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## Intersubjectivity and value reproducibility of outcomes of quantum measurements

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Every measurement determines a single value as its outcome, and yet quantum mechanics predicts it only probabilistically. The Kochen–Specker theorem and Bell’s inequality are often considered to reject a realist view but favor a skeptical view that measuring an observable does not mean ascertaining the value that it has, but producing the outcome, having only a personal meaning. However, precise analysis supporting this view is unknown. Here, we show that a quantum mechanical analysis turns down this view. Supposing that two observers simultaneously measure the same observable, we can well pose the question as to whether they always obtain the same outcome, or whether the probability distributions are the same, but the outcomes are uncorrelated. Contrary to the widespread view in favor of the second, we shall show that quantum mechanics predicts that only the first case occurs. We further show that any measurement establishes a time-like entanglement between the observable to be measured and the meter after the measurement, which causes the space-like entanglement between the meters of different observers. We also show that our conclusion cannot be extended to measurements of so-called ‘generalized’ or ‘unsharp’ observables, suggesting a demand for reconsidering the notion of observables in foundations of quantum mechanics.

**Keywords** Quantum measurements, Probability reproducibility, Repeatability hypothesis, Collapsing hypothesis, Intersubjectivity, Value reproducibility, Observables, POVMs, Von Neumann, Dirac, Schrödinger, Kochen, Specker, Bell

The theorems due to Kochen–Specker<sup>1</sup> and Bell<sup>2</sup>, enforced by the recent loophole-free experimental tests<sup>3–5</sup>, are often considered to defy the correlation between the measurement outcome and the *pre*-measurement value of the measured observable. Accordingly, it is a standard view that the measurement outcome should only correlate to the *post*-measurement value of the measured observable, as Schrödinger stated long ago:

The rejection of realism has logical consequences. In general, a variable has no definite value before I measure it; then measuring it does not mean ascertaining the value that it has. But then what does it mean? [...] Now it is fairly clear; if reality does not determine the measured value, then at least the measured value must determine reality [...] That is, the desired criterion can be merely this: repetition of the measurement must give the same result<sup>6</sup>, p. 329.

The *repeatability hypothesis* mentioned above is formulated as one of the basic axioms of quantum mechanics by von Neumann:

If [a] physical quantity is measured twice in succession in a system, then we get the same value each time<sup>7</sup>, p. 335.

It is well known that this hypothesis is equivalent to the *collapsing hypothesis* formulated by Dirac:

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A measurement always causes the system to jump into an eigenstate of the observable that is being measured, the eigenvalue this eigenstate belongs to being equal to the result of the measurement<sup>8</sup>, p. 36.

The repeatability hypothesis and the collapsing hypothesis formulated as above had been broadly accepted since the inception of quantum mechanics. However, they have been abandoned in the modern formulation treating all the physically realizable quantum measurements. In fact, Davies and Lewis<sup>9</sup> proposed to abandon the repeatability hypothesis and introduced an “operational approach” leading to a more flexible approach to measurement theory based on a mathematical notion of an “instrument”. Subsequently, Yuen<sup>10</sup> proposed the problem of mathematically characterizing all the possible quantum measurements claiming that the Davies–Lewis “operational approach” is too general. Interestingly, Yuen’s problem had been already solved at that time by the present author<sup>11</sup> showing that all the physically realizable quantum measurements are exactly characterized by “completely positive instruments”, instruments state changes of which satisfy complete positivity, nowadays a broadly accepted notion often called as “quantum instruments”.

Von Neumann<sup>7</sup>, p. 440 found a measuring interaction satisfying the repeatability condition for an observable  $A = \sum_a a|\varphi_a\rangle\langle\varphi_a|$ . He showed that such a measurement is described by a unitary operator  $U(\tau)$  such that

$$U(\tau)|\varphi_a\rangle|\xi\rangle = |\varphi_a\rangle|\xi_a\rangle, \quad (1)$$

where  $|\xi\rangle$  is an arbitrary but fixed state of the environment,  $\{|\xi_a\rangle\}$  is an orthonormal basis for the environment, and the meter observable is given by  $M = \sum_a a|\xi_a\rangle\langle\xi_a|$ . If the initial system state is a superposition  $|\psi\rangle = \sum_a c_a|\varphi_a\rangle$ , by linearity we obtain

$$U(\tau)|\psi\rangle|\xi\rangle = \sum_a c_a|\varphi_a\rangle|\xi_a\rangle. \quad (2)$$

Then we have

$$\Pr\{A(\tau) = x, M(\tau) = y\} = \delta_{x,y}|c_x|^2. \quad (3)$$

Thus, this measurement satisfies the *probability reproducibility condition*,

$$\Pr\{M(\tau) = x\} = |c_x|^2, \quad (4)$$

and the *repeatability condition*,

$$\Pr\{A(\tau) = x, M(\tau) = y\} = 0 \quad \text{if } x \neq y; \quad (5)$$

see<sup>7</sup>, p. 440.

According to the above analysis, the measurement outcome is often considered to be created, rather than reproduced, by the act of measurement<sup>12</sup>. Quantum Bayesian interpretation emphasizes its personal nature as one of the fundamental tenets<sup>13</sup> (Abstract):

Along the way, we lay out three tenets of QBism in some detail: [...] 3) Quantum measurement outcomes just are personal experiences for the agent gambling upon them.

However, if we consider the process of measurement from a more general perspective abandoning the repeatability hypothesis, in which only the probability reproducibility condition is required, we confront a puzzling problem.

Suppose that two remote observers, I and II, simultaneously measure the same observable. Then, we can ask whether quantum mechanics predicts that they always obtain the same outcome, or quantum mechanics predicts only that their probability distributions are the same but the outcomes are uncorrelated. In the following we shall show that quantum mechanics predicts that only the first case occurs, in contrast to a common interpretation of the theorems due to Kochen-Specker<sup>1</sup> and Bell<sup>2</sup>.

## Results

### Intersubjectivity of outcomes of quantum measurements

It is fairly well-known that any measurement can be described by an interaction between the system **S** to be measured and the environment **E** including measuring apparatuses and that the outcome of the measurement is obtained by a subsequent observation of a meter observable in the environment by the observer<sup>7,11,14</sup>.

Let  $A$  be an observable to be measured. Let  $M_1$  and  $M_2$  be the meter observables of observers I and II, respectively. We assume that at time 0 the system **S** is in an arbitrary state  $|\psi\rangle$  and the environment **E** is in a fixed state  $|\xi\rangle$ , respectively. In order to measure the observable  $A$  at time 0, observers I and II locally measure their meters  $M_1$  and  $M_2$  at times  $\tau_1 > 0$  and  $\tau_2 > 0$ , respectively.

Then, the time evolution operator  $U(t)$  of the total system **S** + **E** determines the Heisenberg operators  $A(0)$ ,  $M_1(t)$ ,  $M_2(t)$  for any time  $t > 0$ , where  $A(0) = A \otimes I$ ,  $M_1(t) = U(t)^\dagger(I \otimes M_1)U(t)$ , and  $M_2(t) = U(t)^\dagger(I \otimes M_2)U(t)$ . For any observable  $X$ , we denote by  $P^X(x)$  the spectral projection of  $X$  corresponding to  $x \in \mathbb{R}$ , i.e.,  $P^X(x)$  is the projection onto the subspace of vectors  $|\psi\rangle$  satisfying  $X|\psi\rangle = x|\psi\rangle$ .

We pose the following two assumptions.

**Assumption 1 (Locality)** We suppose that  $M_1(\tau_1)$  and  $M_2(\tau_2)$  are mutually commuting and that the joint probability distribution of the outcomes of measurements by observers I and II are given by

$$\Pr\{M_1(\tau_1) = x, M_2(\tau_2) = y\} = \langle \psi, \xi | P^{M_1(\tau_1)}(x) P^{M_2(\tau_2)}(y) | \psi, \xi \rangle \quad (6)$$

for all  $x, y \in \mathbb{R}$ , where  $|\psi, \xi\rangle = |\psi\rangle |\xi\rangle$ .

**Assumption 2** (*Probability reproducibility*) The measurements of the observable  $A$  by observers I and II satisfy the *probability reproducibility condition*, *i.e.*,

$$\Pr\{M_1(\tau_1) = x\} = \Pr\{M_2(\tau_2) = x\} = \Pr\{A(0) = x\} \quad (7)$$

for any  $x \in \mathbb{R}$  in arbitrary  $|\psi\rangle$  and fixed  $|\xi\rangle$ .

Assumption 1 is a natural consequence from the assumption that the two local meter-measurements by observers I and II are space-like separated. In this case, the Local Measurement Theorem<sup>15</sup> (Theorem 5.1) ensures that the joint probability distribution of the outcomes of measurements by observers I and II satisfies Eq. (6), without assuming that the meter measurements satisfy the repeatability or collapsing condition. Thus, the joint probability of their outcomes is well-defined by Eq. (6), and our problem is well-posed.

In Assumption 2 we only require that the outcome of a measurement of an observable should satisfy the Born rule for the measured observable, and we make no assumption on the state change caused by the measurement such as the repeatability condition nor the collapsing condition.

Now we can ask if observers I and II always obtain the same outcome, *i.e.*,

$$\Pr\{M_1(\tau_1) = x, M_2(\tau_2) = y\} = 0 \quad (8)$$

if  $x \neq y$ . We call this condition the *intersubjectivity condition*.

Now we shall show the following.

**Theorem 1** (*Intersubjectivity Theorem*) *The outcomes of simultaneous, probability reproducible measurements of the same observable with two space-like separated meter observables satisfy the intersubjectivity condition.*

A complete proof is given in the “[Methods](#)” section. We call the condition

$$P^{M(\tau)}(x)|\psi\rangle |\xi\rangle = P^{A(0)}(x)|\psi\rangle |\xi\rangle \quad (9)$$

the *time-like entanglement condition*, comparing with the *space-like entanglement condition*

$$P^{M(\tau)}(x)|\psi\rangle |\xi\rangle = P^{A(\tau)}(x)|\psi\rangle |\xi\rangle, \quad (10)$$

which follows from Eq. (2).

Note that an alternative proof for Theorem 1 can be obtained from an advanced structure theorem<sup>16</sup>, Theorem 3.2.1, for joint POVMs (probability operator-valued measures). In fact, if we apply this theorem to the joint POVM

$$\Pi(x, y) = \langle \xi | P^{M_1(\tau_1)}(x) P^{M_2(\tau_2)}(y) | \xi \rangle, \quad (11)$$

we obtain the relation  $\Pi(x, y) = P^A(x)P^A(y)$  and Eq. (8) follows. Nevertheless, the proof of Theorem 1 in this paper clearly shows the following points, which are useful for our later discussions.

- The probability reproducibility condition implies the time-like entanglement condition.
- The intersubjectivity condition is a straightforward consequence from the time-like entanglement condition for two space-like separated meter observables.

It can be easily seen that Theorem 1 can be extended to the assertion for  $n$  observers with any  $n > 2$ . Thus, we conclude that *if two or more mutually space-like separated observers simultaneously measure the same observable, then their outcomes always coincide*.

Example 1 in the “[Methods](#)” section illustrates a typical system-environment interaction to realize simultaneous position measurements of  $n$  observers.

### Non-Intersubjectivity for unconventional generalized observables

We note that the above result, Theorem 1, cannot be extended to an arbitrary ‘generalized observable’  $A$  represented by a POVM, *i.e.*, a family  $\{P^A(x)\}_{x \in \mathbb{R}}$  of positive operators  $P^A(x) \geq 0$ , not necessarily of projections, such that  $\sum_x P^A(x) = I$ . The optical phase is not considered as a quantum observable but typically considered as a physical quantity corresponding to a generalized observable (see Ref<sup>17</sup>, and the references therein).

To immediately see that our conclusion cannot be extended to the class of generalized observables, consider a generalized observable  $A$  defined by  $P^A(x) = \mu(x)I$ , where  $\mu$  is an arbitrary probability distribution, *i.e.*,  $\mu(x) \geq 0$  and  $\sum_x \mu(x) = 1$ . Then, as shown in Example 2 in the “[Methods](#)” section, we can construct continuously parametrized models for which Assumptions 1 and 2 hold, but the intersubjectivity condition, Eq. (8), does not hold.

## Value reproducibility of outcomes of quantum measurements

The intersubjectivity of the measurement outcomes ensures that in quantum mechanics the phrase ‘the outcome of a measurement of an observable  $A$  at time  $t$ ’ has an unambiguous meaning. This may suggest the existence of a correlation between the measurement outcome, or the post-measurement value of the meter, and the pre-measurement value of the measured observable as a common cause for the coincidence of the outcomes.

Now we focus on the measurement carried out by the sole observer, to show that every probability reproducible measurement of an observable is indeed “value reproducible” as precisely formulated below.

Let  $A$  be the observable of the system  $S$  to be measured. Let  $M$  be the meter observable of the observer in the environment. We assume that at time 0 the system  $S$  is in an arbitrary state  $|\psi\rangle$  and the environment  $E$  is in a fixed state  $|\xi\rangle$ . In order to measure the observable  $A$  at time 0, the observer locally measures the meter observable  $M$  at time  $\tau > 0$ . Then, the time evolution operator  $U(\tau)$  of the total system  $S + E$  determines the Heisenberg operators  $A(0)$  and  $M(\tau)$ , where  $A(0) = A \otimes I$ ,  $M(\tau) = U(\tau)^\dagger (I \otimes M) U(\tau)$ . We pose the following assumption. The measurement satisfies the *probability reproducibility condition*, i.e.,

$$\Pr\{M(\tau) = x\} = \Pr\{A(0) = x\} \quad (12)$$

for any vector state  $|\psi\rangle$  of  $S$ .

Since quantum mechanics predicts a relation between values of observables only in the form of probability correlations, the coincidence between the pre-measurement value of the measured observable and the post-measurement value of the meter observable should be best expressed by the relation

$$\Pr\{A(0) = x, M(\tau) = y\} = 0 \quad (13)$$

if  $x \neq y$ . However, this relation shows a difficulty. Since  $A(0)$  and  $M(\tau)$  may not commute in general, the joint probability distribution may not be well-defined.

In what follows, we shall show that under the probability reproducibility condition, the above joint probability is actually well-defined to satisfy Eq. (13) as  $A(0)$  and  $M(\tau)$  commute on the subspace generated by the observables  $A(0)$ ,  $M(\tau)$  and the state  $|\psi\rangle$ .

To see this, recall the notion of *partial commutativity* or *state-dependent commutativity* introduced by von Neumann<sup>7</sup>, p. 230: *If a state  $|\Psi\rangle$  is a superposition of common eigenstates  $|X = x, Y = y\rangle$  of observables  $X$  and  $Y$  of the form*

$$|\Psi\rangle = \sum_{(x,y) \in S} c_{x,y} |X = x, Y = y\rangle, \quad (14)$$

where  $S \subseteq \mathbb{R}^2$ , then the joint probability distribution  $\Pr\{X = x, Y = y\}$  of  $X$  and  $Y$  in  $|\Psi\rangle$  is well-defined as

$$\Pr\{X = x, Y = y\} = \begin{cases} |c_{x,y}|^2 & \text{if } (x, y) \in S, \\ 0 & \text{otherwise.} \end{cases} \quad (15)$$

In this case,  $X$  and  $Y$  actually commute on the subspace  $\mathcal{M}$  generated by  $\{|X = x, Y = y\rangle\}_{(x,y) \in S}$ , and we say that  $X$  and  $Y$  commute in the state  $|\Psi\rangle$ . The observables  $X$  and  $Y$  can be simultaneously measured in the state  $|\Psi\rangle$ , and the joint probability distribution of the outcomes  $X = x$  and  $Y = y$  of the simultaneous measurements satisfies Eq. (15)<sup>7</sup>, p. 230–231. In this case,  $|\Psi\rangle \in \mathcal{M}$  and

$$P^X(x) \wedge P^Y(y) |\Psi\rangle = P^X(x) P^Y(y) |\Psi\rangle = P^Y(y) P^X(x) |\Psi\rangle, \quad (16)$$

where  $\wedge$  denotes the infimum of two projections. The joint probability distribution  $\Pr\{X = x, Y = y\}$  satisfies

$$\Pr\{X = x, Y = y\} = \langle \Psi | P^X(x) \wedge P^Y(y) | \Psi \rangle = \langle \Psi | P^X(x) P^Y(y) | \Psi \rangle = \langle \Psi | P^Y(y) P^X(x) | \Psi \rangle \quad (17)$$

for any  $x, y \in \mathbb{R}$ . Moreover,

$$\langle \Psi | f(X, Y) | \Psi \rangle = \sum_{x,y} f(x, y) \Pr\{X = x, Y = y\} \quad (18)$$

for any real polynomial  $f(X, Y)$  of  $X$  and  $Y$ <sup>18</sup> (Theorem 1).

Based on the above notion of the state-dependent commutativity due to von Neumann<sup>7</sup>, a precise formulation for the value reproducibility is given as follows. A measurement of an observable  $A$  described by the system-environment time evolution  $U(\tau)$  from time  $t = 0$  to  $t = \tau$  with the fixed initial environment state  $|\xi\rangle$  is said to satisfy the *value reproducibility condition* if the pre-measurement observable  $A(0)$  to be measured and the post-measurement meter observable  $M(\tau)$  commute in the initial state  $|\psi\rangle|\xi\rangle$  and the joint probability distribution  $\Pr\{A(0) = x, M(\tau) = y\}$  satisfies Eq. (13) for any state vector  $|\psi\rangle$  of the measured system. The terminology “pre-measurement value of the observable to be measured” and the “post-measurement value of the meter observable” is defined through the well-defined joint probability distribution  $\Pr\{A(0) = x, M(\tau) = y\}$ , where  $x$  is called the pre-measurement value of the observable  $A$  to be measured and  $y$  is called the post-measurement value of the meter observable  $M$ . Thus, the well-defined joint probability distribution in Eq. (13) refers to the probability correlation between the pre-measurement value of the observable to be measured and the post-measurement value of the meter observable.

Then the following theorem holds.

**Theorem 2** (Value Reproducibility Theorem) *Every probability reproducible measurement of an observable is value reproducible.*

A proof is given in the “[Methods](#)” section.

Now, we have seen that the proof of Theorem 2 shows that (i) the time-like entanglement condition implies the value reproducibility condition. We already have seen that the proof of Theorem 1 shows that (ii) the probability reproducibility condition implies the time-like entanglement condition. Thus, we have shown the following implication relations:

$$\boxed{\text{probability reproducibility}} \Rightarrow \boxed{\text{time-like entanglement}} \Rightarrow \boxed{\text{value reproducibility}} \quad (19)$$

We show that the above three conditions are actually all equivalent.

**Theorem 3** *For any measurement of an observable  $A$  in the state  $|\psi\rangle$  with the pre-measurement observable  $A(0)$  and the post-measurement meter observable  $M(\tau)$  in the fixed environment state  $|\xi\rangle$ , the following conditions are all equivalent.*

(i) (Probability reproducibility condition) For any  $x \in \mathbb{R}$  and any system state  $|\psi\rangle$ ,

$$\langle \psi, \xi | P^{M(\tau)}(x) | \psi, \xi \rangle = \langle \psi | P^A(x) | \psi \rangle. \quad (20)$$

(ii) (Time-like entanglement condition) For any  $x \in \mathbb{R}$  and any system state  $|\psi\rangle$ ,

$$P^{M(\tau)}(x) | \psi, \xi \rangle = P^{A(0)}(x) | \psi, \xi \rangle. \quad (21)$$

(iii) (Value reproducibility condition) For any system state  $|\psi\rangle$ , the observables  $A(0)$  and  $M(\tau)$  commute in the initial state  $|\psi\rangle |\xi\rangle$  and satisfy

$$\langle \psi, \xi | P^{A(0)}(x) P^{M(\tau)}(y) | \psi, \xi \rangle = 0 \quad (22)$$

if  $x \neq y$ .

A proof is given in the “[Methods](#)” section.

Since the proof of Theorem 1 have shown that the intersubjectivity condition is a straightforward consequence from the time-like entanglement condition for two space-like separated meter observables, we have eventually shown the following implication relations:

$$\begin{array}{c} \boxed{\text{probability reproducibility condition}} \Leftrightarrow \boxed{\text{time-like entanglement condition}} \Leftrightarrow \boxed{\text{value reproducibility condition}} \\ \Downarrow \\ \boxed{\text{intersubjectivity condition}} \end{array} \quad (23)$$

The implication ‘value reproducibility condition’  $\Rightarrow$  ‘intersubjectivity condition’ can also be obtained by the transitivity of perfect correlations<sup>19</sup>. We say that two observables  $X$  and  $Y$  are perfectly correlated in a state  $\Psi$ , and write  $X =_{|\Psi\rangle} Y$ , if  $\langle \Psi | P^X(x) P^Y(y) | \Psi \rangle = 0$  for  $x \neq y$ . Then it is shown that perfect correlations are transitive, *i.e.*,  $X =_{|\Psi\rangle} Y$  and  $Y =_{|\Psi\rangle} Z$  implies  $X =_{|\Psi\rangle} Z$  for any observables  $X, Y, Z$ <sup>19</sup> (Theorem 4.4). Now we suppose the value reproducibility conditions for the pairs  $(A(0), M_1(\tau_1))$  and  $(A(0), M_2(\tau_2))$ . Then  $M_1(\tau_1) =_{|\Psi\rangle |\xi\rangle} A(0)$  and  $A(0) =_{|\Psi\rangle |\xi\rangle} M_2(\tau_2)$ , so that we obtain the intersubjectivity condition,  $M_1(\tau_1) =_{|\Psi\rangle |\xi\rangle} M_2(\tau_2)$ , by the transitivity of perfect correlations.

Under Assumptions 1 (locality) and 2 (probability reproducibility), we have found the perfect correlation between  $A(0)$  and  $M_1(\tau_1)$  and that between  $A(0)$  and  $M_2(\tau_2)$  from the Value Reproducibility Theorem. Then the perfect correlation between  $M_1(\tau_1)$  and  $M_2(\tau_2)$  found in the Intersubjectivity Theorem is considered to be a straightforward consequence of the transitivity of perfect correlations for the other two pairs. Therefore we can conclude that the pre-measurement value of the observable to be measured is a common cause for the perfect correlation between the outcomes of two or more space-like separated observers, through the perfect correlations with the post-measurement meters realized by the interaction with the environment.

### Value reproducibility of the von Neumann model of repeatable measurements

In the conventional approach to quantum measurements of an observable  $A = \sum_a a |\varphi_a\rangle \langle \varphi_a|$  due to von Neumann<sup>7</sup>, the measurement is required to satisfy both the probability reproducibility condition, Eq. (4), and the repeatability condition, Eq. (5), whereas the rejection of the realism in quantum mechanics has long been considered to defy the value reproducibility condition, Eq. (13), as stated by Schrödinger<sup>6</sup>. However, we should point out that *the conventional analysis of measuring process given by Eq. (2) has failed to unveil the fact that the measurement actually satisfies the value reproducibility condition*, as follows.

Now, we shall show that Eq. (2) can also be rewritten as

$$|\psi\rangle|\xi\rangle = \sum_a c_a |A(0) = a, M(\tau) = a\rangle, \quad (24)$$

which implies that the joint probability distribution  $\Pr\{A(0) = a, M(\tau) = m\}$  is well-defined and

$$\Pr\{A(0) = a, M(\tau) = m\} = \delta_{a,m} |c_a|^2, \quad (25)$$

so that the value reproducibility condition, Eq. (13), holds. To see this, note that we obtain the relations

$$A(0)|\varphi_a\rangle|\xi\rangle = (A \otimes I)|\varphi_a\rangle|\xi\rangle = a|\varphi_a\rangle|\xi\rangle, \quad (26)$$

$$\begin{aligned} M(\tau)|\varphi_a\rangle|\xi\rangle &= U(\tau)^\dagger (I \otimes M) U(\tau)|\varphi_a\rangle|\xi\rangle = U(\tau)^\dagger (I \otimes M)|\varphi_a\rangle|\xi_a\rangle = a U(\tau)^\dagger |\varphi_a\rangle|\xi_a\rangle \\ &= a|\varphi_a\rangle|\xi\rangle \end{aligned} \quad (27)$$

for all  $a$ . Thus, we have

$$P^{M(\tau)}(a)P^{A(0)}(a)|\varphi_a\rangle|\xi\rangle = P^{A(0)}(a)P^{M(\tau)}(a)|\varphi_a\rangle|\xi\rangle = |\varphi_a\rangle|\xi\rangle. \quad (28)$$

It follows that the state  $|\varphi_a\rangle|\xi\rangle$  is a joint eigenstate for observables  $A(0)$  and  $M(\tau)$  with the common eigenvalue  $a$ . Therefore, Eq. (24) follows with  $|A(0) = a, M(\tau) = a\rangle = |\varphi_a\rangle|\xi\rangle$  and we have proved that the value reproducibility condition, Eq. (13), holds.

### Intersubjectivity of some non-repeatable measurement models

It has long been claimed that a measurement entangles the measured observable with the meter, or the measured system with the environment without any precise qualification. However, it is wrong to consider that the intersubjectivity theorem is a consequence from this entangling nature of the measurement, since the entangling nature of the measurement is a consequence of the repeatability condition in addition to the probability reproducibility condition, whereas the intersubjectivity theorem is a consequence of the sole requirement of the probability reproducibility condition. Thus, there is a measurement that does not entangle the measured system with the environment but satisfies the intersubjectivity condition, namely, entangles all the meters.

Recall that the unitary operator  $U(\tau)$  for the von Neumann model given in Eq. (1) has been shown to satisfy both repeatability condition and probability reproducibility condition. To clarify the above point, consider, now instead of  $U(\tau)$ , the unitary operator  $V(\tau)$  satisfying

$$V(\tau)|\varphi_a\rangle|\xi\rangle = |\varphi\rangle|\xi_a\rangle, \quad (29)$$

where  $|\varphi\rangle$  is an arbitrary but fixed state of the measured system. Note that among the qubit measurements, the CNOT gate is a typical example of  $U(\tau)$ , while the SWAP gate is a typical example of  $V(\tau)$ .

If the initial system state is a superposition  $|\psi\rangle = \sum_a c_a |\varphi_a\rangle$  then by linearity we obtain

$$V(\tau)|\psi\rangle|\xi\rangle = \sum_a c_a V(\tau)|\varphi_a\rangle|\xi\rangle = \sum_a c_a |\varphi\rangle|\xi_a\rangle = |\varphi\rangle \left( \sum_a c_a |\xi_a\rangle \right). \quad (30)$$

Thus, this measurement makes no entanglement between the measured observable and the meter, whereas this measurement satisfies the probability reproducibility condition by the relations

$$\Pr\{M(\tau) = x\} = \|(I \otimes |\xi_x\rangle\langle\xi_x|)V(\tau)|\psi\rangle|\xi\rangle\|^2 = \left| \left( |\xi_x\rangle, \sum_a c_a |\xi_a\rangle \right) \right|^2 = |c_x|^2 = \Pr\{A(0) = x\}. \quad (31)$$

An interesting feature of the intersubjectivity theorem is that if such a unitary operator  $V(\tau)$  is extended to two meters then it does not entangle the meters with the measured object either, but entangles the two meters each other. To see this, let  $W(\tau)$  be such that

$$W(\tau)|\varphi_a\rangle|\xi\rangle|\xi'\rangle = |\varphi\rangle|\xi_a\rangle|\xi'_a\rangle, \quad (32)$$

which extends  $V(\tau)$  to another meter  $M' = \sum_a a|\xi'_a\rangle\langle\xi'_a|$  in the initial state  $|\xi'\rangle$ . If the initial system state is a superposition  $|\psi\rangle = \sum_a c_a |\varphi_a\rangle$  then by linearity we obtain

$$W(\tau)|\psi\rangle|\xi\rangle|\xi'\rangle = \sum_a c_a W(\tau)|\varphi_a\rangle|\xi\rangle|\xi'\rangle = \sum_a c_a |\varphi\rangle|\xi_a\rangle|\xi'_a\rangle = |\varphi\rangle \left( \sum_a c_a |\xi_a\rangle|\xi'_a\rangle \right). \quad (33)$$

Thus, this measurement does not entangle the measured observable with none of the two meters either, but both meters satisfy the probability reproducibility condition

$$\Pr\{M(\tau) = x\} = \Pr\{M'(\tau) = x\} = |c_x|^2 = \Pr\{A(0) = x\}, \quad (34)$$

and both are entangled each other to satisfy the intersubjectivity condition

$$\Pr\{M(\tau) = x, M'(\tau) = y\} = 0 \quad (35)$$

if  $x \neq y$ .

Therefore, we conclude that the intersubjectivity theorem is not a consequence of the known type of entangling nature of measurements, while it reveals a new type of entangling nature of measurements that entangles all the meters and that is shared by all the probability reproducible measurements.

### Unconventional generalized observables are not value reproducibly measurable

In the preceding sections, we have shown that any probability reproducible measurement indeed reproduces the pre-measurement value, whether the repeatability is satisfied or not. It is an interesting problem to what extent a probability reproducible measurement of a ‘generalized’ observable can be value reproducible. Here, we answer this question rather surprisingly: only the conventional observables can be measured value reproducibly.

A generalized observable  $A$  on a Hilbert space  $\mathcal{H}$  is called *value reproducibly measurable* if there exists a measuring process  $(\mathcal{K}, |\xi\rangle, U(\tau), M)$  for a Hilbert space  $\mathcal{H}$  satisfying condition (iii) of Theorem 3 (for the generalized observable  $A$ ), where  $P^{A(0)}(x) = P^A(x) \otimes I$ . Then, we have

**Theorem 4** *A generalized observable is value reproducibly measurable if and only if it is an observable (in the conventional sense).*

The proof is given in the “Methods” section.

### Discussion

Schrödinger<sup>6</sup>, p. 329 argued that a measurement does not ascertain the pre-measurement value of the observable and is only required to be repeatable. Since the inception of quantum mechanics, this view has long been supported as one of the fundamental tenets of quantum mechanics. In contrast, we have shown in the Value Reproducibility Theorem (Theorem 2) that any probability reproducible measurement indeed reproduces the pre-measurement value, whether the repeatability is satisfied or not.

It is an interesting problem to what extent a probability reproducible measurement of a ‘generalized’ observable can be value reproducible. Theorem 4 answers this question rather surprisingly as that only conventional observables can be measured value-reproducibly. This suggests a demand for more careful analysis on the notion of observables in foundations of quantum mechanics. In this area, generalized probability theory<sup>20,21</sup> has recently been studied extensively. However, the theory only has the notion of ‘generalized’ observable, but does not have the counter part of ‘conventional’ observables being value-reproducibly measurable.

In this paper, we have considered the notion of measurement of observables ‘state-independently’, and we take it for granted that a measurement of an observable is accurate if and only if it satisfies the probability reproducibility in all states. However, this does not mean that ‘state-dependent’ definition of an accurate measurement of an observable should only require the probability reproducibility in a given state, since requiring the probability reproducibility for all the state is logically and extensionally equivalent to requiring the value reproducibility for all the state as shown in this paper. In the recent debate on the formulation of measurement uncertainty relations, some authors have claimed that the state-dependent approach to this problem is not tenable, based on the state-dependent probability reproducibility requirement<sup>22,23</sup>. In contrast, we have recently shown that state-dependent approach to measurement uncertainty relations is indeed tenable, based on the state-dependent value reproducibility requirement<sup>18</sup>. The debate suggests that the value reproducibility is more reasonable requirement for the state-dependent accuracy of measurements of observables.

The cotextuality in assigning the values to observables shown by the theorems due to Kochen-Specker<sup>1</sup> and Bell<sup>12</sup> is often considered as the rejection of realism. However, it should be emphasized that what is real depends on a particular philosophical premise, and it is not completely determined by physics. Here, we have revealed a new probability correlation, Eq. (8), predicted solely by quantum mechanics, ensuring that the outcome of a measurement of an observable is unambiguously defined in quantum mechanics worth communicating intersubjectively. Further, we have shown that the intersubjectivity of outcomes of measurements is an immediate consequence from another new probability correlation, Eq. (13), the value-reproducibility of measurements. Since the value reproducibility is in an obvious conflict with the conventional understanding of ‘rejection of realism’, it would be an interesting problem to interpret quantum reality taking into account both the contextuality of value-assignments of observables and the intersubjectivity and the value reproducibility of outcomes of measurements.

In this connection, we have discussed the logical characterization of contextual hidden-variable theories based on the recent development of quantum set theory<sup>24</sup>. From this approach, we can conclude that the pre-measurement value of the observable to be measured is an element of reality in the context, in which the post-measurement meter observable is an element of reality to be actually read out by the observer, as a consequence of the perfect correlation between them ensured by the value reproducibility theorem. A. Khrennikov<sup>25</sup> discussed the interpretational implication of the intersubjectivity theorem to the QBism individual agent perspective. We will further discuss interpretational issues of our results of the intersubjectivity and the value reproducibility of outcomes of quantum measurements elsewhere.

### Methods

#### Proof of Theorem 1

**Proof** From Assumption 2, Eq. (7), we have

$$\|P^{M_1(\tau_1)}(x)|\psi\rangle|\xi\rangle\|^2 = \|P^{A(0)}(x)|\psi\rangle|\xi\rangle\|^2. \quad (36)$$

Since  $|\psi\rangle$  is arbitrary, replacing it by  $P^A(y)|\psi\rangle/\|P^A(y)|\psi\rangle\|$  if  $P^A(y)|\psi\rangle \neq 0$ , we obtain

$$\begin{aligned} \|P^{M_1(\tau_1)}(x)P^{A(0)}(y)|\psi\rangle|\xi\rangle\|^2 &= \|P^{M_1(\tau_1)}(x)(P^A(y)|\psi\rangle)|\xi\rangle\|^2 = \|P^{A(0)}(x)(P^A(y)|\psi\rangle)|\xi\rangle\|^2 \\ &= \delta_{x,y}\|P^{A(0)}(y)|\psi\rangle|\xi\rangle\|^2. \end{aligned} \quad (37)$$

Since  $P^{M_1(\tau_1)}(x)$  is a projection, it follows that

$$P^{M_1(\tau_1)}(x)P^{A(0)}(y)|\psi\rangle|\xi\rangle = \delta_{x,y}P^{A(0)}(y)|\psi\rangle|\xi\rangle. \quad (38)$$

Summing up both sides of the above equation for all  $y$ , we obtain

$$P^{M_1(\tau_1)}(x)|\psi\rangle|\xi\rangle = P^{A(0)}(x)|\psi\rangle|\xi\rangle. \quad (39)$$

Similarly,

$$P^{M_2(\tau_2)}(x)|\psi\rangle|\xi\rangle = P^{A(0)}(x)|\psi\rangle|\xi\rangle. \quad (40)$$

Therefore, from Assumption 1, Eq. (6), we have

$$\Pr\{M_1(\tau_1) = x, M_2(\tau_2) = y\} = \langle\psi, \xi|P^{M_1(\tau_1)}(x)P^{M_2(\tau_2)}(y)|\psi, \xi\rangle = \langle\psi, \xi|P^{A(0)}(x)P^{A(0)}(y)|\psi, \xi\rangle = 0 \quad (41)$$

if  $x \neq y$ . Thus, we conclude that the joint probability distribution of the outcomes of the simultaneous measurements of the observable  $A$  by observers I and II satisfies the intersubjectivity condition, Eq. (8), which shows that the outcomes are always identical.

### Intersubjectivity for simultaneous position measurements

**Example 1.** The system  $\mathbf{S}$  to be measured has canonically conjugate observables  $Q, P$  on an infinite dimensional state space with  $[Q, P] = i\hbar$ . Consider the measurement of the observable  $A = Q$ . The environment  $\mathbf{E}$  consists of  $n$  sets of canonically conjugate observables  $Q_j, P_j$  with  $[Q_j, P_k] = \delta_{jk}, [Q_j, Q_k] = [P_j, P_k] = 0$  for  $j, k = 1, \dots, n$ . Here, the part of the actual environment not effectively interacting with  $\mathbf{S}$  can be neglected without any loss of generality. Consider  $n$  observers with their meters  $M_j = Q_j$  for  $j = 1, \dots, n$ . Suppose that the system  $\mathbf{S}$  is in an arbitrary state  $|\psi\rangle$  and the environment  $\mathbf{E}$  is in the joint eigenstate  $|\xi\rangle = |Q_1 = 0, \dots, Q_n = 0\rangle$ . The interaction between  $\mathbf{S}$  and  $\mathbf{E}$  is given by

$$H = KQ \otimes (P_1 + \dots + P_n), \quad (42)$$

where the coupling constant  $K$  is large enough to neglect the other term in the total Hamiltonian of the composite system  $\mathbf{S} + \mathbf{E}$ . Then, we have

$$\frac{d}{dt}Q_j(t) = \frac{1}{i\hbar}[Q_j(t), H(t)] = KQ(t), \quad (43)$$

$$Q_j(t) = Q_j(0) + KtQ(0). \quad (44)$$

Assumptions 1 and 2 are satisfied for  $\tau_j = 1/K$  with  $j = 1, \dots, n$ , i.e.,

$$[M_j(\tau_j), M_k(\tau_k)] = 0, \quad (45)$$

$$\langle\psi, \xi|P^{M_j(\tau_j)}(x)|\psi, \xi\rangle dx = \langle\psi|P^Q(x)|\psi\rangle dx. \quad (46)$$

In this case,  $M_j(t) - M_k(t) = Q_j(t) - Q_k(t)$  are the constant of the motion for all  $j, k$ , i.e.,

$$\frac{d}{dt}(Q_j(t) - Q_k(t)) = \frac{1}{i\hbar}[Q_j(t) - Q_k(t), H(t)] = 0. \quad (47)$$

Thus, the outcomes are identical for all the observers, i.e.,

$$\langle\psi, \xi|P^{M_j(\tau_j)}(x)P^{M_k(\tau_k)}(y)|\psi, \xi\rangle dx dy = \delta(x - y)|\psi(x)\rangle^2 dx dy. \quad (48)$$

### Non-intersubjectivity of unconventional generalized observables

**Example 2.** Let  $A$  be a generalized observable on a system  $\mathbf{S}$  described by a Hilbert space  $\mathcal{H}$  such that  $P^A(x) = \mu(x)I$ , where  $\mu$  is a probability distribution, i.e.,  $\mu(x) \geq 0$  for all  $x \in \mathbb{R}$  and  $\sum_{x \in \mathbb{R}} \mu(x) = 1$ . Let  $X = \{x \in \mathbb{R} \mid \mu(x) > 0\}$ . Suppose that the environment  $\mathbf{E}$  consists of two subsystems so that the environment is described by a Hilbert space  $\mathcal{K} = \mathcal{L} \otimes \mathcal{L}$ , where  $\mathcal{L}$  is a Hilbert space spanned by an orthonormal basis  $\{|x\rangle\}_{x \in X}$ . Suppose that observers I and II measure the meter observables  $M_1(\tau_1) = I \otimes M \otimes I$  and  $M_2(\tau_2) = I \otimes I \otimes M$  on  $\mathcal{H} \otimes \mathcal{K}$ , respectively, which may be constants of motion, where  $M = \sum_x x|x\rangle\langle x|$ , so that Assumption 1 is satisfied. The initial state of the environment  $\mathbf{E}$  can be represented by

$$|\xi\rangle = \sum_{x,y \in X} c_{x,y} |x\rangle |y\rangle. \quad (49)$$

Then, the joint probability distribution of  $M_1(\tau_1)$  and  $M_2(\tau_2)$  is given by

$$\Pr\{M_1(\tau_1) = x, M_2(\tau_2) = y\} = \langle \psi, \xi | P^{M_1(\tau_1)}(x) P^{M_2(\tau_2)}(y) | \psi, \xi \rangle = |c_{x,y}|^2. \quad (50)$$

Thus, Assumption 2 is satisfied if and only if  $\mu(x) = \sum_y |c_{x,y}|^2 = \sum_y |c_{y,x}|^2$ . Thus, under Assumptions 1 and 2, the joint probability distribution  $\mu(x, y) = \Pr\{M_1(\tau) = x, M_2(\tau) = y\}$  can be an arbitrary 2-dimensional probability distribution such that  $\sum_y \mu(x, y) = \sum_y \mu(y, x) = \mu(x)$ . In this case, Eq. (8) is satisfied if and only if  $|c_{x,y}|^2 = \delta_{x,y} \mu(x)$ . Thus, we have continuously parametrized models with  $\mu(x)$  and  $c_{x,y}$  for which Assumptions 1 and 2 are satisfied, *i.e.*,  $\mu(x) = \sum_y |c_{x,y}|^2 = \sum_y |c_{y,x}|^2$  but the intersubjectivity condition, Eq. (8), is not satisfied, *i.e.*,  $|c_{x,y}|^2 \neq \delta_{x,y} \mu(x)$  for some  $(x, y) \in X^2$ .

## Proof of Theorem 2

**Proof** Under the probability reproducibility condition, we can adapt the proof of Theorem 1 for  $M(\tau) = M_1(\tau_1)$  to obtain the time-like entanglement condition

$$P^{M(\tau)}(x) |\psi\rangle |\xi\rangle = P^{A(0)}(x) |\psi\rangle |\xi\rangle. \quad (51)$$

Let  $S = \{(x, y) \in \mathbb{R}^2 \mid P^{A(0)}(x) P^{M(\tau)}(y) |\psi\rangle |\xi\rangle \neq 0\}$ , and  $c_{x,y} = \|P^{A(0)}(x) P^{M(\tau)}(y) |\psi\rangle |\xi\rangle\|$ . For any  $(x, y) \in S$ , define  $|A(0) = x, M(\tau) = y\rangle$  by

$$|A(0) = x, M(\tau) = y\rangle = c_{x,y}^{-1} P^{A(0)}(x) P^{M(\tau)}(y) |\psi\rangle |\xi\rangle. \quad (52)$$

From Eq. (51),

$$\begin{aligned} P^{A(0)}(x) P^{M(\tau)}(y) |\psi\rangle |\xi\rangle &= P^{A(0)}(x) P^{A(0)}(y) |\psi\rangle |\xi\rangle = \delta_{x,y} P^{A(0)}(x) |\psi\rangle |\xi\rangle = \delta_{x,y} P^{M(\tau)}(x) |\psi\rangle |\xi\rangle \\ &= P^{M(\tau)}(y) P^{M(\tau)}(x) |\psi\rangle |\xi\rangle = P^{M(\tau)}(y) P^{A(0)}(x) |\psi\rangle |\xi\rangle. \end{aligned} \quad (53)$$

Thus,  $|A(0) = x, M(\tau) = y\rangle$  is a common eigenstate for  $A(0)$  with eigenvalue  $x$  and  $M(\tau)$  with eigenvalue  $y$  such that

$$|\psi\rangle |\xi\rangle = \sum_{(x,y) \in \mathbb{R}^2} P^{A(0)}(x) P^{M(\tau)}(y) |\psi\rangle |\xi\rangle = \sum_{(x,y) \in S} c_{x,y} |A(0) = x, M(\tau) = y\rangle. \quad (54)$$

Thus, the joint probability distribution  $\Pr\{A(0) = x, M(\tau) = y\}$  is well defined as

$$\Pr\{A(0) = x, M(\tau) = y\} = \|P^{A(0)} P^{M(\tau)}(x) |\psi\rangle |\xi\rangle\|^2. \quad (55)$$

From the second equality in Eq. (53), we have the value reproducibility condition,  $\Pr\{A(0) = x, M(\tau) = y\} = 0$  if  $x \neq y$ .  $\square$

## Proof of Theorem 3

**Proof** The proofs for the implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) have been given in the proofs of Theorems 1 and 2 as discussed after the proof of Theorem 2, and hence it suffices to prove the implication (iii) $\Rightarrow$ (i). From (iii) we have

$$\langle \psi | P^A(x) | \psi \rangle = \Pr\{A(0) = x\} = \sum_y \Pr\{A(0) = x, M(\tau) = y\} = |c_{x,x}|^2, \quad (56)$$

$$\langle \psi, \xi | P^{M(\tau)}(x) | \psi, \xi \rangle = \Pr\{M(\tau) = x\} = \sum_y \Pr\{A(0) = y, M(\tau) = x\} = |c_{x,x}|^2, \quad (57)$$

and hence the probability reproducibility condition (i) follows.  $\square$

## Proof of Theorem 4

**Proof** In the “Results” section it has been shown that every observable is value reproducibly measurable. It suffices to show the converse. Let  $|\psi\rangle \in \mathcal{H}$  be a state vector. Let  $A$  be a generalized observable on  $\mathcal{H}$ , which is value reproducibly measurable with a measuring process  $(\mathcal{K}, |\xi\rangle, U(\tau), M)$  satisfying Eq. (22) if  $x \neq y$ . By the Naimark-Holevo dilation theorem<sup>26</sup>, there exist a Hilbert space  $\mathcal{G}$ , a state vector  $|\eta\rangle \in \mathcal{G}$ , and an observable  $B$  on  $\mathcal{G} \otimes \mathcal{H}$  such that

$$P^A(x) = \langle \eta | P^B(x) | \eta \rangle \quad (58)$$

for all  $x$ . Let  $B(0) = B \otimes I_{\mathcal{K}}$  and  $N(\tau) = I_{\mathcal{G}} \otimes M(\tau)$ . Let  $|\eta_1\rangle, |\eta_2\rangle, \dots$  be an orthonormal basis of  $\mathcal{G}$  such that  $|\eta\rangle = |\eta_1\rangle$ . Let  $|\psi_1\rangle, |\psi_2\rangle, \dots$  be an orthonormal basis of  $\mathcal{H}$  such that  $|\psi\rangle = |\psi_1\rangle$ . Let  $|\xi_1\rangle, |\xi_2\rangle, \dots$  be an orthonormal basis of  $\mathcal{K}$  such that  $|\xi\rangle = |\xi_1\rangle$ . Then, we have

$$\langle \eta_j, \psi_k, \xi_l | P^{N(\tau)}(y) | \eta, \psi, \xi \rangle = 0 \quad (59)$$

if  $j \neq 1$ . Thus, we have

$$\begin{aligned} \langle \eta, \psi, \xi | P^{B(0)}(x) P^{N(\tau)}(y) | \eta, \psi, \xi \rangle &= \sum_{j,k,l} \langle \eta, \psi, \xi | P^{B(0)}(x) | \eta_j, \psi_k, \xi_l \rangle \langle \eta_j, \psi_k, \xi_l | P^{N(\tau)}(y) | \eta, \psi, \xi \rangle \\ &= \sum_{k,l} \langle \eta, \psi, \xi | P^{B(0)}(x) | \eta, \psi_k, \xi_l \rangle \langle \eta, \psi_k, \xi_l | P^{N(\tau)}(y) | \eta, \psi, \xi \rangle \\ &= \sum_{k,l} \langle \psi, \xi | P^{A(0)}(x) | \psi_k, \xi_l \rangle \langle \psi_k, \xi_l | P^{M(\tau)}(y) | \psi, \xi \rangle \\ &= \langle \psi, \xi | P^{A(0)}(x) P^{M(\tau)}(y) | \psi, \xi \rangle. \end{aligned} \quad (60)$$

Let  $|\Psi\rangle = |\eta\rangle |\psi\rangle |\xi\rangle$ . From Eq. (60) and the value reproducibility of  $A$ , we have

$$\langle \Psi | P^{B(0)}(x) P^{N(\tau)}(y) | \Psi \rangle = 0. \quad (61)$$

if  $x \neq y$ . It follows that

$$\langle \Psi | P^{B(0)}(x) P^{N(\tau)}(x) | \Psi \rangle = \| P^{B(0)}(x) | \Psi \rangle \|^2 = \| P^{N(\tau)}(x) | \Psi \rangle \|^2, \quad (62)$$

and hence

$$\| P^{B(0)}(x) | \Psi \rangle - P^{N(\tau)}(x) | \Psi \rangle \|^2 = \| P^{B(0)}(x) | \Psi \rangle \|^2 + \| P^{N(\tau)}(x) | \Psi \rangle \|^2 - 2 \operatorname{Re} \langle \Psi | P^{B(0)}(x) P^{N(\tau)}(x) | \Psi \rangle = 0 \quad (63)$$

Therefore, we obtain

$$P^{B(0)}(x) | \eta \rangle | \psi \rangle | \xi \rangle = P^{N(\tau)}(x) | \eta \rangle | \psi \rangle | \xi \rangle, \quad (64)$$

so that

$$P^{A(0)}(x) | \psi \rangle | \xi \rangle = \langle \eta | P^{B(0)}(x) | \eta \rangle | \psi \rangle | \xi \rangle = \langle \eta | P^{N(\tau)}(x) | \eta \rangle | \psi \rangle | \xi \rangle = P^{M(\tau)}(x) | \psi \rangle | \xi \rangle. \quad (65)$$

Since  $|\psi\rangle$  is arbitrary, replacing  $|\psi\rangle$  in Eq. (65) by  $P^A(x) | \psi \rangle / \| P^A(x) | \psi \rangle \|$  if  $P^A(x) | \psi \rangle \neq 0$ , we have

$$P^{A(0)}(x) (P^A(x) | \psi \rangle) | \xi \rangle = P^{M(\tau)}(x) (P^A(x) | \psi \rangle) | \xi \rangle, \quad (66)$$

which holds even if  $P^A(x) | \psi \rangle = 0$ . From Eq. (65) and Eq. (66), we have

$$\begin{aligned} P^{A(0)}(x)^2 | \psi \rangle | \xi \rangle &= P^{A(0)}(x) (P^A(x) | \psi \rangle) | \xi \rangle = P^{M(\tau)}(x) (P^A(x) | \psi \rangle) | \xi \rangle = P^{M(\tau)}(x) P^{A(0)}(x) | \psi \rangle | \xi \rangle \\ &= P^{M(\tau)}(x)^2 | \psi \rangle | \xi \rangle = P^{M(\tau)}(x) | \psi \rangle | \xi \rangle = P^{A(0)}(x) | \psi \rangle | \xi \rangle. \end{aligned} \quad (67)$$

Therefore, we have

$$P^A(x)^2 | \psi \rangle = P^A(x) | \psi \rangle \quad (68)$$

for all state vector  $|\psi\rangle \in \mathcal{H}$ , so that  $A$  is an observable.  $\square$

## Data availability

No datasets were generated or analyzed during the current study.

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## Author contributions

The author researched, collated, and wrote this paper.

## Competing interests

The author declares no competing interests.

## Additional information

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