communications physics

A Nature Portfolio journal

Article

https://doi.org/10.1038/s42005-025-02199-7

Diagnosing strong-to-weak symmetry breaking via Wightman correlators

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Symmetry plays a fundamental role in quantum many-body physics, and a central concept is spontaneous symmetry breaking. Recent developments highlight new symmetry-breaking patterns known as strong-to-weak spontaneous symmetry breaking, characterized by two different order parameters: the fidelity correlator and the Rényi-2 correlator which are inequivalent. In this work, we propose the Wightman correlator as an alternative diagnostic tool. This construction relies on the introduction of the thermofield double state for a generic density matrix, which maps the strong symmetry of the density matrix to the doubled symmetry of the pure state, allowing the Wightman correlator to emerge naturally as a standard probe of symmetry breaking. We prove the equivalence between the Wightman correlator and the fidelity correlator, and examine explicit examples. Additionally, we discuss a susceptibility interpretation of the Wightman correlator. The validity of Wightman correlator has wide applications for understanding strong-to-weak spontaneous symmetry breaking.

Understanding quantum phases and phase transitions is one of the most important topics in quantum many-body physics. A celebrated paradigm is spontaneous symmetry breaking[1], which posits that different phases are classified based on their symmetry properties, as indicated by the long-range correlation of order parameters $\langle \hat{O}_i \hat{O}_j^\dagger \rangle$. Recent advancements in the study of symmetry and phase transitions in open systems have brought new insights into this fundamental question. An important observation is that the symmetry of density matrices can be defined in two different ways[2–5]. As an example, when the particle numbers of the system and the bath are conserved separately, the density matrix ρ can show strong symmetry with $U\rho = e^{i\theta}\rho$, where U generates the symmetry transformation. On the other ha , if the system and bath can exchange particles, the density matrix displays only weak symmetry, fulfilling $U\rho U^\dagger = \rho$. These fruitful symmetry features of open quantum systems promise exciting avenues for exploring novel phases of matter.

Special attention has been given to the spontaneous breaking of strong symmetry down to weak symmetry, a new symmetry-breaking pattern that has no direct counterpart in pure states[6–15]. In the seminal work by Lessa et al.[9], the fidelity correlator is proposed as a universal order parameter for strong-to-weak spontaneous symmetry breaking (SWSSB). It is defined as the fidelity $F(i,j) = \mathrm{tr}\left[\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right]$ between the original density matrix ρ and the decorated density matrix $\sigma = \hat{O}_i\hat{O}_j^\dagger\rho\hat{O}_i^\dagger\hat{O}_j$ with charged operator O_i. When the fidelity correlator is non-vanishing for $|i-j| \to \infty$, it is concluded that the system exhibits SWSSB. The SWSSB serves as an obstacle to the disentangling of quantum states with symmetric low-depth quantum channels[9], and can be understood as a divergence of fidelity susceptibility of local charge dephasing[15]. For systems with strong U(1) symmetry, it is demonstrated that SWSSB implies the hydrodynamic behavior of charges[7,8,11,12].

On the other hand, Rényi-2 correlator has been proposed as an alternative candidate for SWSSB, and it has been studied in many examples[6,9,10]. One intriguing reason for using the Rényi-2 correlator is that its construction is based on the Choi-Jamiolkowski isomorphism[16,17] of the density matrix, which maps the density matrix ρ to a pure state $|\rho\rangle\rangle$ in the doubled Hilbert space. Consequently, the symmetry-breaking patterns of the density matrix reduce to those of some well-established pure state. Nevertheless, case studies show that the Rényi-2 correlator leads to different transition points compared to the fidelity correlator. Given the series of favorable properties satisfied by the fidelity correlator[9], the Rényi-2 correlator becomes less appealing.

In this work, we introduce the use of the Wightman function to diagnose SWSSB, as illustrated in Fig. 1. Our approach involves mapping the input density matrix to the thermofield double state, where the strong symmetry is transformed into a doubled symmetry, represented by U and $\tilde{U}$. The Wightman function naturally arises as a correlator of charged operators within this framework. By establishing two-sided bounds, we demonstrate the equivalence between the Wightman correlator and the

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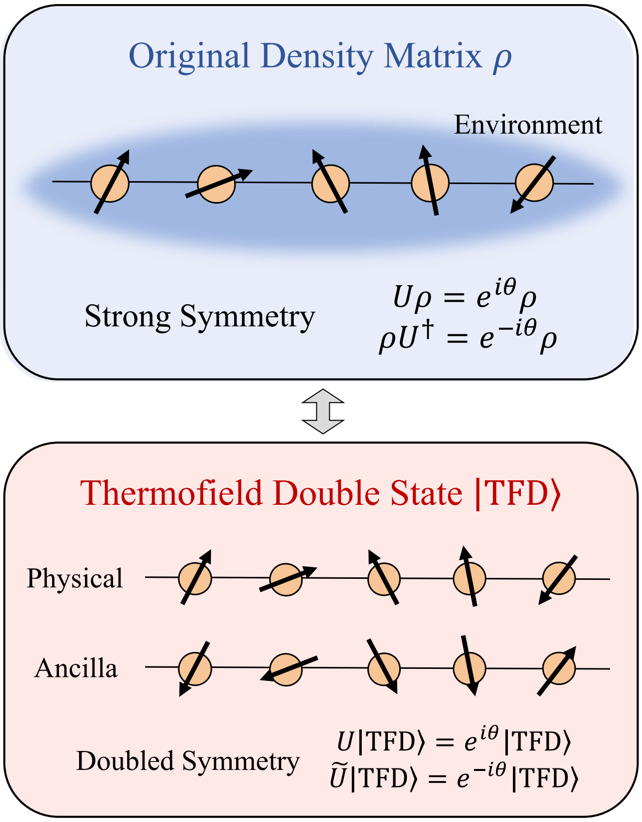

Fig. 1 | Schematic of our proposal. We present a schematic of our proposal for using the Wightman correlator to diagnose strong-to-weak symmetry breaking of the density matrix. The strong symmetry of the density matrix is mapped onto the doubled symmetry of the thermofield double state, with each symmetry acting individually on the physical and auxiliary subsystems.

fidelity correlator, with several illustrative examples provided. We also draw an analogy with spin-glass susceptibility to propose a susceptibility interpretation of the Wightman correlator.

Results

Wightman correlator

Our primary aim is to reuse the pure-state perspective of SWSSB. Unlike the Choi-Jamiolkowski isomorphism, which directly employs the operator-state mapping, we consider a specific purification of the density matrix, typically referred to as the thermofield double (TFD) state for thermal density matrices[18,19]. Here, we extend this terminology to apply to any generic density matrix. We focus on quantum many-body systems that contain N qubits, with Pauli operators $\{X_i, Y_i, Z_i\}$ for $i = 1, 2, \ldots, N$. The generalization to arbitrary local Hilbert space dimensions and dimensions is straightforward. Now, we introduce N auxiliary qubits with Pauli operators $\{\tilde{X}_i, \tilde{Y}_i, \tilde{Z}_i\}$, and prepare the full system in the (unnormalized) maximally entangled state between the original and the auxiliary system $|\,\mathrm{EPR}\,\rangle = \otimes_i(|\uparrow\downarrow\rangle_i - |\downarrow\uparrow\rangle_i)$. Given any density matrix ρ, the corresponding TFD state is defined as

$$|\,\mathrm{TFD}\,\rangle = (\sqrt{\rho}\otimes I)|\,\mathrm{EPR}\,\rangle. \quad (1)$$

Here, $\sqrt{\rho}$ only operates on the original quantum system. It is straightforward to check that tracing out the auxiliary system reproduces a reduced density matrix ρ.

The symmetry property of the TFD state is inherited from the original density matrix ρ. For strongly symmetric density matrices, diagonalization yields $\rho = \sum_a \lambda_a |\psi_a\rangle\langle\psi_a|$, where all states $|\psi_a\rangle$ possess the same symmetry charge. This leads to

$$|\,\mathrm{TFD}\,\rangle = \sum_a \sqrt{\lambda_a}|\psi_a\rangle \otimes |\tilde{\psi}_a\rangle. \quad (2)$$

Here, $|\tilde{\psi}_a\rangle := \langle \psi_a |\,\mathrm{EPR}\,\rangle$ is the state in the auxiliary system. To be concrete, let us focus on either the Z_2 symmetry with $U = \prod_i X_i$ or U(1) symmetry with $U = \exp(i\phi\sum_i Z_i)$. Using the fact that both $(P_i + \tilde{P}_i)|\,\mathrm{EPR}\,\rangle = 0$ for an arbitrary Pauli operator P, we find $U|\psi_a\rangle = e^{i\theta}|\psi_a\rangle$ implies $\tilde{U}|\tilde{\psi}_a\rangle = e^{-i\theta}|\tilde{\psi}_a\rangle$ with unitary transformations on the auxiliary system $\tilde{U} = \prod_i(-1)^N\tilde{X}_i$ or $\tilde{U} = \exp(i\phi\sum_i\tilde{Z}_i)$. As a consequence, the physical system and the auxiliary system exhibit doubled symmetry U and $\tilde{U}$.

The breaking of symmetry is probed by long-range correlations between charged operators on the TFD state, which can be supported in both physical and auxiliary systems. Let us first consider the scenario where the doubled symmetry is completely broken. Taking the U(1) case as an example, we expect both the charged operator $O_i = S_i^+$ and the auxiliary operator $\tilde{O}_i = \tilde{S}_i^-$ to exhibit long-range correlations. More generally, we have

$$C(i,j) = \left\langle O_i O_j^\dagger \right\rangle_{\mathrm{TFD}} = \left\langle \tilde{O}_j \tilde{O}_i^\dagger \right\rangle_{\mathrm{TFD}} = \mathrm{tr}\,[\rho O_i O_j^\dagger], \quad (3)$$

which corresponds to the standard two-point function of the density matrix ρ. Now, if the strong symmetry is broken to weak symmetry, we should choose an operator that carries the charge of U but does not carry the charge of $U\otimes\tilde{U}$. A natural choice is $O_i\tilde{O}_i$, leading to

$$C^W(i,j) = \left\langle O_i\tilde{O}_i O_j^\dagger \tilde{O}_j^\dagger \right\rangle_{\mathrm{TFD}}. \quad (4)$$

We expect the SWSSB corresponds to having $C^W(i,j) \neq 0$ for $|i-j| \to \infty$. This can also be written in the physical Hilbert space as

$$C^W(i,j) = tr\left(\sqrt{\rho}O_i O_j^\dagger \sqrt{\rho} O_i^\dagger O_j\right), \quad (5)$$

which is non-negative. Again, we adopt the terminology for thermal density matrices and refer to $C^W(i,j)$ as the Wightman correlator. This correlator has been studied in special cases to probe the SWSSB in ref. 11. Compared to the Rényi-2 correlator, there are two key differences. Firstly, it replaces ρ in the Rényi-2 correlator by $\sqrt{\rho}$. Secondly, no normalization factor is needed since the TFD state is automatically normalized. When the charged operator is unitary, the Wightman correlator becomes equivalent to the "Holevo just-as-good fidelity" since $F_H(i,j) = \left(\mathrm{tr}\left[\sqrt{\rho}\sqrt{\sigma}\right]\right)^2 = C^W(i,j)^2$[20,21]. In the following sections, we prove that the SWSSB defined by the Wightman correlator is equivalent to the definition based on fidelity and provide illustrative examples from several perspectives.

Equivalence

We first prove that for operators with bounded operator norm $||O||_\infty$, the fidelity correlator is lower bounded by the Wightman correlator with the same operator $O_iO_j^\dagger$. To see this, we first apply Hölder's inequality[22]

$$C^W(i,j) \le ||\sqrt{\rho}O_iO_j^\dagger\sqrt{\rho}||_p \times ||O_i^\dagger O_j||_q, \quad (6)$$

for $1/p + 1/q = 1$. Here, $||A||_p := [\mathrm{tr}(A^\dagger A)^{\frac{p}{2}}]^{1/p}$ denotes the p-norm of the operator A. We take the limit that $p \to 1$ and $q \to \infty$. $||O_i^\dagger O_j||_q$ becomes the $||O||_\infty^2$, while $||\sqrt{\rho}O_iO_j^\dagger\sqrt{\rho}||_1$ exactly matches the fidelity correlator. Therefore, we find the inequality

$$Bound1: \quad C^W(i,j) \le F(i,j)\,||O_i||_\infty^2. \quad (7)$$

In particular, for Pauli operators, $||O_i||_\infty = 1$ and we obtain $C^W(i,j) \le F(i,j)$. We proceed to derive how the fidelity correlator serves as a lower bound for the Wightman correlator. To achieve this, we apply the generalized Hölder's inequality, which states that $||ABC||_1 \le ||A||_p||B||_q||C||_r$ with the constraint $1/p + 1/q + 1/r = 1$. In this context, we set $A = C = \rho^{1/4}$ and $B = \rho^{1/4}O_iO_j^\dagger\rho^{1/4}$. By substituting these definitions into the inequality, we obtain

$$F(i,j) \le ||\rho^{1/4}||_p \times ||\rho^{1/4}||_r \times ||\rho^{1/4}O_iO_j^\dagger\rho^{1/4}||_q. \tag{8}$$

Let us choose $p = r = 4$ and $q = 2$. Interestingly, we find $||\rho^{1/4}||_4 = (\mathrm{tr}[\rho])^{1/4} = 1$, and, by definition, $||\rho^{1/4}O_iO_j^\dagger\rho^{1/4}||_2 = \sqrt{C^W(i,j)}$. Therefore, the inequality becomes

$$Bound2: \quad F(i,j) \le \sqrt{C^W(i,j)}. \tag{9}$$

By combining both bounds from Eq. (7) and Eq. (9), we conclude that the SWSSB defined by the Wightman correlator is equivalent to the definition using the fidelity correlator.

This relation indicates that the Wightman correlator satisfies all the favorable conditions satisfied by the fidelity correlator[9]. For example, the stability condition states that if a density matrix ρ has a finite Wightman correlator $C^W(i,j)$ for $|i-j| \to \infty$ and $\mathcal{E}$ is a strongly symmetric finite-depth local quantum channel, then $\mathcal{E}[\rho]$ also has finite Wightman correlator for $|i-j| \to \infty$. We also propose several generalizations of our inequality, including two classes of generalized Wightman correlators

$$\begin{aligned} F_\alpha(i,j) &:= \left\| \rho^{\alpha/2}O_iO_j^\dagger\rho^{\alpha/2} \right\|_{1/\alpha} \quad \alpha \in (0,1], \\ C_\alpha^W(i,j) &:= \mathrm{tr}\,[\rho^\alpha O_iO_j^\dagger\rho^{1-\alpha}O_i^\dagger O_j] \quad \alpha \in (0,1). \end{aligned} \tag{10}$$

As elaborated in the Supplementary Note 1, both generalizations are equivalent to the fidelity correlator and the Wightman correlator in diagnosing SWSSB. In particular, we have $F_1(i,j) = F(i,j)$ and $F_{1/2}(i,j) = C_{1/2}^W(i,j) = C^W(i,j)$.

Examples

Having established generic bounds on the Wightman correlator, we now provide a few examples that illustrate how the Wightman correlator enhances our understanding of SWSSB by simplifying calculations and discovering interesting connections. More examples can be found in the Supplementary Note 2.

Spin glass. It is known that the SWSSB has a close relationship with traditional discussions of the spin glass[23]. We consider special density matrices in which the charged operator cannot couple different states that diagonalize the density matrix $\langle\psi_b|O_iO_j^\dagger|\psi_a\rangle \propto \delta_{ab}$. The fidelity correlator in this scenario becomes $F(i,j) = \sum_a \lambda_a |\langle\psi_a|\hat{O}_i\hat{O}_j^\dagger|\psi_a\rangle|$, which matches the standard definition of the Edwards-Anderson (EA) order parameter[9]. For the Wightman correlator, a straightforward calculation shows that

$$C^W(i,j) = \sum_a \lambda_a |\langle\psi_a|\hat{O}_i\hat{O}_j^\dagger|\psi_a\rangle|^2. \tag{11}$$

By averaging over i and j, we find $N^{-2}\sum_{ij}C^W(i,j) := \chi_{SG}/N$, where χ_{SG} is known as the spin glass susceptibility[23], a universal probe of spin-glass order. It measures the response of a random magnetic field. In the paramagnetic phase, χ_{SG} is finite, leading to a vanishing averaged Wightman correlator in the thermodynamic limit. In contrast, in the spin-glass phase, the susceptibility scales with the system size N, implying a finite Wightman correlator. Furthermore, higher order moments of $|\langle\psi_a|\hat{O}_i\hat{O}_j^\dagger|\psi_a\rangle|$ can also be probed by generalized Wightman correlators $F_\alpha(i,j)$, while $C_\alpha^W(i,j)$ becomes independent of α.

Thermal ensemble. It has been conjectured that the finite-temperature canonical ensemble $\rho_{\beta,c}$ of a local Hamiltonian with charge conservation exhibits SWSSB[9]. Here, we examine the Wightman correlator. Assuming no spontaneous breaking of the weak symmetry, the Wightman function for $|i-j| \gg 1$ can be approximated as

$$\begin{aligned} C^W(i,j) &= \mathrm{tr}\left(\sqrt{\rho_{\beta,c}}O_iO_j^\dagger\sqrt{\rho_{\beta,c}}O_i^\dagger O_j\right) \\ &\approx \mathrm{tr}\left(\sqrt{\rho_{\beta,gc}}O_iO_j^\dagger\sqrt{\rho_{\beta,gc}}O_i^\dagger O_j\right) \approx C_{O_i}^W C_{O_j}^W. \end{aligned} \tag{12}$$

In the first approximation, we replace the canonical ensemble with the grand canonical ensemble $\rho_{\beta,gc}$, as both yield the same predictions within a fixed charge sector. In the second approximation, we neglect the connected part between operators at sites i and j. Here, we have introduced the Wightman function of a single charged operator as $C_{O_i}^W = \mathrm{tr}\,(\sqrt{\rho_{\beta,gc}}O_i\sqrt{\rho_{\beta,gc}}O_i^\dagger)$, which matches the imaginary-time Green's function of O_i with a time separation $\tau = \beta/2$. Since $C_{O_i}^W > 0$ for any finite temperature ensemble, we conclude that $\rho_{\beta,c}$ exhibits the SWSSB. For systems with a charge gap Δ, we expect $C_{O_i}^W \sim e^{-\beta\Delta/2}$, which leads to $C^W(i,j) \sim e^{-\beta\Delta}$. Interestingly, this matches the result of the fidelity correlator from a replica calculation in[9]. More generally, we have $F_\alpha(i,j) \sim e^{-\beta\Delta}$ and $C_\alpha^W(i,j) \sim e^{-2\tau\Delta}$, with $\tau/\beta = \min\{\alpha, 1-\alpha\}$. Finally, we note that the locality requirement can be relaxed in physical models with all-to-all interactions. A celebrated example is the complex Sachdev-Ye-Kitaev model[24–28], in which the Wightman function has been extensively studied for the calculation of out-of-time-order correlators (OTOC).

Decohered Ising model. Another concrete example of SWSSB is the decohered Ising model in 2D[9], which is dual to the toric code under bit-flip errors[29–33]. We consider a 2D square lattice where each site is occupied by a qubit. The system is initialized in the product state along the x-direction $\rho_0 = \otimes_i|+x\rangle\langle+x|$. Then, a nearest neighbor Ising channel is applied to the system with

$$\mathcal{E} = \prod_{\langle ij\rangle}\mathcal{E}_{ij}, \quad \mathcal{E}_{ij}[\rho_0] = (1-p)\rho_0 + pZ_iZ_j\rho_0Z_jZ_i. \tag{13}$$

Here, $p \in [0, 1/2]$. The decohered density matrix $\rho = \mathcal{E}[\rho_0]$ exhibits strong symmetry with $U = \prod_i X_i$. To calculate the Wightman correlator using the replica trick, we first introduce $C^{W,(n)}(i,j) = \mathrm{tr}\,(\rho^n Z_iZ_j\rho^n Z_iZ_j)$. The strategy is to derive results for generic n, and perform the analytical continuation to $n = 1/2$. Generalizing the analysis in ref. 9, we find that $C^{W,(n)}(i,j)$ is described by the correlators similar to the fidelity correlator

$$C^{W,(n)}(i,j) = \left\langle \prod_{\alpha=1}^n \sigma_i^{(\alpha)}\sigma_j^{(\alpha)} \right\rangle_{H_{\mathrm{eff}}}. \tag{14}$$

with

$$H_{\mathrm{eff}} = -\sum_{\langle ij\rangle}\left(\sum_{\alpha=1}^{2n-1}\sigma_i^{(\alpha)}\sigma_j^{(\alpha)} + \prod_{\alpha=1}^{2n-1}\sigma_i^{(\alpha)}\sigma_j^{(\alpha)}\right), \tag{15}$$

at an effective temperature β determined by $\tanh\beta = p/(1-p)$. In the single replica limit $n \to 1/2$, the effective model describes the random bond Ising model along the Nishimori line[32], which undergoes a ferromagnetic-to-paramagnetic phase transition. The SWSSB for ρ occurs when the random bond Ising model becomes ordered at $p > p_c \approx 0.109$. Here, we emphasize the significance of the single replica limit, enabled by the use of the TFD state, or equivalently, by having $\sqrt{\rho}$ in Eq. (5). This also implies that the same transition point applies to all generalized Wightman correlators.

Susceptibility interpretation

Motivated by its close relation to spin glass susceptibility, we present an interpretation of the Wightman correlator in terms of susceptibility in generic setups. There is a similar proposal for the fidelity correlator[15]. We consider the Wightman function $C^W_{O_i} = \langle O_i \tilde{O}_i^\dagger \rangle_{\mathrm{TFD}}$ of a single charged operator O_i for a generic density matrix. When ρ exhibits strong symmetry, the Wightman function vanishes due to the charge constraint. Now, we introduce a perturbation to the TFD state by coupling the physical system and the auxiliary system with a direct hopping

$$|\psi\rangle \sim \exp\left(\frac{\epsilon}{2}\sum_j \left[O_j^\dagger \tilde{O}_j + O_j \tilde{O}_j^\dagger\right]\right)|\,\mathrm{TFD}\,\rangle. \tag{16}$$

Physically, this perturbation enables charge fluctuations. We then measure the change of the Wightman function $C^W_{O_i}$. Perturbatively, the result is given by

$$C^W_{O_i}/\epsilon := \chi^W = \sum_j C^W(i,j). \tag{17}$$

Therefore, the susceptibility χ^W remains finite when the strong symmetry is unbroken but diverges if SWSSB occurs. This is closely analogous to the spin glass case. To leading order in ϵ, we can translate the perturbation into the language of quantum channel $\mathcal{N} = \prod_i \mathcal{N}_i$ with:

$$\mathcal{N}_i[\rho] = \left(1 - \frac{\epsilon^2}{2}\right)\rho + \frac{\epsilon^2}{4} O_i^\dagger \rho O_i + \frac{\epsilon^2}{4} O_i \rho O_i^\dagger. \tag{18}$$

Here, we assume the charged operator O_i is unitary. This quantum channel perturbs the density matrix by a homogeneous charge creation.

We can also interpret the generalized Wightman functions as susceptibilities using a similar construction. Here, we briefly discuss an alternative strategy for $C^W_\alpha(i,j)$ with $\alpha \to 0$, showing a close relation to the response of von Neumann entropy. Assuming that O_i is unitary, we have

$$C^W_\alpha(i,j) = C^W_0(i,j) - \alpha\ \mathrm{tr}\,[H_M \delta\rho], \tag{19}$$

to the leading order in α. Here, we introduce the modular Hamiltonian $H_M = -\ln\rho$ and $\delta\rho = [O_i O_j^\dagger \rho O_i^\dagger O_j - \rho]$. The second term can be interpreted as computing the difference in von Neumann entropy between the original density matrix ρ and the decohered density matrix

$$\mathcal{M}[\rho] = (1-\alpha)\rho + \alpha O_i O_j^\dagger \rho O_i^\dagger O_j = \rho + \alpha\delta\rho, \tag{20}$$

in the limit of $\alpha \to 0$. For systems without SWSSB, $C^W_0(i,j) > 0$ while $C^W_\alpha(i,j) = 0$ for $\alpha > 0$, indicating a divergence in the entropy response $\mathrm{tr}[H_M \delta\rho]$. This further suggests that the modular Hamiltonian of density matrices without SWSSB should exhibit spooky properties, possibly with operator sizes scaling with the system size or an unbounded spectrum. This relation also provides solid proof that the thermal state of physical Hamiltonians always exhibits SWSSB, for which the entropy response function is bounded.

Discussion

The validity of the pure-state perspective has significant implications for understanding SWSSB. It is well-known that order-to-disorder transitions can be probed by the expectation value of disorder operators, which remain non-zero in the disordered phase[34]. We anticipate that a similar disorder operator can be introduced straightforwardly within our TFD setup. Furthermore, recent studies highlight the singularity of the full counting statistics of charges in a subregion during the superfluid phase[35]. Applied to the TFD state, this framework suggests the singularity of a correlator involving charge twist operators acting on both the physical and auxiliary systems. Another intriguing direction is exploring SWSSB from the perspective of the modular flow. For a system with SWSSB, our analysis of entropy response suggests the modular Hamiltonian is well-behaved. This facilitates the study of modular flow, which defines the intrinsic dynamics of the density matrix. For instance, evolving the operator O_j in Eq. (5) to a later time t results in an OTOC. This framework could pave the way for further classification of density matrices, allowing for the distinction between those with chaotic and non-chaotic modular Hamiltonians.

Data availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Received: 13 January 2025; Accepted: 22 June 2025;
Published online: 02 July 2025

References

1. Landau, L. D. & Lifshitz, E. M. *Statistical Physics: Volume 5* (Elsevier, Amsterdam, 2013).
2. Buca, B. & Prosen, T. A note on symmetry reductions of the Lindblad equation: transport in constrained open spin chains. *New J. Phys.* **14**, 073007 (2012).
3. Albert, V. V. & Jiang, L. Symmetries and conserved quantities in Lindblad master equations. *Phys. Rev. A* **89**, 022118 (2014).
4. Albert, V. V. *Lindbladians with multiple steady states: theory and applications.* (Ph.D. thesis, Yale Univ. 2017).
5. Lieu, S. et al. Symmetry breaking and error correction in open quantum systems. *Phys. Rev. Lett.* **125**, 240405 (2020).
6. Lee, J. Y., Jian, C.-M. & Xu, C. Quantum criticality under decoherence or weak measurement. *PRX Quantum* **4**, 030317 (2023).
7. Ogunnaike, O., Feldmeier, J. & Lee, J. Y. Unifying emergent hydrodynamics and Lindbladian low-energy spectra across symmetries, constraints, and long-range interactions. *Phys. Rev. Lett.* **131**, 220403 (2023).
8. Moudgalya, S. & Motrunich, O. I. Symmetries as ground states of local superoperators: hydrodynamic implications. *PRX Quantum* **5**, 040330 (2024).
9. Lessa, L. A. et al. Strong-to-weak spontaneous symmetry breaking in mixed quantum states. *PRX Quantum* **6**, 010344 (2025).
10. Sala, P., Gopalakrishnan, S., Oshikawa, M. & You, Y. Spontaneous strong symmetry breaking in open systems: purification perspective. *Phys. Rev. B* **110**, 155150 (2024).
11. Huang, X., Qi, M., Zhang, J.-H. & Lucas, A. Hydrodynamics as the effective field theory of strong-to-weak spontaneous symmetry breaking. *Phys. Rev. B* **111**, 125147 (2025).
12. Gu, D., Wang, Z. & Wang, Z. Spontaneous symmetry breaking in open quantum systems: strong, weak, and strong-to-weak. Preprint at https://arxiv.org/abs/2406.19381 (2024).
13. Kuno, Y., Orito, T. & Ichinose, I. Strong-to-weak symmetry breaking states in stochastic dephasing stabilizer circuits. *Phys. Rev. B* **110**, 094106 (2024).
14. Zhang, C. et al. Strong-to-weak spontaneous breaking of 1-form symmetry and intrinsically mixed topological order. *Phys. Rev. B* **111**, 115137 (2025).
15. Zhang, J.-H., Xu, C. & Xu, Y. Fluctuation-dissipation theorem and information geometry in open quantum systems. Preprint at https://arxiv.org/abs/2409.18944 (2024).
16. Jamiołkowski, A. Linear transformations which preserve trace and positive semidefiniteness of operators. *Rept. Math. Phys.* **3**, 275–278 (1972).
17. Choi, M.-D. Completely positive linear maps on complex matrices. *Linear Algebra Appl.* **10**, 285–290 (1975).
18. Israel, W. Thermo field dynamics of black holes. *Phys. Lett. A* **57**, 107–110 (1976).
19. Maldacena, J. M. Eternal black holes in anti-de Sitter. *JHEP* **04**, 021 (2003).

20. Kholevo, A. S. On quasiequivalence of locally normal states. *Teor. Mat. Fiz.* **13**, 184–199 (1972).
21. Wilde, M. M. Recoverability for Holevo's just-as-good fidelity. In *Proceedings of the 2018 International Symposium on Information Theory* 2331–2335 (2018) https://doi.org/10.1109/ISIT.2018.8437346.
22. Bhatia, R. *Matrix Analysis*, Vol. 169 (Springer, New York, 1997).
23. Mezard, M. & Montanari, A. *Information, Physics, and Computation* (Oxford University Press, Oxford, 2009).
24. Maldacena, J. & Stanford, D. Remarks on the Sachdev-Ye-Kitaev model. *Phys. Rev. D* **94**, 106002 (2016).
25. Kitaev, A. A simple model of quantum holography. Talks at KITP http://online.kitp.ucsb.edu/online/entangled15/kitaev/ and http://online.kitp.ucsb.edu/online/entangled15/kitaev2/ (2015).
26. Chowdhury, D., Georges, A., Parcollet, O. & Sachdev, S. Sachdev-Ye-Kitaev models and beyond: window into non-Fermi liquids. *Rev. Mod. Phys.* **94**, 035004 (2022).
27. Davison, R. A. et al. Thermoelectric transport in disordered metals without quasiparticles: the Sachdev-Ye-Kitaev models and holography. *Phys. Rev. B* **95**, 155131 (2017).
28. Gu, Y., Kitaev, A., Sachdev, S. & Tarnopolsky, G. Notes on the complex Sachdev-Ye-Kitaev model. *JHEP* **02**, 157 (2020).
29. Dennis, E., Kitaev, A., Landahl, A. & Preskill, J. Topological quantum memory. *J. Math. Phys.* **43**, 4452–4505 (2002).
30. Wang, C., Harrington, J. & Preskill, J. Confinement Higgs transition in a disordered gauge theory and the accuracy threshold for quantum memory. *Ann. Phys.* **303**, 31–58 (2003).
31. Chen, Y.-H. & Grover, T. Separability transitions in topological states induced by local decoherence. *Phys. Rev. Lett.* **132**, 170602 (2024).
32. Fan, R., Bao, Y., Altman, E. & Vishwanath, A. Diagnostics of mixed-state topological order and breakdown of quantum memory. *PRX Quantum* **5**, 020343 (2024).
33. Bao, Y., Fan, R., Vishwanath, A. & Altman, E. Mixed-state topological order and the errorfield double formulation of decoherence-induced transitions. Preprint at https://arxiv.org/abs/2301.05687 (2023).
34. Fradkin, E. Disorder operators and their descendants. *J. Statist. Phys.* **167**, 427 (2017).
35. Wang, C.-Y., Zhou, T.-G., Zhou, Y.-N. & Zhang, P. Distinguishing quantum phases through cusps in full counting statistics. *Phys. Rev. Lett.* **133**, 083402 (2024).

Acknowledgements
We thank Ning Sun, Zijian Wang, Zhaoyi Zeng and Tian-Gang Zhou for helpful discussions. We are especially grateful to Jian-Hao Zhang for bringing this topic to our attention and for offering invaluable explanations and discussions. This project is supported by the NSFC under grant number 12374477 and by the Innovation Program for Quantum Science and Technology under grant number 2024ZD0300101.

Author contributions
P.Z. proposed the study and wrote the manuscript. Z.L., L.C., Y.Z. and P.Z. conducted the theoretical calculations. S.Z. performed the numerical simulations. All authors discuss the results and contribute to the manuscript.

Competing interests
The authors declare no competing interests.

Additional information
Supplementary information The online version contains supplementary material available at https://doi.org/10.1038/s42005-025-02199-7.

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Peer review information *Communications Physics* thanks the anonymous reviewers for their contribution to the peer review of this work. A peer review file is available.