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Minimal evolution time and quantum speed limit of non-Markovian open systems

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We derive a sharp bound as the quantum speed limit (QSL) for the minimal evolution time of quantum open systems in the non-Markovian strong-coupling regime with initial mixed states by considering the effects of both renormalized Hamiltonian and dissipator. For a non-Markovian quantum open system, the possible evolution time between two arbitrary states is not unique, among the set of which we find that the minimal one and its QSL can decrease more steeply by adjusting the coupling strength of the dissipator, which thus provides potential improvements of efficiency in many quantum physics and quantum information areas.

As a fundamental bound for the evolution time of quantum systems, the quantum speed limit (QSL) (also referred to as quantum evolution time limit) plays an important role in tremendous areas of quantum physics and quantum information, such as quantum computation and communication^{1,2}, quantum metrology³, cavity quantum electrodynamics⁴, quantum control⁵, etc. The derivation of QSL is most required for the purpose of simplification and/or optimization in theoretical analysis, since in most quantum-cases one only needs to derive a lower bound on the minimal time of evolution without solving the exact equation to see the dominant factors in evolution and/or optimize our demand. For closed quantum systems, two types of QSL have been derived at the start: the Mandelstam-Tamm (MT) bound $\tau \geq \pi\hbar/(2\Delta E)$ ⁶ and the Margolus-Levitin (ML) bound $\tau \geq \pi\hbar/(2\langle E \rangle)$ ⁷. Since then, further investigations are launched into QSL^{8–12}. As the energy of a closed system is conserved, the QSL of a closed system is decided by the variance of energy ΔE or the mean energy $\langle E \rangle$, related only to the unitary Hamiltonian. Recently, the QSL for quantum open systems¹³ draws wide attention with several bounds^{14–18} being found. Because there is energy and/or coherence exchange between system and environment for quantum open systems, the evolution generator therein contains not only a time-dependent Hamiltonian H_t but also a dissipator $\mathcal{D}_t(\rho_t)$ (a trace-preserving term referring to dissipation behaviors)¹³. In quantum open systems, non-Markovianity is valuable in practice and highly emphasized for its particular characteristics of memory effect, negative energy/population flow and singularity of the state evolution^{19,20}. The latter two characteristics are commonly found in the strong-coupling regime, where the system and environment are strongly coupled and the non-Markovianity becomes a non-negligible strong effect²¹. Typically, a strong-coupling regime can be achieved and temporarily maintained in high-Q optical micro-cavities²² and quantum circuits²³. In spite of recent breakthrough on measurement methods for non-Markovianity^{24–28}, the strong-coupling regime still remains as an open question. Also, QSL issue becomes more complicated than it was considered¹⁶, since in such a regime the possible evolution time between two arbitrary states is not unique, while only the QSL for the minimal one does matter. In addition to non-Markovianity, the evolution of mixed states in quantum open systems also attracts concern. It is therefore of great significance to derive a sharp bound on evolution time for general conditions, i.e., for mixed states in different non-Markovian coupling regimes.

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In this report, we study the non-Markovian problem by using geometric methods and derive a sharp bound for the minimal evolution time for quantum open systems with initial mixed states. We define the minimal evolution time $\hat{\tau}$ for non-Markovian quantum open systems as the minimal possible evolution time between two arbitrary states before we study its relevant QSL using new mathematical inequality tools. A steeper decrease of QSL than previous result¹⁶ caused by strong non-Markovianity is observed in the examples of two-level models, indicating that a much smaller evolution time can be achieved in the strong-coupling regime. It is implied that the evolution of quantum physical process and computation involving strong-coupling interactions can be more effective.

Results

Geometric fidelity. To quantify the geometric distance between two general quantum states, the Bures fidelity²⁹ $\mathcal{F}_B(\rho_1, \rho_2) = \|\sqrt{\rho_1}\sqrt{\rho_2}\|_{\text{tr}}$ with the Bures angle $\Theta_B = \cos^{-1} \mathcal{F}_B$ was usually used, where ρ is the density operator of a general quantum state. Here, however, we introduce the *relative-purity fidelity* $\mathcal{F}_R(\rho_1, \rho_2) = \|\sqrt{\rho_1}\sqrt{\rho_2}\|_{\text{HS}} / \sqrt{\|\rho_1\|_{\text{HS}}\|\rho_2\|_{\text{HS}}}$ with $\Theta_R = \cos^{-1} \mathcal{F}_R$. This one derived from the so-called relative purity³⁰ is more useful in studying QSL³¹. It is easy to prove that $\mathcal{F}_B(\rho, \rho) = \mathcal{F}_R(\rho, \rho) = 1$, and, if ρ_2 is a pure state, then one has $\mathcal{F}_B(\rho_1, \rho_2) = \mathcal{F}_R(\rho_1, \rho_2)$.

From the von Neumann trace inequality³²,

$$\begin{aligned} \|\sqrt{\rho_1}\sqrt{\rho_2}\|_{\text{HS}}^2 &= \text{Tr}\{\rho_1\rho_2\} \leq \sum_i \sigma_{1,i} \cdot \sigma_{2,i} \\ &\leq \sqrt{\sum_i \sigma_{1,i}^2} \cdot \sqrt{\sum_i \sigma_{2,i}^2} \leq \sqrt{\sum_i \sigma_{2,i}^2} \\ &= \|\rho_2\|_{\text{HS}}. \end{aligned}$$

Hence, we have $0 \leq \mathcal{F}_R(\rho_1, \rho_2) \leq 1$ and $\Theta_R = \cos^{-1} \mathcal{F}_R$ is valid. In addition, compared with another recently used fidelity¹⁸ $\mathcal{F}'(\rho_1, \rho_2) = \|\sqrt{\rho_1}\sqrt{\rho_2}\|_{\text{HS}} / \sqrt{\|\rho_1\|_{\text{HS}} \cdot \|\rho_2\|_{\text{HS}}}$, $\mathcal{F}_R(\rho_1, \rho_2)$ can guarantee a perfect and simple linear relationship (as we shall see later) at the expense of good symmetry between ρ_1 and ρ_2 .

Minimal evolution time. The minimal evolution time $\hat{\tau}$ of a quantum evolution is defined in the following: given a predefined quantum evolution $\dot{\rho}_t = \mathcal{L}_t(\rho_t)$, then, a predetermined state ρ_τ , one has $\hat{\tau} = \min\{\tau | \tau \in \mathbb{T}\}$, where \mathbb{T} stands for the set of all the actual possible driving time τ that the evolution from ρ_0 to ρ_τ may take. One should notice that τ is not unique, especially in the non-Markovian strong-coupling regime.

Quantum speed limit. In order to derive a lower bound as the QSL for driving time τ , the square of the relative-purity fidelity

$$[\mathcal{F}_R(\rho_t, \rho_0)]^2 = \frac{\text{Tr}\{|\sqrt{\rho_t}\sqrt{\rho_0}|^2\}}{\|\rho_0\|_{\text{HS}}} = \frac{\text{Tr}\{\rho_0\rho_t\}}{\|\rho_0\|_{\text{HS}}}$$

is used, which is simply linear with ρ_t . The same linear relationship for $[\mathcal{F}_B(\rho_t, \rho_0)]^2$ is not true unless ρ_0 is a pure state. Taking time derivatives of $\Theta_R = \cos^{-1} \mathcal{F}_R$ yields

$$\dot{\Theta}_R = \frac{\text{Tr}\{\rho_0\dot{\rho}_t\}}{-\|\rho_0\|_{\text{HS}} \sin 2\Theta_R} \leq \frac{|\text{Tr}\{\rho_0\mathcal{L}_t(\rho_t)\}|}{\|\rho_0\|_{\text{HS}} \sin 2\Theta_R}.$$

The dynamical map of a general quantum system reads $\mathcal{L}_t(\rho_t) = (-i/\hbar)[H_t, \rho_t] + \mathcal{D}_t(\rho_t)$ ¹³, where the renormalized Hamiltonian $H_t = H_0 + H_t^{\text{LS}}$ contains a time-dependent Lamb shift term H_t^{LS} . For a Markovian system, the super-operator \mathcal{L} takes a Lindblad form and is time-independent, hence $|\text{Tr}\{\rho_0\mathcal{L}(\rho_t)\}| = |\text{Tr}\{\mathcal{L}^\dagger(\rho_0)\rho_t\}|$, where \mathcal{L}^\dagger obeys the adjoint master equation¹⁵. However, this is invalid for a non-Markovian system¹³. To derive the lower bound for a non-Markovian case, we divide \mathcal{L}_t into two parts using the triangle inequality

$$|\text{Tr}\{\rho_0\mathcal{L}_t(\rho_t)\}| \leq \frac{1}{\hbar} |\text{Tr}\{[H_t, \rho_t]\rho_0\}| + |\text{Tr}\{\mathcal{D}_t(\rho_t)\rho_0\}|.$$

The absolute trace inequality³² reads $|\text{Tr}\{AB\}| \leq \min\{\sigma_1^A \sum_i \sigma_i^B, \sigma_1^B \sum_i \sigma_i^A\}$. Since $\|\rho_0\|_{\text{tr}} = 1$, one has

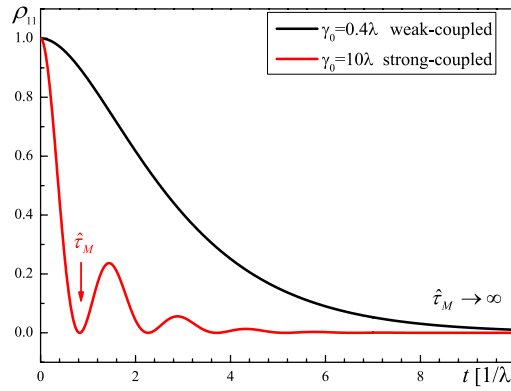


Figure 1. Solutions of the population of the damped Jaynes-Cummings model¹³ in the weak- (black line) and strong-coupling regime (red line), with $\gamma_0 = 0.4$ and $\gamma_0 = 10$, respectively, and $\lambda = 1$ for both. $\hat{\tau}_M$ is when the maximum of geometric distance is reached ($\rho_{11}(\hat{\tau}_M) = 0$).

$$\|\rho_0\|_{\text{HS}} \Theta_R \sin 2\Theta_R \leq \frac{1}{\hbar} \| [H_t, \rho_t] \|_{\text{op}} + \|\mathcal{D}_t(\rho_t)\|_{\text{op}}. \quad (1)$$

As H_t and ρ_t are both positive (by shifting the ground energy of H_t) and Hermitian operators, we take the commutator inequality³³, i.e., $\|AB - BA\|_{\text{op}} \leq (\sigma_1^A - \sigma_N^A)(\sigma_1^B - \sigma_N^B)/2$, where $\sigma_N = \min\{\sigma_i\}$ and N is the rank of the operator. For convenience, we denote $\sigma_1^A - \sigma_N^A = \|A\|_{\Delta}$. It is worth noting that this inequality is sharp, e.g., if $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$, then $\|A\|_{\Delta} = \|B\|_{\Delta} = 1$ and $\|AB - BA\|_{\text{op}} = 1/2$. Since $\|\rho_t\|_{\Delta} \leq 1$, for simplicity we have $\|H_t \rho_t - \rho_t H_t\|_{\text{op}} \leq \|H_t\|_{\Delta}/2$. Substituting it into Eq. (1) and integrating t from 0 to τ then yield

$$\tau \geq \frac{\|\rho_0\|_{\text{HS}} \sin^2 \Theta_R}{\frac{1}{2\hbar} \langle \|H_t\|_{\Delta} \rangle_{\tau} + \langle \|\mathcal{D}_t(\rho_t)\|_{\text{op}} \rangle_{\tau}}, \quad (2)$$

where $\langle A \rangle_{\tau} = \tau^{-1} \int_0^{\tau} A dt$. It is manifested that Eq. (2) is determined by both the renormalized Hamiltonian H_t (system) and the dissipator $\mathcal{D}_t(\rho_t)$ (environment). Also, this bound can reduce to the previous result¹⁶ when ρ_0 is a pure state and $H_t \equiv 0$.

Non-Markovianity. To investigate the minimal evolution time $\hat{\tau}$ in more detail, we use the damped Jaynes-Cummings model as an example, which describes the coupling between a two-level system and a single cavity mode with the background of cavity-QED¹³. Within a resonant Lorentzian spectral density of environment that $J(\omega) = (2\pi)^{-1} \gamma_0 \lambda^2 / [(\omega_0 - \omega)^2 + \lambda^2]$, the exact Hamiltonians read¹³

$$H_0 = \hbar \omega_0 \sigma_+ \sigma_-, \quad H_t^{LS} = 0,$$

where $\hbar \omega_0$ is the energy difference and $\sigma_{\pm} = (\sigma_x \pm i \sigma_y)/2$ are Pauli operators. The exact dissipator reads

$$\mathcal{D}_t(\rho_t) = \gamma(t) (\sigma_- \rho_t \sigma_+ - \frac{1}{2} \{\sigma_+ \sigma_-, \rho_t\}),$$

with $\gamma(t) = 2\gamma_0(\lambda/D) \tan(Dt/2) / [1 + (\lambda/D) \tan(Dt/2)]$, in which λ is the spectral width, γ_0 the coupling strength, and $D = \sqrt{2\gamma_0 \lambda - \lambda^2}$. When $\gamma_0 < \lambda/2$, the system and environment are weakly coupled and evolve subexponentially; the degree of non-Markovianity $\mathcal{N} = 0$ ²⁴. When $\gamma_0 > \lambda/2$, D is real; the system and environment are strongly coupled with oscillatory characteristics¹³ and $\mathcal{N} > 0$ (see Fig. 2). The initial environment is chosen to be a vacuum state and the initial system $\rho_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ fully excited to make the model simpler. Consequently, we only need to consider the dissipator $\mathcal{D}_t(\rho_t)$, for the exact solution¹³ implies ρ_t a diagonal operator so that $[H_t, \rho_t] \equiv 0$ in Eq. (1). It is useful to introduce a special minimal evolution time $\hat{\tau}_M = \max\{\hat{\tau}\}$, $\forall \mathbb{T}$, i.e., $\hat{\tau}_M$ is the minimal evolution time for the maximum of

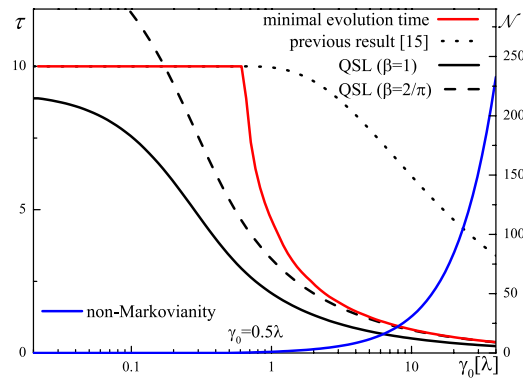


Figure 2. Minimal evolution time (red solid line) of the same model and its different QSL bounds (black lines) as a function of γ_0 . The bounds are derived from the previous result¹⁶ (dotted), Eq. (4) (solid), and Eq. (5) (dashed). Also indicated here is the degree of non-Markovianity²⁴ (blue solid line). We set $\lambda = 1$ and $\tau = 10$.

Θ_R . It is worth noting that $\hat{\tau}_M$ depends strongly on different coupling regimes (see Fig. 1): in the weak-coupling regime, we have $\hat{\tau}_M \rightarrow \infty$, but in the strong-coupling case, $\hat{\tau}_M$ is finite, which is caused by the oscillatory characteristics of the population. Like $\hat{\tau}_M$, it is worth noting for $\hat{\tau}$ itself that it will be smaller in the strong-coupling regime than that in the weak-coupling regime¹⁶. Although $\hat{\tau}$ is equal to the only possible driving time $\tau \in \mathbb{T}$ when it is weakly coupled, it is not the case for the strong-coupling regime. From numerical solution of $\hat{\tau}$ we find that $\hat{\tau}$ has a first derivative singular point at $\gamma_0 \approx \lambda/2$ and a steep decrease in the strong-coupling regime, which cannot be implied from \mathcal{N} (see Fig. 2). A decrease of the QSL for $\hat{\tau}$ was also suggested in the previous result¹⁶, but the decreasing slope with γ_0 deviates from the minimal evolution time as shown in Fig. 2.

As the energy of an open system is not conserved, the average of the dissipator $\mathcal{D}_t(\rho_t)$ decreases with time and $\langle \|\mathcal{D}_t(\rho_t)\|_{\text{op}} \rangle_{\infty} = 0$; as a result, we have $\langle \|\mathcal{D}_t(\rho_t)\|_{\text{op}} \rangle_{\tau} \leq \langle \|\mathcal{D}_t(\rho_t)\|_{\text{op}} \rangle_{\hat{\tau}}$ since $\tau \geq \hat{\tau}$. $\hat{\tau}$ depends on the short duration from 0 to $\hat{\tau}_M$ at most, so we can simply replace the time average by the maximum and eliminate the subscript $\hat{\tau}$,

$$\langle \|\mathcal{D}_t(\rho_t)\|_{\text{op}} \rangle_{\hat{\tau}} \leq \max \left\{ \|\mathcal{D}_t(\rho_t)\|_{\text{op}} \right\}. \quad (3)$$

Substituting Eq. (3) into Eq. (2), the final bound for $\hat{\tau}$ yields

$$\hat{\tau} \geq \frac{\|\rho_0\|_{\text{HS}} \sin^2 \Theta_R}{\frac{1}{2\hbar} \langle \|H_t\|_{\Delta} \rangle_{\hat{\tau}} + \max \left\{ \|\mathcal{D}_t(\rho_t)\|_{\text{op}} \right\}}, \quad (4)$$

which is valid for general quantum systems, regardless of whether they are closed or open and how strong the coupling is. It is found that the QSL Eq. (4) in the strong-coupling regime has a fitting decreasing slope as shown in Fig. 2. However, this bound is not asymptotic when $\gamma_0/\lambda \rightarrow \infty$. To derive a sharper QSL, we notice that Eq. (3) can take an approximation,

$$\langle \|\mathcal{D}_t(\rho_t)\|_{\text{op}} \rangle_{\hat{\tau}} \approx \beta \max \{ \|\mathcal{D}_t(\rho_t)\|_{\text{op}} \}, \quad (5)$$

where the parameter β introduced as a metric of the time average rests upon specific models, and the rough bound of Eq. (3) can also be treated as $\beta = 1$. For this case, we consider that in the strong-coupling regime when $\lambda/D \rightarrow 0$,

$$\frac{\gamma(t)}{2\gamma_0} = \frac{\lambda}{D} \tan\left(\frac{Dt}{2}\right) - \left[\frac{\lambda}{D} \tan\left(\frac{Dt}{2}\right) \right]^2 + \dots.$$

The first-order approximation yields $\gamma(t) \approx D \tan(Dt/2)$, since $2\gamma_0\lambda \approx D^2$. The exact solution of $\dot{\rho}_t = \mathcal{D}_t(\rho_t)$ yields $\mathcal{D}_t(\rho_t) = \mathcal{D}_t \left[\mathcal{T}_{\leftarrow} \exp \left(\int_0^t dt' \mathcal{D}_{t'} \right) \right] \rho_0$. Here \mathcal{T}_{\leftarrow} is the chronological super-operator which orders the t' arguments to increase from right to left³⁴. Hence,

$$\|\mathcal{D}_t(\rho_t)\|_{\text{op}} \approx \max\{\|\mathcal{D}_t(\rho_t)\|_{\text{op}}|\sin(Dt)|, \quad (6)$$

where $\max\{\|\mathcal{D}_t(\rho_t)\|_{\text{op}}\} \approx D/2$. Generally speaking, for a continuous evolution, strong coupling between the system and environment certainly involves a non-Markovian bidirectional flow of energy and/or coherence, which can always be characterized as oscillator(s). Therefore, a general form like Eq. (6) can provide a reasonable approximation of oscillation for other models.

It is found that the parameter β depends typically on the relation between $\hat{\tau}$ and $\hat{\tau}_M$ in the strong-coupling limit. With different $\hat{\tau}/\hat{\tau}_M$ (when $\gamma_0/\lambda \rightarrow \infty$), the time average in the left term of Eq. (5) will take a different time period, and β thus changes in the range from 0 to 1. In this case of the damped Jaynes-Cummings model, we have $\hat{\tau}/\hat{\tau}_M \rightarrow 1$ as $\gamma_0/\lambda \rightarrow \infty$, suggesting that the time average in Eq. (5) should take nearly a $\pi/2$ period. Taking Eq. (6) into Eq. (5) immediately indicates $\beta = 2/\pi$ then. From Fig. 2, it is clear that this bound is sharp, but is not valid when it comes into the weak-coupling regime since the approximation $\gamma(t) \approx D \tan(Dt/2)$ is invalid there.

Renormalized Hamiltonian. To verify our result and manifest the influence of the renormalized Hamiltonian term in Eq. (2), we introduce another two-level system containing a two-band environment as the second example. This model can simulate the interaction between a spin and a single-particle quantum dot^{35,36}, of which the total Hamiltonian is $H = H_0 + V$ where $H_0 = \Delta E \sigma_z + \sum_{n_1} (\delta\varepsilon/N_1) n_1 |n_1\rangle \langle n_1| + \sum_{n_2} (\Delta E + \delta\varepsilon/N_2) n_2 |n_2\rangle \langle n_2|$ with σ_z the Pauli operator. The lower energy band contains N_1 levels and the upper N_2 levels, with the same band width $\delta\varepsilon$ and the inter-bands distance ΔE in resonance with the spin. V represents the interaction that $V = \lambda \sum_{n_1, n_2} c(n_1, n_2) \sigma_+ |n_1\rangle \langle n_2| + \text{h.c.}$, with λ the coupling coefficient and $c(n_1, n_2)$ complex Gaussian random variables. At the beginning, we numerically solve the model concerning the minimal evolution time problem and identify the same singularity at $\lambda \approx 0.0072$ and steep decrease when $\lambda > 0.0072$ like those shown in Fig. 2. To demonstrate the influence of renormalized Hamiltonian, first we set $(\rho \otimes \rho_{E0}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes |n_1\rangle \langle n_1|$ with a driving time $\tau = 8.0$, from which one derives $\Theta_R \approx 0.7707$ and $\hat{\tau} \approx 2.0$ (see Fig. 3(a,b)). As $\rho_\infty = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$, $\hat{\tau}/\hat{\tau}_M \rightarrow 1$ now³⁵. It is recalculated from Eq. (5) that $\beta \approx [1 - \cos(3\pi/4)]/(3\pi/4) \approx 0.72$. Further calculation shows that the previous QSL¹⁶ $\hat{\tau}_o^{\text{QSL}} = 5.1757$ is too large, while Eqs. (3) and (5) indicate $\hat{\tau}_{\beta=1}^{\text{QSL}} = 1.4421$ and $\hat{\tau}_{\beta=0.72}^{\text{QSL}} = 1.9905$. Both of them stay valid while the latter is sharp. Second, we set $\Delta E = 10\hbar$, $(\rho \otimes \rho_{E0}) = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \otimes |n_1\rangle \langle n_1|$ and $\tau = 8.0$, from which one derives $\Theta_R \approx 0.7832$ and $\hat{\tau} \approx 0.2$ (see Fig. 3(c,d)). Since ρ_0 is not diagonal, $[H_t, \rho_t] \neq 0$ and $\langle \|H_t\|_\Delta \rangle_\tau$ should be considered. Further calculation shows that the previous QSL¹⁶ $\hat{\tau}_o^{\text{QSL}} = 1.0242$ is too large, while $\hat{\tau}_{\beta=1}^{\text{QSL}} = 0.1130$ and $\hat{\tau}_{\beta=0.72}^{\text{QSL}} = 0.1196$. The mere difference between $\hat{\tau}_{\beta=1}^{\text{QSL}}$ and $\hat{\tau}_{\beta=0.72}^{\text{QSL}}$ implies that the renormalized Hamiltonian H_t is dominant in Eq. (4). As $\langle \|H_t\|_\Delta \rangle_\tau$ added, $\hat{\tau}^{\text{QSL}}$ becomes smaller, which apparently follows the time-energy uncertainty relation.

Discussion

Only Hamiltonian was considered in some of previous investigations^{6–12}, while for an open system, the coupling strength of its dissipator also has an influence on QSL^{15,16}. However, it is demonstrated in our study that in non-Markovian case such influence could be more significant than it was thought. Therefore, to achieve a high speed of evolution⁵, it is more probable that we only focus on improving the coupling interaction instead of increasing the energy. This implies that the power consumption can stay a low level for cavity-QED process while high efficiency can still be achieved. Previously it was always thought that a strong coupling with environment should be prevented due to its enhanced decoherence effect on qubits. However, as a trade-off, the operation time for transforming and/or erasing qubits for example can also be remarkably reduced in the strong-coupling regime. It is thus possible to make quantum computation more feasible and achievable by adjusting the coupling strength in a well-chosen pattern.

In summary, we derive a sharp bound as the quantum speed limit of open systems available for mixed initial states. Considering the non-Markovian feature, we find that the minimal evolution time of the two two-level examples considered here has singularity nearly at the cross-point of regimes and a steep decrease in the strong-coupling regime. This result may lead to high-efficiency quantum information research and engineering. As the time-energy uncertainty relation dictates, renormalized Hamiltonian will also contribute to the final quantum speed limit bound as manifested in the quantum dot model in detail. We expect our result to be used for quantum time analysis and optimal control, as well as in pertinent topics on general physics.

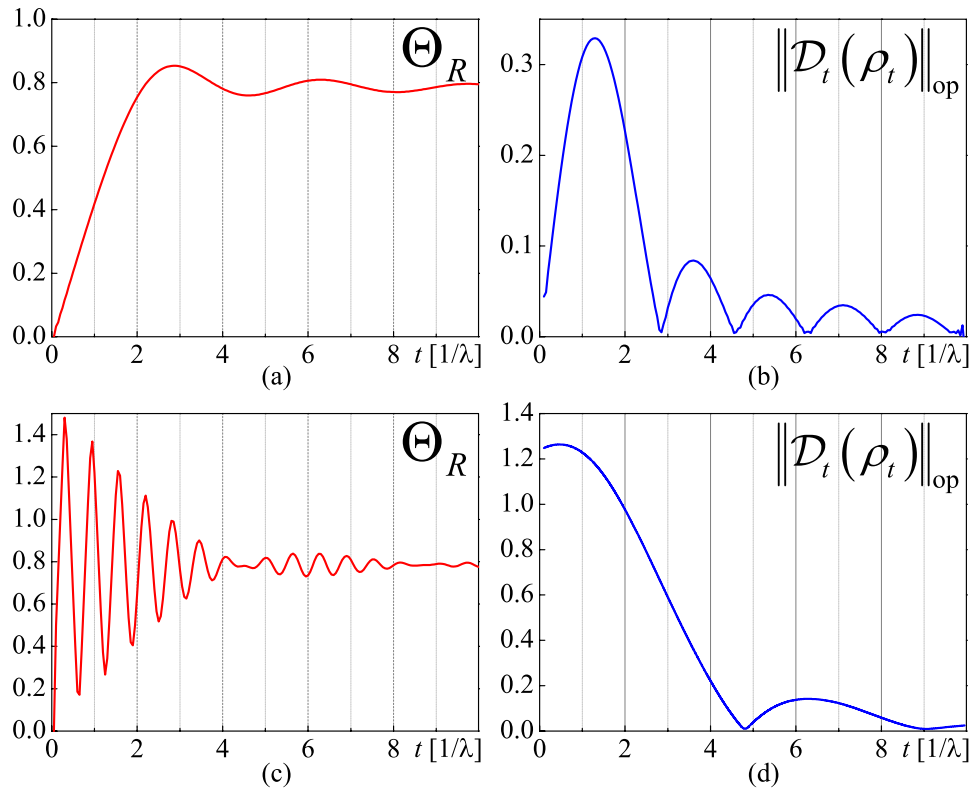


Figure 3. Numerical solution of the relative-purity fidelity (red lines) and the dissipator (blue lines) of the quantum dot model³⁵. $\lambda = 0.02$ which represents the strong-coupling regime, with $N_1 = N_2 = 500$ and $\delta\varepsilon = 0.5\hbar$. The initial states are (a), (b): $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and (c), (d): $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$, respectively.

Methods

Norms of operators. A general Schatten p -norm of an operator A is $\|A\|_p = (\sum_i \sigma_i^p)^{1/p}$, where singular values $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_i \geq \dots \geq 0$ are the eigenvalues of $|A| = \sqrt{A^\dagger A}$, and $\|A\|_{\text{op}} = \|A\|_{p \rightarrow \infty} = \sigma_1$, $\|A\|_{\text{tr}} = \|A\|_{p=1}$ and $\|A\|_{\text{HS}} = \|A\|_{p=2}$ as the operator norm, trace norm and Hilbert-Schmidt norm of A , respectively³⁷.

Approximation for the dissipator $\|\mathcal{D}_t(\rho_t)\|_{\text{op}}$ of the damped Jaynes-Cummings model in the strong-coupling regime. With

$$\mathcal{D}_t(\rho_t) = \gamma(t) \left(\sigma_- \rho_t \sigma_+ - \frac{1}{2} \{ \sigma_+ \sigma_-, \rho_t \} \right)$$

given¹³, the exact solution of $\dot{\rho}_t = \mathcal{D}_t(\rho_t)$ yields

$$\begin{aligned} \mathcal{D}_t(\rho_t) &= \mathcal{D}_t \left[\mathcal{T}_- \exp \left(\int_0^t dt' \mathcal{D}_{t'} \right) \right] \rho_0 \\ &= -\gamma(t) \mathcal{T}_- \exp \left(- \int_0^t dt' \gamma(t') \right) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

We have $\gamma(t) \approx D \tan(Dt/2)$ in the strong-coupling regime. As a result,

$$\begin{aligned} \|\mathcal{D}_t(\rho_t)\|_{\text{op}} &\approx \left| D \tan \left(\frac{Dt}{2} \right) \exp \left[-D \int_0^t \tan \left(\frac{Dt'}{2} \right) dt' \right] \right| \\ &= \frac{D}{2} |\sin(Dt)| \propto \max\{\|\mathcal{D}_t(\rho_t)\|_{\text{op}}\} |\sin(Dt)|, \end{aligned}$$

which yields the result of Eq. (6). In this case, we also have $\hat{\tau} \rightarrow \hat{\tau}_M$ as γ_0 increases, which implies that the time average in Eq. (5) takes nearly a $\pi/2$ period. Taking Eq. (6) into Eq. (5) then indicates $\beta = 2/\pi$.

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Author Contributions

X.M., C.W. and H.G. calculated and analyzed the results. X.M. and H.G. co-wrote the paper. All authors reviewed the manuscript and agreed with the submission.

Additional Information

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