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Characterizing the Jellinek curve using a discounting model with time deformations

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This paper deals with the relationship between addiction/self-control and discount rates. In effect, it can be shown that the evaluation of temporal discounting is a candidate behavioral marker for addiction. More specifically, high discount rates are associated with several unhealthy behaviors (drinking, smoking and taking drugs, among them). There are also empirical studies that used a measure of delay discounting in order to predict abstinence or relapse. The main objective of this paper is to present a discounting model that describes the phases of addiction, vicious circles and recovery, present in the so-called Jellinek curve. To do this, we will start from the exponential discount function, which will be deformed, one or more times, by using the well-known Weber-Fechner and Stevens' laws, in order to model the three aforementioned phases, specifically the relapse periods which characterize the vicious circles of the Jellinek curve. In this paper, we propose novel discount functions able to describe the behavior of addictions in all their phases and to fit intertemporal choice data in order to obtain the individuals' discount rates.

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Introduction

Addictions can be considered forms of impulsive behavior. In effect, the existing literature has shown a clear relationship between varieties of impulsive and addiction-related behaviors, such as drug and alcohol consumption (Jentsch et al., 2014). More generally, impulsive behaviors have been observed in a wide range of psychiatric disorders, including substance use, bipolar, attention-deficit hyperactivity, antisocial and borderline personality, gambling, and eating disorders.

The relevance of impulsivity to relapses was presented by Adinoff et al. (2007), with a focus on three different neurocognitive constructs: automaticity, response inhibition and decision-making. Specifically, decision-making deficits contribute to relapses through a poorly considered assessment of the consequences of drug use.

In this research line, Ida and Goto (2009) concluded that the higher the time-preference rate and the lower the risk aversion coefficient becomes, the more likely individuals smoke, drink frequently, and gamble on pachinko (a popular Japanese form of pinball gambling), and the horses. Bretteville-Jensen (1999) found that active injectors of heroin and amphetamine exhibit a higher discount rate than a group of individuals who have never used the substances. Additionally, they found that the discount rate among active and former users differs significantly. These findings raise the question of whether a high time-preference rate leads to addiction or whether the onset of an addiction itself alters people's intertemporal equilibrium.

Impulsivity, expressed as impulsive choice or inhibitory failure, plays a role in several key transition phases of drug abuse. Perry and Carroll (2008) point out that (1) increased levels of impulsivity lead to the acquisition of drug abuse and a subsequent escalation or dysregulation of drug intake, and (2) drugs of abuse may increase impulsivity, which is an additional contributor to the aforementioned escalation or dysregulation. As indicated by Perry and Carroll (2008), relating drug abuse and impulsivity in phases of addiction through the former hypotheses provides a heuristic model from which future experimental questions can be addressed. Indeed, this provides some more context to the present, thus emphasizing the real-world application of this work.

Consequently, this paper discusses the relationship between addiction/self-control and discount rates, highlighting how high discount rates are associated with unhealthy behaviors like drinking, smoking, and drug use. It also mentions that the delay discounting can be used to predict abstinence or relapse in individuals with addiction. Specifically, the Jellinek curve is used as a model of addiction, which details the progressive stages of alcoholism and other forms of substance abuse. To do this, the paper introduces various discounting models, such as exponential discounting, hyperbolic discounting, and exponentiated hyperbolic discounting, and discusses how these models can be deformed by using laws like Weber-Fechner and Stevens' laws to describe the phases of addiction, vicious circles, and recovery present in the Jellinek curve.

The main contribution of this manuscript lies in the behavioral implications of discount functions and time deformations in the context of addictions. Specifically, it discusses how changes in impatience described by discount functions can reflect the phases of addiction, such as increasing impatience in the crucial phase, and decreasing impatience in the rehabilitation phase.

The relevance of exponential and hyperbolic discounting in intertemporal choice is well known. The exponential discount function has been the paradigm of intertemporal choices from the paper of Samuelson (1937) where the Discounted Utility (DU) model assumes a constant discount rate (also referred to as impatience) which is associated with rational and consistent behaviors. However, empirical studies have shown that

individuals show decreasing discount rates. This has made the hyperbolic discounting preferred over the exponential. However, none of these discounting models exhibits a change in the monotonicity of the discount rate¹. In effect, the existing literature on this topic has analyzed monotone (increasing or decreasing) discount rates, leading to inconsistent behaviors. For example, linear discounting ($F(t) = 1 - kt$, $k > 0$) exhibits an increasing discount rate:

$$\delta(t) = \frac{k}{1 - kt},$$

whilst the generalized hyperbolic discounting (Takahashi et al., 2008) ($F(t) = \frac{1}{(1+\alpha t)^\beta}$, $\alpha > 0$, $\beta > 0$) exhibits a decreasing discount rate:

$$\delta(t) = \frac{\alpha}{1 + \alpha t}.$$

However, little has been said about those discount functions whose discount rate is increasing in a certain interval and decreasing otherwise. This is the case of the exponentiated hyperbolic discounting (Cruz Rambaud et al., 2017) ($F(t) = \frac{1}{1+\alpha t^\beta}$, $\alpha > 0$, $\beta > 0$) whose discount rate is given by:

$$\delta(t) = \frac{\alpha\beta t^{\beta-1}}{1 + \alpha t^\beta}.$$

This is very important to describe the three main phases into which the Jellinek curve has been divided: the crucial phase, the chronic phase, and the rehabilitation phase, as each phase represents different aspects of addiction and recovery.

From a behavioral point of view, Cruz Rambaud et al. (2017) linked the monotonicity of impatience to both the relapse and recovery phases of a substance abuser. Analogously, Bickel et al. (1999) proposed the evaluation of temporal discounting as "a candidate behavioral marker for addiction". In the same line, Story et al. (2014) systematic review found that high discount rates were associated with several unhealthy behaviors (drinking, smoking and taking drugs, among them). Complementarily, some empirical studies used a measure of delay discounting in order to predict abstinence or relapse (Stevens et al., 2015). This is why the so-called Jellinek curve (Jellinek, 1952), which shows successive periods of recovery and relapse of an individual with one or several addictions (see Fig. 1), requires the introduction of a discount function able to describe the changing behavior of a substance abuser, but unfortunately, the aforementioned functions do not adapt to this cyclical situation.

A first attempt to model this changing behavior was introduced by Cruz Rambaud et al. (2017) when justifying the so-called exponentiated hyperbolic discount function:

$$F(t) = \frac{1}{1 + it^k}, \quad i > 0, \quad k > 1,$$

to describe the passage from a relapse to a recovery phase of an addicted individual.

More generally, the Jellinek curve is a model of addiction trying to identify the progressive stages of alcoholism (though it can be used for most forms of substance abuse), detailing very specific events and circumstances that come as a result of addiction throughout each phase.

Thus, it can also be used as a tool to track progress within the context of Alcoholics Anonymous², but really, it can complement any modality of treatment. One can use it to not only track those progressive phases of alcohol addiction, but also most any substance that may induce a mental or physical dependence.

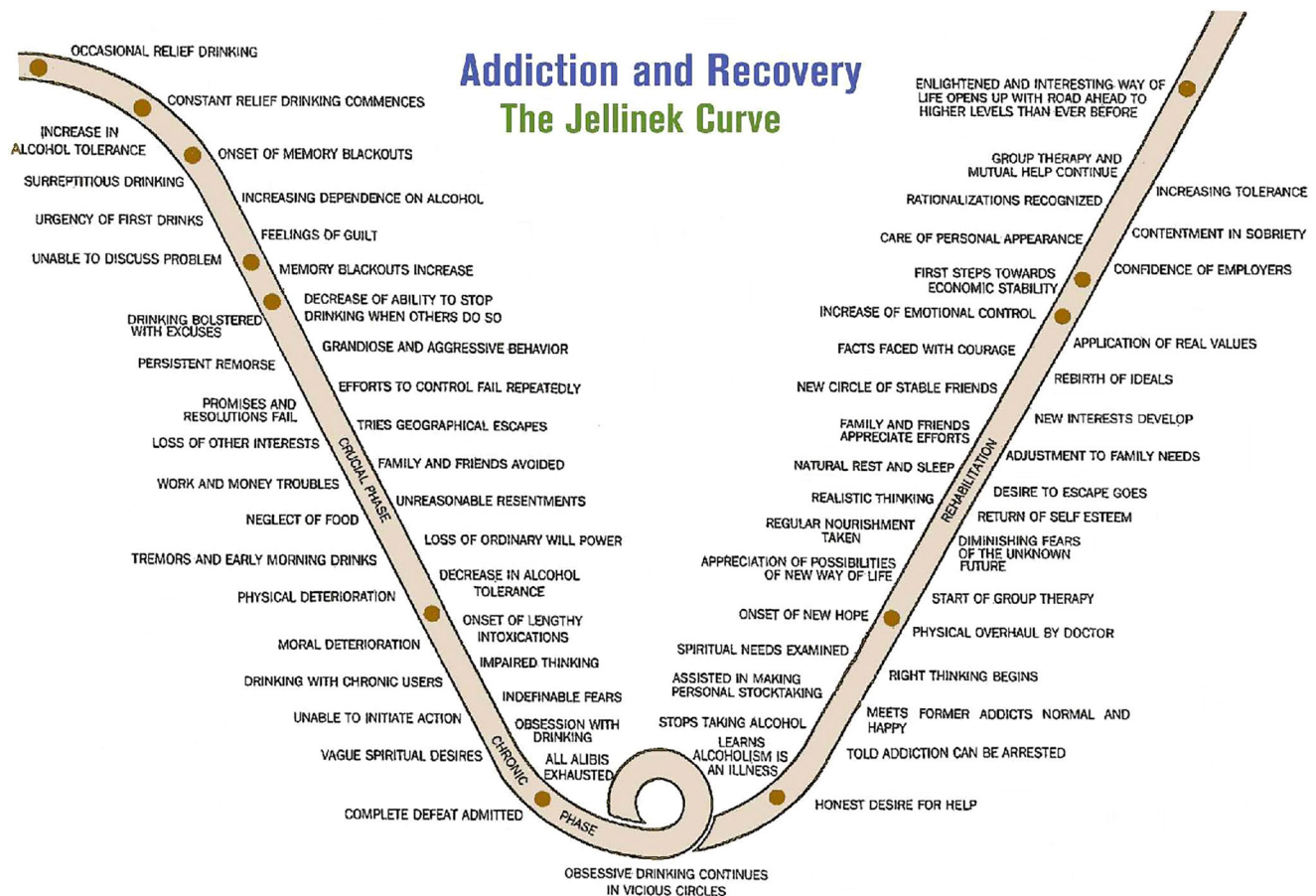


Fig. 1 The Jellinek curve. Source: Jellinek (1952).

The Jellinek curve describes the typical pattern of how people with addiction experience addiction and recovery. Specifically, it can be separated into three main phases:

- The crucial phase.
- The chronic phase.
- The rehabilitation phase.

The crucial phase. The crucial phase represents the time during which addiction begins, usually with social consumption of alcohol before moving towards occasional consumption of relief. This phase is often followed by increased dependence and feelings of guilt, social avoidance and loss of willpower. This results in increased periods of intoxication, which may include excessive alcohol consumption and an inability to moderate or restrict consumption, ultimately leading to the lowest section of the curve.

The chronic phase. There is a loop at the bottom of the Jellinek curve where people often get caught up in the cyclical nature of addiction, spiraling deeper and deeper into obsessive patterns of alcohol or drug use. This happens during this vicious cycle of dependence. This is when external help is really needed to start and go ahead with the recovery process.

The rehabilitation phase. Starting with a sincere desire for help, the road begins to curve upwards with a steady slope. After one stops drinking, confused thinking becomes clear, and thoughts of a new life begin to arise as one is freed from the agony of addiction. Self-esteem is rebuilt, new connections are forged, and

courage and a strong support network encourage people to continue on the road to recovery. The rehabilitation process to lead a sober life is not easy and may take some time, but the end result is well worth it. Keeping that hopeful but realistic thinking is part of using the Jellinek curve.

The main objective of this paper is to present a discounting model able to describe the different phases of addiction, vicious circles and recovery, present in the so-called Jellinek curve. To do this, we will start from the exponential discount function, which will be deformed, one or more times, by using the well-known Weber-Fechner and “Power” Stevens’ laws, in order to model the three aforementioned phases, specifically the relapse periods (Mao et al., 2024), which characterize the vicious circles of the chronic phase in the Jellinek curve. As a result of this research, the novelty of the proposed procedure is the proposal of new discount functions able to describe the behavior of addictions in all their phases and to fit intertemporal choice data in order to obtain the individuals’ discount rates. More specifically, we have used the generalized hyperbolic discount function and deformed it multiple times (successively) by using Stevens’ time deformation. The first round of deformation gives the (generalized) exponentiated hyperbolic discount function (Cruz Rambaud et al., 2017), which in turn characterizes the “crucial” phase in the Jellinek curve. The multiple deformations thereafter characterize the vicious cycles in the “chronic” phase and, finally, the “rehabilitation” phase. The discount rates corresponding to these functions, in addition to other measures, could have a predictive role and be useful at different stages in the evolution of clinical treatments.

This paper is organized as follows. After this Introduction, the section “Preliminary definitions” introduces some definitions

(discount function and time deformation) that will be needed for the development of this manuscript. Section “Results” presents the first results and provides a nice result (Theorem 1), which will be essential for the discussion of the particular cases presented in the subsequent sections. In effect, section “Deformation with Weber-Fechner laws” presents the application of the general results obtained in section “Results” to deformations ruled by the Weber-Fechner law. Analogously, section “Deformation with Stevens’ laws” applies Theorem 1 to the time deformation of a hyperbolic and an exponential discount function by using the “Power” Stevens’ law. This section will provide Corollary 9, which will be crucial for a complete interpretation of the Jellinek curve. Finally, the section “Discussion and conclusions” summarizes and concludes.

Preliminary definitions

In this preliminary section, we are going to introduce the following definitions.

Definition 1. A positive and continuous real-valued function

$$F: X \times T \rightarrow X$$

such that

$$(x, t) \mapsto F(x, t)$$

is said to be a *stationary discount function* if F is strictly increasing with respect to x , strictly decreasing with respect to t , and satisfies the following conditions:

1. $F(0, t) = 0$, for every $t \in T$.
2. $F(x, 0) = x$, for every $x \in X$.

$F(x, t)$ represents the amount equivalent at 0 of $\$x$ available at t .

On the other hand, Definition 2 sets the structure of the discount function to be employed in this paper, where amount and time behave as separate variables in the expression of the discount function.

Definition 2. A stationary discount function $F(x, t)$ is said to be *separable* if $F(x, t)$ can be expressed as $u(x)F(t)$, where u is a utility function, F is strictly decreasing, and $F(0) = 1$.

The following definition is based on Cruz Rambaud and Ventre (2017) and Cruz Rambaud et al. (2018).

Definition 3. A *time distortion* (or *time deformation*) is a continuous real-valued function, $g(t)$, defined in an interval $[0, t_0)$ (t_0 can be $+\infty$), satisfying the following conditions:

1. $g(0) = 0$.
2. $g(t)$ is strictly increasing.

If $t_0 = +\infty$, it is reasonable to require that $\lim_{t \rightarrow +\infty} g(t) = +\infty$. The main time distortions in the existing literature on the topic of intertemporal choice are the following functions:

- The so-called Weber-Fechner law (Takahashi et al., 2008), defined as $g(t) = \alpha \ln(1 + \beta t)$, where $\alpha > 0$ and $\beta > 0$. In this case, $g'(t) = \frac{\alpha\beta}{1+\beta t}$.
- The well-known Stevens’ “power” law (Stevens, 1957), defined as $g(t) = \alpha t^\beta$, where $\alpha > 0$ and $\beta > 0$. In this case, $g'(t) = \alpha\beta t^{\beta-1}$.

Results

As indicated in the Introduction, this paper proposes a description of the Jellinek curve through the composition of a discount function F with a time deformation function g . The key idea behind the work lies in proving that the composition $F_g := F \circ g$ (recall that “ $:=$ ” means “equal by definition”) could lead to a

variation in the degree of its impatience, measured by the instantaneous discount rate:

$$\delta_g = -(\ln F_g)'.$$

In what follows, we will use the notation $f_g := -\ln F_g$ (F passes to f to express the natural logarithm of F which minus sign and the subscript indicates the deformation by means of g). Thus, the objective of this paper is to analyze the changes in the monotonicity (i.e., changes either from increase to decrease or from decrease to increase) of the discount rate (with the new notation, $\delta_g := f'_g$), that is to say, the changes in the sign of $\delta'_g = f''_g$ or, equivalently, the changes of the kind of convexity of f_g . With this notation, one has the following cases:

- If $f''_g > 0$, the impatience described by F_g is increasing.
- If $f''_g < 0$, the impatience described by F_g is decreasing.

In this sense, we are proposing that pathological attitudes of an individual could be described by directly considering a discount function F not applied to “objective” time but to subjective time described by g (Cruz Rambaud and Sánchez García, 2023). Since the Jellinek curve is characterized by three phases, we specifically propose that:

- The crucial phase is described by $f''_g > 0$.
- The chronic phase is described by several changes in the sign of f''_g .
- The rehabilitation phase is described by $f''_g < 0$.

Proposition 1 discusses two single cases where F_g does not change its kind of convexity, as a previous step to analyze the absence of changes in the convexity of f_g .

Proposition 1. Assume that the discount function F and the time deformation g have different convexity. Then the deformed discount function F_g does not change its kind of convexity.

Proof. In effect, by the well-known Chain Rule for differentiation, one has:

$$F'_g = (F \circ g)' = (F' \circ g)g'$$

and

$$F''_g = (F'' \circ g)(g')^2 + (F' \circ g)g''.$$

In the former second derivative, observe that:

- If F is convex and g is concave, then $F'' \circ g > 0$, $(g')^2 > 0$, $F' \circ g < 0$ and $g'' < 0$. Therefore, $F''_g > 0$ and so F_g is convex.
- If F is concave and g is convex, then $F'' \circ g < 0$, $(g')^2 > 0$, $F' \circ g < 0$ and $g'' > 0$. Therefore, $F''_g < 0$ and so F_g is concave.

Obviously, in any case, F_g does change its kind of convexity. $\square$

The results obtained in Proposition 1 can be summarized in Table 1:

Corollary 1. Assume that the discount function F is concave and the time deformation g is convex. Then the function f_g does not change its kind of convexity.

Table 1 Summarizing sufficient conditions for $F''_g \neq 0$.

		Deformation g	
		Convex	Concave
Discount function F	Convex	?	convex
	Concave	concave	?

Source: Own elaboration.

Table 2 Summarizing sufficient conditions for $f_g'' \neq 0$.

		Deformation g	
		Convex	Concave
Discount function F	Convex	?	?
	Concave	convex	?

Source: Own elaboration.

Table 3 Summarizing necessary conditions for $f_g'' \neq 0$.

		Deformation g	
		Convex	Concave
Discount function F	Convex	✓	×
	Concave	×	✓

Source: Own elaboration.

Proof. Take into account that the second derivative of the function f_g is:

$$f_g'' = (-\ln F_g)'' = -\frac{F_g'' F_g - (F_g')^2}{F_g^2}.$$

As $F'' < 0$ (see the second item in the proof of Proposition 1), then $f_g'' > 0$ and so f_g is convex and, consequently, f_g does not change its kind of convexity. $\square$

Table 2 summarizes the result obtained in Corollary 1 and describes the unique “sure” case where the function f_g does not change its kind of convexity:

Corollaries 2 and 3 give a clue about those discount functions and time deformations where the equation $f_g'' = 0$ may have a solution, and so are “good candidates” for F_g changing its type of impatience.

Corollary 2. Let F be a concave discount function and let g be a time deformation. A necessary condition for F_g changing the type of impatience is that g is also concave at least at the instants satisfying $f_g'' = 0$.

Proof. It is obvious by taking into account that this corollary is the contrapositive implication of Corollary 1. $\square$

Analogously, one can enunciate the following result.

Corollary 3. Let F be a discount function and let g be a convex time deformation. A necessary condition for F_g changing the type of impatience is that F is also convex at least at the instants satisfying $f_g'' = 0$. $\square$

The former results about the existence of possible solutions of $f_g'' = 0$ can be summarized in Table 3.

Theorem 1 is going to provide a fundamental result for the description of the three phases exhibited by the Jellinek curve, as it gives a necessary and sufficient condition for the change in the type of impatience (increase or decrease) of F_g to occur. But, before, let us see some notation. Prelec (2004) demonstrated that the degree of decreasing impatience exhibited by a discount function can be measured by the Arrow-Pratt degree of convexity of the natural logarithm of such a function. Thus,

$$DI(t) := -\frac{[\ln F(t)]''}{[\ln F(t)]'}$$

could serve as a measure of decreasing impatience. More specifically, $DI(t) > 0$ means decreasing impatience, whilst $DI(t) < 0$ means increasing impatience. On the other hand, the degree of decreasing relative impatience is captured by the degree of convexity of the discount function (Rohde, 2009):

$$DRI(t) = -\frac{F''(t)}{F'(t)}.$$

The following lemma relates both concepts:

Lemma 1. For every t , the following equality holds:

$$DRI(t) = DI(t) + \delta(t),$$

where

$$\delta(t) := -\frac{F'(t)}{F(t)}$$

is the instantaneous discount rate of F at time t .

Proof. See Theorem 4.2 in Rohde (2009). $\square$

Theorem 1. Let F be a discount function and let g be a time deformation. Then the deformed discount function F_g changes the type of impatience if, and only if, the equation

$$DI(g(t)) = -(g^{-1})'(g(t))g'(t) \quad (1)$$

has at least a solution.

Proof. In effect,

$$\delta_g = f_g' = -\frac{(F' \circ g)g'}{F \circ g}$$

and

$$\delta_g' = f_g'' = -\frac{[(F'' \circ g)(g')^2 + (F' \circ g)g''](F \circ g) - (F' \circ g)^2(g')^2}{(F \circ g)^2}.$$

Observe that, in the former expression, $(g')^2 > 0$, $F' \circ g < 0$, $F \circ g > 0$ and $(F' \circ g)^2 > 0$. Therefore, the equation

$$\delta_g' = 0 \quad (2)$$

may have at least a solution depending on the kind of convexity of both F and g . Simple algebra shows that this equation is successively equivalent to:

$$\frac{(F'' \circ g)(g')^2 + (F' \circ g)g''}{(F' \circ g)^2(g')^2}(F \circ g) = 1,$$

$$\left[\frac{F'' \circ g}{(F' \circ g)^2} + \frac{g''}{(F' \circ g)(g')^2} \right](F \circ g) = 1,$$

$$\left[\frac{F'' \circ g}{F' \circ g} + \frac{g''}{(g')^2} \right] \frac{F \circ g}{F' \circ g} = 1,$$

$$\left[-DRI \circ g - \left(\frac{1}{g'} \right)' \right] \left(-\frac{1}{\delta \circ g} \right) = 1. \quad (3)$$

As, in general, the derivative of the inverse of any function h is given by $(h^{-1})'(t) = \frac{1}{h'[h^{-1}(t)]}$, one has:

$$\left[\frac{1}{g'(t)} \right]' = \left[\frac{1}{g'(g^{-1}(g(t)))} \right]' = (g^{-1})''(g(t))g'(t).$$

Therefore, Equation (3) leads to:

$$DRI(g(t)) + (g^{-1})''(g(t))g'(t) = \delta(g(t)). \quad (4)$$

Finally, by using Lemma 1, one has the desired result. $\square$

Lemma 2. Every discount function G can be written as a time deformation of the exponential discount function.

Proof. In effect, every discount function F can be written as:

$$G = \exp\{-(-\ln G)\}.$$

Obviously, it can be shown that $g := -\ln G$ is a time deformation and so G is the time deformation of the exponential discount function by means of the so-defined function g . $\square$

Observe that this result coincides with the one obtained by Ventre and Martino (2022). In order to check the accuracy of Equation (1), we are going to apply it to the particular case of the exponential discount function $F(t) = \exp\{-t\}$, deformed by $g(t) = -\ln G(t)$ (which, by Lemma 2, gives rise to the discount function $G(t)$). Therefore, it is expected that Equation (1) must lead to the usual equation $\delta'_G = 0$, where δ_G is the discount rate of G .

Deformation with Weber-Fechner laws

In the first statement of this section, we will consider that F is an exponential discount function and g is a Weber-Fechner time deformation. In this case, as F is convex and g is concave, by Corollary 1, it is not possible to state anything about the existence of possible solutions of Equation (1). Therefore, in the following corollary, we are going to analyze this specific case.

Corollary 4. Let F be an exponential discount function and let g be a Weber-Fechner time deformation. Then Equation (1) has no solution and so the deformed discount function F_g does not change the type of impatience.

Proof. In effect, let us analyze each component of Equation (1):

- $F(t) = \exp\{-kt\}$, where $k > 0$.
- $g(t) = \alpha \ln(1 + \beta t)$, where $\alpha > 0$ and $\beta > 0$.
- $DI(g(t)) = DRI(t) = 0$.
- $-(g^{-1})''(g(t))g'(t) = -\frac{1}{\alpha}$.

Therefore, obviously, Equation (1) does not have any solution, and so, by Theorem 1, the deformed discount function F_g does not change the type of impatience. In effect,

$$F_g = \frac{1}{(1 + \beta t)^{ka}}$$

and

$$\delta_g = \frac{k\alpha\beta}{1 + \beta t},$$

whereby

$$\delta'_g = -\frac{k\alpha\beta^2}{(1 + \beta t)^2} < 0$$

and so $F_g(t)$ does not change the type of impatience, i.e., the discount rate does not change neither from increasing to decreasing nor from decreasing to increasing. $\square$

Now, we are going to consider that F is the generalized hyperbolic discount function.

Corollary 5. Let F be a generalized hyperbolic discount function and let g be a Weber-Fechner time deformation. Then Equation (1) has no solution and so the deformed discount function F_g does not change the type of impatience.

Proof. The proof is analogous to that of Corollary 4. $\square$

Deformation with Stevens' laws

Theorem 2. Let F be a concave discount function and let g be a Stevens' time deformation. Then a necessary condition for F_g changing the type of impatience is $0 < \beta < 1$.

Proof. If F_g changes the type of impatience, then, by Corollary 2, g is non-convex, whereby $g'' \leq 0$. As, in this case, $g(t) = \alpha t^\beta$, where $\alpha > 0$ and $0 < \beta < 1$, then $g''(t) = \alpha\beta(\beta - 1)t^{\beta-2}$ and, consequently, $0 < \beta \leq 1$. $\square$

As in section "Deformation with Weber-Fechner laws", it will be subsequently assumed that F is an exponential (Corollary 6) and a generalized hyperbolic discount function in the next corollaries 7 and 8. In these cases, as F is convex, by Corollary 1, nothing can be stated about the existence of possible solutions of Equation (1). Therefore, in the following corollary, we are going to analyze these cases.

Corollary 6. Let $F(t) = \exp\{-kt\}$, $k > 0$, be an exponential discount function and let $g(t) = \alpha t^\beta$, $\alpha > 0$ and $\beta > 1$, be a Stevens' time deformation. Then Equation (1) has no solution and so the deformed discount function F_g does not change the type of impatience.

Proof. In this case, one has:

$$DI(g(t)) = 0$$

and

$$-(g^{-1})''(g(t))g'(t) = \frac{\beta - 1}{\alpha\beta t^{\beta-1}},$$

and so Equation (1) has no solution. In effect, this conclusion can be reinforced by taking into account that $F_g = \exp\{-kat^\beta\}$ and then:

$$\delta_g(t) = k\alpha\beta t^{\beta-1}$$

and

$$\delta'_g(t) = k\alpha\beta(\beta - 1)t^{\beta-2},$$

which does not change its sign. $\square$

Corollary 7. (Cruz Rambaud and Sánchez García, 2023) Let $F(t) = \frac{1}{(1+it)^k}$ ($i > 0$ and $k > 0$) be a generalized hyperbolic discount function and let $g(t) = \alpha t^\beta$ be a Stevens' time deformation, where $\beta > 1$. Then Equation (1) has one solution, and so the deformed discount function F_g can change the type of impatience, giving rise to the so-called exponentiated hyperbolic discount function.

Proof. In effect, in this case, one has:

- $F_g(t) = \frac{1}{(1+i\alpha t^\beta)^k}$.
- $\delta_g(t) = k\alpha\beta \frac{t^{\beta-1}}{1+i\alpha t^\beta}$.
- $\delta'_g(t) = k\alpha\beta \frac{(\beta-1)t^{\beta-2} - i\alpha t^{2\beta-2}}{(1+i\alpha t^\beta)^2} = k\alpha\beta t^{\beta-2} \frac{(\beta-1) - i\alpha t^\beta}{(1+i\alpha t^\beta)^2}$.

Therefore, F_g changes the type of impatience at instant $t_0 = \left(\frac{\beta-1}{i\alpha}\right)^{1/\beta}$. $\square$

Corollary 8. Let $F(t) = \frac{1}{(1+it)^k}$ ($i > 0$ and $k > 0$) be a generalized hyperbolic discount function and let $g_i(t) = \alpha_i t^{\beta_i}$ ($i = 1, 2, \dots, n$) be Stevens' time deformations, where $\alpha_i > 0$ and $\beta_i > 1$. Then the successive application of these n deformations over the convexity interval of the resulting discount functions makes that Equation (1) has n increasing solutions, and so the deformed discount function F_g changes n times the type of impatience.

Proof. In effect, the composition of the first Stevens' time deformation g_1 and the generalized hyperbolic discount function F (Fig. 2: curve in red) gives, as a result, the so-called generalized, exponentiated hyperbolic discount function F_1 (Fig. 2: curve in green) which is concave in a neighborhood of time 0 and convex

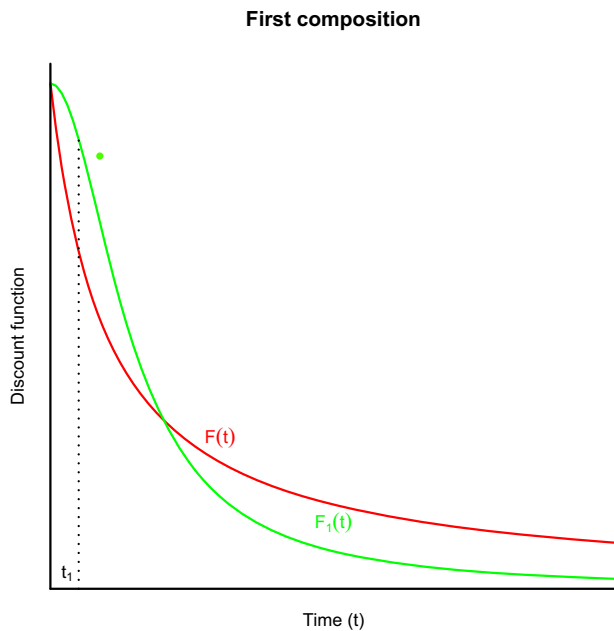


Fig. 2 Discount function corresponding to the first composition. Source: Own elaboration.

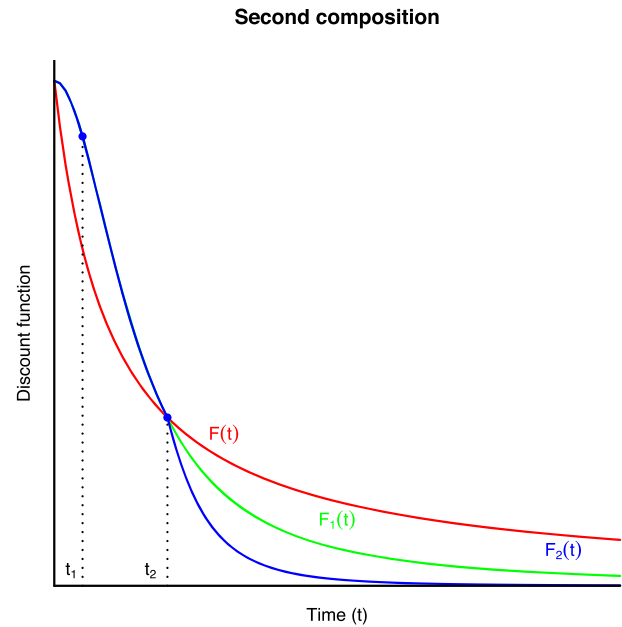


Fig. 3 Discount function corresponding to the second composition. Source: Own elaboration.

in the rest of its domain. Observe that the inflection instant is:

$$t_1 := \left[\frac{\beta_1 - 1}{i\alpha_1(\beta_1 + 1)} \right]^{1/\beta_1}$$

and the instant at which the monotonicity of the impatience changes (i.e., the discount rate changes from increasing to decreasing or from decreasing to increasing) is:

$$t'_1 := \left(\frac{\beta_1 - 1}{i\alpha_1} \right)^{1/\beta_1}.$$

Observe that, obviously, $t_1 < t'_1$, that is to say, the change from increasing to decreasing impatience (i.e., the variation of convexity of $f = -\ln F$) is later than the change of convexity of F . Let t_2 be an instant later than t'_1 . Now, we are going to consider the Stevens' time deformation g_2 starting from t_2 ($t_2 > t'_1$), i.e., the continuous and differentiable deformation h_2 defined by:

$$h_2(t) := \begin{cases} t, & \text{if } 0 < t \leq t_2 \\ t + \alpha_2(t - t_2)^{\beta_2}, & \text{if } t > t_2 \end{cases}$$

The composition of h_2 with F_1 gives, as a result, the curve in blue F_2 (see Fig. 3), which overlaps the graphic representation of F_1 from 0 to t_2 . Now, we can determine again an instant t'_2 ($t_2 < t'_2$) at which this function changes from increasing impatience to decreasing impatience. Let t_3 be an instant later than t'_2 . Subsequently, we are going to consider the Stevens' time deformation g_3 starting from t_3 ($t_3 > t'_2$), i.e., the deformation h_3 defined by:

$$h_3(t) := \begin{cases} t, & \text{if } 0 < t \leq t_3 \\ t + \alpha_3(t - t_3)^{\beta_3}, & \text{if } t > t_3 \end{cases}$$

Once again, the composition of h_3 with F_2 gives, as a result, the curve in black F_3 (see Fig. 4) which overlaps F_2 from 0 to t_3 , and so on. Summarizing, we have obtained n instants $t'_1, t'_2, \dots, t'_n$ ($t'_1 < t'_2 < \dots < t'_n$) where the deformed discount function

Discount function of the Jellinek curve

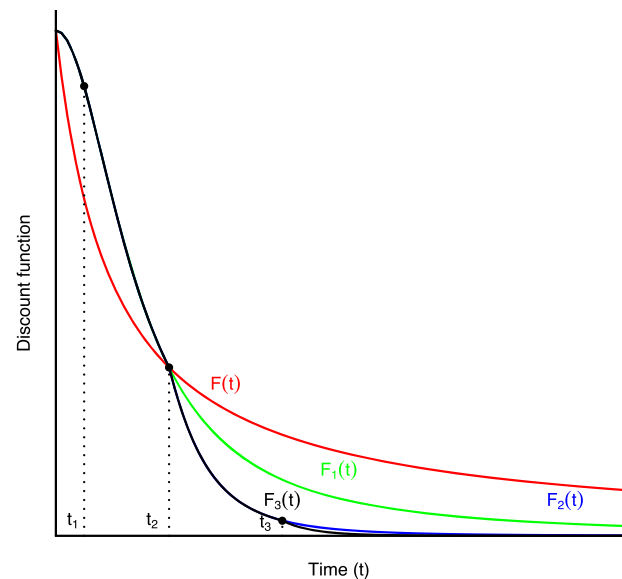


Fig. 4 Discount function corresponding to the Jellinek curve with three changes from increasing to decreasing impatience. Source: Own elaboration.

$F_{h_1 \circ h_2 \circ \dots \circ h_n} := F \circ h_1 \circ h_2 \circ \dots \circ h_n$ changes n times the type of increasing impatience. $\square$

The relevance of Corollary 8 lies in the application of n Stevens' deformations, which give rise to the so-called Jellinek curve. In effect, the crucial phase of the Jellinek curve is characterized by increasing impatience, i.e., $DI < 0$. Then the chronic phase is described by n changes of the sign of DI as a consequence of implementing n time deformations. Finally, this process results in the rehabilitation phase characterized by $DI > 0$.

Discussion and conclusions

This paper has dealt with the issue of describing the three phases (crucial, chronic and rehabilitation) of the so-called Jellinek curve as a result of deforming time, several times, in an exponential discount function. As stated in the proof of Corollary 4, the generalized hyperbolic discount function results from the deformation of the exponential discount function with a Weber-Fechner law. Subsequently, the deformation of a generalized hyperbolic discount function by a Stevens' law can be considered as the deformation of an exponential discount function by the composition of one Weber-Fechner law and one Stevens' law. By Corollary 5, the Weber-Fechner law does not contribute any solution, whilst Stevens' law may provide one solution to Equation (1). In this case, F_g changes the type of impatience, giving rise to the so-called exponentiated hyperbolic discount function.

Observe that this last discount function exhibits $DI < 0$ in a neighborhood of 0 so describing the crucial phase of the Jellinek curve. Indeed, this is the role of the first Weber-Fechner and the first Stevens' law, which together convert the exponential discount function into an exponentiated hyperbolic discount function where $DI < 0$ in a neighborhood of 0. The successive $n - 1$ deformations provided by the rest of Stevens' laws describe the changes of DI characteristic of the chronic phase of the Jellinek curve. Finally, if the discount function is regular, there is a neighborhood of infinity where $DI > 0$, which describes the rehabilitation phase of the Jellinek curve.

More specifically, let $F(t) = \frac{1}{(1+it)^k}$ be the generalized discount function and $g(t) = \alpha t^\beta$ be the Stevens' time deformation function. The exponentiated hyperbolic discount function can be generated by means of the following process:

- $(F \circ g)(t) = F_g(t) = \frac{1}{(1+i\alpha t^\beta)^k}$.
- $\delta_g(t) = k\alpha\beta \frac{t^{\beta-1}}{1+i\alpha t^\beta}$.
- $\delta'_g(t) = k\alpha\beta t^{\beta-2} \frac{(\beta-1)-i\alpha t^\beta}{(1+i\alpha t^\beta)^2}$.

The above-deformed discount function exhibits:

- a. Crucial phase: increasing impatience, i.e., $DI < 0$ in a neighborhood of 0. In this case, when $\beta \geq 2$,

$$\lim_{t \rightarrow 0} \delta'(t) = 0.$$

- b. Chronic phase: changes in impatience with successive application of n deformations.
- c. Rehabilitation phase: decreasing impatience, i.e., $DI > 0$ in a neighborhood of infinity, satisfying:

$$\lim_{t \rightarrow +\infty} \delta'(t) = 0.$$

Let us see whether these conclusions coincide with the description of each phase from an intuitive point of view. The crucial stage, where one underestimates their ability to postpone future gratification and then increases their chances of growing addiction, should be characterized by decreasing patience (i.e., increasing impatience).

Conversely, the rehabilitation phase, where individuals show a greater tendency to postpone addictive substance use while still experiencing withdrawal symptoms, should be characterized by increasing patience (i.e., decreasing impatience or hyperbolic discounting). In effect, decreasing impatience implies greater patience in more distant trade-offs, meaning an individual's preference for immediate rewards over future rewards diminishes as time progresses.

Finally, from a behavioral point of view, observe that each subsequent deformation in the process described by Corollary 8 corresponds to different moods of the substance addict.

The methodology employed in this paper could be of interest to detect the degree of impatience of a substance (alcohol, tobacco, drugs, etc.) abuser. The relevance of this information lies in the correspondence between high levels of impatience and critical periods of substance consumption. Moreover, periods of decreasing/increasing impatience are associated with periods of recovery/relapse of substance abusers. Therefore, these results could be applied to the detection and treatment of drug consumption when periods of recovery and relapse alternate in time.

Data availability

The manuscript has no associated data.

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Notes

- 1 The term "monotonicity of the discount rate" or "monotonicity of the impatience" refers to the increasing or decreasing behavior of the discount rate.
- 2 According to "The AA Structure Handbook for Great Britain 2023" (<https://www.alcoholics-anonymous.org.uk/>), "Alcoholics Anonymous is a fellowship of men and women who share their experience, strength and hope with each other that they may solve their common problem and help others to recover from alcoholism. The only requirement for membership is a desire to stop drinking."

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Author contributions

All authors contributed to the study conception and design. Material preparation and analysis were performed by Salvador Cruz Rambaud, Fabrizio Maturo and Viviana Ventre. The first draft of the manuscript was written by Roberta Martino and Annamaria Porreca, and all authors commented on previous versions of the manuscript. All authors read and approved the final manuscript.

Competing interests

The authors declare no competing interests.

Ethical approval

Ethical approval was not required as the study did not involve human participants.

Informed consent

The study does not involve human participants or their data.

Additional information

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