of the largest nonuniform sampled-data inequality constrained invariant subset, if $I(\widehat{\mathscr{E}}_e) \neq \emptyset$, $\delta_{2^{n\varsigma}}^{\alpha} \in E_{\Sigma_{i=0}^{\chi-1}\tau_i}(I(\widehat{\mathscr{E}}_e))$, then $\delta_{2^{n\varsigma}}^{\alpha} \in E_{\Sigma_{i=0}^{q}\tau_i}(I(\widehat{\mathscr{E}}_e)), \forall\, q \geq \chi - 1$. Hence, we have the following result on the inequality constrained reachable sets.

Proposition 3 *If $I(\widehat{\mathscr{E}}_e) \neq \emptyset$, then $E_{\Sigma_{i=0}^{\chi-1}\tau_i}(I(\widehat{\mathscr{E}}_e)) \subseteq E_{\Sigma_{i=0}^{\chi}\tau_i}(I(\widehat{\mathscr{E}}_e))$ holds for any $\chi \in \mathbb{Z}_+$.*

Finally, based on the inequality constrained reachable sets, we provide the result on the inequality constrained set stabilization.

Theorem 2 *System (11) is nonuniform sampled-data inequality constrained set stabilizable to $\widehat{\mathscr{E}}_e$ under a time-variant state feedback NSDC (10), if and only if*

(i) $I(\widehat{\mathscr{E}}_e) \neq \emptyset$;
(ii) there exists an integer $1 \leq \chi \leq \beta^{\varsigma}$ such that $E_{\Sigma_{i=0}^{\chi-1}\tau_i}(I(\widehat{\mathscr{E}}_e)) = C_y$.

Proof (Necessity) Obviously, from Definition 3, (i) holds. Now, we prove (ii).

From Definition 3, there exist an integer $T > 0$ and a time-variant state feedback NSDC such that

$$y(t) \in I(\widehat{\mathscr{E}}_e), \forall\, t \geq T,\ \forall\, y(0) \in C_y. \tag{23}$$

Take $\hat{T}$ to represent the smallest integer $T > 0$ that satisfies (23). By reduction to absurdity, we prove $\hat{T} < \beta^{\varsigma}\overline{\tau}$. If $\hat{T} \geq \beta^{\varsigma}\overline{\tau}$, we have $y(t_h) \notin I(\widehat{\mathscr{E}}_e), h = 0, \cdots, \beta^{\varsigma}$. However, since system (11) with the state inequality constraint has at most β^{ς} different states, there exist different $h_1, h_2 \in \{0, 1, \cdots, \beta^{\varsigma}\}$ such that $y(t_{h_1}) = y(t_{h_2})$. Hence, under the NSDC, starting from $y'(0) = y(t_{h_1})$, the state trajectory forms a cycle, which contradicts the fact that system (11) is inequality constrained set stabilizable to $\widehat{\mathscr{E}}_e$ under the NSDC.

Setting $\chi = \lfloor \hat{T}/\overline{\tau} \rfloor + 1 \leq \beta^{\varsigma}$, we have $E_{\sum_{i=0}^{\chi-1}\tau_i}(I(\widehat{\mathscr{E}}_e)) = C_y$. Thus, (ii) holds.

(Sufficiency) Assume that (i) and (ii) hold. For each $y(0) = \delta_{2^{n\varsigma}}^{\alpha} \in E_{\Sigma_{i=0}^{\chi-1}\tau_i}(I\,(\widehat{\mathscr{E}}_e)) = C_y$, there exists an integer $0 < \xi_\alpha \leq 2^m$ such that $E_{\Sigma_{i=1}^{\chi-1}\tau_i}(I(\widehat{\mathscr{E}}_e))$ is inequality constrained set reachable from $\delta_{2^{n\varsigma}}^{\alpha}$ in τ_0 steps under the NSDC $v(t_0) = \delta_{2^m}^{\xi_\alpha}$. Set $G(t_0) = \delta_{2^m}[\eta_1^{t_0}\ \cdots\ \eta_{2^{n\varsigma}}^{t_0}]$, where

$$\eta_i^{t_0} \in \begin{cases} \{\xi_\alpha\}, & \text{if } i = \alpha, \alpha = \mu_1, \cdots, \mu_{\beta^{\varsigma}}, \\ \{1, \cdots, 2^m\}, & \text{otherwise.} \end{cases} \tag{24}$$

Under the NSDC $v(t_0) = G(t_0)y(0)$, let

$$\begin{aligned}\Upsilon_{\tau_0}(y(0)) &= \Big\{y(t_1)\big|y(t_1) = (\overline{K}v(t_0))^{\tau_0}y(0) \in C_y, y(0) \in E_{\sum_{i=0}^{\chi-1}\tau_i}(\widehat{\mathscr{E}}_e)\Big\} \\ &:= \Big\{\delta_{2^{n\varsigma}}^{\alpha_1^1}, \cdots, \delta_{2^{n\varsigma}}^{\alpha_{\varrho_1}^1}\Big\} \subseteq E_{\Sigma_{i=1}^{\chi-1}\tau_i}(I(\widehat{\mathscr{E}}_e)).\end{aligned}$$

For each $\delta_{2^{n\varsigma}}^{\alpha_j^1} \in \Upsilon_{\tau_0}(y(0)), j = 1, \cdots, \varrho_1$, there exists an integer $1 \leq \xi_{\alpha_j^1} \leq 2^m$ such that $E_{\Sigma_{i=2}^{\chi-1}\tau_i}(I(\widehat{\mathscr{E}}_e))$ is inequality constrained set reachable from $y(t_1) = \delta_{2^{n\varsigma}}^{\alpha_j^1}$ in τ_1 steps under the NSDC $v(t_1) = \delta_{2^m}^{\xi_{\alpha_j^1}}$. Set

$G(t_1) = \delta_{2^m}[\eta_1^{t_1}\ \cdots\ \eta_{2^{n\varsigma}}^{t_1}]$, where

$$\eta_i^{t_1} \in \begin{cases} \{\xi_{\alpha_j^1}\}, & \text{if } i = \alpha_j^1,\ j = 1, \cdots, \varrho_1, \\ \{1, \cdots, 2^m\}, & \text{otherwise.} \end{cases} \tag{25}$$

Under the NSDC $v(t_1) = G(t_1)y(t_1)$, let

$$\begin{aligned}\Upsilon_{\tau_1+\tau_0}(y(0)) &= \Big\{y(t_2)\big|y(t_2) = (\overline{K}v(t_1))^{\tau_1}y(t_1) \in C_y, y(t_1) \in \Upsilon_{\tau_0}(y(0))\Big\} \\ &:= \Big\{\delta_{2^{n\varsigma}}^{\alpha_1^2}, \cdots, \delta_{2^{n\varsigma}}^{\alpha_{\varrho_2}^2}\Big\} \subseteq E_{\Sigma_{i=2}^{\chi-1}\tau_i}(I(\widehat{\mathscr{E}}_e)).\end{aligned}$$

Keeping this procedure going, we have $G(t_i) = \delta_{2^m}[\eta_1^{t_i} \ \cdots \ \eta_{2^{n\varsigma}}^{t_i}],\ i = 0, \cdots, \chi - 1$. Under the NSDC

$$v(t_{\chi-1}) = G(t_{\chi-1})y(t_{\chi-1}),$$

let

$$\Upsilon_{\Sigma_{i=0}^{\chi-1}\tau_i}(y(0)) = \left\{y(t_\chi)\middle|y(t_\chi) = (\overline{K}v(t_{\chi-1}))^{\tau_{\chi-1}}y(t_{\chi-1}) \in C_y, y(t_{\chi-1}) \in \Upsilon_{\Sigma_{i=0}^{\chi-2}\tau_i}(y(0))\right\} \subseteq I(\widehat{\mathscr{E}}_e).$$

From (i) and the largest nonuniform sampled-data inequality constrained invariant subset, for each $\delta_{2^{n\varsigma}}^{\alpha} \in I(\widehat{\mathscr{E}}_e)$, there exists an integer $0 < \xi_\alpha \leq 2^m$ such that $y(t) = (\overline{K}\delta_{2^m}^{\xi_\alpha})^{t-t_h}\delta_{2^{n\varsigma}}^{\alpha} \in I(\widehat{\mathscr{E}}_e), \forall\ t_h \leq t \leq t_{h+1}, \forall\ h \geq \chi$. Set $G(t_h) = \delta_{2^m}[\eta_1^{t_\chi} \ \cdots \ \eta_{2^{n\varsigma}}^{t_\chi}]$, where

$$\eta_i^{t_\chi} \in \left\{ \begin{array}{l} \{\xi_\alpha\},\ \text{if}\ i = \alpha, \\ \{1, \cdots, 2^m\},\ \text{otherwise}. \end{array} \right. \tag{26}$$

For each $y(t_h) \in I(\widehat{\mathscr{E}}_e)$, under the NSDC $v(t_h) = \delta_{2^m}[\eta_1^{t_\chi} \ \cdots \ \eta_{2^{n\varsigma}}^{t_\chi}]y(t_h)$, we have $y(t) \in I(\widehat{\mathscr{E}}_e), \forall\ t_h \leq t \leq t_{h+1},\ h \geq \chi$.

Thus, we obtain the time-variant state feedback NSDC as follows:

$$v(t) = G(t_h)y(t_h) = \left\{ \begin{array}{l} \delta_{2^m}[\eta_1^{t_h} \ \cdots \ \eta_{2^{n\varsigma}}^{t_h}]y(t_h),\ h = 0, \cdots, \chi - 1, \\ \delta_{2^m}[\eta_1^{t_\chi} \ \cdots \ \eta_{2^{n\varsigma}}^{t_\chi}]y(t_h),\ h \geq \chi. \end{array} \right. \tag{27}$$

$t \in [t_h, t_{h+1})|_{\mathbb{Z}}$, under which system (11) is nonuniform sampled-data inequality constrained set stabilizable to $\widehat{\mathscr{E}}_e$. $\square$

Corollary 1 *For $\mathscr{E}_e \subseteq C_x$ given in Definition 2, DBCN (2) is nonuniform sampled-data inequality constrained set stabilizable to $\mathscr{E}_e$ by a time-variant state feedback NSDC (7), if and only if (i) and (ii) of Theorem 2 hold.*

Remark 2 The state constraints in BNs are generally directly given a state constraint set, while state inequality constraints require solving the state constraint set based on the constraints satisfied by the state. Meanwhile, this paper provides a method for determining the state constraint set based on the state inequality constraints. This provides technical support for studying the stabilization problem of BNs under different constraint conditions.

Illustrative example

Example 3 Consider the apoptosis network (5):

$$\left\{ \begin{array}{l} x_1(t+1) = \neg x_2(t-1) \wedge v(t), \\ x_2(t+1) = \neg x_1(t-1) \wedge x_3(t-1), \\ x_3(t+1) = x_2(t-1) \vee v(t), \end{array} \right. \tag{28}$$

with the state time delay $\varsigma = 2$ and the state inequality constraint

$$2 \leq 2^0 x_1(i) + 2^1 x_2(i) + 2^2 x_3(i) \leq 4,\ i = -1, 0, 1, \cdots. \tag{29}$$

Setting $x(t) = \ltimes_{i=1}^3 x_i(t), y(t) = x(t-1) \ltimes x(t)$, from system (11), we have

$$y(t+1) = \overline{K}v(t)y(t), \tag{30}$$

where

$$\begin{aligned} \overline{K} = \delta_{64}[&7\ 15\ 23\ 31\ 39\ 47\ 55\ 63\ 7\ 15\ 23\ 31\ 39\ 47\ 55\ 63\ 3\ 11\ 19\ 27\ 35\ 43\ 51\ 59\ 3\ 11\ 19\ 27\ 35\ 43\ 51\ 59 \\ &5\ 13\ 21\ 29\ 37\ 45\ 53\ 61\ 7\ 15\ 23\ 31\ 39\ 47\ 55\ 63\ 1\ \ 9\ 17\ 25\ 33\ 41\ 49\ 57\ 3\ 11\ 19\ 27\ 35\ 43\ 51\ 59 \\ &7\ 15\ 23\ 31\ 39\ 47\ 55\ 63\ 7\ 15\ 23\ 31\ 39\ 47\ 55\ 63\ 8\ 16\ 24\ 32\ 40\ 48\ 56\ 64\ 8\ 16\ 24\ 32\ 40\ 48\ 56\ 64 \\ &5\ 13\ 21\ 29\ 37\ 45\ 53\ 61\ 7\ 15\ 23\ 31\ 39\ 47\ 55\ 63\ 6\ 14\ 22\ 30\ 38\ 46\ 54\ 62\ 8\ 16\ 24\ 32\ 40\ 48\ 56\ 64]. \end{aligned}$$

By Algorithm 1, we have $C_x = \{\delta_8^2, \delta_8^6, \delta_8^7\}$.

Next, we study the nonuniform sampled-data inequality constrained set stabilization of system (28) with $\mathscr{E}_e = \{\delta_8^6, \delta_8^7\}$

and

$$\tau_h = \left\{ \begin{array}{l} 2,\ h = 2i, \\ 1,\ h = 2i + 1, \end{array} \right.\ i \in \mathbb{N}. \tag{31}$$

From (12) and (13), we have $C_y = \{\delta_{64}^{10}, \delta_{64}^{14}, \delta_{64}^{15}, \delta_{64}^{42}, \delta_{64}^{46}, \delta_{64}^{47}, \delta_{64}^{50}, \delta_{64}^{54}, \delta_{64}^{55}\}$ and $\widehat{\mathscr{E}}_e = \{\delta_{64}^{46}, \delta_{64}^{47}, \delta_{64}^{54}, \delta_{64}^{55}\}$.

By the proof process of Theorem 1, we have $I(\widehat{\mathscr{E}}_e) = \widehat{\mathscr{E}}_e$ and $E_{\tau_0}(I(\widehat{\mathscr{E}}_e)) = C_y$. From Corollary 1, system (28) is inequality constrained set stabilizable to $\mathscr{E}_e$. In addition, by Theorem 2, the time-variant state feedback sampled-data gain matrix is designed as

$$G(t_0) = \delta_2[\eta_1^{t_0} \ \cdots \ \eta_{64}^{t_0}], \ G(t_h) = \delta_2[\eta_1^{t_h} \ \cdots \ \eta_{64}^{t_h}], \tag{32}$$

where

$$\eta_i^{t_0} \in \begin{cases} \{2\}, \ i = 47, 50, 54, 55 \\ \{1, 2\}, \ \text{otherwise} \end{cases}, \ \eta_i^{t_h} \in \begin{cases} \{2\}, \ i = 47, 54, 55 \\ \{1, 2\}, \ \text{otherwise} \end{cases}, \quad h \geq 1.$$

Remark 3 According to Example 3, the convergence speed of the proposed control algorithm can be controlled by (17). Using the traditional control[15], it needs two state feedback controllers to make all states reach $\mathscr{E}_e$. However, using the nonuniform sampled-data control, it only need one state feedback controller to make all states reach $\mathscr{E}_e$. Therefore, it reduces the frequency of controller updates, and the amount of calculation will be reduced.

Conclusions

In this paper, we have analyzed the nonuniform sampled-data set stabilization of DBCNs with state inequality constraints via time-variant state feedback NSDC. We have presented an effective criterion for the nonuniform sampled-data inequality constrained set reachability of DBCNs under NSDC by constructing an inequality constrained controllability matrix. By virtue of the inequality constrained reachable set and the largest inequality constrained invariant subset, we have proposed a procedure to design time-variant state feedback nonuniform sampled-data stabilizers for DBCNs. In future works, we will further investigate the stabilization and synchronization of stochastic Boolean networks with state inequality constraints by establishing a new algebraic representation. It is worth pointing out that stochastic Boolean networks with state inequality constraints have more possibilities in the state transition process, which will bring greater challenges to research.

Data availability

Data is provided within the manuscript or supplementary information files.

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Author contributions
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Competing interests
The authors declare no competing interests.

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