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Anti-windup control of discrete time switched delay systems with actuator saturation and failures

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The problem of L_2 -gain analysis and anti-windup (AW) fault-tolerant controller design of a class of time-varying delay discrete-time switched systems with actuator saturation and external disturbances is investigated by using the multiple Lyapunov functionals method. Firstly, for each subsystem, we construct an AW fault-tolerant controller consisting of a dynamic state feedback (DSF) controller and an AW compensator, such that the closed-loop system with actuator faults can meet the disturbance attenuation performance index and ensure that the state trajectories of the closed-loop system are bounded under the action of external disturbances. Then, the problem of estimating the allowable interference capacity is transformed into a constrained optimization problem. Next, a sufficient condition on the existence of the restricted L_2 -gain is established, and the minimum upper bound of the restricted L_2 -gain is obtained by solving the constrained optimization problem. Finally, the DSF controller gain and AW compensator gain of the AW fault-tolerant controller are obtained by solving the above two optimization problems which have been further handled. A numerical example is given to show the effectiveness and feasibility of the proposed method.

Keywords Multiple lyapunov functionals, Time-varying delay, Switched systems, Anti-windup fault-tolerant controllers, L_2 -gain

Switched systems are a very important class of hybrid systems that consist of a finite number of subsystems and a switching signal for autonomously selecting a particular subsystem at different moments^{1–3}. Due to the wide application of switched system in practical engineering, in recent years, it has attracted extensive attention from scholars^{4–6}. The main analysis tools for studying switched systems are multiple Lyapunov functions method⁷, the switched Lyapunov function method⁸, and the average dwell time scheme⁹. The multiple Lyapunov functions method was used to study the fault estimation problem of non-uniformly sampled systems with actuator faults and bounded disturbances in⁷. By utilizing switched Lyapunov function method⁸, considered the control problem of dynamically disturbed partially unknown nonlinear switched systems. By using the slow state feedback control method, the persistent dwell time (PDT) switching law was first proposed for a class of discrete-time singularly perturbed switched systems in⁹. Specifically, the multiple Lyapunov functions method has been demonstrated as a more powerful and effective tool for identifying a class of useful switching laws.

What's more, switched systems will inevitably encounter various external disturbances in practice, so L_2 -gain analysis and design of switched systems have become a very meaningful research topic¹⁰. Ref¹¹. considered the multiple Lyapunov functions method to study the stability and L_2 -gain analysis of a class of switched linear systems. The multiple Lyapunov functions method was used to study the finite-time analysis of stability and L_2 -gain for mode unstable switched systems in¹². Meanwhile, it is well known that actuator saturation is unavoidable for practical control systems, because the device as the actuator is subjected to a certain physical limit or the output amplitude of the actuator reaches the limit, which affects the performance of the system and even leads to the instability of the closed-loop system^{13–16}. In recent years, with the deep research of the switched systems, it is necessary to consider actuator saturation in the L_2 -gain analysis and control design of the switched systems. Therefore, more and more scholars pay attention to the L_2 -gain problem of switched system with actuator saturation^{17,18}. Ref¹⁹. investigated the L_2 -gain analysis and synthesis problem for a class of uncertain switched systems with actuator saturation, by using the switched Lyapunov function approach²⁰. investigated the problem of L_2 -gain analysis and anti-windup control for a class of switched systems with actuator saturation by using the single Lyapunov function method.

On the other hand, the effect of actuator failure on system stability is huge, or even brings disastrous consequences. Thus, the fault-tolerant control of switched systems has received extensive attention and

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the fault-tolerant control strategy is proposed from many scholars in recent years^{21–24}. In²⁵ a fault tolerant control strategy for uncertain switched systems that can be used in the case of additive actuator faults was proposed by using the multiple Lyapunov functions method²⁶, considered fault-tolerant control of a class of uncertain switched nonlinear systems against partial actuator failures by using the common Lyapunov function technique²⁷, used convex combination technology and linear matrix inequality to design the reliable controller and the corresponding switched strategy. In²⁸, an average dwell time method for fault-tolerant control of a class of uncertain switched linear time-delay systems was proposed, and the proposed control strategy ensures exponential stabilization of this type of time-delay systems with actuator faults.

In addition, time delay occurs frequently in practical engineering systems, and it is often a major factor causing system instability and performance decay. In²⁹, static anti-windup compensators for a class of saturated constrained nonlinear systems with both state and input time-varying delays were designed, by using the augmented Lyapunov–Krasovskii and sector conditions³⁰, was concerned with the anti-windup design problem for linear time-delay systems with actuator saturation by utilizing the information of the time delay sufficiently and the generalized delay-dependent sector conditions. Meanwhile, the control problem of time-delay switched systems has attracted the attention of scholars and some beneficial research results are obtained. For instance³¹, This article is concerned with the stability and stabilization of switched time-delay systems with exponential uncertainty based on an improved state-dependent switching strategy³², studied the problem of fault estimation for a class of discrete-time switched nonlinear systems with mixed time delay by using the multiple Lyapunov–Krasovskii functionals and the average dwell time methods. In³³, stability analysis and stabilization control of discrete-time impulsive switched time-delay systems with all unstable subsystems are discussed by the mode-dependent interval dwell-time switching rule.

To sum up, many control problems have been studied for switched systems by scholars. At present, there is less research on the problem of L_2 -gain analysis and anti-windup fault-tolerant control of discrete switched systems with time-varying delay and actuator saturation. However, the performance analysis and design of switched systems have become more complicated when switching signal, actuator failure, time delay and actuator saturation interact, which poses a great challenge to analysis and controller design. Switching signals can amplify the effects of actuator faults and saturation, particularly in the presence of time delays. Actuator faults, such as partial or complete loss of effectiveness, reduce the available control authority, while saturation limits the control input, leading to compounded performance degradation and potential instability. Time delays further exacerbate these issues by introducing additional dynamics that may destabilize the system or obscure fault symptoms. These interactions create a complex, nonlinear, and time-varying control environment, making it challenging to ensure stability and performance.

Thereby, we will study the L_2 -gain analysis and anti-windup fault-tolerant control of a class of discrete-time switched systems with time-varying and actuator saturation by using the multiple Lyapunov functionals approach. Firstly, the AW fault-tolerant controller consisting of a DSF controller and an AW compensator is constructed for each subsystem, and then a sufficient condition for the system state to be bounded with actuator faults and external disturbances is derived by using the multiple Lyapunov functionals method, and the maximum allowable level of disturbances is obtained by solving the constrained optimization problem. Then on this basis, the constrained L_2 -gain is analyzed for the closed-loop system. Next, the DSF controller and the anti-windup compensator are designed to ensure the maximization of the system tolerant interference capability and the minimization of the upper bound of the restricted L_2 -gain. Finally, a numerical example is given to show the effectiveness and feasibility of the proposed method.

Problem formulation and preliminaries

Consider a class of discrete-time time-varying delay switched systems with actuator saturation and external disturbances as follows

$$\begin{cases} x(k+1) = A_\sigma x(k) + A_{d\sigma} x(k-d(k)) + B_{1\sigma} \text{sat}(u(k)) + H_{1\sigma} w(k) \\ z(k) = C_\sigma x(k) \\ x(\theta) = \varphi(\theta), \theta = -d_2, d_2+1, \dots, 0 \end{cases} \quad (1)$$

where $\varphi(\theta)$ is the vector-valued initial function, $d(k)$ is a time-varying delay satisfying $0 \leq d_1 \leq d(k) \leq d_2$. A_σ , $A_{d\sigma}$, $B_{1\sigma}$, $H_{1\sigma}$ and C_σ are constant matrices with appropriate dimensions, $x(k) \in \mathbb{R}^n$ is the system state vector, $u(k) \in \mathbb{R}^q$ is the control input vector, $w(k) \in \mathbb{R}^h$ is the external disturbance input vector and $z(k) \in \mathbb{R}^p$ denotes the controlled output vector. $\sigma: [0, \infty) \rightarrow I_N = \{1, \dots, N\}$ denotes switching signal and it is a piecewise right continuous constant, and $\sigma = i$ means that the i -th subsystem is activated.

It is well known that the L_2 -gain can be used to analyze the disturbance attenuation ability of systems. However, for systems with actuator saturation, the large external perturbations will result in unbounded states and thus we make the following assumption

$$W_\beta^2 := \left\{ w: \mathbb{R}_+ \rightarrow \mathbb{R}^h, \sum_{k=0}^{\infty} w^T(k)w(k) \leq \beta \right\}, \quad (2)$$

where β is a positive number which indicates the allowable interference capability of the system. $\text{sat}: \mathbb{R}^q \rightarrow \mathbb{R}^q$ is the standard vector-valued saturation function. The definition is as follows

$$\begin{cases} \text{sat}(u) = [\text{sat}(u^1), \dots, \text{sat}(u^q)]^T, \\ \text{sat}(u^j) = \text{sign}(u^j) \min \{1, |u^j|\}, \\ \forall j \in V_q = \{1, \dots, q\}. \end{cases} \quad (3)$$

Obviously, without loss of generality, we assume unit saturation limits, since non-standard saturation functions can always be obtained by altering the matrix by employing appropriate transformations, and for simplicity, and in accordance with the notation prevalent in the literature, we adopt the notation $\text{sat}(\bullet)$ to denote both scalar and vector saturation functions.

Considering actuator failures, we introduce a fault matrix M_i , in which

$$M_i = \text{diag} \{m_{i1}, m_{i2}, \dots, m_{iq}\}, \quad 0 \leq m_{ijl} \leq m_{ij} \leq m_{iju} \leq 1, j \in V_q, \quad (4)$$

where m_{ijl} and m_{iju} are given constants, $m_{ij} = 0$ means that the j -th actuator is completely disabled, and $m_{ij} = 1$ indicates that the j -th actuator is normal. Further $0 < m_{iju} < m_{ij} < m_{ijl} < 1$ represents the j -th actuator partial failure.

$$\text{Let } M_{0i} = \text{diag} \{m_{0i1}, m_{0i2}, \dots, m_{0iq}\}, \quad m_{0ij} = \frac{m_{ijl} + m_{iju}}{2},$$

$$L_i = \text{diag} \{l_{i1}, l_{i2}, \dots, l_{iq}\}, \quad l_{ij} = \frac{m_{ij} - m_{0ij}}{m_{0ij}}.$$

Then, the resulting matrix M_i can be written as

$$M_i = M_{0i}(I + L_i), \quad |L_i| \leq I, \quad (5)$$

where $|L_i| = \text{diag} \{|l_{i1}|, |l_{i2}|, \dots, |l_{iq}|\}$.

From the above Eq. (5), it can be seen that for the subsystem i , the matrix M_{0i} is a constant matrix and the uncertainty of fault matrix M_i is only related to the matrix L_i , then the closed-loop system containing actuator faults is

$$\begin{cases} x(k+1) = A_i x(k) + A_{di} x(k-d(k)) + B_{1i} M_i \text{sat}(u(k)) + H_{1i} w(k), \\ z(k) = C_i x(k). \end{cases} \quad (6)$$

From³⁴, we construct the following AW fault-tolerant controller containing DSF controller and AW compensator

$$u(k+1) = F_i x + G_i u + E_{ci} M_i (\text{sat}(u(k)) - u(k)), \quad (7)$$

where $u(k)$ is the controller state, $E_{ci} \in R^{q \times q}$ is the AW compensator gain, and matrices $F_i \in R^{q \times n}$ and $G_i \in R^{q \times q}$ are the DSF controller gain.

From (6) and (7) the above closed-loop system can be rewritten as the following form

$$\begin{cases} x(k+1) = A_i x(k) + A_{di} x(k-d(k)) + B_{1i} M_i \text{sat}(u(k)) + H_{1i} w(k), \\ z(k) = C_i x(k), \\ u(k+1) = F_i x + G_i u + E_{ci} M_i (\text{sat}(u) - u). \end{cases} \quad (8)$$

Then, we define some new variables and matrices as follows

$$\begin{aligned} \zeta &= \begin{bmatrix} u \\ x \end{bmatrix} \in R^{q+n}, \quad \hat{A}_i = \begin{bmatrix} 0 & 0 \\ B_i M_{0i} & A_i \end{bmatrix}, \quad \hat{B}_i = \begin{bmatrix} 0 \\ B_i \end{bmatrix}, \quad \hat{A}_{di} = \begin{bmatrix} 0 & 0 \\ 0 & A_{di} \end{bmatrix}, \\ \hat{G}_i &= \frac{I + L_i}{2}, \quad \hat{G}_i^T \hat{G}_i \leq I, \quad \hat{H}_{1i} = \begin{bmatrix} 0 \\ H_{1i} \end{bmatrix}, \quad \hat{C}_i = [0 \quad C_i], \\ I_R &= \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad K_i = [G_i \quad F_i], \quad \hat{N}_i = [I \quad 0]. \end{aligned}$$

The resulting closed-loop system can be rewritten as

$$\begin{cases} \zeta(k+1) = (\hat{A}_i + I_R K_i + \hat{B}_i M_{0i} L_i I_R^T) \zeta(k) + \hat{A}_{di} \zeta(k-d(k)) \\ \quad + 2(\hat{B}_i + I_R E_{ci}) M_{0i} \hat{G}_i \psi(v) + \hat{H}_{1i} w(k), \\ z(k) = \hat{C}_i \zeta(k), \\ \zeta(\theta) = \hat{\varphi}(\theta), \quad \theta = -d_2, -d_2+1, \dots, 0, \end{cases} \quad (9)$$

where $\psi(v) = (\text{sat}(v) - v)$, $v = \hat{N}_i \zeta = u(k)$, and the vector-valued initial function $\hat{\varphi}(\theta)$ is magnitude and difference bounded or say that the initial function $\hat{\varphi}(\theta)$ belongs to the following domain.

$$\hat{\chi}_\rho = \left\{ \hat{\phi}(\theta) : \max_{[-d_2, 0]} \|\hat{\phi}(\theta)\|^2 \leq \kappa_{11}, \max_{[-d_2, 0]} \|\hat{\phi}(\theta + 1) - \hat{\phi}(\theta)\|^2 \leq \kappa_{22} \right\}$$

The following definitions and mathematical notation will be used for system (9):

Definition 1 Ref^{35,36}: Given $\gamma > 0$. The system (9) is said to have a restricted L_2 -gain less than γ , if there exists a switching law σ , the following condition is satisfied under the zero initial condition,

$$\sum_{k=0}^{\infty} z^T(k)z(k) < \gamma^2 \sum_{k=0}^{\infty} w^T(k)w(k), \quad (10)$$

for all non-zero $w(k) \in W_\beta^2$.

Definition 2 Ref³⁷: Let $P \in \mathbb{R}^{(q+n) \times (q+n)}$ represent the positive definite matrix and define the ellipsoid:

$$\Omega(P, \rho) = \left\{ \zeta \in \mathbb{R}^{q+n} : \zeta^T P \zeta \leq \rho, \rho > 0 \right\}. \quad (11)$$

Definition 3 Ref³⁷: For the matrix $\hat{N}_i, H_i \in \mathbb{R}^{q \times (q+n)}$, we define the following symmetric polyhedron.

$$L(\hat{N}_i, H_i) = \left\{ \zeta \in \mathbb{R}^{q \times n} : |(\hat{N}_i^j - H_i^j)\zeta| \leq 1, i \in I_N, j \in V_q \right\}, \quad (12)$$

where \hat{N}_i^j, H_i^j , denote the j -th row of the matrices \hat{N}_i and H_i , respectively.

Lemma 1 Ref³⁸: For the symmetric matrix.

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix},$$

where $S_{11} \in \mathbb{R}^{n \times n}, S_{12} = S_{21}^T \in \mathbb{R}^{n \times q}, S_{22} \in \mathbb{R}^{q \times q}$, the following three conditions are equivalent

- (1) $S < 0$
- (2) $S_{11} < 0, S_{22} - S_{12}^T S_{11}^{-1} S_{12} < 0$
- (3) $S_{22} < 0, S_{11} - S_{12} S_{22}^{-1} S_{12}^T < 0$

Lemma 2 Ref³⁹: Consider the function $\psi(v)$ defined above, if $\zeta \in L(\hat{N}_i, H_i)$, then the relation.

$$\psi(\hat{N}_i \zeta) J_i (\psi(\hat{N}_i \zeta) - H_i \zeta) \leq 0, \forall i \in I_N,$$

holds for any matrix $J_i \in \mathbb{R}^{q \times q}$ diagonal and positive definite.

Lemma 3 Ref⁴⁰: Let Y, U and V , be given matrices of appropriate dimensions, then for any matrix Γ satisfying $\Gamma^T \Gamma \leq I$,

$$Y + U \Gamma V + V^T \Gamma^T U^T < 0,$$

if and only if there exists a constant $\lambda > 0$ such that

$$Y + \lambda U U^T + \lambda^{-1} V^T V < 0. \quad (13)$$

Disturbance tolerance

In this section, by using the multiple Lyapunov functionals approach, a sufficient condition for the state trajectory of the closed-loop system (9) to be bounded is given under the assumption that the AW fault-tolerant controller gains are known. How to design the AW fault-tolerant controller problem will be given in section "Control synthesis solution".

Theorem 1: Consider the closed-loop switched system (9). For a given positive scalar ρ_1 , if there exist positive definite matrices $P_i, Q_1, Q_2, Q_3, Z_1, Z_2$, positive definite diagonal matrices J_i , and a set of scalars $\beta_{ir} > 0$, such that.

$$\begin{bmatrix} \Lambda_{i11} & \Lambda_{i12} & 0 & 0 & \Lambda_{i15} & \Lambda_{i16} \\ * & \Lambda_{i22} & 0 & 0 & \Lambda_{i25} & \Lambda_{i26} \\ * & * & -Q_1 & 0 & 0 & 0 \\ * & * & * & -Q_2 & 0 & 0 \\ * & * & * & * & \Lambda_{i55} & \Lambda_{i56} \\ * & * & * & * & * & \Lambda_{i66} \end{bmatrix} < 0, \quad (14)$$

$$i \in I_N,$$

and

$$\Omega(P_i, \rho_1 + \beta) \cap \Phi_i \subset L(\hat{N}_i, H_i), \quad i \in I_N, \quad (15)$$

where

$$\begin{aligned} \tilde{A}_i &= \hat{A}_i + I_R K_i + \hat{B}_i M_{0i} L_i I_R^T, \\ V_{1i} &= 2(\hat{B}_i + I_R E_{ci}) M_{0i} \hat{G}_i, \\ \Pi &= d_2 Z_1 + (d_2 - d_1) Z_2, \\ \Lambda_{i11} &= \tilde{A}_i^T P_i \tilde{A}_i - P_i + Q_1 + Q_2 + (d_2 - d_1 + 1) Q_3 + (\tilde{A}_i - I)^T \Pi (\tilde{A}_i - I) + \sum_{r=1, r \neq i}^N \beta_{ir} (P_r - P_i), \\ \Lambda_{i12} &= \tilde{A}_i^T P_i \hat{A}_{di} + (\tilde{A}_i - I)^T \Pi \hat{A}_{di}, \\ \Lambda_{i15} &= \tilde{A}_i^T P_i V_{1i} + (\tilde{A}_i - I)^T \Pi V_{1i} + H_i^T J_i^T, \\ \Lambda_{i16} &= \tilde{A}_i^T P_i \hat{H}_{1i} + (\tilde{A}_i - I)^T \Pi \hat{H}_{1i}, \\ \Lambda_{i22} &= \hat{A}_{di}^T P_i \hat{A}_{di} - Q_3 + \hat{A}_{di}^T \Pi \hat{A}_{di}, \\ \Lambda_{i25} &= \hat{A}_{di}^T P_i V_{1i} + \hat{A}_{di}^T \Pi V_{1i}, \\ \Lambda_{i26} &= \hat{A}_{di}^T P_i \hat{H}_{1i} + \hat{A}_{di}^T \Pi \hat{H}_{1i}, \\ \Lambda_{i55} &= V_{1i}^T P_i V_{1i} + V_{1i}^T \Pi V_{1i} - 2J_i, \\ \Lambda_{i56} &= V_{1i} P_i \hat{H}_{1i}^T + V_{1i} \Pi \hat{H}_{1i}^T, \\ \Lambda_{i66} &= \hat{H}_{1i}^T P_i \hat{H}_{1i} + \hat{H}_{1i}^T \Pi \hat{H}_{1i} - I, \\ \Phi_i &= \{\zeta \in R^{q \times n} : \zeta^T (P_r - P_i) \zeta \geq 0, \forall r \in I_N, r \neq i\}, \end{aligned}$$

and the initial conditions are satisfied

$$\begin{aligned} \rho_1 &\geq \kappa_{11} [\lambda_1 + d_1 \lambda_4 + d_2 \lambda_5 + 0.5 (d_2 - d_1 + 1) (d_2 + d_1) \lambda_6] \\ &+ \kappa_{22} [0.5 d_2 (d_2 + 1) \lambda_2 + 0.5 (d_2 - d_1) (d_2 + d_1 + 1) \lambda_3], \end{aligned} \quad (16)$$

where $\lambda_\nu, \nu = 1, 2, 3 \dots 6$, are positive scalars satisfying

$$\lambda_1 I \geq P_i, \quad \lambda_2 I \geq Z_1, \quad \lambda_3 I \geq Z_2, \quad \lambda_4 I \geq Q_1, \quad \lambda_5 I \geq Q_2, \quad \lambda_6 I \geq Q_3, \quad \forall i \in I_N,$$

then any trajectory of the system (9) starting from the region $\cup_{i=1}^N (\Omega(P_i, \rho_1) \cap \Phi_i)$ will remain inside the region $\cup_{i=1}^N (\Omega(P_i, \rho_1 + \beta) \cap \Phi_i)$ for every $\forall w \in W_\beta^2$, under the sate-dependent switching law

$$\sigma = \arg \min \{\zeta^T P_i \zeta, i \in I_N\}. \quad (17)$$

Proof: By condition (15), if $\forall \zeta \in \Omega(P_i, \rho_1 + \beta) \cap \Phi_i$, then $\zeta \in L(\hat{N}_i, H_i)$, therefore in view of Lemma 2, for $\forall \zeta \in \Omega(P_i, \rho_1 + \beta) \cap \Phi_i$ it follows that $\psi(v) = (sat(v) - v)$ satisfies $\psi(\hat{N}_i \zeta) J_i (\psi(\hat{N}_i \zeta) - H_i \zeta) \leq 0, \forall i \in I_N$.

In view of the switching law (17), for $\zeta \in \Omega(P_i, \rho_1 + \beta) \cap \Phi_i \subset L(\hat{N}_i, H_i)$, the i -th subsystem is active.

Choose the Lyapunov-Krasovskii functionals of the closed-loop system (9) as

$$V(\zeta(k)) = V_\sigma = V_{1\sigma}(\zeta(k)) + V_2(\zeta(k)) + V_3(\zeta(k)) + V_4(\zeta(k)) \quad (18)$$

where

$$\begin{aligned}
V_{1\sigma}(\zeta(k)) &= \zeta^T(k) P_{\sigma} \zeta(k), \\
V_2(\zeta(k)) &= \sum_{l=-d_1}^{-1} \zeta^T(k+l) Q_1 \zeta(k+l) + \sum_{l=-d_2}^{-1} \zeta^T(k+l) Q_2 \zeta(k+l), \\
V_3(\zeta(k)) &= \sum_{\theta=-d_2+1}^{-d_1+1} \sum_{l=\theta-1}^{-1} \zeta^T(k+l) Q_3 \zeta(k+l), \\
V_4(\zeta(k)) &= \sum_{\theta=-d_2+1}^0 \sum_{l=\theta-1}^{-1} y^T(k+l) Z_1 y(k+l) + \sum_{\theta=-d_2+1}^{-d_1} \sum_{l=\theta-1}^{-1} y^T(k+l) Z_2 y(k+l), \\
y(k) &= \zeta(k+1) - \zeta(k).
\end{aligned}$$

Case 1: $\sigma(k+1) = \sigma(k) = i$, for $\forall \zeta(k) \in \Omega(P_i, \rho_1 + \beta) \cap \Phi_i \subset L(\hat{N}_i, H_i)$. It follows that

$$\begin{aligned}
\Delta V_{1i}(\zeta(k)) &= V_{1i}(\zeta(k+1)) - V_{1i}(\zeta(k)) = \zeta^T(k+1) P_i \zeta(k+1) - \zeta^T(k) P_i \zeta(k), \\
\Delta V_2(\zeta(k)) &= V_2(\zeta(k+1)) - V_2(\zeta(k)) \\
&= \zeta^T(k) (Q_1 + Q_2) \zeta(k) - \zeta^T(k-d_1) Q_1 \zeta(k-d_1) - \zeta^T(k-d_2) Q_2 \zeta(k-d_2), \\
\Delta V_3(\zeta(k)) &= V_3(\zeta(k+1)) - V_3(\zeta(k)) = (d_2 - d_1 + 1) \zeta^T(k) Q_3 \zeta(k) - \sum_{\theta=-d_2}^{-d_1} \zeta^T(k+\theta) Q_3 \zeta(k+\theta) \\
&\leq (d_2 - d_1 + 1) \zeta^T(k) Q_3 \zeta(k) - \zeta^T(k-d(k)) Q_3 \zeta(k-d(k)), \\
\Delta V_4(\zeta(k)) &= V_4(\zeta(k+1)) - V_4(\zeta(k)) = y^T(k) \Pi y(k) - \sum_{\theta=-d(k)}^{-1} y^T(k+\theta) Z_1 y(k+\theta) \\
&\quad - \sum_{\theta=-d(k)}^{-d_1-1} y^T(k+\theta) Z_2 y(k+\theta) - \sum_{\theta=-d_2}^{-d(k)-1} y^T(k+\theta) (Z_1 + Z_2) y(k+\theta), \\
\Delta V(\zeta(k)) &= \Delta V_{1i}(\zeta(k)) + \Delta V_2(\zeta(k)) + \Delta V_3(\zeta(k)) + \Delta V_4(\zeta(k)) \\
&\leq \max_{i \in I_N} \zeta^T(k) \left[\tilde{A}_i^T P_i \tilde{A}_i - P_i + Q_1 + Q_2 + (d_2 - d_1 + 1) Q_3 + (\tilde{A}_i - I)^T \Pi (\tilde{A}_i - I) \right] \zeta(k) \\
&\quad + 2\zeta^T(k) \left[\tilde{A}_i^T P_i \hat{A}_{di} + (\tilde{A}_i - I)^T \Pi \hat{A}_{di} \right] \zeta(k-d(k)) \\
&\quad + 2\zeta^T(k) \left[\tilde{A}_i^T P_i V_{1i} + (\tilde{A}_i - I)^T \Pi V_{1i} \right] \psi(k) \\
&\quad + 2\zeta^T(k) \left[\tilde{A}_i^T P_i \hat{H}_{1i} + (\tilde{A}_i - I)^T \Pi \hat{H}_{1i} \right] w(k) \\
&\quad + \zeta^T(k-d(k)) \left[\hat{A}_{di}^T P_i \hat{A}_{di} + \hat{A}_{di}^T \Pi \hat{A}_{di} - Q_3 \right] \zeta(k-d(k)) \\
&\quad + 2\zeta^T(k-d(k)) \left[\hat{A}_{di}^T P_i V_{1i} + \hat{A}_{di}^T \Pi V_{1i} \right] \psi(k) \\
&\quad + 2\zeta^T(k-d(k)) \left[\hat{A}_{di}^T P_i \hat{H}_{1i} + \hat{A}_{di}^T \Pi \hat{H}_{1i} \right] w(k) \\
&\quad + \psi^T(k) \left[V_{1i}^T P_i V_{1i} + V_{1i}^T \Pi V_{1i} \right] \psi(k) \\
&\quad + 2\psi^T(k) \left[V_{1i}^T P_i \hat{H}_{1i} + V_{1i}^T \Pi \hat{H}_{1i} \right] w(k) \\
&\quad + w^T(k) \left[\hat{H}_{1i}^T P_i H_{1i} + \hat{H}_{1i}^T \Pi H_{1i} \right] w(k) \\
&\quad + \zeta^T(k-d_1) (-Q_1) \zeta(k-d_1) + \zeta^T(k-d_2) (-Q_2) \zeta(k-d_2) \\
&\quad - \sum_{\theta=-d(k)}^{-1} y^T(k+\theta) Z_1 y(k+\theta) - \sum_{\theta=-d(k)}^{-d_1-1} y^T(k+\theta) Z_2 y(k+\theta) \\
&\quad - \sum_{\theta=-d_2}^{-d(k)-1} y^T(k+\theta) (Z_1 + Z_2) y(k+\theta) - 2\psi^T(\hat{N}_i \zeta(k)) J_i \left[\psi(\hat{N}_i \zeta(k)) - H_i \zeta(k) \right].
\end{aligned}$$

Then, combining with Lemma 2 and condition (15), we have

$$\begin{aligned}
\Delta V(\zeta(k)) &= \Delta V_{1i}(\zeta(k)) + \Delta V_2(\zeta(k)) + \Delta V_3(\zeta(k)) + \Delta V_4(\zeta(k)) \\
&\leq \max_{i \in I_N} \zeta^T(k) \left[\tilde{A}_i^T P_i \tilde{A}_i - P_i + Q_1 + Q_2 + (d_2 - d_1 + 1) Q_3 + (\tilde{A}_i - I)^T \Pi (\tilde{A}_i - I) \right] \zeta(k) \\
&\quad + 2\zeta^T(k) \left[\tilde{A}_i^T P_i \hat{A}_{di} + (\tilde{A}_i - I)^T \Pi \hat{A}_{di} \right] \zeta(k - d(k)) \\
&\quad + 2\zeta^T(k) \left[\tilde{A}_i^T P_i V_{1i} + (\tilde{A}_i - I)^T \Pi V_{1i} \right] \psi(v) \\
&\quad + 2\zeta^T(k) \left[\tilde{A}_i^T P_i \hat{H}_{1i} + (\tilde{A}_i - I)^T \Pi \hat{H}_{1i} \right] w(k) \\
&\quad + \zeta^T(k - d(k)) \left[\hat{A}_{di}^T P_i \hat{A}_{di} + \hat{A}_{di}^T \Pi \hat{A}_{di} - Q_3 \right] \zeta(k - d(k)) \\
&\quad + 2\zeta^T(k - d(k)) \left[\hat{A}_{di}^T P_i V_{1i} + \hat{A}_{di}^T \Pi V_{1i} \right] \psi(v) \\
&\quad + 2\zeta^T(k - d(k)) \left[\hat{A}_{di}^T P_i \hat{H}_{1i} + \hat{A}_{di}^T \Pi \hat{H}_{1i} \right] w(k) \\
&\quad + \psi^T(v) \left[V_{1i}^T P_i V_{1i} + V_{1i}^T \Pi V_{1i} \right] \psi(v) \\
&\quad + 2\psi^T(v) \left[V_{1i}^T P_i \hat{H}_{1i} + V_{1i}^T \Pi \hat{H}_{1i} \right] w(k) \\
&\quad + w^T(k) \left[\hat{H}_{1i}^T P_i H_{1i} + \hat{H}_{1i}^T \Pi H_{1i} \right] w(k) \\
&\quad + \zeta^T(k - d_1) (-Q_1) \zeta(k - d_1) + \zeta^T(k - d_2) (-Q_2) \zeta(k - d_2) \\
&\quad - \sum_{\theta=-d(k)}^{-1} y^T(k + \theta) Z_1 y(k + \theta) - \sum_{\theta=-d(k)}^{-d_1-1} y^T(k + \theta) Z_2 y(k + \theta) \\
&\quad - \sum_{\theta=-d_2}^{-d(k)-1} y^T(k + \theta) (Z_1 + Z_2) y(k + \theta).
\end{aligned}$$

Multiplying (14) from the left by $\begin{bmatrix} \zeta(k) \\ \zeta(k - d(k)) \\ \zeta(k - d_1) \\ \zeta(k - d_2) \\ \psi(v) \\ w(k) \end{bmatrix}^T$ and then from the right by $\begin{bmatrix} \zeta(k) \\ \zeta(k - d(k)) \\ \zeta(k - d_1) \\ \zeta(k - d_2) \\ \psi(v) \\ w(k) \end{bmatrix}$, we have.

$$\begin{aligned}
&\zeta^T(k) \left[\tilde{A}_i^T P_i \tilde{A}_i - P_i + Q_1 + Q_2 + (d_2 - d_1 + 1) Q_3 + (\tilde{A}_i - I)^T \Pi (\tilde{A}_i - I) \right] \zeta(k) \\
&\quad + 2\zeta^T(k) \left[\tilde{A}_i^T P_i \hat{A}_{di} + (\tilde{A}_i - I)^T \Pi \hat{A}_{di} \right] \zeta(k - d(k)) \\
&\quad + 2\zeta^T(k) \left[\tilde{A}_i^T P_i V_{1i} + (\tilde{A}_i - I)^T \Pi V_{1i} \right] \psi(v) \\
&\quad + 2\zeta^T(k) \left[\tilde{A}_i^T P_i \hat{H}_{1i} + (\tilde{A}_i - I)^T \Pi \hat{H}_{1i} \right] w(k) \\
&\quad + \zeta^T(k - d(k)) \left[\hat{A}_{di}^T P_i \hat{A}_{di} + \hat{A}_{di}^T \Pi \hat{A}_{di} - Q_3 \right] \zeta(k - d(k)) \\
&\quad + 2\zeta^T(k - d(k)) \left[\hat{A}_{di}^T P_i V_{1i} + \hat{A}_{di}^T \Pi V_{1i} \right] \psi(v) \\
&\quad + 2\zeta^T(k - d(k)) \left[\hat{A}_{di}^T P_i \hat{H}_{1i} + \hat{A}_{di}^T \Pi \hat{H}_{1i} \right] w(k) \\
&\quad + \psi^T(v) \left[V_{1i}^T P_i V_{1i} + V_{1i}^T \Pi V_{1i} \right] \psi(v) \\
&\quad + 2\psi^T(v) \left[V_{1i}^T P_i \hat{H}_{1i} + V_{1i}^T \Pi \hat{H}_{1i} \right] w(k) \\
&\quad + w^T(k) \left[\hat{H}_{1i}^T P_i H_{1i} + \hat{H}_{1i}^T \Pi H_{1i} \right] w(k) \\
&\quad + \zeta^T(k - d_1) (-Q_1) \zeta(k - d_1) + \zeta^T(k - d_2) (-Q_2) \zeta(k - d_2) \\
&\quad - \sum_{\theta=-d(k)}^{-1} y^T(k + \theta) Z_1 y(k + \theta) - \sum_{\theta=-d(k)}^{-d_1-1} y^T(k + \theta) Z_2 y(k + \theta) \\
&\quad - \sum_{\theta=-d_2}^{-d(k)-1} y^T(k + \theta) (Z_1 + Z_2) y(k + \theta) - w^T(k) w(k) + \sum_{r=1, r \neq i}^N \beta_{ir} \zeta^T(k) (P_r - P_i) \zeta(k) < 0.
\end{aligned}$$

Then, according to the above two inequalities, we can obtain

$$\Delta V(\zeta(k)) < w^T(k) w(k) - \sum_{r=1, r \neq i}^N \beta_{ir} \zeta^T(k) (P_r - P_i) \zeta(k),$$

By switching law (17) again, we can get

$$\sum_{r=1, r \neq i}^N \beta_{ir} \zeta^T(k) (P_r - P_i) \zeta(k) \geq 0,$$

therefore

$$\Delta V(\zeta(k)) < w^T(k) w(k). \quad (19)$$

Case 2: $\sigma(k) = i$, $\sigma(k+1) = r$ and $i \neq r$, for $\forall \zeta(k) \in \Omega(P_i, \rho_1 + \beta) \cap \Phi_i \subset L(\hat{N}_i, H_i)$. By switching law (17), we can get

$$\begin{aligned} \Delta V(\zeta(k)) &= V_r(\zeta(k+1)) - V_i(\zeta(k)) \\ &\leq V_i(\zeta(k+1)) - V_i(\zeta(k)) \\ &< w^T(k) w(k) - \sum_{r=1, r \neq i}^N \beta_{ir} \zeta^T(k) (P_r - P_i) \zeta(k). \end{aligned}$$

From the switching law (17) again, it follows that

$$\sum_{r=1, r \neq i}^N \beta_{ir} \zeta^T(k) (P_r - P_i) \zeta(k) \geq 0,$$

which indicates

$$\Delta V(\zeta(k)) < w^T(k) w(k). \quad (20)$$

By combining Eqs. (19) and (20), we can obtain

$$\begin{aligned} \Delta V(\zeta(k)) &= V(\zeta(k+1)) - V(\zeta(k)) < w^T(k) w(k), \\ \forall \zeta(k) &\in \cup_{i=1}^N (\Omega(P_i, \rho_1 + \beta) \cap \Phi_i). \end{aligned} \quad (21)$$

Thus, it follows that

$$\sum_{t=0}^k \Delta V(\zeta(t)) < \sum_{t=0}^k w^T(t) w(t),$$

which in turn gives

$$V(\zeta(k+1)) < V(\zeta(0)) + \sum_{t=0}^k w^T(t) w(t), \quad \forall k \geq 0.$$

Next, it is easy to see that the following inequalities hold.

$$\begin{aligned} \zeta^T(0) P_i \zeta(0) &\leq \max_{l \in [-d_2, 0]} \|\zeta(l)\|^2 \max_{i \in I_N} \lambda_{\max}(P_i) \leq \kappa_{11} \lambda_1 \\ \sum_{l=-d_1}^{-1} \zeta^T(l) Q_1 \zeta(l) &= \zeta^T(-d_1) Q_1 \zeta(-d_1) + \zeta^T(-d_1+1) Q_1 \zeta(-d_1+1) \\ &\quad + \cdots + \zeta^T(-1) Q_1 \zeta(-1) \\ &\leq \max_{l \in [-d_2, 0]} \|\zeta(l)\|^2 \lambda_{\max}(Q_1) + \max_{l \in [-d_2, 0]} \|\zeta(l)\|^2 \lambda_{\max}(Q_1) \\ &\quad + \cdots + \max_{l \in [-d_2, 0]} \|\zeta(l)\|^2 \lambda_{\max}(Q_1) \\ &\leq \kappa_{11} d_1 \lambda_4 \end{aligned}$$

$$\begin{aligned}
& \sum_{l=-d_1}^{-1} \zeta^T(l) Q_2 \zeta(l) = \zeta^T(-d_2) Q_2 \zeta(-d_2) + \zeta^T(-d_2+1) Q_2 \zeta(-d_2+1) \\
& + \cdots + \zeta^T(-1) Q_2 \zeta(-1) \\
& \leq \max_{l \in [-d_2, 0]} \|\zeta(l)\|^2 \lambda_{\max}(Q_2) + \max_{l \in [-d_2, 0]} \|\zeta(l)\|^2 \lambda_{\max}(Q_2) \\
& + \cdots + \max_{l \in [-d_2, 0]} \|\zeta(l)\|^2 \lambda_{\max}(Q_2) \\
& \leq \kappa_{11} d_2 \lambda_5 \\
& \sum_{\theta=-d_2+1}^{-d_1+1} \sum_{l=\theta-1}^{-1} \zeta^T(l) Q_3 \zeta(l) = \sum_{l=-d_2}^{-1} \zeta^T(l) Q_3 \zeta(l) + \sum_{l=-d_2+1}^{-1} \zeta^T(l) Q_3 \zeta(l) + \cdots + \sum_{l=-d_1}^{-1} \zeta^T(l) Q_3 \zeta(l) \\
& \leq \max_{l \in [-d_2, 0]} \|\zeta(l)\|^2 d_2 \lambda_{\max}(Q_3) + \max_{l \in [-d_2, 0]} \|\zeta(l)\|^2 (d_2-1) \lambda_{\max}(Q_3) \\
& + \cdots + \max_{l \in [-d_2, 0]} \|\zeta(l)\|^2 d_1 \lambda_{\max}(Q_3) \\
& \leq \kappa_{11} \frac{1}{2} (d_1 + d_2) (d_2 - d_1 + 1) \lambda_6 \\
& \sum_{\theta=-d_2+1}^0 \sum_{l=\theta-1}^{-1} y^T(l) Z_1 y(l) = \sum_{l=-d_2}^{-1} y^T(l) Z_1 y(l) + \sum_{l=-d_2+1}^{-1} y^T(l) Z_1 y(l) + \cdots + \sum_{l=-1}^{-1} y^T(l) Z_1 y(l) \\
& \leq \max_{l \in [-d_2, 0]} \|y(l)\|^2 d_2 \lambda_{\max}(Z_1) + \max_{l \in [-d_2, 0]} \|y(l)\|^2 (d_2-1) \lambda_{\max}(Z_1) \\
& + \cdots + \max_{l \in [-d_2, 0]} \|y(l)\|^2 \lambda_{\max}(Z_1) \\
& \leq \kappa_{22} \frac{1}{2} (1 + d_2) d_2 \lambda_2 \\
& \sum_{\theta=-d_2+1}^{-d_1} \sum_{l=\theta-1}^{-1} y^T(l) Z_2 y(l) = \sum_{l=-d_2}^{-1} y^T(l) Z_2 y(l) + \sum_{l=-d_2+1}^{-1} y^T(l) Z_2 y(l) + \cdots + \sum_{l=-d_1-1}^{-1} y^T(l) Z_2 y(l) \\
& \leq \max_{l \in [-d_2, 0]} \|y(l)\|^2 d_2 \lambda_{\max}(Z_2) + \max_{l \in [-d_2, 0]} \|y(l)\|^2 (d_2-1) \lambda_{\max}(Z_2) \\
& + \cdots + \max_{l \in [-d_2, 0]} \|y(l)\|^2 (d_1+1) \lambda_{\max}(Z_2) \\
& \leq \kappa_{22} \frac{1}{2} (1 + d_1 + d_2) (d_2 - d_1) \lambda_3,
\end{aligned}$$

Thus, we have

$$\begin{aligned}
V(\zeta(0)) &= \zeta^T(0) P_i \zeta(0) + \sum_{l=-d_1}^{-1} \zeta^T(l) Q_1 \zeta(l) + \sum_{l=-d_2}^{-1} \zeta^T(l) Q_2 \zeta(l) + \sum_{\theta=-d_2+1}^{-d_1+1} \sum_{l=\theta-1}^{-1} \zeta^T(l) Q_3 \zeta(l) \\
&+ \sum_{\theta=-d_2+1}^0 \sum_{l=\theta-1}^{-1} y^T(l) Z_1 y(l) + \sum_{\theta=-d_2+1}^{-d_1} \sum_{l=\theta-1}^{-1} y^T(l) Z_2 y(l) \\
&\leq \kappa_{11} [\lambda_1 + d_1 \lambda_4 + d_2 \lambda_5 + 0.5 (d_2 - d_1 + 1) (d_2 + d_1) \lambda_6] \\
&+ \kappa_{22} [0.5 d_2 (d_2 + 1) \lambda_2 + 0.5 (d_2 - d_1) (d_2 + d_1 + 1) \lambda_3], \\
&= \mathbb{Q}(\zeta(l)).
\end{aligned}$$

Due to $\sum_{k=0}^{\infty} w^T(k) w(k) \leq \beta$, we have

$$V(\zeta(k+1)) < \mathbb{Q}(\zeta(l)) + \beta. \quad (22)$$

Thus, in light of $\mathbb{Q}(\zeta(l)) \leq \rho_1$, it is easy to see that $\zeta^T(0) P_i \zeta(0) \leq \rho_1$, and the control constraints $L(\hat{N}_i, H_i)$ are also satisfied owing to (15). Thus, all the trajectories of closed loop system (9) that start from $\cup_{i=1}^N (\Omega(P_i, \rho_1) \cap \Phi_i)$ remain in set $\cup_{i=1}^N (\Omega(P_i, \rho_1 + \beta) \cap \Phi_i)$. This proof is completed.

From the above conclusions, we know that the disturbance tolerance capacity of the closed loop system should be estimated before analyzing the restricted L_2 -gain. It is obvious that the larger the β , the larger the disturbance tolerance capacity is. Thus, the largest disturbance tolerance level β^* for the closed-loop system (14) can be obtained by solving the following optimization problem

$$\begin{aligned} & \sup_{P_i, Q_1, Q_2, Q_3, Z_1, Z_2, J_i, \beta_{ir}} \beta, \\ & \text{s.t. (a) inequality (14), } i \in I_N, \\ & \text{(b) } \Omega(P_i, \rho_1 + \beta) \cap \Phi_i \subset L(\hat{N}_i, H_i), i \in I_N. \end{aligned} \quad (23)$$

Applying the lemma 1 to (14), we get

$$\begin{bmatrix} v_i & * & * & * & * & * & * & * & * \\ 0 & -Q_3 & * & * & * & * & * & * & * \\ 0 & 0 & -Q_1 & * & * & * & * & * & * \\ 0 & 0 & 0 & -Q_2 & * & * & * & * & * \\ J_i H_i & 0 & 0 & 0 & -2J_i & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & -I & * & * & * \\ P_i \hat{A}_i & P_i \hat{A}_{di} & 0 & 0 & P_i V_{1i} & P_i \hat{H}_{1i} & -P_i & * & * \\ \sqrt{d_2} P_i (\hat{A}_i - I) & \sqrt{d_2} P_i \hat{A}_{di} & 0 & 0 & \sqrt{d_2} P_i V_{1i} & \sqrt{d_2} P_i \hat{H}_{1i} & 0 & -P_i Z_1^{-1} P_i & * \\ d_1^* P_i (\hat{A}_i - I) & d_1^* P_i \hat{A}_{di} & 0 & 0 & d_1^* P_i V_{1i} & d_1^* P_i \hat{H}_{1i} & 0 & 0 & -P_i Z_2^{-1} P_i \end{bmatrix} < 0, \quad (24)$$

where

$$v_i = -P_i + Q_1 + Q_2 + d_2^* Q_3 + \sum_{r=1, r \neq i}^N \beta_{ir} (P_r - P_i), \quad d_1^* = \sqrt{d_2 - d_1}, \quad d_2^* = d_2 - d_1 + 1.$$

Due to $\tilde{A}_i = \hat{A}_i + I_R K_i + \hat{B}_i M_{0i} L_i I_R^T V_{1i} = 2(\hat{B}_i + I_R E_{ci}) M_{0i} \hat{G}_i$, the inequality (24) can be arranged as

$$\begin{aligned} & \begin{bmatrix} v_i & * & * & * & * & * & * & * & * \\ 0 & -Q_3 & * & * & * & * & * & * & * \\ 0 & 0 & -Q_1 & * & * & * & * & * & * \\ 0 & 0 & 0 & -Q_2 & * & * & * & * & * \\ J_i H_i & 0 & 0 & 0 & -2J_i & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & -I & * & * & * \\ P_i (\hat{A}_i + I_R K_i) & P_i \hat{A}_{di} & 0 & 0 & 2P_i (\hat{B}_i + I_R E_{ci}) M_{0i} \hat{G}_i & P_i \hat{H}_{1i} & -P_i & * & * \\ \sqrt{d_2} P_i \begin{pmatrix} \hat{A}_i \\ + I_R K_i \\ -I \end{pmatrix} & \sqrt{d_2} P_i \hat{A}_{di} & 0 & 0 & 2\sqrt{d_2} P_i (\hat{B}_i + I_R E_{ci}) M_{0i} \hat{G}_i & \sqrt{d_2} P_i \hat{H}_{1i} & 0 & -P_i Z_1^{-1} P_i & * \\ d_1^* P_i \begin{pmatrix} \hat{A}_i \\ + I_R K_i \\ -I \end{pmatrix} & d_1^* P_i \hat{A}_{di} & 0 & 0 & 2d_1^* P_i (\hat{B}_i + I_R E_{ci}) M_{0i} \hat{G}_i & d_1^* P_i \hat{H}_{1i} & 0 & 0 & -P_i Z_2^{-1} P_i \end{bmatrix} \\ & + \begin{bmatrix} I_R \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} L_i^T \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} M_{0i}^T \hat{B}_i^T P_i^T \quad \sqrt{d_2} M_{0i}^T \hat{B}_i^T P_i^T \quad d_1^* M_{0i}^T \hat{B}_i^T P_i^T \\ & + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} L_i \begin{bmatrix} I_R^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} < 0. \end{aligned} \quad (25)$$

From Lemma 3, (25) holds if and only if the following matrix inequality holds

$$\begin{aligned}
 & \begin{bmatrix} v_i & * & * & * & * & * & * & * & * \\ 0 & -Q_3 & * & * & * & * & * & * & * \\ 0 & 0 & -Q_1 & * & * & * & * & * & * \\ 0 & 0 & 0 & -Q_2 & * & * & * & * & * \\ J_i H_i & 0 & 0 & 0 & -2J_i & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & -I & * & * & * \\ P_i(\hat{A}_i + I_R K_i) & P_i \hat{A}_{di} & 0 & 0 & 2P_i(\hat{B}_i + I_R E_{ci})M_{0i}\hat{G}_i & P_i \hat{H}_{1i} & -P_i & * & * \\ \sqrt{d_2}P_i \begin{pmatrix} \hat{A}_i \\ +I_R K_i \\ -I \end{pmatrix} & \sqrt{d_2}P_i \hat{A}_{di} & 0 & 0 & 2\sqrt{d_2}P_i(\hat{B}_i + I_R E_{ci})M_{0i}\hat{G}_i & \sqrt{d_2}P_i \hat{H}_{1i} & 0 & -P_i Z_1^{-1}P_i & * \\ d_1^* P_i \begin{pmatrix} \hat{A}_i \\ +I_R K_i \\ -I \end{pmatrix} & d_1^* P_i \hat{A}_{di} & 0 & 0 & 2d_1^* P_i(\hat{B}_i + I_R E_{ci})M_{0i}\hat{G}_i & d_1^* P_i \hat{H}_{1i} & 0 & 0 & -P_i Z_2^{-1}P_i \end{bmatrix} + \\
 & \lambda_{1i} \begin{bmatrix} I_R \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} I_R^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \\
 & \lambda_{1i}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \\
 & \begin{bmatrix} P_i \hat{B}_i M_{0i} \\ \sqrt{d_2}P_i \hat{B}_i M_{0i} \\ d_1^* P_i \hat{B}_i M_{0i} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & M_{0i}^T \hat{B}_i^T P_i^T & \sqrt{d_2}M_{0i}^T \hat{B}_i^T P_i^T & d_1^* M_{0i}^T \hat{B}_i^T P_i^T \end{bmatrix} < 0,
 \end{aligned} \tag{26}$$

where $\lambda_{1i} > 0$.

Then, using Lemma 1 to (26), we have

$$\begin{aligned}
 & \begin{bmatrix} v_i + \lambda_{1i} I_R I_R^T & * & * & * & * \\ 0 & -Q_3 & * & * & * \\ 0 & 0 & -Q_1 & * & * \\ 0 & 0 & 0 & -Q_2 & * \\ J_i H_i & 0 & 0 & 0 & -2J_i \\ 0 & 0 & 0 & 0 & 0 \\ P_i(\hat{A}_i + I_R K_i) & P_i \hat{A}_{di} & 0 & 0 & 2P_i(\hat{B}_i + I_R E_{ci})M_{0i}\hat{G}_i \\ \sqrt{d_2}P_i(\hat{A}_i + I_R K_i - I) & \sqrt{d_2}P_i \hat{A}_{di} & 0 & 0 & 2\sqrt{d_2}P_i(\hat{B}_i + I_R E_{ci})M_{0i}\hat{G}_i \\ d_1^* P_i(\hat{A}_i + I_R K_i - I) & d_1^* P_i \hat{A}_{di} & 0 & 0 & 2d_1^* P_i(\hat{B}_i + I_R E_{ci})M_{0i}\hat{G}_i \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
 & \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ -I & * & * & * & * \\ P_i \hat{H}_{1i} & -P_i & * & * & P_i \hat{B}_i M_{0i} \\ \sqrt{d_2}P_i \hat{H}_{1i} & 0 & -P_i Z_1^{-1}P_i & * & \sqrt{d_2}P_i \hat{B}_i M_{0i} \\ d_1^* P_i \hat{H}_{1i} & 0 & 0 & -P_i Z_2^{-1}P_i & d_1^* P_i \hat{B}_i M_{0i} \\ 0 & * & * & * & -\lambda_{1i} I \end{bmatrix} < 0,
 \end{aligned} \tag{27}$$

Then, the inequality (27) can be again arranged as

$$\begin{aligned}
& \begin{bmatrix} v_i + \lambda_{1i} I_R I_R^T & * & * & * & * & * & * & * & * & * \\ 0 & -Q_3 & * & * & * & * & * & * & * & * \\ 0 & 0 & -Q_1 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & -Q_2 & * & * & * & * & * & * \\ J_i H_i & 0 & 0 & 0 & -2J_i & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & -I & * & * & * & * \\ P_i(\hat{A}_i + I_R K_i) & P_i \hat{A}_{di} & 0 & 0 & 0 & P_i \hat{H}_{1i} & -P_i & * & * & P_i \hat{B}_i M_{0i} \\ \sqrt{d_2} P_i \begin{pmatrix} \hat{A}_i \\ +I_R K_i \\ -I \end{pmatrix} & \sqrt{d_2} P_i \hat{A}_{di} & 0 & 0 & 0 & \sqrt{d_2} P_i \hat{H}_{1i} & 0 & -P_i Z_1^{-1} P_i & * & \sqrt{d_2} P_i \hat{B}_i M_{0i} \\ d_1^* P_i \begin{pmatrix} \hat{A}_i \\ +I_R K_i \\ -I \end{pmatrix} & d_1^* P_i \hat{A}_{di} & 0 & 0 & 0 & d_1^* P_i \hat{H}_{1i} & 0 & 0 & -P_i Z_2^{-1} P_i & d_1^* P_i \hat{B}_i M_{0i} \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & -\lambda_{1i} I \end{bmatrix} \\
& + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \hat{G}_i^T \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 2M_{0i}^T(\hat{B}_i + I_R E_{ci})^T P_i^T & 2\sqrt{d_2} M_{0i}^T(\hat{B}_i + I_R E_{ci})^T P_i^T & 2d_1^* M_{0i}^T(\hat{B}_i + I_R E_{ci})^T P_i^T & 0 \end{bmatrix} \\
& + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \hat{G}_i \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \end{bmatrix} < 0.
\end{aligned} \tag{28}$$

From Lemma 3, (28) holds if and only if the following matrix inequality holds

$$\begin{aligned}
& \begin{bmatrix} v_i + \lambda_{1i} I_R I_R^T & * & * & * & * & * & * & * & * & * \\ 0 & -Q_3 & * & * & * & * & * & * & * & * \\ 0 & 0 & -Q_1 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & -Q_2 & * & * & * & * & * & * \\ J_i H_i & 0 & 0 & 0 & -2J_i & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & -I & * & * & * & * \\ P_i(\hat{A}_i + I_R K_i) & P_i \hat{A}_{di} & 0 & 0 & 0 & P_i \hat{H}_{1i} & -P_i & * & * & P_i \hat{B}_i M_{0i} \\ \sqrt{d_2} P_i \begin{pmatrix} \hat{A}_i \\ +I_R K_i \\ -I \end{pmatrix} & \sqrt{d_2} P_i \hat{A}_{di} & 0 & 0 & 0 & \sqrt{d_2} P_i \hat{H}_{1i} & 0 & -P_i Z_1^{-1} P_i & * & \sqrt{d_2} P_i \hat{B}_i M_{0i} \\ d_1^* P_i \begin{pmatrix} \hat{A}_i \\ +I_R K_i \\ -I \end{pmatrix} & d_1^* P_i \hat{A}_{di} & 0 & 0 & 0 & d_1^* P_i \hat{H}_{1i} & 0 & 0 & -P_i Z_2^{-1} P_i & d_1^* P_i \hat{B}_i M_{0i} \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & -\lambda_{1i} I \end{bmatrix} \\
& + \lambda_{2i} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \end{bmatrix} + \lambda_{2i}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
& \times \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 2M_{0i}^T(\hat{B}_i + I_R E_{ci})^T P_i^T & 2\sqrt{d_2} M_{0i}^T(\hat{B}_i + I_R E_{ci})^T P_i^T & 2d_1^* M_{0i}^T(\hat{B}_i + I_R E_{ci})^T P_i^T & 0 \end{bmatrix} < 0,
\end{aligned} \tag{29}$$

where $\lambda_{2i} > 0$.

Then, in the light of Lemma 1, we can get

$$\begin{bmatrix} \prod_{i11} & * \\ \prod_{i21} & \prod_{i22} \end{bmatrix} < 0, \tag{30}$$

where

$$\prod_{i11} = \begin{bmatrix} v_i + \lambda_{1i} I_R I_R^T & * & * & * & * & * \\ 0 & -Q_3 & * & * & * & * \\ 0 & 0 & -Q_1 & * & * & * \\ 0 & 0 & 0 & -Q_2 & * & * \\ J_i H_i & 0 & 0 & 0 & -2J_i + \lambda_{2i} I^T I & * \\ 0 & 0 & 0 & 0 & 0 & -I \end{bmatrix},$$

$$\prod_{i21} = \begin{bmatrix} P_i(\hat{A}_i + I_R K_i) & P_i \hat{A}_{di} & 0 & 0 & 0 & P_i \hat{H}_{1i} \\ \sqrt{d_2} P_i (\hat{A}_i + I_R K_i - I) & \sqrt{d_2} P_i \hat{A}_{di} & 0 & 0 & 0 & \sqrt{d_2} P_i \hat{H}_{1i} \\ d_1^* P_i (\hat{A}_i + I_R K_i - I) & d_1^* P_i \hat{A}_{di} & 0 & 0 & 0 & d_1^* P_i \hat{H}_{1i} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\prod_{i22} = \begin{bmatrix} -P_i & * & * & * & * \\ 0 & -P_i Z_1^{-1} P_i & * & * & * \\ 0 & 0 & -P_i Z_2^{-1} P_i & * & * \\ M_{0i}^T \hat{B}_i^T P_i^T & \sqrt{d_2} M_{0i}^T \hat{B}_i^T P_i^T & d_1^* M_{0i}^T \hat{B}_i^T P_i^T & -\lambda_{1i} I & * \\ 2M_{0i}^T (\hat{B}_i & 2\sqrt{d_2} M_{0i}^T (\hat{B}_i & 2d_1^* M_{0i}^T (\hat{B}_i + I_R E_{ci})^T P_i^T & 0 & -\lambda_{2i} I \\ + I_R E_{ci})^T P_i^T & + I_R E_{ci})^T P_i^T & \end{bmatrix}.$$

Next, let $\varepsilon = (\rho_1 + \beta)^{-1}$ and we will explain that the constraint (15) can be transformed into the following matrix inequality

$$\begin{bmatrix} \varepsilon & \hat{N}_i^j - H_i^j \\ * & P_i - \sum_{r=1, r \neq i}^N \delta_{ir} (P_r - P_i) \end{bmatrix} \geq 0, \quad (31)$$

where $\delta_{ir} > 0$, and \hat{N}_i^j, H_i^j denotes the j -th row of \hat{N}_i and H_i respectively.

Let $G_i = P_i - \sum_{r=1, r \neq i}^N \delta_{ir} (P_r - P_i)$. For $\forall \zeta(k) \in \Omega(P_i, \rho_1 + \beta) \cap \Phi_i$, then from switching law (17), it is clear that

$$\sum_{r=1, r \neq i}^N \delta_{ir} \zeta^T(k) (P_r - P_i) \zeta(k) \geq 0,$$

and

$$\zeta^T(k) P_i \zeta(k) \leq (\rho_1 + \beta) = \varepsilon^{-1}.$$

Then, we get

$$\zeta^T G_i \zeta \leq \varepsilon^{-1}, \quad (32)$$

$$G_i - (\hat{N}_i^j - H_i^j)^T \varepsilon^{-1} (\hat{N}_i^j - H_i^j) \geq 0. \quad (33)$$

Finally, we have

$$\zeta^T (\hat{N}_i^j - H_i^j)^T \varepsilon^{-1} (\hat{N}_i^j - H_i^j) \zeta \leq \zeta^T G_i \zeta \leq \varepsilon^{-1},$$

or

$$|(\hat{N}_i^j - H_i^j) \zeta| \leq 1. \quad (34)$$

(34) implies that if $\forall \zeta(k) \in \Omega(P_i, \rho_1 + \beta) \cap \Phi_i$, then $\zeta(k) \in L(\hat{N}_i, H_i)$. As a result, constraint condition $\Omega(P_i, \rho_1 + \beta) \cap \Phi_i \subset L(\hat{N}_i, H_i)$, $i \in I_N$, can be transformed into (31).

Thus, we can rewrite the optimization problem (23) as follows:

$$\begin{aligned} & \inf_{P_i, Q_1, Q_2, Q_3, Z_1, Z_2, J_i, \beta_{ir}} \varepsilon, \\ & \text{s.t. (a) inequality (30), } i \in I_N, \\ & \text{(b) inequality (31), } i \in I_N, j \in V_q. \end{aligned} \quad (35)$$

If we do not consider the anti-windup compensation cases, we can get the corollary 1.

Corollary 1: Consider the closed-loop switched system (9). For a given positive scalar ρ_1 , if there exist positive definite matrices $P_i, Q_1, Q_2, Q_3, Z_1, Z_2$, positive definite diagonal matrices J_i , and a set of scalars $\beta_{ir} > 0$, such that.

$$\begin{bmatrix} \Lambda_{i11} & \Lambda_{i12} & 0 & 0 & \Lambda_{i15} & \Lambda_{i16} \\ * & \Lambda_{i22} & 0 & 0 & \Lambda_{i25} & \Lambda_{i26} \\ * & * & -Q_1 & 0 & 0 & 0 \\ * & * & * & -Q_2 & 0 & 0 \\ * & * & * & * & \Lambda_{i55} & \Lambda_{i56} \\ * & * & * & * & * & \Lambda_{i66} \end{bmatrix} < 0,$$

$$i \in I_N,$$

and

$$\Omega(P_i, \rho_1 + \beta) \cap \Phi_i \subset L(\hat{N}_i, H_i), \quad i \in I_N,$$

where

$$\begin{aligned} \tilde{A}_i &= \hat{A}_i + I_R K_i + \hat{B}_i M_{0i} L_i I_R^T, \\ V_{1i} &= 2\hat{B}_i M_{0i} \hat{G}_i, \\ \Pi &= d_2 Z_1 + (d_2 - d_1) Z_2, \\ \Lambda_{i11} &= \tilde{A}_i^T P_i \tilde{A}_i - P_i + Q_1 + Q_2 + (d_2 - d_1 + 1) Q_3 + (\tilde{A}_i - I)^T \Pi (\tilde{A}_i - I) + \sum_{r=1, r \neq i}^N \beta_{ir} (P_r - P_i), \\ \Lambda_{i12} &= \tilde{A}_i^T P_i \hat{A}_{di} + (\tilde{A}_i - I)^T \Pi \hat{A}_{di}, \\ \Lambda_{i15} &= \tilde{A}_i^T P_i V_{1i} + (\tilde{A}_i - I)^T \Pi V_{1i} + H_i^T J_i^T, \\ \Lambda_{i16} &= \tilde{A}_i^T P_i \hat{H}_{1i} + (\tilde{A}_i - I)^T \Pi \hat{H}_{1i}, \\ \Lambda_{i22} &= \hat{A}_{di}^T P_i \hat{A}_{di} - Q_3 + \hat{A}_{di}^T \Pi \hat{A}_{di}, \\ \Lambda_{i25} &= \hat{A}_{di}^T P_i V_{1i} + \hat{A}_{di}^T \Pi V_{1i}, \\ \Lambda_{i26} &= \hat{A}_{di}^T P_i \hat{H}_{1i} + \hat{A}_{di}^T \Pi \hat{H}_{1i}, \\ \Lambda_{i55} &= V_{1i}^T P_i V_{1i} + V_{1i}^T \Pi V_{1i} - 2J_i, \\ \Lambda_{i56} &= V_{1i}^T P_i \hat{H}_{1i} + V_{1i}^T \Pi \hat{H}_{1i}, \\ \Lambda_{i66} &= \hat{H}_{1i}^T P_i \hat{H}_{1i} + \hat{H}_{1i}^T \Pi \hat{H}_{1i} - I, \\ \Phi_i &= \{\zeta \in R^{q \times n} : \zeta^T (P_r - P_i) \zeta \geq 0, \forall r \in I_N, r \neq i\}, \end{aligned}$$

then for the closed-loop system (9), under the state-dependent switching law (17) and the initial conditions (16), the state trajectory starting from the region $\cup_{i=1}^N (\Omega(P_i, \rho_1) \cap \Phi_i)$ will remain inside the region $\cup_{i=1}^N (\Omega(P_i, \rho_1 + \beta) \cap \Phi_i)$ for every $\forall w \in W_\beta^2$.

L₂4. -Gain analysis

In this section, we will use the multiple Lyapunov functionals method to solve the restricted L₂-gain problem for the system (9), based on Theorem 1, that is, the state trajectories of the system are bounded. Similarly, the design method of anti-windup fault-tolerant controller will be proposed in section "Control synthesis solution".

Theorem 2: Consider the closed-loop system (9). For a given constant $\beta \in (0, \beta^*]$ and $\gamma > 0$, suppose there exist positive definite matrices $P_i, Q_1, Q_2, Q_3, Z_1, Z_2$, diagonal positive definite matrices J_i , and a set of scalars $\beta_{ir} > 0$, such that.

$$\begin{bmatrix} \Upsilon_{i11} & \Upsilon_{i12} & 0 & 0 & \Upsilon_{i15} & \Upsilon_{i16} \\ * & \Upsilon_{i22} & 0 & 0 & \Upsilon_{i25} & \Upsilon_{i26} \\ * & * & -Q_1 & 0 & 0 & 0 \\ * & * & * & -Q_2 & 0 & 0 \\ * & * & * & * & \Upsilon_{i55} & \Upsilon_{i56} \\ * & * & * & * & * & \Upsilon_{i66} \end{bmatrix} < 0,$$

$$i \in I_N,$$
(36)

and

$$\Omega(P_i, \rho_1 + \beta) \cap \Phi_i \subset L(\hat{N}_i, H_i), \quad i \in I_N,$$
(37)

where

$$\begin{aligned}
 \tilde{A}_i &= \hat{A}_i + I_R K_i + \hat{B}_i M_{0i} L_i I_R^T, \\
 V_{1i} &= 2(\hat{B}_i + I_R E_{ci}) M_{0i} \hat{G}_i, \\
 \Upsilon_{i11} &= \tilde{A}_i^T P_i \tilde{A}_i - P_i + Q_1 + Q_2 + (d_2 - d_1 + 1) Q_3 + (\tilde{A}_i - I)^T \Pi (\tilde{A}_i - I) + \sum_{r=1, r \neq i}^N \beta_{ir} (P_r - P_i) + \gamma^{-2} \hat{C}_i^T \hat{C}_i, \\
 \Upsilon_{i12} &= \tilde{A}_i^T P_i \hat{A}_{di} + (\tilde{A}_i - I)^T \Pi \hat{A}_{di}, \\
 \Upsilon_{i15} &= \tilde{A}_i^T P_i V_{1i} + (\tilde{A}_i - I)^T \Pi V_{1i} + H_i^T J_i^T, \\
 \Upsilon_{i16} &= \tilde{A}_i^T P_i \hat{H}_{1i} + (\tilde{A}_i - I)^T \Pi \hat{H}_{1i}, \\
 \Upsilon_{i22} &= \hat{A}_{di}^T P_i \hat{A}_{di} - Q_3 + \hat{A}_{di}^T \Pi \hat{A}_{di}, \\
 \Upsilon_{i25} &= \hat{A}_{di}^T P_i V_{1i} + \hat{A}_{di}^T \Pi V_{1i}, \\
 \Upsilon_{i26} &= \hat{A}_{di}^T P_i \hat{H}_{1i} + \hat{A}_{di}^T \Pi \hat{H}_{1i}, \\
 \Upsilon_{i55} &= V_{1i}^T P_i V_{1i} + V_{1i}^T \Pi V_{1i} - 2J_i, \\
 \Upsilon_{i56} &= V_{1i}^T P_i \hat{H}_{1i} + V_{1i}^T \Pi \hat{H}_{1i}, \\
 \Upsilon_{i66} &= \hat{H}_{1i}^T P_i \hat{H}_{1i} + \hat{H}_{1i}^T \Pi \hat{H}_{1i} - I.
 \end{aligned}$$

Then the restricted L_2 -gain from w to z over $w \in W_\beta^2$ is less than γ under the state-dependent switching law

$$\sigma = \arg \min \{ \zeta^T P_i \zeta, i \in I_N \}.$$

Proof: Using the similar method as for proving Theorem 1, we choose the Lyapunov–Krasovskii functionals of the closed-loop system (9) as.

$$V(\zeta(k)) = V_{1\sigma}(\zeta(k)) + V_2(\zeta(k)) + V_3(\zeta(k)) + V_4(\zeta(k)),$$

where

$$\begin{aligned}
 V_{1\sigma}(\zeta(k)) &= \zeta^T(k) P_\sigma \zeta(k), \\
 V_2(\zeta(k)) &= \sum_{l=-d_1}^{-1} \zeta^T(k+l) Q_1 \zeta(k+l) + \sum_{l=-d_2}^{-1} \zeta^T(k+l) Q_2 \zeta(k+l), \\
 V_3(\zeta(k)) &= \sum_{\theta=-d_2+1}^{-d_1+1} \sum_{l=\theta-1}^{-1} \zeta^T(k+l) Q_3 \zeta(k+l), \\
 V_4(\zeta(k)) &= \sum_{\theta=-d_2+1}^0 \sum_{l=\theta-1}^{-1} y^T(k+l) Z_1 y(k+l) + \sum_{\theta=-d_2+1}^{-d_1} \sum_{l=\theta-1}^{-1} y^T(k+l) Z_2 y(k+l), \\
 y(k) &= \zeta(k+1) - \zeta(k).
 \end{aligned}$$

Here, we still divide the proof into two parts.

Case 1: $\sigma(k+1) = \sigma(k) = i$, for $\forall \zeta(k) \in \Omega(P_i, \rho_1 + \beta) \cap \Phi_i \subset L(\hat{N}_i, H_i)$. It follows that:

$$\begin{aligned}
 \Delta V_{1i}(\zeta(k)) &= V_{1i}(\zeta(k+1)) - V_{1i}(\zeta(k)) = \zeta^T(k+1) P_i \zeta(k+1) - \zeta^T(k) P_i \zeta(k), \\
 \Delta V_2(\zeta(k)) &= V_2(\zeta(k+1)) - V_2(\zeta(k)) \\
 &= \zeta^T(k) (Q_1 + Q_2) \zeta(k) - \zeta^T(k-d_1) Q_1 \zeta(k-d_1) - \zeta^T(k-d_2) Q_2 \zeta(k-d_2), \\
 \Delta V_3(\zeta(k)) &= V_3(\zeta(k+1)) - V_3(\zeta(k)) = (d_2 - d_1 + 1) \zeta^T(k) Q_3 \zeta(k) - \sum_{\theta=-d_2}^{-d_1} \zeta^T(k+\theta) Q_3 \zeta(k+\theta) \\
 &\leq (d_2 - d_1 + 1) \zeta^T(k) Q_3 \zeta(k) - \zeta^T(k-d(k)) Q_3 \zeta(k-d(k)),
 \end{aligned}$$

$$\begin{aligned}\Delta V_4(\zeta(k)) &= V_4(\zeta(k+1)) - V_4(\zeta(k)) \\ &= y^T(k) \Pi y(k) - \sum_{\theta=-d(k)}^{-1} y^T(k+\theta) Z_1 y(k+\theta) \\ &\quad - \sum_{\theta=-d(k)}^{-d_1-1} y^T(k+\theta) Z_2 y(k+\theta) - \sum_{\theta=-d_2}^{-d(k)-1} y^T(k+\theta) (Z_1 + Z_2) y(k+\theta).\end{aligned}$$

Thereby,

$$\begin{aligned}\Delta V(\zeta(k)) &= \Delta V_{1i}(\zeta(k)) + \Delta V_2(\zeta(k)) + \Delta V_3(\zeta(k)) + \Delta V_4(\zeta(k)) \\ &\leq \max_{i \in I_N} \zeta^T(k) \left[\tilde{A}_i^T P_i \tilde{A}_i - P_i + Q_1 + Q_2 + (d_2 - d_1 + 1) Q_3 + (\tilde{A}_i - I)^T \Pi (\tilde{A}_i - I) \right] \zeta(k) \\ &\quad + 2\zeta^T(k) \left[\tilde{A}_i^T P_i \hat{A}_{di} + (\tilde{A}_i - I)^T \Pi \hat{A}_{di} \right] \zeta(k - d(k)) \\ &\quad + 2\zeta^T(k) \left[\tilde{A}_i^T P_i V_{1i} + (\tilde{A}_i - I)^T \Pi V_{1i} \right] \psi(v) \\ &\quad + 2\zeta^T(k) \left[\tilde{A}_i^T P_i \hat{H}_{1i} + (\tilde{A}_i - I)^T \Pi \hat{H}_{1i} \right] w(k) \\ &\quad + \zeta^T(k - d(k)) \left[\hat{A}_{di}^T P_i \hat{A}_{di} + \hat{A}_{di}^T \Pi \hat{A}_{di} - Q_3 \right] \zeta(k - d(k)) \\ &\quad + 2\zeta^T(k - d(k)) \left[\hat{A}_{di}^T P_i V_{1i} + \hat{A}_{di}^T \Pi V_{1i} \right] \psi(v) \\ &\quad + 2\zeta^T(k - d(k)) \left[\hat{A}_{di}^T P_i \hat{H}_{1i} + \hat{A}_{di}^T \Pi \hat{H}_{1i} \right] w(k) \\ &\quad + \psi^T(v) \left[V_{1i}^T P_i V_{1i} + V_{1i}^T \Pi V_{1i} \right] \psi(v) \\ &\quad + 2\psi^T(v) \left[V_{1i}^T P_i \hat{H}_{1i} + V_{1i}^T \Pi \hat{H}_{1i} \right] w(k) \\ &\quad + w^T(k) \left[\hat{H}_{1i}^T P_i \hat{H}_{1i} + \hat{H}_{1i}^T \Pi \hat{H}_{1i} \right] w(k) \\ &\quad + \zeta^T(k - d_1) (-Q_1) x(k - d_1) \\ &\quad + \zeta^T(k - d_2) (-Q_2) \zeta(k - d_2) \\ &\quad - \sum_{\theta=-d(k)}^{-1} y^T(k+\theta) Z_1 y(k+\theta) \\ &\quad - \sum_{\theta=-d(k)}^{-d_1-1} y^T(k+\theta) Z_2 y(k+\theta) \\ &\quad - \sum_{\theta=-d_2}^{-d(k)-1} y^T(k+\theta) (Z_1 + Z_2) y(k+\theta).\end{aligned}$$

Then, combining with Lemma 2 and condition (37), we have

$$\begin{aligned}
\Delta V(\zeta(k)) &= \Delta V_{1i}(\zeta(k)) + \Delta V_2(\zeta(k)) + \Delta V_3(\zeta(k)) + \Delta V_4(\zeta(k)) \\
&\leq \max_{i \in I_N} \zeta^T(k) \left[\tilde{A}_i^T P_i \tilde{A}_i - P_i + Q_1 + Q_2 + (d_2 - d_1 + 1) Q_3 + (\tilde{A}_i - I)^T \Pi (\tilde{A}_i - I) \right] \zeta(k) \\
&\quad + 2\zeta^T(k) \left[\tilde{A}_i^T P_i \hat{A}_{di} + (\tilde{A}_i - I)^T \Pi \hat{A}_{di} \right] \zeta(k - d(k)) \\
&\quad + 2\zeta^T(k) \left[\tilde{A}_i^T P_i V_{1i} + (\tilde{A}_i - I)^T \Pi V_{1i} \right] \psi(k) \\
&\quad + 2\zeta^T(k) \left[\tilde{A}_i^T P_i \hat{H}_{1i} + (\tilde{A}_i - I)^T \Pi \hat{H}_{1i} \right] w(k) \\
&\quad + \zeta^T(k - d(k)) \left[\hat{A}_{di}^T P_i \hat{A}_{di} + \hat{A}_{di}^T \Pi \hat{A}_{di} - Q_3 \right] \zeta(k - d(k)) \\
&\quad + 2\zeta^T(k - d(k)) \left[\hat{A}_{di}^T P_i V_{1i} + \hat{A}_{di}^T \Pi V_{1i} \right] \psi(k) \\
&\quad + 2\zeta^T(k - d(k)) \left[\hat{A}_{di}^T P_i \hat{H}_{1i} + \hat{A}_{di}^T \Pi \hat{H}_{1i} \right] w(k) \\
&\quad + \psi^T(k) \left[V_{1i}^T P_i V_{1i} + V_{1i}^T \Pi V_{1i} \right] \psi(k) \\
&\quad + 2\psi^T(k) \left[V_{1i}^T P_i \hat{H}_{1i} + V_{1i}^T \Pi \hat{H}_{1i} \right] w(k) \\
&\quad + w^T(k) \left[\hat{H}_{1i}^T P_i H_{1i} + \hat{H}_{1i}^T \Pi H_{1i} \right] w(k) \\
&\quad + \zeta^T(k - d_1) (-Q_1) x(k - d_1) + \zeta^T(k - d_2) (-Q_2) \zeta(k - d_2) \\
&\quad - \sum_{\theta=-d(k)}^{-1} y^T(k + \theta) Z_1 y(k + \theta) - \sum_{\theta=-d(k)}^{-d_1-1} y^T(k + \theta) Z_2 y(k + \theta) \\
&\quad - \sum_{\theta=-d_2}^{-d(k)-1} y^T(k + \theta) (Z_1 + Z_2) y(k + \theta) - 2\psi^T(\hat{N}_i \zeta(k)) J_i \left[\psi(\hat{N}_i \zeta(k)) - H_i \zeta(k) \right].
\end{aligned}$$

Multiplying (36) from the left by $\begin{bmatrix} \zeta(k) \\ \zeta(k - d(k)) \\ \zeta(k - d_1) \\ \zeta(k - d_2) \\ \psi(v) \\ w(k) \end{bmatrix}^T$ and then from the right by $\begin{bmatrix} \zeta(k) \\ \zeta(k - d(k)) \\ \zeta(k - d_1) \\ \zeta(k - d_2) \\ \psi(v) \\ w(k) \end{bmatrix}$, we have

$$\begin{aligned}
&\zeta^T(k) \left[\tilde{A}_i^T P_i \tilde{A}_i - P_i + Q_1 + Q_2 + (d_2 - d_1 + 1) Q_3 + (\tilde{A}_i - I)^T \Pi (\tilde{A}_i - I) \right] \zeta(k) \\
&\quad + 2\zeta^T(k) \left[\tilde{A}_i^T P_i \hat{A}_{di} + (\tilde{A}_i - I)^T \Pi \hat{A}_{di} \right] \zeta(k - d(k)) \\
&\quad + 2\zeta^T(k) \left[\tilde{A}_i^T P_i V_{1i} + (\tilde{A}_i - I)^T \Pi V_{1i} \right] \psi(v) \\
&\quad + 2\zeta^T(k) \left[\tilde{A}_i^T P_i \hat{H}_{1i} + (\tilde{A}_i - I)^T \Pi \hat{H}_{1i} \right] w(k) \\
&\quad + \zeta^T(k - d(k)) \left[\hat{A}_{di}^T P_i \hat{A}_{di} + \hat{A}_{di}^T \Pi \hat{A}_{di} - Q_3 \right] \zeta(k - d(k)) \\
&\quad + 2\zeta^T(k - d(k)) \left[\hat{A}_{di}^T P_i V_{1i} + \hat{A}_{di}^T \Pi V_{1i} \right] \psi(v) \\
&\quad + 2\zeta^T(k - d(k)) \left[\hat{A}_{di}^T P_i \hat{H}_{1i} + \hat{A}_{di}^T \Pi \hat{H}_{1i} \right] w(k) \\
&\quad + \psi^T(v) \left[V_{1i}^T P_i V_{1i} + V_{1i}^T \Pi V_{1i} \right] \psi(v) \\
&\quad + 2\psi^T(v) \left[V_{1i}^T P_i \hat{H}_{1i} + V_{1i}^T \Pi \hat{H}_{1i} \right] w(k) \\
&\quad + w^T(k) \left[\hat{H}_{1i}^T P_i H_{1i} + \hat{H}_{1i}^T \Pi H_{1i} \right] w(k) \\
&\quad + \zeta^T(k - d_1) (-Q_1) x(k - d_1) + \zeta^T(k - d_2) (-Q_2) \zeta(k - d_2) \\
&\quad - \sum_{\theta=-d(k)}^{-1} y^T(k + \theta) Z_1 y(k + \theta) - \sum_{\theta=-d(k)}^{-d_1-1} y^T(k + \theta) Z_2 y(k + \theta) \\
&\quad - \sum_{\theta=-d_2}^{-d(k)-1} y^T(k + \theta) (Z_1 + Z_2) y(k + \theta) - w^T(k) w(k) \\
&\quad + \gamma^{-2} \zeta^T(k) \hat{C}_i^T \hat{C}_i \zeta(k) + \sum_{r=1, r \neq i}^N \beta_{ir} \zeta^T(k) (P_r - P_i) \zeta(k) < 0.
\end{aligned}$$

Then, according to the above two inequalities, we can obtain.

$\Delta V(\zeta(k)) < w^T(k)w(k) - \gamma^{-2}\zeta^T(k)\hat{C}_i^T\hat{C}_i\zeta(k) - \sum_{r=1, r \neq i}^N \beta_{ir}\zeta^T(k)(P_r - P_i)\zeta(k)$. By switching law (38), we can get

$$\sum_{r=1, r \neq i}^N \beta_{ir}\zeta^T(k)(P_r - P_i)\zeta(k) \geq 0.$$

Therefore

$$\Delta V(\zeta(k)) < w^T(k)w(k) - \gamma^{-2}z^T(k)z(k). \quad (40)$$

Case 2: $\sigma(k) = i$, $\sigma(k+1) = r$ and $i \neq r$, for $\forall \zeta(k) \in \Omega(P_i, \rho_1 + \beta) \cap \Phi_i \subset L(\hat{N}_i, H_i)$. By switching law (38), we can get

$$\begin{aligned} \Delta V(\zeta(k)) &= V_r(\zeta(k+1)) - V_i(\zeta(k)) \\ &\leq V_i(\zeta(k+1)) - V_i(\zeta(k)) \\ &< w^T(k)w(k) - \gamma^{-2}z^T(k)z(k) - \sum_{r=1, r \neq i}^N \beta_{ir}\zeta^T(k)(P_r - P_i)\zeta(k). \end{aligned}$$

From the switching law (38), it follows that

$$\sum_{r=1, r \neq i}^N \beta_{ir}\zeta^T(k)(P_r - P_i)\zeta(k) \geq 0,$$

which indicates

$$\Delta V(\zeta(k)) < w^T(k)w(k) - \gamma^{-2}z^T(k)z(k). \quad (41)$$

By combining Eqs. (40) and (41), we can obtain

$$\begin{aligned} \Delta V(\zeta(k)) &= V(\zeta(k+1)) - V(\zeta(k)) < w^T(k)w(k) - \gamma^{-2}z^T(k)z(k), \\ \forall \zeta(k) &\in \cup_{i=1}^N (\Omega(P_i, \rho_1 + \beta) \cap \Phi_i). \end{aligned} \quad (42)$$

Thus, it follows

$$\sum_{k=0}^{\infty} \Delta V(\zeta(k)) < \sum_{k=0}^{\infty} w^T(k)w(k) - \sum_{k=0}^{\infty} \gamma^{-2}z^T(k)z(k),$$

which in turn gives

$$V(\zeta(\infty)) < V(\zeta(0)) + \sum_{k=0}^{\infty} w^T(k)w(k) - \sum_{k=0}^{\infty} \gamma^{-2}z^T(k)z(k). \quad (43)$$

Due to $\zeta(0) = 0$ and $V(\zeta(\infty)) \geq 0$, we obtain, we obtain

$$\sum_{k=0}^{\infty} z^T(k)z(k) < \gamma^2 \sum_{k=0}^{\infty} w^T(k)w(k). \quad (44)$$

(44) means that the switched system (9) has the restricted L_2 -gain from w to z over $w \in W_\beta^2$ less than γ . This proof is complete.

Just as (15) is treated as (31), (37) can also be treated as (31), similar to the processing of (15) into (31), (37) can also be processed into (31).

It follows from Theorem 2 that for each given $\beta \in (0, \beta^*]$, the minimum upper bound on the restricted L_2 -gain of the closed-loop system (9) can be obtained by solving the following optimization problem

$$\begin{aligned} &\inf_{P_i, Q_1, Q_2, Q_3, Z_1, Z_2, J_i, G_{1i}, G_{2i}, \beta_{ir}} \gamma^2, \\ &\text{s.t. (a) inequality (36), } i \in I_N, \\ &\quad \text{(b) inequality (31), } i \in I_N, j \in V_q. \end{aligned} \quad (45)$$

Then, we adopt the similarly method for converting optimization problem (23) into optimization problem (30). If the following matrix inequality holds, then matrix inequality (36) holds.

$$\begin{bmatrix} \Theta_{i11} & * \\ \Theta_{i21} & \Theta_{i22} \end{bmatrix} < 0, \quad (46)$$

where

$$\Theta_{i11} = \begin{bmatrix} v_i + \lambda_{1i} I_R I_R^T & * & * & * & * & * \\ 0 & -Q_3 & * & * & * & * \\ 0 & 0 & -Q_1 & * & * & * \\ 0 & 0 & 0 & -Q_2 & * & * \\ J_i H_i & 0 & 0 & 0 & -2J_i + \lambda_{2i} I^T I & * \\ 0 & 0 & 0 & 0 & 0 & -I \end{bmatrix},$$

$$\Theta_{i21} = \begin{bmatrix} P_i(\hat{A}_i + I_R K_i) & P_i \hat{A}_{di} & 0 & 0 & 0 & P_i \hat{H}_{1i} \\ \sqrt{d_2} P_i (\hat{A}_i + I_R K_i - I) & \sqrt{d_2} P_i \hat{A}_{di} & 0 & 0 & 0 & \sqrt{d_2} P_i \hat{H}_{1i} \\ d_1^* P_i (\hat{A}_i + I_R K_i - I) & d_1^* P_i \hat{A}_{di} & 0 & 0 & 0 & d_1^* P_i \hat{H}_{1i} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hat{C}_i & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Theta_{i22} = \begin{bmatrix} -P_i & * & * & * & * & * \\ 0 & -P_i Z_1^{-1} P_i & * & * & * & * \\ 0 & 0 & -P_i Z_2^{-1} P_i & * & * & * \\ M_{0i}^T \hat{B}_i^T P_i^T & \sqrt{d_2} M_{0i}^T \hat{B}_i^T P_i^T & d_1^* M_{0i}^T \hat{B}_i^T P_i^T & -\lambda_{1i} I & * & * \\ 2M_{0i}^T (\hat{B}_i + I_R E_{ci})^T P_i^T & 2\sqrt{d_2} M_{0i}^T (\hat{B}_i + I_R E_{ci})^T P_i^T & 2d_1^* M_{0i}^T (\hat{B}_i + I_R E_{ci})^T P_i^T & 0 & -\lambda_{2i} I & * \\ 0 & 0 & 0 & 0 & 0 & -\varsigma I \end{bmatrix},$$

where, $\varsigma = \gamma^2$.

Then, optimization problem (45) can be expressed as

$$\begin{aligned} & \inf_{P_i, Q_1, Q_2, Q_3, Z_1, Z_2, J_i, G_{1i}, G_{2i}, \beta_{ir}} \varsigma, \\ & \text{s.t. (a) inequality (46), } i \in I_N, \\ & \text{(b) inequality (31), } i \in I_N, j \in V_q. \end{aligned} \quad (47)$$

Control synthesis solution

In this section, we consider the anti-windup compensator gain E_{ci} and the dynamic state feedback controller gain K_i as the variables to be designed, which allows the performance of the closed-loop system (9) to be further improved.

Firstly, set $P_i^{-1} = X_i$, $F_i = H_i X_i, J_i^{-1} = S_i$, $Q_1^{-1} = U$, $Q_2^{-1} = U_1$, $Q_3^{-1} = U_2$, $Z_1^{-1} = U_3$, $Z_2^{-1} = U_4, K_i X_i = Y_i$. Multiplying both sides of the inequality (24) by the diagonal matrix $\{P_i^{-1}, Q_3^{-1}, Q_1^{-1}, Q_2^{-1}, J_i^{-1}, I, P_i^{-1}, P_i^{-1}, P_i^{-1}, I, I\}$ respectively, and using lemma 1, we have

$$\begin{bmatrix} \nabla_{i11} & * \\ \nabla_{i21} & \nabla_{i22} \end{bmatrix} < 0, \quad (48)$$

where

$$\nabla_{i11} = \begin{bmatrix} -X_i - \sum_{r=1, r \neq i}^N \beta_{ir} X_i & * & * & * & * & * & * & * & * & * \\ 0 & -U_2 & * & * & * & * & * & * & * & * \\ 0 & 0 & -U & * & * & * & * & * & * & * \\ 0 & 0 & 0 & -U_1 & * & * & * & * & * & * \\ F_i & 0 & 0 & 0 & -2S_i & * & * & * & * & * \\ 0 & 0 & 0 & 0 & +\lambda_{2i} I^T I & * & * & * & * & * \\ \hat{A}_i X_i & \hat{A}_{di} U_2 & 0 & 0 & 0 & \hat{H}_{1i} & -X_i & * & * & * \\ +I_R Y_i & & & & & & & & & \\ \sqrt{d_2} \hat{A}_i X_i & \sqrt{d_2} \hat{A}_{di} U_2 & 0 & 0 & 0 & \sqrt{d_2} \hat{H}_{1i} & 0 & -U_3 & * & * \\ +I_R Y_i & & & & & & & & & \\ -X_i & & & & & & & & & \\ d_1^* \hat{A}_i X_i & d_1^* \hat{A}_{di} U_2 & 0 & 0 & 0 & d_1^* \hat{H}_{1i} & 0 & 0 & -U_4 & * \\ +I_R Y_i & & & & & & & & & \\ -X_i & & & & & & & & & \end{bmatrix},$$

$$\nabla_{i21} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & M_{0i}^T \hat{B}_i^T & \sqrt{d_2} M_{0i}^T \hat{B}_i^T & d_1^* M_{0i}^T \hat{B}_i^T \\ 0 & 0 & 0 & 0 & 0 & 0 & 2M_{0i}^T (\hat{B}_i + I_R E_{ci})^T & 2\sqrt{d_2} M_{0i}^T (\hat{B}_i + I_R E_{ci})^T & 2d_1^* M_{0i}^T (\hat{B}_i + I_R E_{ci})^T \\ X_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ X_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{d_2^*} X_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I_R^T X_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ X_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ X_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ X_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\nabla_{i22} = \begin{bmatrix} -\lambda_{1i} I & * & * & * & * & * & * & * & * \\ 0 & -\lambda_{2i} I & * & * & * & * & * & * & * \\ 0 & 0 & -U & * & * & * & * & * & * \\ 0 & 0 & 0 & -U_1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & -U_2 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & -\lambda_{1i}^{-1} I & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & -\beta_{i1}^{-1} X_1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta_{iN}^{-1} X_N \end{bmatrix}.$$

A similar process is applied to (31), then we also obtain

$$\begin{bmatrix} X_i + \sum_{r=1, r \neq i}^N \delta_{ir} X_i & * & * & * & * \\ \hat{N}_i^j X_i - H_i^j & \varepsilon & * & * & * \\ X_i & 0 & \delta_{i1}^{-1} X_1 & * & * \\ X_i & 0 & 0 & \ddots & * \\ X_i & 0 & 0 & 0 & \delta_{iN}^{-1} X_N \end{bmatrix} \geq 0, \quad (49)$$

where, \hat{N}_i^j, H_i^j denotes the j -th row of \hat{N}_i and H_i respectively.

Thus, the maximum allowable disturbance level β^* can be established by solving the following optimization problems.

$$\begin{aligned} & \inf_{X_i, J, J_1, J_2, J_3, J_4, Y_i, \beta_{ir}, \lambda_{1i}, \lambda_{2i}, \delta_{ir}} \varepsilon, \\ & \text{s.t. (a) inequality (48), } i \in I_N, \\ & \text{(b) inequality (49), } i \in I_N, j \in V_q. \end{aligned} \quad (50)$$

Next, multiplying both sides of the inequality (46) by the diagonal matrix $\{P_i^{-1}, Q_3^{-1}, Q_1^{-1}, Q_2^{-1}, J_i^{-1}, I, P_i^{-1}, P_i^{-1}, P_i^{-1}, I, I\}$ respectively, and using lemma 1, we have

$$\begin{bmatrix} \Xi_{i11} & * \\ \Xi_{i21} & \Xi_{i22} \end{bmatrix} < 0, \quad (51)$$

where

$$\Xi_{i11} = \begin{bmatrix} -X_i - \sum_{r=1, r \neq i}^N \beta_{ir} X_i & * & * & * & * & * & * & * & * & * \\ 0 & -U_2 & * & * & * & * & * & * & * & * \\ 0 & 0 & -U & * & * & * & * & * & * & * \\ 0 & 0 & 0 & -U_1 & * & * & * & * & * & * \\ F_i & 0 & 0 & 0 & -2S_i + \lambda_{2i} I^T I & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & -I & * & * & * & * \\ \hat{A}_i X_i + I_R Y_i & \hat{A}_{di} U_2 & 0 & 0 & 0 & \hat{H}_{1i} & -X_i & * & * & * \\ \sqrt{d_2} \hat{A}_i X_i & \sqrt{d_2} \hat{A}_{di} U_2 & 0 & 0 & 0 & \sqrt{d_2} \hat{H}_{1i} & 0 & -U_3 & * & * \\ + I_R Y_i - X_i & & & & & & & & & \\ d_1^* \hat{A}_i X_i & d_1^* \hat{A}_{di} U_2 & 0 & 0 & 0 & d_1^* \hat{H}_{1i} & 0 & 0 & -U_4 & * \\ + I_R Y_i - X_i & & & & & & & & & \end{bmatrix},$$

$$\Xi_{i21} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & M_{0i}^T \hat{B}_i^T & \sqrt{d_2} M_{0i}^T \hat{B}_i^T & d_1^* M_{0i}^T \hat{B}_i^T \\ 0 & 0 & 0 & 0 & 0 & 0 & 2M_{0i}^T (\hat{B}_i + I_R E_{ci})^T & 2\sqrt{d_2} M_{0i}^T (\hat{B}_i + I_R E_{ci})^T & 2d_1^* M_{0i}^T (\hat{B}_i + I_R E_{ci})^T \\ X_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ X_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{d_2^*} X_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I_R^T X_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ X_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ X_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ X_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hat{C}_i X_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Xi_{i22} = \begin{bmatrix} -\lambda_{1i} I & * & * & * & * & * & * & * & * & * \\ 0 & -\lambda_{2i} I & * & * & * & * & * & * & * & * \\ 0 & 0 & -U & * & * & * & * & * & * & * \\ 0 & 0 & 0 & -U_1 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & -U_2 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & -\lambda_{1i}^{-1} I & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & -\beta_{i1}^{-1} X_1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta_{iN}^{-1} X_N & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\tau I \end{bmatrix},$$

where, $\tau = \gamma^2$.

Therefore, the minimum upper bound on the restricted L_2 -gain of the closed-loop system (9) can then be obtained by solving the following optimization problem

$$\begin{aligned} & \inf_{X_i, J, J_1, J_2, J_3, J_4, Y_i, \beta_{ir}, \lambda_i, \lambda_{1i}, \delta_{ir}} \tau, \\ & \text{s.t. (a) inequality (51), } i \in I_N, \\ & \text{(b) inequality (49), } i \in I_N, j \in V_q. \end{aligned} \quad (52)$$

Then, we can obtain the anti-windup compensator gain matrices E_{ci} by solving the above two optimization problems (50) and (52), and the corresponding dynamic state feedback controller gain matrices can be calculated as $K_i = Y_i X_i^{-1}$.

Remark 1. A key feature of the proposed anti-windup approach is that the anti-windup compensator remains inactive when saturation does not occur, thereby preserving the nominal design performance of the system. In practice, systems operate in non-saturated conditions most of the time, meaning that the design performance is unaffected in the majority of cases. The anti-windup compensator only becomes active when saturation occurs, ensuring system stability and performance during saturation. As a result, the anti-windup method has strong practical engineering relevance for the switched systems. However, in the convex hull approach^{36,38}, as the control input dimension and subsystem count increase, the number of linear matrix inequalities (LMIs) grows exponentially, leading to a prohibitively high computational burden.

Numerical simulation

This section presents two numerical examples to demonstrate the effectiveness of the proposed method. We consider the following a class of time-varying delay discrete-time switched systems with actuator saturation and external disturbances.

$$\begin{cases} x(k+1) = A_\sigma x(k) + A_{d\sigma} x(k-d(k)) + B_{1\sigma} \text{sat}(u(k)) + H_{1\sigma} w(k), \\ z(k) = C_\sigma x(k), \\ x(k_0 + \theta) = \varphi(\theta), \theta = -d_2, -d_2 + 1, \dots, 0, \end{cases} \quad (53)$$

Example 1

where $\sigma \in I_2 = \{1, 2\}$,

$$\begin{aligned} A_1 &= \begin{bmatrix} 0.415 & 0.53 \\ 0.21 & 2.65 \end{bmatrix}, A_2 = \begin{bmatrix} 0.63 & 0.38 \\ 0.1 & 0.05 \end{bmatrix}, A_{d1} = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0.1 \end{bmatrix}, A_{d2} = \begin{bmatrix} 0.1 & 0.1 \\ 0 & 0.1 \end{bmatrix}, \\ B_1 &= \begin{bmatrix} 0.6 & 0.99 \\ 0.46 & 0.807 \end{bmatrix}, B_2 = \begin{bmatrix} 0.4 & 0 \\ 1 & 0.39 \end{bmatrix}, H_{11} = \begin{bmatrix} -0.2 & 0 \\ 0 & 0.1 \end{bmatrix}, H_{12} = \begin{bmatrix} 0.3 & 0 \\ 0 & -0.1 \end{bmatrix}. \end{aligned}$$

Assuming that the range of actuator faults is $0.1 \leq m_{ij} \leq 0.9$, and according to the continuous fault model we can get

$$M_{01} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, M_{02} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}.$$

Let $M_1 = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.5 \end{bmatrix}$, $M_2 = \begin{bmatrix} 0.6 & 0 \\ 0 & 0.5 \end{bmatrix}$, $d_1 = 1, d_2 = 9, \rho_1 = 0.5, \lambda_{11} = 30, \lambda_{12} = 0.033$.

Then, by solving the optimization problem (50) we can obtain the following feasible solutions

$$\varepsilon^* = 0.0782, \beta^* = 12.2797, \lambda_{21} = 3.8568 \times 10^4, \lambda_{22} = 3.8470 \times 10^4.$$

$$X_1 = \begin{bmatrix} 5.2427 & -5.6586 & -1.6548 & 0.6425 \\ * & 16.6402 & 1.1196 & -2.4512 \\ * & * & 3.5092 & -0.4379 \\ * & * & * & 0.4815 \end{bmatrix},$$

$$X_2 = \begin{bmatrix} 5.2003 & -8.3629 & -2.2349 & 1.1820 \\ * & 23.9339 & 2.8067 & -3.6175 \\ * & * & 3.5731 & -0.7306 \\ * & * & * & 0.7273 \end{bmatrix},$$

$$Y_1 = \begin{bmatrix} 0.8387 & -2.8544 & -0.8276 & 0.7040 \\ -0.8796 & 2.1680 & 1.5216 & -1.6516 \end{bmatrix},$$

$$Y_2 = \begin{bmatrix} 1.2164 & 1.3004 & -0.8668 & -0.2484 \\ -5.3976 & -3.0753 & 0.7083 & 0.9669 \end{bmatrix},$$

and the AW compensator gains for the closed-loop system are

$$E_{c1} = \begin{bmatrix} 0.3496 & -3.2979 \\ 0.6331 & -5.7500 \end{bmatrix}, E_{c2} = \begin{bmatrix} 1.4758 & -7.1687 \\ 0.6242 & -2.6969 \end{bmatrix}.$$

Then, the dynamic state controller gains of the closed-loop system are computed as

$$K_1 = Y_1 X_1^{-1} = [G_1 \quad F_1] = \begin{bmatrix} 0.1627 & 0.2957 & 0.1013 & 2.8426 \\ -1.6428 & -2.8636 & -1.5807 & -17.2541 \end{bmatrix},$$

$$K_2 = Y_2 X_2^{-1} = [G_2 \quad F_2] = \begin{bmatrix} 0.7072 & 0.3003 & -0.0448 & -0.0423 \\ -3.4430 & -1.2946 & -1.0559 & -0.5748 \end{bmatrix}.$$

Then, we select $\omega(t) = \sqrt{2\beta^*}e^{-t}$ as external interference input for simulation. Figure 1 shows the state response curve of the switched system (53). The state response of the controller of the switched system (53) is shown in Fig. 2. Figure 3 shows the switching signal of the closed-loop system. The control input signal for the switched system (53) is shown in Fig. 4. Figure 5 shows the variation of Lyapunov function values of the switched system (53). It can be seen through Fig. 5 that the values of the Lyapunov function for the switched system (53) are consistently less than $\beta^* = 12.2797$. This indicates that the state trajectories of the switched system (53) with the initial conditions always remains within this bounded set.

Then, for each given $\beta \in (0, \beta^*]$, we estimate an upper bound on the restricted L_2 -gain of the closed-loop system (53). Therefore, we can consider the following scenarios.

1. If $\beta = 0.5$, we obtain, $\gamma = 23.0997$,

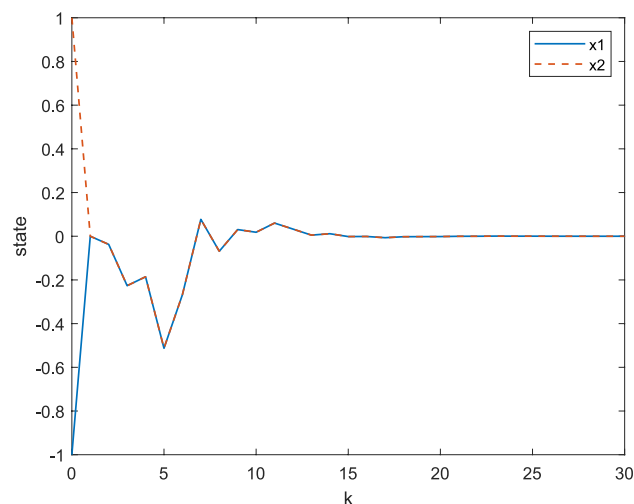


Fig. 1. State response of the switched system (53).

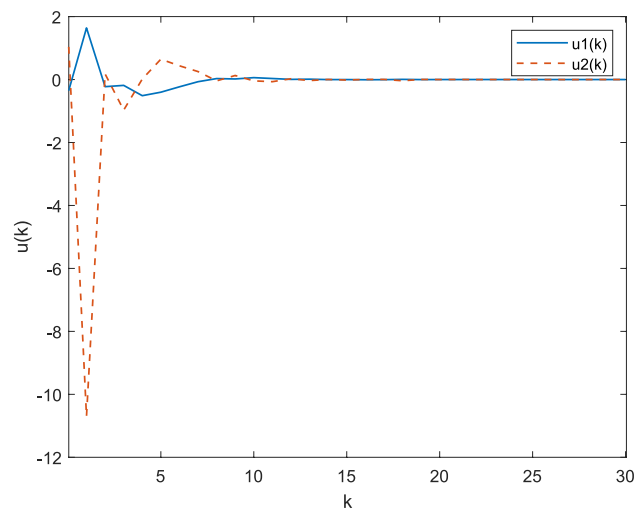


Fig. 2. State response of the controller of the switched system (53).

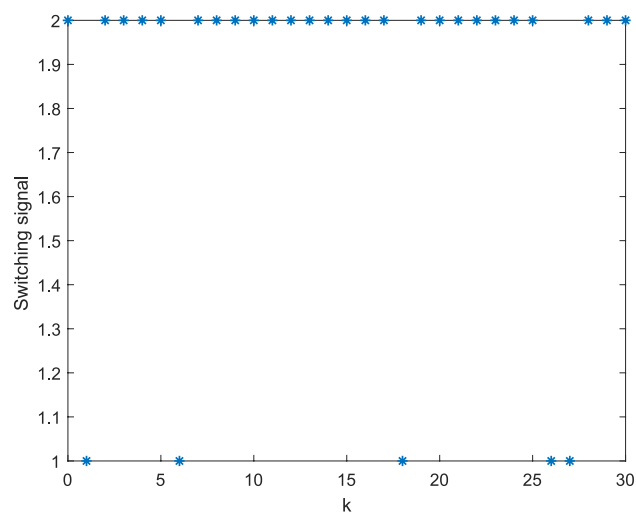


Fig. 3. The switched signal for the switched system (53).

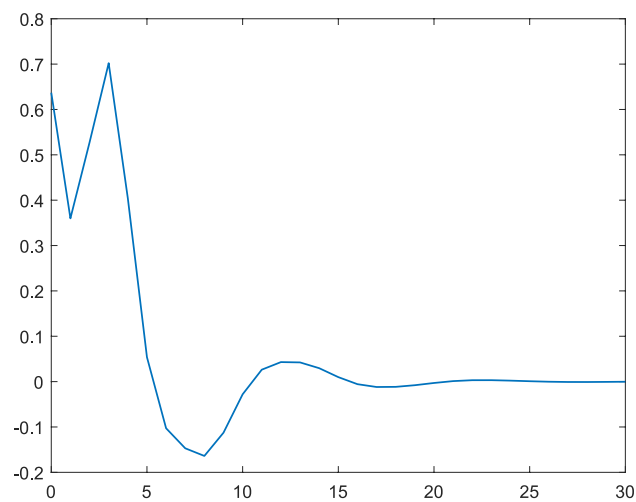


Fig. 4. The input signal of the closed loop system (53).

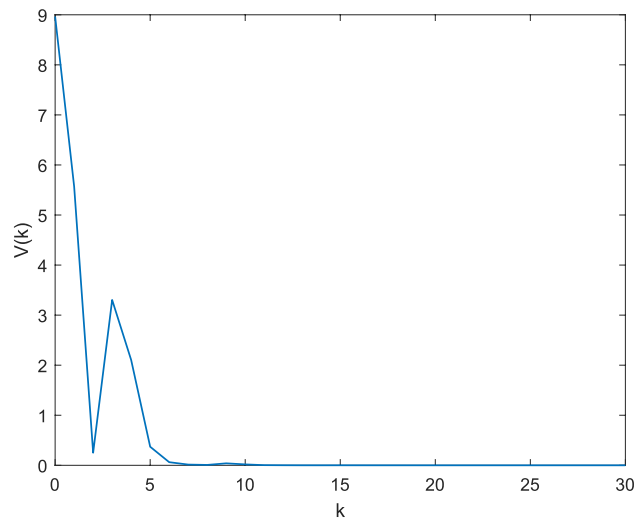


Fig. 5. Lyapunov function values for the switched system (53).

$$K_1 = \begin{bmatrix} 0.1633 & 0.2960 & 0.1014 & 2.8474 \\ -1.6428 & -2.8632 & -1.5806 & -17.2502 \end{bmatrix}, E_{c1} = \begin{bmatrix} 0.3791 & -3.3134 \\ 0.6843 & -5.7767 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} 0.7071 & 0.3001 & -0.0454 & -0.0424 \\ -3.4434 & -1.2942 & -1.0575 & -0.5768 \end{bmatrix}, E_{c2} = \begin{bmatrix} 1.5081 & -7.1632 \\ 0.6368 & -2.6950 \end{bmatrix}.$$

2. If $\beta = 1$, we obtain $\gamma = 23.1065$,

$$K_1 = \begin{bmatrix} 0.1626 & 0.2949 & 0.1000 & 2.8581 \\ -1.6436 & -2.8645 & -1.5814 & -17.2529 \end{bmatrix}, E_{c1} = \begin{bmatrix} 0.3855 & -3.3244 \\ 0.6956 & -5.7955 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} 0.7086 & 0.3010 & -0.0475 & -0.0443 \\ -3.4567 & -1.2986 & -1.0657 & -0.5794 \end{bmatrix}, E_{c2} = \begin{bmatrix} 1.5000 & -7.1100 \\ 0.6343 & -2.6733 \end{bmatrix}.$$

3. If $\beta = 7.5$, we obtain $\gamma = 23.1960$,

$$K_1 = \begin{bmatrix} 0.1630 & 0.2955 & 0.1008 & 2.8534 \\ -1.6431 & -2.8636 & -1.5808 & -17.2503 \end{bmatrix}, E_{c1} = \begin{bmatrix} 0.3851 & -3.3222 \\ 0.6949 & -5.7918 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} 0.7078 & 0.3005 & -0.0464 & -0.0435 \\ -3.4477 & -1.2956 & -1.0602 & -0.5768 \end{bmatrix}, E_{c2} = \begin{bmatrix} 1.5047 & -7.1100 \\ 0.6357 & -2.6742 \end{bmatrix}.$$

$$\begin{cases} x(k+1) = A_\sigma x(k) + A_{d\sigma} x(k-d(k)) + B_{1\sigma} \text{sat}(u(k)) + H_{1\sigma} w(k), \\ z(k) = C_\sigma x(k), \\ x(k_0 + \theta) = \varphi(\theta), \theta = -d_2, -d_2 + 1, \dots, 0, \end{cases} \quad (54)$$

Example 2

where $\sigma \in I_2 = \{1, 2\}$,

$$A_1 = \begin{bmatrix} 1.195 & -1.0 \\ 0.93 & 0.60 \end{bmatrix}, A_2 = \begin{bmatrix} 0.80 & 0.90 \\ 0.06 & 0.49 \end{bmatrix}, A_{d1} = \begin{bmatrix} 0.3 & 0 \\ 0.3 & 0.3 \end{bmatrix}, A_{d2} = \begin{bmatrix} 0.3 & 0.3 \\ 0 & 0.3 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.75 & 0.83 \\ 0.74 & 0.15 \end{bmatrix}, B_2 = \begin{bmatrix} 0.45 & 0.93 \\ 0.61 & 0.83 \end{bmatrix}, H_{11} = \begin{bmatrix} -0.3 & 0 \\ 0 & 0.2 \end{bmatrix}, H_{12} = \begin{bmatrix} 0.4 & 0 \\ 0 & -0.2 \end{bmatrix}.$$

Assuming that the range of actuator faults is $0.2 \leq m_{ij} \leq 0.4$, and according to the continuous fault model, we can get

$$M_{01} = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}, M_{02} = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}.$$

Let $M_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix}, M_2 = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.3 \end{bmatrix}, d_1 = d_2 = 1, \rho_1 = 0.5, \lambda_{11} = 20, \lambda_{12} = 0.5$.

Then, by solving the optimization problem (50) we can obtain the following feasible solutions.

$$\varepsilon^* = 0.2400, \beta^* = 3.6674, \lambda_{21} = 3.8079 \times 10^5, \lambda_{22} = 3.8077 \times 10^5.$$

$$\begin{aligned}
X_1 &= \begin{bmatrix} 12.8955 & -1.2251 & -2.8140 & -0.5712 \\ * & 6.9433 & -0.6051 & -0.0206 \\ * & * & 0.9814 & 0.0454 \\ * & * & * & 0.4674 \end{bmatrix}, \\
X_2 &= \begin{bmatrix} 12.6572 & -2.8802 & -2.2339 & -0.3790 \\ * & 7.2063 & -0.3242 & -0.8016 \\ * & * & 0.9452 & -0.0406 \\ * & * & * & 0.6740 \end{bmatrix}, \\
Y_1 &= \begin{bmatrix} 0.9701 & -1.6553 & -1.7354 & 1.5678 \\ 0.1572 & -0.5219 & -0.2819 & 0.3946 \end{bmatrix}, \\
Y_2 &= \begin{bmatrix} 2.6507 & -3.1484 & -0.5810 & -1.0833 \\ -1.1405 & -1.7150 & 0.4720 & -0.2846 \end{bmatrix},
\end{aligned}$$

and the AW compensator gains for the closed-loop system (54) are

$$E_{c1} = \begin{bmatrix} -3.9974 & -0.6218 \\ -3.2007 & -0.6329 \end{bmatrix}, \quad E_{c2} = \begin{bmatrix} -2.2091 & -1.2181 \\ -4.0736 & -1.8668 \end{bmatrix}.$$

Then, the dynamic state controller gains of the closed-loop system (54) are computed as

$$\begin{aligned}
K_1 &= Y_1 X_1^{-1} = [G_1 \quad F_1] = \begin{bmatrix} -1.1943 & -0.9556 & -5.8941 & 2.4252 \\ -0.1884 & -0.1915 & -0.9779 & 0.7006 \end{bmatrix}, \\
K_2 &= Y_2 X_2^{-1} = [G_2 \quad F_2] = \begin{bmatrix} -0.6670 & -1.2297 & -2.7678 & -3.6112 \\ -0.3657 & -0.5598 & -0.6141 & -1.3306 \end{bmatrix}.
\end{aligned}$$

If we set $E_{c1} = E_{c2} = 0$, then the optimal solution obtained is $\beta^* = 1.6372$ based on corollary 1. Obviously, this shows that the interference tolerance of the closed-loop system (54) is enhanced under the action of the anti-windup compensator.

Then, we select $\omega(t) = \sqrt{2\beta^*}e^{-t}$ as external interference input for simulation. Figure 6 shows the state response curve of the switched system (54). The state response of the controller of the switched system (54) is shown in Fig. 7.

In the event of actuator failure, the state response is depicted in Figs. 8 and 9 for switched system (54) under the conventional anti-windup controller. Note that the conventional anti-windup controller here refers to a design that does not account for actuator failures. As evident from Figs. 8 and 9, the system states become unbounded when actuator failure occurs under this controller. In contrast, the anti-windup fault-tolerant controller ensures that the state trajectory remains within a bounded region, as demonstrated in Figs. 6 and 7. This confirms that the anti-windup fault-tolerant controller, designed using our proposed method, exhibits superior fault tolerance performance.

Additionally, Table 1 presents the relationship between the maximum allowable disturbance β^* and varying d values, obtained by solving optimization problem (50) (where $d = d_1 = d_2$). The results demonstrate that the disturbance tolerance of closed-loop system (54) decreases as d increases.

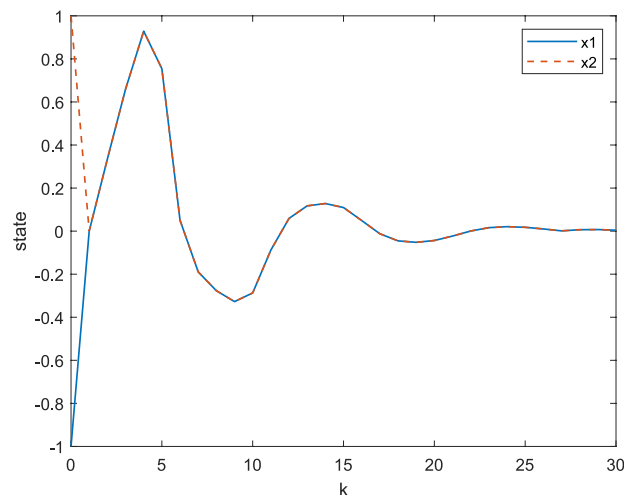


Fig. 6. State response of the switched system (54).

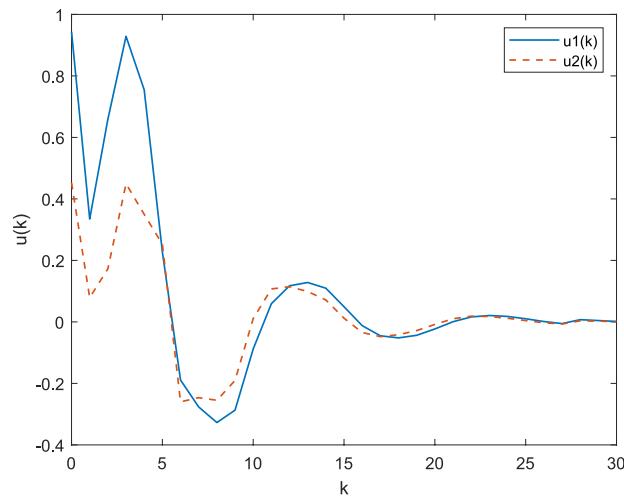


Fig. 7. State response of the controller of the switched system (54).

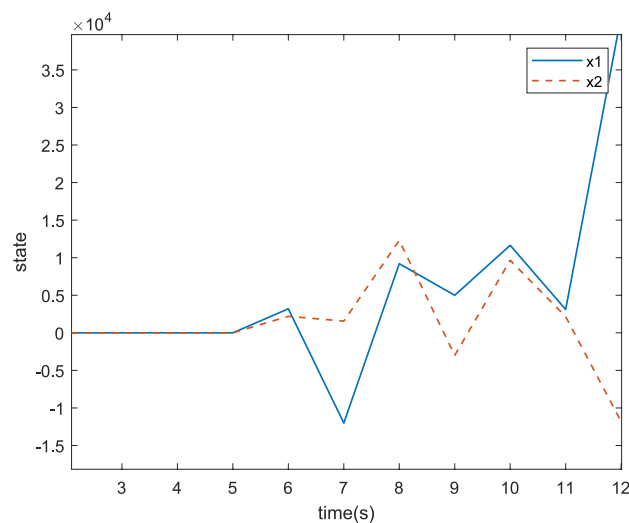


Fig. 8. State response of system (54) under conventional anti-windup controller.

Conclusion

In this paper, the problem of L_2 -gain analysis and anti-windup fault-tolerant controller design of a class of discrete-time switched systems with time-varying delay and actuator saturation is investigated by using the multiple Lyapunov functionals method. Firstly, with the anti-windup fault-tolerant controller given in advance, the sufficient conditions to be state bounded under the action of actuator fault and external disturbance are given for the closed-loop system, and then disturbance tolerance problem is transformed into a constrained optimization problem. Then, based on this condition, the constrained L_2 -gain of a closed-loop system is analyzed and the minimum upper bound of the restricted L_2 -gain is presented by solving a convex optimization problem. Finally, in order to obtain better performance of the closed-loop system, we design the AW controller to maximize the allowable disturbance capability of the closed-loop system as well as to minimize the restricted L_2 -gain upper bound. A numerical simulation example is given to verify the effectiveness of the proposed method in this paper.

In this paper, we presuppose the availability of certain unknown parameters beforehand and subsequently reformulate the associated optimization challenges into problems featuring LMI constraints. Nonetheless, in real-world applications, these parameters inevitably influence the outcomes of the optimization processes. Consequently, determining the optimal selection of these parameters to achieve the highest possible level of disturbance tolerance and the lowest feasible upper limit of the constrained L_2 -gain presents a fascinating and complex area for investigation. Employing advanced optimization techniques, including genetic algorithms and ant colony optimization, could prove effective in identifying the ideal values for these parameters, meriting additional exploration and study. In addition, the Zeno phenomenon induced by minimum switching laws can cause significant issues, potentially resulting in severe harm to real-world engineering systems. As a result, in

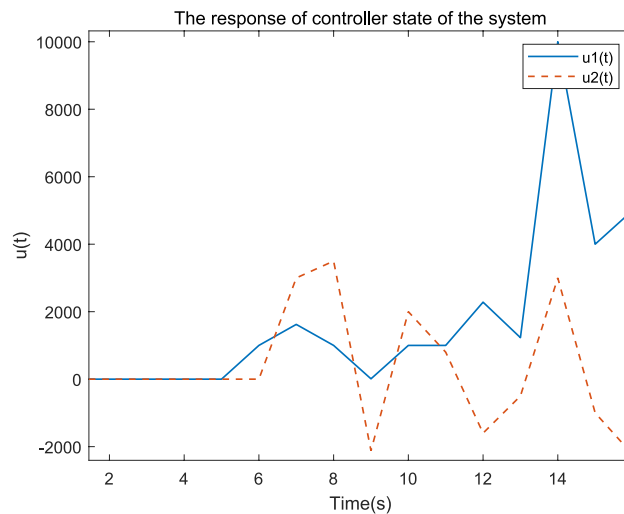


Fig. 9. State response of conventional anti-windup controller.

d	β^*
1.5	2.8818
2	1.5917
5	0.2435
7	—

Table 1. The values of β^* at different values of d .

our forthcoming research, we intend to employ the dwell time approach to explore the fault-tolerant control challenges in switched systems experiencing actuator saturation and failure. The switching law developed using the dwell time method is expected to successfully prevent the occurrence of the Zeno phenomenon.

Data availability

All data generated or analysed during this study are included in this published article.

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Author contributions

Mengxiang Li and Xinquan Zhang wrote the main manuscript text and Mengxiang Li and Xinquan Zhang prepared figures. All authors reviewed the manuscript.

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Declarations

Competing interests

The authors declare no competing interests.

Additional information

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