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Analysis of a class of fractal hybrid fractional differential equation with application to a biological model

Thabet Abdeljawad^{1,2,5}, Muhammad Sher³, Kamal Shah^{1,3}, Muhammad Sarwar^{1,3}, Inas Amacha²✉, Manar Alqudah⁴ & Asma Al-Jaser⁴

Recently, the area devoted to fractional calculus has given much attention by researchers. The reason behind such huge attention is the significant applications of the mentioned area in various disciplines. Different problems of real world processes have been investigated by using the concepts of fractional calculus and important and applicable outcomes were obtained. Because, there has been a lot of interest in fractional differential equations. It is brought on by both the extensive development of fractional calculus theory and its applications. The use of linear and quadratic perturbations of nonlinear differential equations in mathematical models of a variety of real-world problems has received a lot of interest. Therefore, motivated by the mentioned importance, this research work is devoted to analyze in detailed, a class of fractal hybrid fractional differential equation under Atangana- Baleanu- Caputo ABC derivative. The qualitative theory of the problem is examined by using tools of non-linear functional analysis. The Ulam-Hyer's (U-H) type stability criteria is also applied to the consider problem. Further, the numerical solution of the model is developed by using powerful numerical technique. Lastly, the Wazewska-Czyzewska and Lasota Model, a well-known biological model, verifies the results. Several graphical representations by using different fractals fractional orders values are presented. The detailed discussion and explanations are given at the end.

Keywords Fractal Fractional derivative, Existence theory, U-H stability, Numerical results

Fractional calculus (FC) is an important discipline of mathematics which is mainly investigate non-integer order integrals and derivatives. For the first time in history FC was discussed by two great mathematician Leibnitz and L. Hospital by asking half order derivative of a function¹. Because of its complexity researchers did not study the subject at that time. With the passage of time when the technology get advancement and new function have been introduced, the subject got proper attention from the mathematician it deserved. Currently, FC has attracted attention of many researchers in the fields of science and engineering. For example author have used FC for the solutions of problem in signal processing², complex systems related to control theory³, diffusion problem in physics⁴, problem related economic terms like interest rates, commodity prices and dynamics of market fluctuations⁵ and accurate dimensional analysis of image processing⁶. Many applications of FC can also be found in biology, chemistry and other diverse fields of sciences. For instance, authors⁶ studied a super-twisting sliding mode control of robotic manipulator using the concepts of FC. Shah, et.al⁷ studied an epidemic model of dengue fever disease using fractional non-singular derivative. Ahmad, et.al⁸ investigated an adaptive control of a robotic manipulator with a delay in the input by applying the fractional order derivative. Shaikh, et.al⁹ conducted a detailed mathematical analysis of HIV/AIDS disease by using non-singular type derivative. In the same way, authors¹⁰ applied FC concepts to study mathematical of COVID-19 using real data of Nigeria. Proceeding with the mentioned process, researchers¹¹ studied mathematical model for the evolution of COVID-19 outbreak in India.

¹Department of Mathematics and Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia. ²Department of Medical Research, China Medical University, Taichung 40402, Taiwan. ³Department of Mathematics, University of Malakand, Chakdara Dir(L), 18000 Khyber Pakhtunkhwa, Pakistan. ⁴Department of Mathematical Science, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia. ⁵Department of Mathematics and Applied Mathematics, School of Science and Technology, Sefako Makgatho Health Sciences University, Ga-Rankuwa, South Africa. ✉email: amasha.inas@gmail.com

One of the interesting property of fractional derivative is that it does not have a unique definition. Various definitions, like the Riemann-Liouville (R-L) and the Caputo derivatives¹², the conformable derivative¹³, the Hilfer derivative¹⁴, the Harmard derivative¹⁵ and some other are available in literature. These ideas have recently been used as the foundation for breaking down numerous mathematical problems, we refer to¹⁶, and¹⁷. Problem related to heterogeneities in materials can not be studied using the above mentioned concepts¹⁸. Researchers¹⁹ used modified Caputo definition by replacing the singular kernel by non-singular kernel and named the new operator is Caputo-Fabrizio derivative(CFD). Authors¹⁸ studied some important properties of the aforesaid operators. Many problems have been analyzed by using mentioned derivative for existence theory and introduced various properties for the mentioned operator. For some recent work on the mentioned topic, we refer²⁰, and²¹ and references therein. In CFD researchers identified property of locality in the kernel used in the operator. To overcome this difficulty, Atangana and Baleanu²² in 2016 replaced the kernel of exponential by Mittag-Leffler in the CFD. The newly defined definition was called ABC derivative and applied by different authors for the solution of real world problem. For instance, Xu et al.²³ discussed HIV-1 model, authors²⁴ explained in details the Klein-Gordan equation using ABC differential operators. Using ABC derivative with fractional order, authors²⁵ computed the numerical results for parabolic partial differential equations. In additional, researchers²⁶ applied ZZ Transform to compute the approximate solution of Fokker Plank equations using fuzzy concepts with ABC derivative.

Furthermore, fractional differential operators, as previously stated, are insufficient to adequately explain problems involving irregular forms and geometry. To describe these kinds of behaviours, the fractal notion has been introduced in the literature. Historically, fractal was mainly used from the 17th century and there is a disagreement among mathematicians on how to define fractals formally. Weierstrass in 1872 presented a function with graph which nowadays called fractal. Different mathematician define fractal in different ways like a fractal is a set for which the Hausdorff-Besicovitch dimension strictly exceeds the topological dimension. In other words, fractal is a fragmented geometrical shapes that may be split into figments, each figment represent a copy of the whole shape. Because they appear in the geometric representations of the majority of chaotic processes, fractals have thus been noted by authors as being significant to the theory of chaos. Atangana²⁷ has made a significant contribution to this field by bridging the gap between FC and fractal calculus. The aforementioned scholar established numerous definitions that are now widely used and made a substantial contribution to the subject of fractals fractional theory. Newly proposed operators can help to explain fractal behaviour of complicated physical systems and nonlocal phenomena. For instant, Hu and He²⁸ explained fractals space time and dimensions. In addition, authors²⁹. Furthermore, Qureshi and Atangana³⁰ have studied diarrheal illness models using fractal operators. In the same way, researchers³¹ studied mathematical model of Ebola virus disease using the the mentioned operator. The use of fractal theory to issues including fractal heat exchangers, heat transport in porous media, etc., is growing. Here is where some outstanding accomplishments related to the properties and uses of fractals in porous media should be mentioned. The authors investigated a fractal model for capillary flow through a single tortuous capillary with roughened surfaces in fibrous porous media³². An analytical model for the transverse permeability of the gas diffusion layer in proton exchange membrane fuel cells with electrical double layer effects was carried out by Liang et al.³³. Yu, et.al³⁴ investigated the Characterization of the behaviour of water migration during spontaneous imbibition in coal by using fractals fractional model. Recently, Ahmad, et.al³⁵ studied a malaria disease mathematical model using fractals fractional concepts. Author³⁶ investigated a dynamical problem by using generalized fractal-fractional derivative with Mittag-Leffler kernel. Recently, authors³⁷ have used fractals-fractional derivative to study re-infection model of COVID-19. Author³⁸ studied a class of hybrid problems by using confirmable fractal fractional derivative. Researchers³⁹ have studied a mathematical model by using ABC fractals fractional derivative.

Delay differential equations (DDEs) are a unique class of differential equations where the unknown function depends on both its past and present times. The physical phenomenon has a time delay as a result of these issues. DDEs have been used in a number of disciplines, most notably computational physics and electrodynamics. It has also been successfully used to challenges in the domains of economics, chemistry, engineering, and infectious diseases (delay can be added for the time a disease takes to display its symptoms). It is very helpful for modelling issues with time delay systems or memory effects. Examples include population dynamics with time delays in birth or death rates, control systems with communication delays, and chemical reactions with delayed consequences. Many scholars have devoted a tremendous amount of time to finding solutions to delay-type problems. Authors⁴⁰ explained how DDEs are using in mathematical modeling of life sciences phenomenon. Author⁴¹ gave a comprehensive analysis of DDEs. The qualitative analysis of DDEs related to the existence theory of solutions and their properties, we refer to⁴². Recently, authors⁴³ have published a detailed work on computations of numerical results for fractals fractional DDEs.

Investigating the existence literature, we found that very rare work has been done for the class of hybrid fractals HFDEs. Recently, Shafiqullah, et.al⁴⁴ deduced some computational and theoretical results for a class of fractals HFDEs involving power law kernel by using fixed point analysis and numerical algorithm. But the mentioned area still has not explored very well under various fractals fractional differential operators. Therefore, it was needed to investigate a class of fractals HFDEs under the Mittag-Leffler type kernel. Hence, motivated by the above discussion, in this study a class of fractal HFDEs of linear perturbation type is selected for qualitative analysis and approximate solution. Here we remark that it is possible to model and describe non-homogeneous physical processes that occur in their form using HFDEs. Because the mentioned area incorporate different dynamical systems as specific examples. This category of differential equations comprises the hybrid derivative of an unknown function that depends on nonlinearity. Here a continuous function for fractals fractional derivatives and a piecewise continuous or discrete function needs for fractals fractional integration and such combination make an hybrid problem. For some numerous applications of the said area, we refer to⁴⁵.

To the best of our knowledge no one has considered the general problem defined as follow:

$$\begin{cases} {}^{FAB}D_{\theta}^{\sigma, \xi} [\Phi(\theta) - \mathbf{f}(\theta, \Phi(\theta))] = \mathbf{g}(\theta, \Phi(\lambda\theta)), \theta \in I = [0, T], \\ \Phi(0) = \Phi_0 + \mathbf{h}(\theta, \Phi(\theta)), \end{cases} \quad (1)$$

where $0 < \sigma \leq 1$, $0 < \lambda < 1$, $\mathbf{f}, \mathbf{g} \in C[I \times R, R]$. Proportional delay term is shown by the symbol λ . Pantograph equations are differential equations that have proportional delay terms. Hence, special kind of functional differential equation with proportional delay is known as the pantograph equation. It appears in a variety of pure and practical mathematics domains, including probability, number theory, quantum physics, electrodynamics, and control systems. Certain kinds of the said equations are using in modeling various process of astrophysics. In the concerned problem (1), if $\lambda = 1$, the said equation will reduce to usual fractals HFDE. If $\lambda > 1$, the said problem (1) become illposed. The detail theory about the pantograph equations and various applications were discussed in⁴⁶. Here ξ stands for fractal dimension. Additionally, let $\xi = 1$ then, the fractional differential operator that is typically used becomes the considered operator. In past, hybrid differential equations have been studied by using ordinary or usual fractional derivatives. But investigation of aforementioned problems by using fractals fractional derivatives were found very rare in literature. On the other hand, the concerned area has numerous applications in modelling real world problems. Therefore, it was needed to investigate hybrid problems by using the fractals fractional derivatives. Stability theory is an important aspects of the qualitative theory which play important role in constructing various numerical schemes. Researchers have studied various kinds of stabilities for different problems. In this research work, we are interested to investigate the Ulam-Hyers (U-H) stability theory for our considered problem. The aforementioned stability was introduced by Ulam⁴⁷ and explained by Hyers⁴⁸. In addition, further the aforesaid stability was utilized for various functional problems by Rassias⁴⁹. Recently, the mentioned stability was investigated for the solution of stochastic⁵⁰, impulsive⁵¹⁻⁵³, and piecewise⁵⁴ fractional differential problems. By using hybrid fixed point theory, we will establish some important results related to existence theory and stability theory to the mentioned problem. Then by using Lagrange's interpolation method a general numerical scheme will be developed. At the final stage, an examples from biological sciences is given to elaborate our theoretical and numerical results. Here, we remark that numerical analysis under fractals fractional concepts for hybrid problems has not well studied yet.

Preliminaries

Some basic results are recalled from^{12,22,28,30}. Let, $J = [0, T]$, and $\Omega = C(J, R)$ stands for Banach space endowed with norm $\|\cdot\|$ on Ω as follow:

$$\|\Phi\| = \sup_{\theta \in J} |\Phi(\theta)|.$$

Definition 2.1 The R-L fractional integral of order $0 < \sigma \leq 1$ is defined as follows:

$${}^{RL}I^{\sigma} \Phi(\theta) = \int_0^{\theta} (\theta - s)^{\sigma-1} \Phi(s) \frac{1}{\Gamma(\sigma)} ds, \quad \theta \in [0, T].$$

Definition 2.2 ABC type arbitrary order integral with order $\sigma > 0$ is defined as follow:

$${}^{ABC}I_{0+}^{\sigma} \Phi(\theta) = \frac{1-\sigma}{M(\sigma)} \Phi(\theta) + \frac{\sigma}{\Gamma(\sigma)M(\sigma)} \int_0^{\theta} (\theta - s)^{\sigma-1} \Phi(s) dx,$$

such that the right side exists.

Definition 2.3 ABC type arbitrary order derivative of order σ ($0 < \sigma \leq 1$) is defined as follow:

$${}^{ABC}D_0^{\sigma} \Phi(\theta) = \frac{M(\sigma)}{(1-\sigma)} \int_0^{\theta} E_{\sigma} \left[-\frac{\sigma}{1-\sigma} (\theta - s)^{\sigma} \right] \Phi'(s) dx.$$

The functions $M(\sigma)$ in the definition is called normalization obeys $M(0) = M(1) = 1$, and E_{σ} is Mittag-Leffler function.

Definition 2.4 Fractal type arbitrary order integral of order $\sigma > 0$ is defined as follow:

$${}^FI_{0+}^{\sigma} \Phi(\theta) = \sigma \int_0^{\theta} s^{\sigma-1} \Phi(s) ds. \quad (2)$$

Definition 2.5 Fractal fractional ABC integral with fractals order $\xi \in (0, 1]$, and fractional order $\sigma \in (0, 1]$ is defined as follow:

$${}^{FAB}I_{0+}^{\sigma, \xi} \Phi(\theta) = \frac{\xi(1-\sigma)\theta^{\xi-1}}{M(\sigma)} \Phi(\theta) + \frac{\sigma\xi}{\Gamma(\sigma)M(\sigma)} \int_0^{\theta} (\theta - s)^{\sigma-1} s^{\xi-1} \Phi(s) ds.$$

Definition 2.6 Fractal fractional ABC derivative with fractals order $\xi \in (0, 1]$, and fractional order $\sigma \in (0, 1]$ is defined as follow:

$${}^{FAB}D_0^{\sigma,\xi} \Phi(\theta) = \frac{M(\sigma)}{(1-\sigma)} \int_0^\theta E_\sigma[-\frac{\sigma}{1-\sigma}(\theta-s)^\sigma] \frac{du(s)}{ds^\xi} ds.$$

Lemma 2.7 *If $x \in L[0, T]$ and $x(0) = 0$, then the solution of*

$$\begin{aligned} {}^{FAB}D_0^{\sigma,\xi} \Phi(\theta) &= x(\theta), \text{ with } \sigma \in (0, 1], \\ \Phi(0) &= \Phi_0, \end{aligned}$$

is described as follows:

$$\Phi(\theta) = \Phi_0 + \frac{\xi(1-\sigma)\theta^{\xi-1}}{M(\sigma)} x(\theta) + \frac{\sigma\xi}{\Gamma(\sigma)M(\sigma)} \int_0^\theta (\theta-s)^{\sigma-1} s^{\xi-1} x(s) ds.$$

Existence theory

In this part, qualitative theory of problem (1) will be discussed.

Lemma 3.1 *Let $\mathbf{f}(\theta, \Phi(\theta))$, and $\mathbf{g}(\theta, \Phi(\lambda\theta))$ be continuous functions on Ω , then the solution of fractal-fractional differential equation described by*

$$\begin{cases} {}^{FAB}D_0^{\sigma,\xi} [\Phi(\theta) - \mathbf{f}(\theta, \Phi(\theta))] = \mathbf{g}(\theta, \Phi(\lambda\theta)), \theta \in I = [0, T], \\ \Phi(0) = \Phi_0 + \mathbf{h}(\theta, \Phi(\theta)) \end{cases} \tag{3}$$

is given by

$$\Phi(\theta) = \begin{cases} \Phi_0 + \mathbf{h}(\theta, \Phi(\theta)) + \mathbf{f}(\theta, \Phi(\theta)) + \frac{\xi(1-\sigma)\theta^{\xi-1}}{M(\sigma)} \mathbf{g}(\theta, \Phi(\lambda\theta)) \\ + \frac{\sigma\xi}{\Gamma(\sigma)M(\sigma)} \int_0^\theta (\theta-s)^{\sigma-1} s^{\xi-1} \mathbf{g}(s, \Phi(\lambda s)) ds. \end{cases} \tag{4}$$

Proof Using Lemma 2.7, we get the solution as follows:

$$\Phi(\theta) = \begin{cases} \Phi_0 + \mathbf{h}(\theta, \Phi(\theta)) + \mathbf{f}(\theta, \Phi(\theta)) + \frac{\xi(1-\sigma)\theta^{\xi-1}}{M(\sigma)} \mathbf{g}(\theta, \Phi(\lambda\theta)) \\ + \frac{\sigma\xi}{\Gamma(\sigma)M(\sigma)} \int_0^\theta (\theta-s)^{\sigma-1} s^{\xi-1} \mathbf{g}(s, \Phi(\lambda s)) ds. \end{cases} \tag{5}$$

□

We define the operator as follows: $\mathbf{Z} : \Omega \rightarrow \Omega$ by:

$$\mathbf{Z}\Phi(\theta) = \begin{cases} \Phi_0 + \mathbf{h}(\theta, \Phi(\theta)) + \mathbf{f}(\theta, \Phi(\theta)) + \frac{\xi(1-\sigma)\theta^{\xi-1}}{M(\sigma)} \mathbf{g}(\theta, \Phi(\lambda\theta)) \\ + \frac{\sigma\xi}{\Gamma(\sigma)M(\sigma)} \int_0^\theta (\theta-s)^{\sigma-1} s^{\xi-1} \mathbf{g}(s, \Phi(\lambda s)) ds. \end{cases} \tag{6}$$

Divide the above operator Eq. (6) into two sub operators as follow:

$$\mathbf{Z}_1\Phi(\theta) = \Phi_0 + \mathbf{h}(\theta, \Phi(\theta)) + \mathbf{f}(\theta, \Phi(\theta)) + \frac{\xi(1-\sigma)\theta^{\xi-1}}{M(\sigma)} \mathbf{g}(\theta, \Phi(\lambda\theta)) \tag{7}$$

and

$$\mathbf{Z}_2\Phi(\theta) = \frac{\sigma\xi}{\Gamma(\sigma)M(\sigma)} \int_0^\theta (\theta-s)^{\sigma-1} s^{\xi-1} \mathbf{g}(s, \Phi(\lambda s)) ds. \tag{8}$$

From Eq.(7) and Eq.(8), $\mathbf{Z}(\Phi)$ can be written as follows:

$$\mathbf{Z}(\Phi) = \mathbf{Z}_1(\Phi) + \mathbf{Z}_2(\Phi). \tag{9}$$

For additional investigation, the ensuing presumptions are required.

A₁ There is a constant $K_1 > 0$, such that

$$|\mathbf{h}(\theta, \Phi(\theta)) - \mathbf{h}(\theta, \bar{\Phi}(\theta))| \leq K_1 |\Phi(\theta) - \bar{\Phi}(\theta)|.$$

A₂ One has a constant $K_2 > 0$, such that

$$|f(\theta, \Phi(\theta)) - f(\theta, \bar{\Phi}(\theta))| \leq K_2 |\Phi(\theta) - \bar{\Phi}(\theta)|.$$

A₃ One has a constant $K_3 > 0$, such that

$$|g(\theta, \Phi(\theta)) - g(\theta, \bar{\Phi}(\theta))| \leq K_3 |\Phi(\theta) - \bar{\Phi}(\theta)|.$$

A₄ Corresponding to function $g(\theta, \Phi(\theta))$, $\exists K_4 > 0$, and $K_5 > 0$, which yield

$$\|g\| \leq K_4 \|\Phi\| + K_5.$$

Theorem 3.2 Our suggested problem Eq.(1) has one solution at most under the assumptions **A₁ – A₄**, provided that $\mathbf{K} = \left(K_1 + K_2 + \frac{K_3 \xi (1-\sigma) T^{\xi-1}}{M(\sigma)} \right) < 1$.

Proof Here it is needed to prove that Z_1 is a contraction and the operator Z_2 is equi-continuous in order to verify the aforementioned claim. $\mathbf{D} = \{U \in B : \|\Phi\| \leq \sigma\}$ is a bounded, closed, and convex subset of B for this define. It's obvious that Z_1 is continuous. Assuming that $\Phi, \bar{\Phi} \in \mathbf{D}$, taking

$$\begin{aligned} \|Z_1(\Phi) - Z_1(\bar{\Phi})\| &= \sup_{\theta \in [0, T]} \left\{ \left| \Phi_0 + h(\theta, \Phi(\theta)) + f(\theta, \Phi(\theta)) + \frac{\xi(1-\sigma)\theta^{\xi-1}}{M(\sigma)} g(\theta, \Phi(\lambda\theta)) \right. \right. \\ &\quad \left. \left. - \left(\bar{\Phi}_0 + h(\theta, \bar{\Phi}(\theta)) + f(\theta, \bar{\Phi}(\theta)) + \frac{\xi(1-\sigma)\theta^{\xi-1}}{M(\sigma)} g(\theta, \bar{\Phi}(\lambda\theta)) \right) \right| \right\}, \\ &\leq \sup_{\theta \in [0, T]} \left\{ |h(\theta, \Phi(\theta)) - h(\theta, \bar{\Phi}(\theta))| + |f(\theta, \Phi(\theta)) - f(\theta, \bar{\Phi}(\theta))| \right. \\ &\quad \left. + \frac{\xi(1-\sigma)\theta^{\xi-1}}{M(\sigma)} |g(\theta, \Phi(\lambda\theta)) - g(\theta, \bar{\Phi}(\lambda\theta))| \right\}, \\ &\leq \sup_{\theta \in [0, T]} \left\{ K_1 |\Phi(\theta) - \bar{\Phi}(\theta)| + K_2 |\Phi(\theta) - \bar{\Phi}(\theta)| \right. \\ &\quad \left. + \frac{K_3 \xi (1-\sigma) \theta^{\xi-1}}{M(\sigma)} |\Phi(\lambda\theta) - \bar{\Phi}(\lambda\theta)| \right\}, \\ &\leq \left[K_1 + K_2 + \frac{K_3 \xi (1-\sigma) T^{\xi-1}}{M(\sigma)} \right] \|\Phi - \bar{\Phi}\|, \\ &= \mathbf{K} \|\Phi - \bar{\Phi}\|. \end{aligned}$$

As a result, Z_1 is contracted. Taking $\Phi \in \mathbf{D}$, we can now demonstrate that Z_2 is equi-continuous.

$$\begin{aligned} \|Z_2(\Phi)\| &= \sup_{\theta \in [0, T]} \left\{ \left| \frac{\sigma \xi}{\Gamma(\sigma)M(\sigma)} \int_0^\theta (\theta - s)^{\sigma-1} s^{\xi-1} g(s, \Phi(\lambda s)) ds \right| \right\}, \\ &\leq \frac{\sigma \xi \|g\|}{M(\sigma)} \sup_{\theta \in [0, T]} \left\{ \left| \int_0^\theta \frac{(\theta - s)^{\sigma-1}}{\Gamma(\sigma)} s^{\xi-1} ds \right| \right\}, \\ &\leq \left(\frac{\sigma \Gamma(\xi + 1) T^{\sigma+\xi-1}}{M(\sigma) \Gamma(\sigma + \xi)} \right) \|g\|, \\ &< \infty. \end{aligned}$$

As a result, $Z_2(\Phi)$ has a bound. $Z_2(\Phi)$ is continuous, just as f . Additionally, if $\theta_1 < \theta_2 \in [0, T]$,

$$\begin{aligned} \|Z_2\Phi(\theta_2) - Z_2\Phi(\theta_1)\| &= \sup_{\theta_1, \theta_2 \in [0, T]} \left\{ \left| \frac{\sigma \xi}{\Gamma(\sigma)M(\sigma)} \int_0^{\theta_2} (\theta_2 - s)^{\sigma-1} s^{\xi-1} g(s, \Phi(\lambda s)) ds \right. \right. \\ &\quad \left. \left. - \int_0^{\theta_1} (\theta_1 - s)^{\sigma-1} s^{\xi-1} g(s, \Phi(\lambda s)) ds \right| \right\}, \\ &\leq \frac{\sigma \xi \|g\|}{\Gamma(\sigma)M(\sigma)} \left| \int_0^{\theta_2} (\theta_2 - s)^{\sigma-1} s^{\xi-1} ds \right. \\ &\quad \left. - \int_0^{\theta_1} (\theta_1 - s)^{\sigma-1} s^{\xi-1} ds \right| \\ &\leq \frac{\sigma \Gamma(\xi + 1) \|g\|}{M(\sigma) \Gamma(\sigma + \xi)} \left(\theta_2^{\sigma+\xi-1} - \theta_1^{\sigma+\xi-1} \right) \rightarrow 0, \text{ as } \theta_1 \rightarrow \theta_2. \end{aligned}$$

As a result, Z_2 is bounded and equi-continuous. Since the aforementioned operator is compact, it can be made reasonably compact and consequently entirely continuous by applying the Arzelá-Ascoli theorem. Schauder's fixed point theorem thus guarantees that Z_2 has a minimum of one fixed point. Therefore, there is at least one solution to problem Eq. (1). □

Theorem 3.3 Assuming $A_1 - A_3$, there exists a unique solution to our problem if the condition

$$K' = \left[K_1 + K_2 + \left(\frac{K_3 T^{\xi-1}}{M(\sigma)} \right) \left(\xi(1 - \sigma) + \frac{\sigma \Gamma(\xi + 1) T^\sigma}{\Gamma(\sigma + \xi)} \right) \right] < 1,$$

holds.

Proof Take $\Phi, \bar{\Phi} \in D$, we have

$$\begin{aligned} \|Z(\Phi) - Z(\bar{\Phi})\| &\leq \|Z_1(\Phi) - Z_1(\bar{\Phi})\| + \|Z_2(\Phi) - Z_2(\bar{\Phi})\|, \\ &\leq \sup_{\theta \in [0, T]} \left\{ \left| \Phi_0 + \mathbf{h}(\theta, \Phi(\theta)) + \mathbf{f}(\theta, \Phi(\theta)) + \frac{\xi(1 - \sigma)\theta^{\xi-1}}{M(\sigma)} \mathbf{g}(\theta, \Phi(\lambda\theta)) \right. \right. \\ &\quad \left. \left. - \left(\bar{\Phi}_0 + \mathbf{h}(\theta, \bar{\Phi}(\theta)) + \mathbf{f}(\theta, \bar{\Phi}(\theta)) + \frac{\xi(1 - \sigma)\theta^{\xi-1}}{M(\sigma)} \mathbf{g}(\theta, \bar{\Phi}(\lambda\theta)) \right) \right| \right\} \\ &\quad + \frac{\sigma \xi}{\Gamma(\sigma)M(\sigma)} \sup_{\theta \in [0, T]} \left\{ \left| \int_0^\theta (\theta - s)^{\sigma-1} s^{\xi-1} \mathbf{g}(s, \Phi(\lambda s)) ds \right. \right. \\ &\quad \left. \left. - \left(\int_0^\theta (\theta - s)^{\sigma-1} s^{\xi-1} \mathbf{g}(s, \bar{\Phi}(\lambda s)) ds \right) \right| \right\}, \\ &\leq \sup_{\theta \in [0, T]} \left[K_1 |\Phi(\theta) - \bar{\Phi}(\theta)| + K_2 |\Phi(\theta) - \bar{\Phi}(\theta)| \right. \\ &\quad \left. + \frac{\xi(1 - \sigma)\theta^{\xi-1}}{M(\sigma)} K_3 |\Phi(\theta) - \bar{\Phi}(\lambda\theta)| \right] \\ &\quad + \frac{\sigma \xi K_3}{\Gamma(\sigma)M(\sigma)} \left[\int_0^\theta (\theta - s)^{\sigma-1} s^{\xi-1} |\Phi(\theta) - \bar{\Phi}(\lambda\theta)| ds \right], \\ &\leq \left(K_1 + K_2 + \frac{K_3 \xi(1 - \sigma) T^{\xi-1}}{M(\sigma)} \right) \|\Phi - \bar{\Phi}\| \\ &\quad + \frac{\sigma \Gamma(\xi + 1) K_3 T^{\sigma+\xi-1}}{M(\sigma)\Gamma(\sigma + \xi)} \|\Phi - \bar{\Phi}\|, \\ &= K' \|\Phi - \bar{\Phi}\|, \end{aligned}$$

where

$$K' = \left[K_1 + K_2 + \left(\frac{K_3 T^{\xi-1}}{M(\sigma)} \right) \left(\xi(1 - \sigma) + \frac{\sigma \Gamma(\xi + 1) T^\sigma}{\Gamma(\sigma + \xi)} \right) \right].$$

So, our problem under consideration has a unique solution according to Banach's contraction principle. \square

UH stability

In this section, we will establish the conditions required for U-H type stability. The numerical solution of a problem depends on this kind of stability. What conditions allow the approximation of a problem to be almost equal to the exact answer is the fundamental question about an approximation. In the 1940s, Ulam offered a thorough analysis of this important topic (see⁴⁷). The stability that was previously discussed was extended and generalized by Hyer and Rassias to become generalized U-H and U-H-Rassias stability (see^{48,49}).

Examine the matching perturbation issue of Eq. (1) as

$$\begin{cases} {}^{FAB}D_{\theta^{\sigma,\xi}}[\Phi(\theta) - \mathbf{f}(\theta, \Phi(\theta))] = \mathbf{g}(\theta, \Phi(\lambda\theta)) + \mathbf{h}(\theta), \theta \in I = [0, T], \\ \Phi(0) = \Phi_0 + \mathbf{h}(\theta, \Phi(\theta)). \end{cases} \tag{10}$$

Here, $\mathbf{h}(\theta) \in C([0, T], R)$, \exists , for $\epsilon > 0$, and $|\mathbf{h}(\theta)| \leq \epsilon$. Corresponding to any solution Φ of problem Eq. (10), one has

$$Z(\Phi(\theta)) = \Phi(\theta) + \mathbf{h}(\theta). \tag{11}$$

The solution of Eq.(10) is given as follows:

$$\Phi(\theta) = \begin{cases} \Phi_0 + \mathbf{h}(\theta, \Phi(\theta)) + \mathbf{f}(\theta, \Phi(\theta)) + \frac{\xi(1 - \sigma)\theta^{\xi-1}}{M(\sigma)} [\mathbf{g}(\theta, \Phi(\lambda\theta)) + \mathbf{h}(\theta)] \\ + \frac{\sigma \xi}{\Gamma(\sigma)M(\sigma)} \int_0^\theta (\theta - s)^{\sigma-1} s^{\xi-1} [\mathbf{g}(s, \Phi(\lambda s)) + \mathbf{h}(s)] ds. \end{cases} \tag{12}$$

Using Eq. (6), and Eq. (12), we have

$$|Z(\Phi(\theta)) - \Phi(\theta)| \leq \left[\frac{\xi T^{\xi-1}}{M(\sigma)} \left((1-\sigma) + \frac{\sigma \Gamma(\xi+1)}{\Gamma(\sigma+\xi)} T^\sigma \right) \right] \epsilon, \\ \leq \Pi \epsilon,$$

where

$$\Pi = \left[\frac{\xi T^{\xi-1}}{M(\sigma)} \left((1-\sigma) + \frac{\sigma \Gamma(\xi+1)}{\Gamma(\sigma+\xi)} T^\sigma \right) \right].$$

Theorem 4.1 System Eq. (1) has a U-H and generalized U-H stable solution if

$$K' = \left[K_1 + K_2 + \left(\frac{K_3 T^{\xi-1}}{M(\sigma)} \right) \left(\xi(1-\sigma) + \frac{\sigma \Gamma(\xi+1) T^\sigma}{\Gamma(\sigma+\xi)} \right) \right] < 1$$

holds.

Proof Given that $\bar{\Phi} \in \Omega$ represents a unique solution to problem Eq. (10), and $\Phi \in \Omega$ is any solution of problem Eq. (1), then

$$\|\Phi - \bar{\Phi}\| = \|\Phi - Z\bar{\Phi}\| \leq \|\Phi - Z\Phi\| + \|Z\Phi - Z\bar{\Phi}\| \\ \leq \Pi \epsilon + K' \|\Phi - \bar{\Phi}\| \\ \leq \frac{\Pi \epsilon}{1 - K'}.$$
(13)

Thus the solution is U-H stable. Further, let there exists a nondecreasing function say $\zeta : (0, T) \rightarrow R$, such that $\zeta(\epsilon) = \epsilon$, then from Eq.(13), we can write

$$\|\Phi - \bar{\Phi}\| \leq \frac{\Pi}{1 - K'} \zeta(\epsilon),$$
(14)

thus, Eq.(14) indicates that the solution is generalized U-H stable. □

Numerical approximation

We will use a numerical approach in this section of the manuscript to solve our considered problem. The Lagrange interpolation method will be used to build the numerical scheme, in accordance with the numerical method⁵⁵. The integral form equivalent to our suggested problem, Eq. (1), is provided by

$$\Phi(\theta) = \begin{cases} \Phi_0 + \mathbf{h}(\theta, \Phi(\theta)) + \mathbf{f}(\theta, \Phi(\theta)) + \frac{\xi(1-\sigma)\theta^{\xi-1}}{M(\sigma)} \mathbf{g}(\theta, \Phi(\lambda\theta)) \\ + \frac{\sigma \xi}{\Gamma(\sigma)M(\sigma)} \int_0^\theta (\theta-s)^{\sigma-1} s^{\xi-1} \mathbf{g}(s, \Phi(\lambda s)) ds. \end{cases}$$
(15)

Using $\theta = \theta_{n+1}$, $n = 0, 1, 2, 3, \dots$, Eq. (15) implies

$$\Phi(\theta_{n+1}) = \begin{cases} \Phi_0 + \mathbf{h}(\theta_n, \Phi(\theta_n)) + \mathbf{f}(\theta_n, \Phi(\theta_n)) + \frac{\xi(1-\sigma)\theta_n^{\xi-1}}{M(\sigma)} \mathbf{g}(\theta_n, \Phi(\lambda\theta_n)) \\ + \frac{\sigma \xi}{\Gamma(\sigma)M(\sigma)} \int_0^{\theta_n} (\theta_n-s)^{\sigma-1} s^{\xi-1} \mathbf{g}(s, \Phi(\lambda s)) ds. \end{cases}$$
(16)

Now approximating function \mathbf{g} on $[\theta_{k-1}, \theta_k]$ with $\mathbf{h} = \frac{\theta_k - \theta_{k-1}}{n}$, and using the following interpolations

$$P(\theta) = \frac{(s - \theta_k)}{\mathbf{h}} \mathbf{g}(\theta_k, \Phi(\lambda\theta_k)) - \frac{(s - \theta_{k-1})}{\mathbf{h}} \mathbf{g}(\theta_{k-1}, \Phi(\lambda\theta_{k-1})),$$
(17)

we obtain

$$\Phi(\theta_{n+1}) = \begin{cases} \Phi_0 + \mathbf{h}(\theta_n, \Phi(\theta_n)) + \mathbf{f}(\theta_n, \Phi(\theta_n)) + \frac{\xi(1-\sigma)\theta_n^{\xi-1}}{M(\sigma)} \mathbf{g}(\theta_n, \Phi(\lambda\theta_n)) \\ + \frac{\sigma\xi}{\Gamma(\sigma)M(\sigma)} \sum_{k=0}^n \left[\int_{\theta_{k-1}}^{\theta_k} (\theta_k - s)^{\sigma-1} s^{\xi-1} \frac{(s-\theta_k)}{\mathbf{h}} \mathbf{g}(\theta_k, \Phi(\lambda\theta_k)) ds \right. \\ \left. - \int_{\theta_{k-1}}^{\theta_k} (\theta_k - s)^{\sigma-1} s^{\xi-1} \frac{(s-\theta_{k-1})}{\mathbf{h}} \mathbf{g}(\theta_{k-1}, \Phi(\lambda\theta_{k-1})) ds \right]. \\ \Phi_0 + \mathbf{h}(\theta_n, \Phi(\theta_n)) + \mathbf{f}(\theta_n, \Phi(\theta_n)) + \frac{\xi(1-\sigma)\theta_n^{\xi-1}}{M(\sigma)} \mathbf{g}(\theta_n, \Phi(\lambda\theta_n)) \\ + \frac{\sigma\xi}{\Gamma(\sigma)M(\sigma)} \sum_{k=0}^n \left[\mathbf{g}(\theta_k, \Phi(\lambda\theta_k)) \int_{\theta_{k-1}}^{\theta_k} (\theta_k - s)^{\sigma-1} s^{\xi-1} \frac{(s-\theta_k)}{\mathbf{h}} ds \right. \\ \left. - \mathbf{g}(\theta_{k-1}, \Phi(\lambda\theta_{k-1})) \int_{\theta_{k-1}}^{\theta_k} (\theta_k - s)^{\sigma-1} s^{\xi-1} \frac{(s-\theta_{k-1})}{\mathbf{h}} ds \right]. \\ \Phi_0 + \mathbf{h}(\theta_n, \Phi(\theta_n)) + \mathbf{f}(\theta_n, \Phi(\theta_n)) + \frac{\xi(1-\sigma)\theta_n^{\xi-1}}{M(\sigma)} \mathbf{g}(\theta_n, \Phi(\lambda\theta_n)) \\ + \frac{\sigma\xi}{\Gamma(\sigma)M(\sigma)} \sum_{k=0}^n \left[\theta_k^{\xi-1} \mathbf{g}(\theta_k, \Phi(\lambda\theta_k))(n+1-k)^{\sigma+1}(n+2-k+\sigma) \right. \\ \left. - (-k+n)^\sigma(-k+n+2(1+\sigma)) - \theta_{k-1}^{\xi-1} \mathbf{g}(\theta_{k-1}, \Phi(\lambda\theta_{k-1}))(n+1-k)^{\sigma+1} \right. \\ \left. - (-k+n)^\sigma(n+1-k+\sigma) \right]. \end{cases}$$

Demonstration of our analysis

Example 6.1 To verify our analysis and to demonstrate our numerical scheme, we consider the following problem called Regulation of Haematopoiesis⁵⁶. Haematopoiesis is the biological process by which precursor stem cells reappear and divide to form mature blood cells. The body’s process of producing blood cells is known as hemopoiesis regulation. Red and white blood cells are produced in the bone marrow and subsequently reach the bloodstream. The primary factor in the production of red blood cells are unique hormones found in the kidney called erythropoiesis. If the blood function fails to supply enough oxygen, renal tubular epithelial cells emit about 90% of the erythropoiesis hormones. Because of this problem, the blood’s oxygen content drops and a chemical is released, which causes the bone marrow to produce more blood cells. A message that blood components deliver to the marrow is what causes haematological illness.

Here, we will apply our chosen findings to a popular hematopoiesis biological model known as “Wazewska – Czyzewska and Lasota Model”. The concerned model has numerous applications in real world problems, we refer few as^{57,58}. Time delays have been incorporated into various biological models to illustrate resource regeneration durations, maturation intervals, feeding schedules, reaction times, and so forth. The model in question can be expressed numerically as follows:

$$\begin{cases} \frac{du(\theta)}{dt} = -\mu\Phi(\theta) + pe^{-\gamma\Phi(\theta-\tau)}, \\ \Phi(0) = \Phi_0. \end{cases} \tag{18}$$

In this case, $\Phi(\theta)$ denotes the quantity of red blood cells over time. Red blood cell death rates are indicated by θ and μ , while red blood cell formation rates are indicated by p and γ , which are positive constants. Furthermore, τ denotes the amount of time needed to produce one red blood cell.

There are some modifications included in the model mentioned above. Here, we added a linear perturbation term to the left side of problem Eq. (18) and substituted the proportional delay term for the discrete delay term. The updated model is provided by

$$\begin{cases} {}^{FAB}D_{\theta}^{\sigma,\xi}[\Phi(\theta) - \kappa e^{-\Phi(\theta)}] = -\mu\Phi(\lambda\theta) + pe^{-\gamma\Phi(\lambda\theta)}, \\ \Phi(0) = 1 + \delta\Phi(\theta). \end{cases} \tag{19}$$

From Eq.(19), we have

$$\mathbf{f}(\theta, \Phi(\theta)) = \kappa e^{-\Phi(\theta)}, \mathbf{g}(\theta, \Phi(\lambda\theta)) = -\mu\Phi(\lambda\theta) + pe^{-\gamma\Phi(\lambda\theta)},$$

and

$$\mathbf{h}(\theta, \Phi(\theta)) = \delta \Phi(\theta).$$

Now we deduce the assumptions (A₁ – A₄) by performing the given process

$$|\mathbf{h}(\theta, \Phi(\theta)) - \mathbf{h}(\theta, \bar{\Phi}(\theta))| \leq \delta |\Phi(\theta) - \bar{\Phi}(\theta)|,$$

$$|\mathbf{f}(\theta, \Phi(\theta)) - \mathbf{f}(\theta, \bar{\Phi}(\theta))| \leq \kappa |\Phi(\theta) - \bar{\Phi}(\theta)|.$$

Additionally, we compute

$$\begin{aligned} |\mathbf{g}(\theta, \Phi(\lambda\theta)) - \mathbf{g}(\theta, \bar{\Phi}(\lambda\theta))| &= \left| -\mu \Phi(\lambda\theta) + pe^{-\gamma \Phi(\lambda\theta)} - \left(-\mu \bar{\Phi}(\lambda\theta) + pe^{-\gamma \bar{\Phi}(\lambda\theta)} \right) \right| \\ &\leq (\mu + p\gamma) |\Phi(\lambda\theta) - \bar{\Phi}(\lambda\theta)| \end{aligned}$$

and

$$\begin{aligned} \|\mathbf{g}\| &= \sup_{\theta \in [0,1]} \{ |\mathbf{g}(\theta, \Phi(\lambda\theta))| \} = \sup_{\theta \in [0,1]} \left\{ \left| -\mu \Phi(\lambda\theta) + pe^{-\gamma \Phi(\lambda\theta)} \right| \right\} \\ &\leq (\mu + p\gamma) \|\Phi\|. \end{aligned}$$

Therefore, **f**, **g**, and **h** are functions.fulfil each of the related presumptions. From (A₁ – A₄), using constants $K_1 = \delta$, $K_2 = \kappa$, $K_3 = \gamma - p$, $K_4 = \gamma - p$ and $K_5 = 0$.

Moreover $M(\sigma) = 1 - \sigma(1 - \frac{1}{\Gamma(\sigma+1)})$, $\delta = 0.0020$, $T = 10.0$, $\kappa = 0.10$, $\mu = 0.050$, $p = 0.0010$, and $\gamma = 0.250$, we obtain

$$\begin{aligned} \mathbf{K} &= \left(K_1 + \frac{K_3 \xi (1 - \sigma) T^{\xi-1}}{M(\sigma)} + K_2 \right), \\ &= \left(\delta + \kappa + \frac{(\mu + p\gamma) \xi (1 - \sigma) T^{\xi-1}}{M(\sigma)} \right), \\ &< 1, \text{ for all } \sigma, \xi \in (0, 1]. \end{aligned}$$

and

$$\begin{aligned} \mathbf{K}' &= \left[K_1 + \left(\frac{K_3 T^{\xi-1}}{M(\sigma)} \right) \left(\xi(1 - \sigma) + \frac{\sigma \Gamma(\xi + 1) T^\sigma}{\Gamma(\sigma + \xi)} \right) + K_2 \right], \\ &= \left[\delta + \kappa + \left(\frac{(\mu + p\gamma) T^{\xi-1}}{M(\sigma)} \right) \left(\xi(1 - \sigma) + \frac{\sigma \Gamma(\xi + 1) T^\sigma}{\Gamma(\sigma + \xi)} \right) \right], \\ &< 1, \text{ for all } \sigma, \xi \in (0, 1]. \end{aligned}$$

In Fig. 1, we display the geometrical behaviours of **K**, **K'** as:

As can be seen in Fig. 1, both $\mathbf{K} < 1$, $\mathbf{K}' < 1$. Thus, Theorem 3.2, Theorem 3.3, and Theorem 4.1 hold in their entirety. Consequently, there is at least one solution for Eq. (19). The requirement for the solution's uniqueness is also met. Additionally, for [0, 10], the solution is U-H and generalized U-H stable. To illustrate the dynamic behaviour, we here provide the approximate solutions of Example 6.1 for different fractals fractional orders in Figs. 2, 3, 4, 5, and 6, respectively.

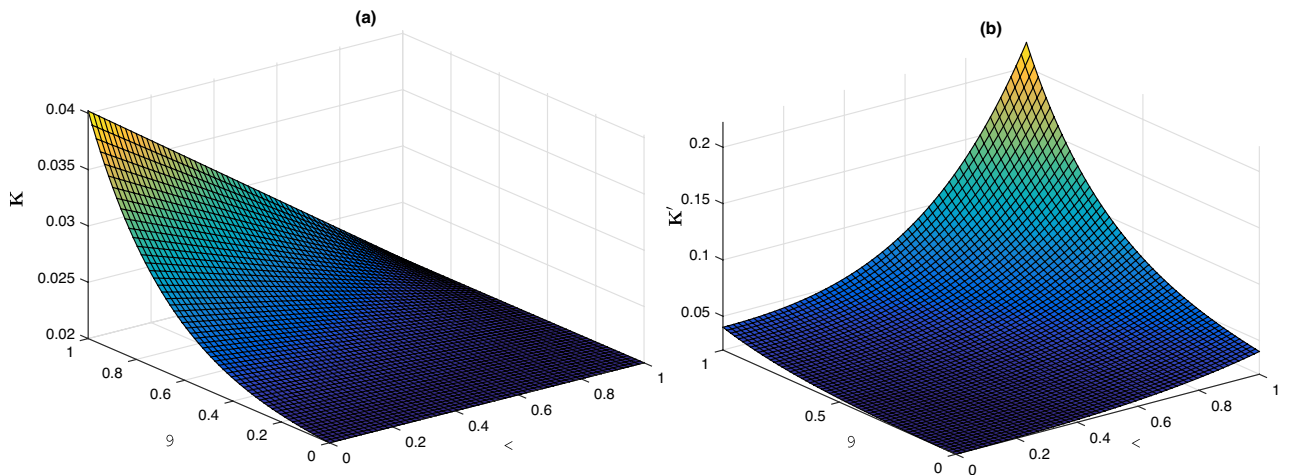


Figure 1. (a). Fractional and fractals behaviour of **K**. (b). Fractional and fractals behaviour of **K'**.

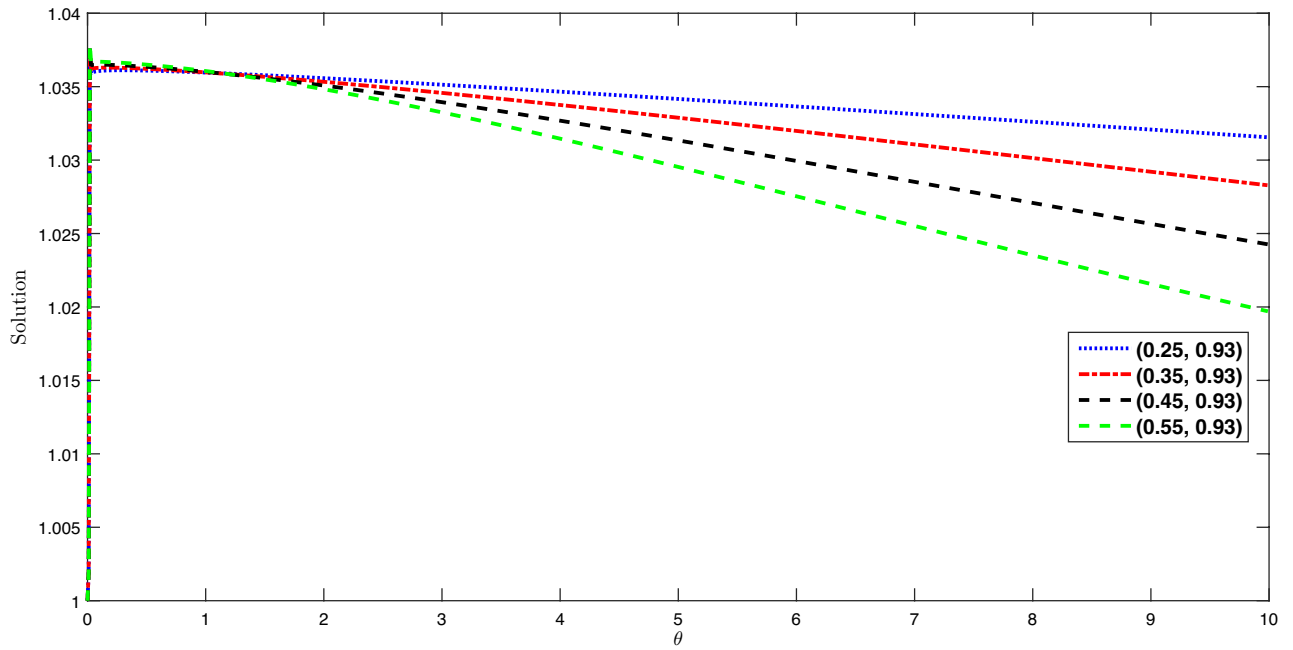


Figure 2. Utilizing first set of different fractional and fractals orders to present the solution graphically of Example 6.1.

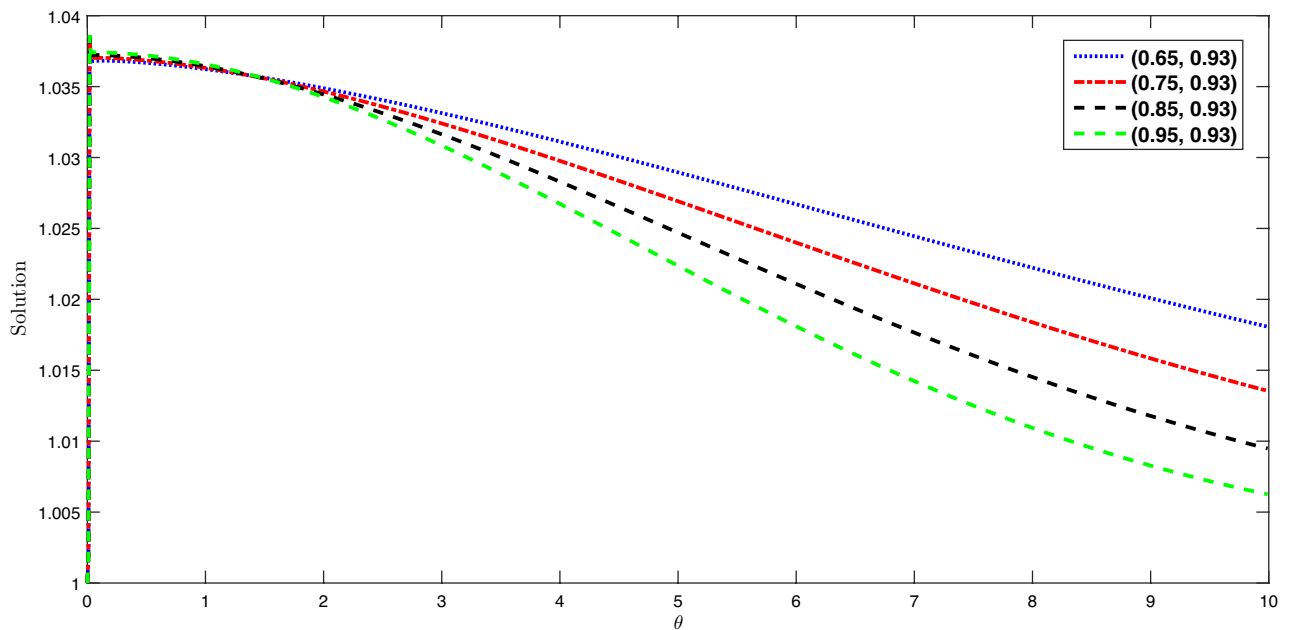


Figure 3. Utilizing second set of different fractional and fractals orders to present the solution graphically of Example 6.1.

In Figs. 2, 3, 4, 5, and 6, we have simulated the numerical results graphically for different fractals fractional order values. We see that in start the density of red-blood cells is increasing exponentially and then start to decrease. The decrease in the density of the said amount of red-blood cells is different. Because under different fractals fractional order the decay processes will be different. Usually the decay process is faster on smaller fractional order and greater fractals order and vice versa. In the same way, the growth phenomenon in dynamical system under fractals fractional order is also affected. The said processes is faster at larger fractional order values. Here, we use the fractional order values in (0, 0.25) and fractals values in (0.90, 1.0) and present the results graphically in Fig. 7.

Here, we use the fractional order values in (0, 0.1) and fractals values in (0.94, 1.0] and present the results graphically in Fig. 8.

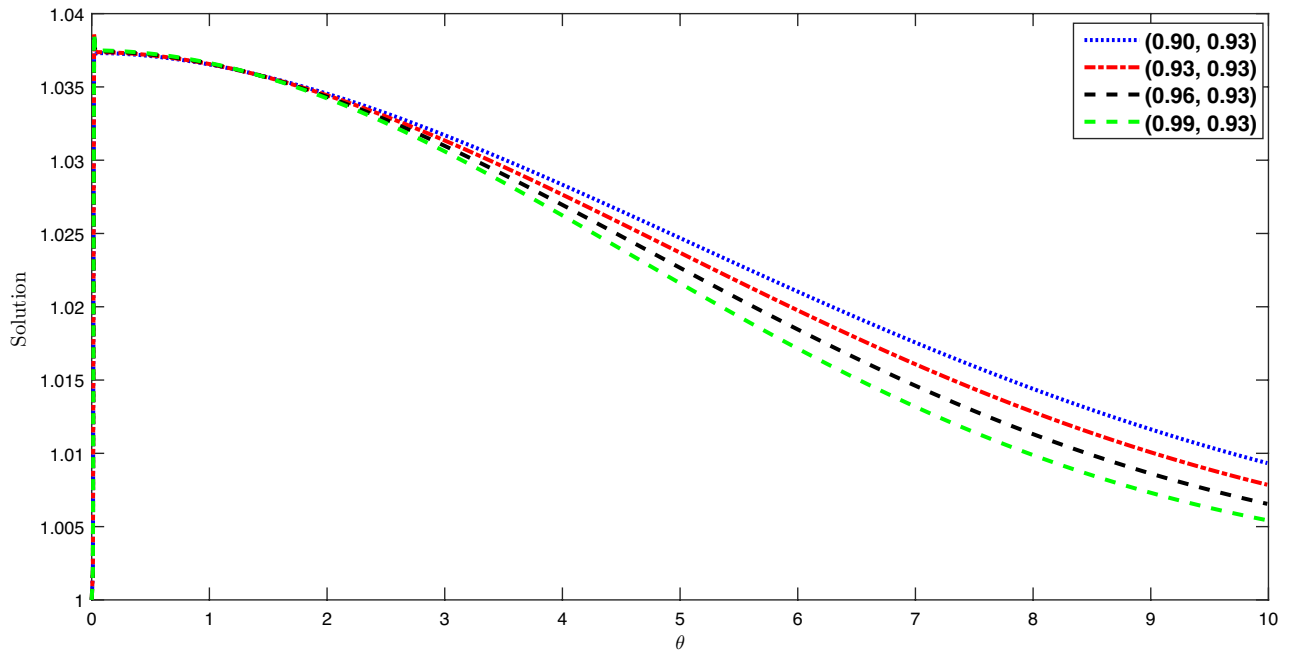


Figure 4. Utilizing third set of different fractional and fractals orders to present the solution graphically of Example 6.1.

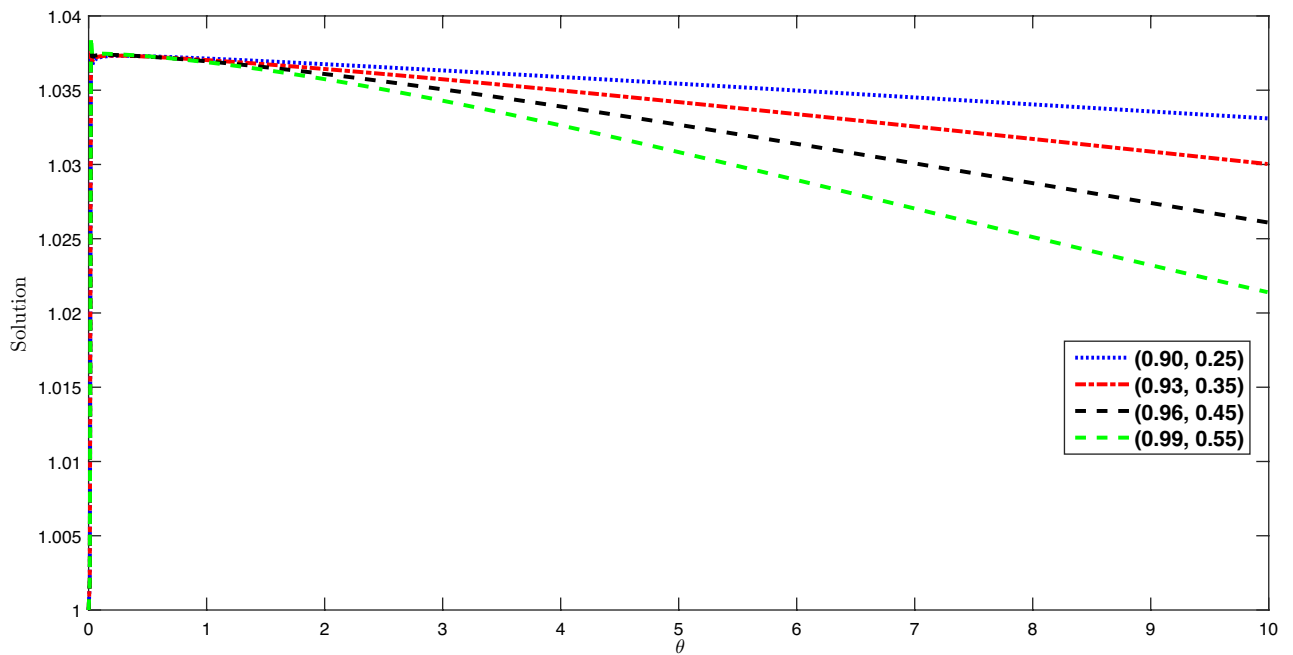


Figure 5. Utilizing fourth set of different fractional and fractals orders to present the solution graphically of Example 6.1.

From Figs. 7 and 8 we observe that fractional and fractals orders have significant impact on the dynamical behavior of the problem. At lower fractional order values the growth in population for some time is very fast like exponential growth and then becomes stable.

Conclusion

Here, we remark that as non-local operators of differentiation can incorporate increasingly complicated natural phenomena into mathematical equations. The said area has gotten much attention from number of researchers of nearly every field in the sciences, technology, and engineering. The FC has become interested very well for all researchers of science and technology. Since area devoted to fractals HFDEs has not well investigated using non singular fractals fractional differential operators. Therefore, it was needed to provide a sophisticated analysis

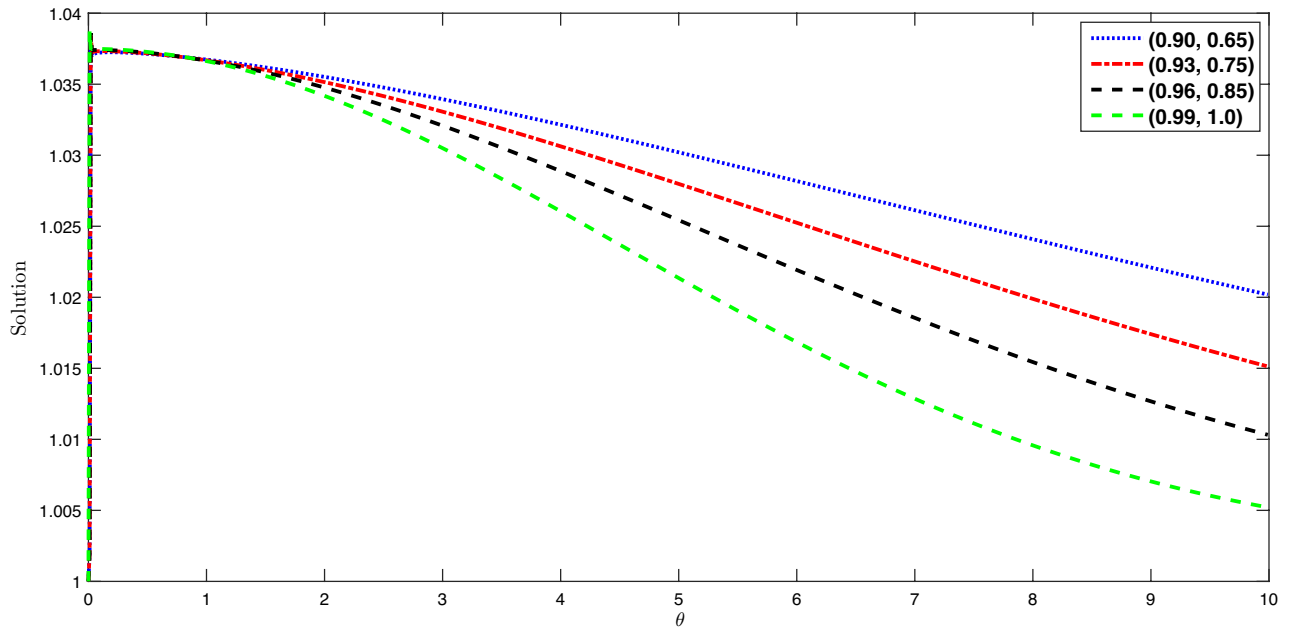


Figure 6. Utilizing fifth set of different fractional and fractals orders to present the solution graphically of Example 6.1.

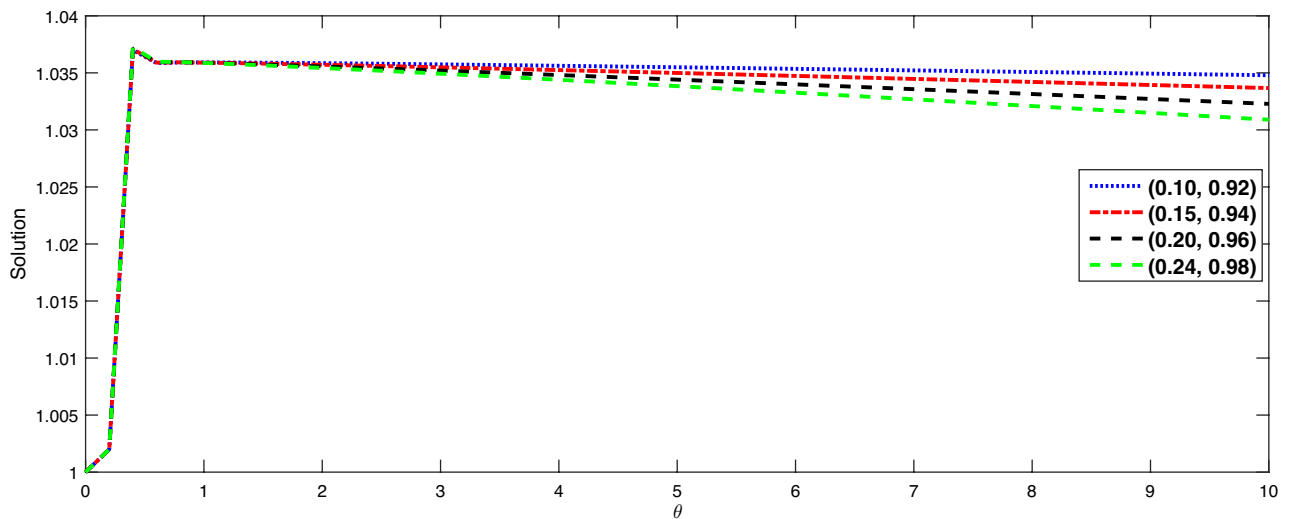


Figure 7. Utilizing another set of different fractional and fractals orders to present the solution graphically of Example 6.1.

and numerical results for young researchers to extend their knowledge in this direction. In this research work we have used *ABC* fractals fractional derivative. Here, we remark that the said operator has all the characteristics which have by other fractional differential operators. This study presents a theoretical and numerical investigation of a class of fractals HFDEs with *ABC* fractals fractional derivative. The issue under consideration was a hybrid problem involving linear perturbation. Using some fixed point analysis, sufficient requirements were inferred for the existence and uniqueness of the solution. Since stability theory is an important requirement for approximate solutions to nonlinear problems. Because with the help of stability theory we deduce the stable behaviour of solution and methodology we use. Different concepts in this regards were given in literature for stability analysis. U-H concept is one of the powerful procedure to be used to investigate stability results for different problems. Therefore, U-H stability requirements were developed to solve the aforementioned issue. To interpret the results numerically, a potent interpolation-based numerical technique was developed. Our previously mentioned research was applied to an intriguing example Lasota-Ważewska system in order to illustrate our findings. Every theoretical and numerical result was tested with success. We plan to utilize this research in the future for system of fractals HFDEs using more complex dynamical systems addressing real-world problems.

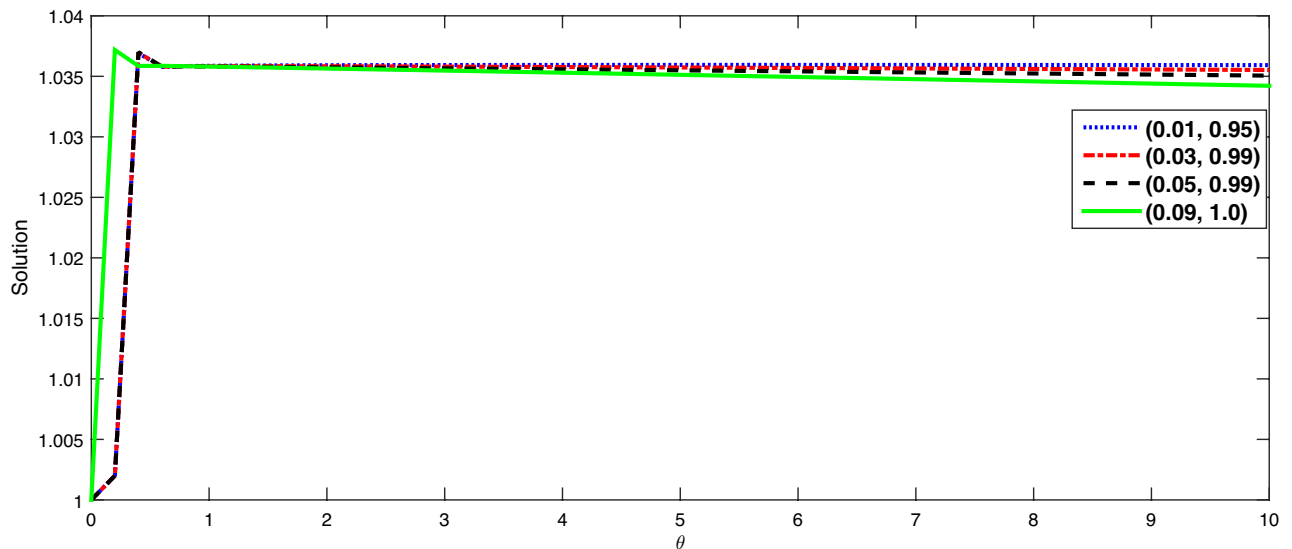


Figure 8. Utilizing lower values set of different fractional and fractals orders to present the solution graphically of Example 6.1.

Data availability

All the data used in this work is included in the paper.

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References

- Almeida, R., Malinowska, A. B. & Monteiro, M. T. T. Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications. *Mathe. Methods Appl. Sci.* **41**(1), 336–352 (2018).
- Das, S. & Pan, I. *Fractional order signal processing: introductory concepts and applications* (Springer Science & Business Media, 2011).
- Araz, S. İ. Analysis of a Covid-19 model: optimal control, stability and simulations. *Alexandria Eng. J.* **60**(1), 647–658 (2021).
- Awadalla, M. & Yameni, Y. Modeling exponential growth and exponential decay real phenomena by ψ -Caputo fractional derivative. *J. Adv. Mathe. Comput. Sci.* **28**(2), 1–13 (2018).
- Atangana, A. & İğret Araz, S. Mathematical model of COVID-19 spread in Turkey and South Africa: theory, methods, and applications. *Adv. Diff. Equ.* **2020**(1), 1–89 (2020).
- Ahmed, S., Ahmed, A., Mansoor, I., Junejo, F. & Saeed, A. Output feedback adaptive fractional-order super-twisting sliding mode control of robotic manipulator. *Iran. J. Sci. Technol. Trans. Elect. Eng.* **45**, 335–347 (2021).
- Shah, K., Jarad, F. & Abdeljawad, T. On a nonlinear fractional order model of dengue fever disease under Caputo-Fabrizio derivative. *Alex. Eng. J.* **59**(4), 2305–2313 (2020).
- Ahmed, S., Wang, H., Aslam, M. S., Ghous, I. & Qaisar, I. Robust adaptive control of robotic manipulator with input time-varying delay. *Int. J. Control Autom. Syst.* **17**(9), 2193–2202 (2019).
- Shaikh, A., Nisar, K. S., Jadhav, V., Elagan, S. K. & Zakarya, M. Dynamical behaviour of HIV/AIDS model using fractional derivative with Mittag-Leffler kernel. *Alex. Eng. J.* **61**(4), 2601–2610 (2022).
- Peter, O. J. *et al.* Analysis and dynamics of fractional order mathematical model of COVID-19 in Nigeria using atangana-baleanu operator. *Comput. Mater. Continua* **66**(2), 1823–1848 (2021).
- Shaikh, A. S., Shaikh, I. N. & Nisar, K. S. A mathematical model of COVID-19 using fractional derivative: outbreak in India with dynamics of transmission and control. *Adv. Differ. Equ.* **2020**(1), 373 (2020).
- Teodoro, G. S., Machado, J. T. & De Oliveira, E. C. A review of definitions of fractional derivatives and other operators. *J. Comput. Phys.* **388**, 195–208 (2019).
- Khalil, R., Al Horani, M., Yousef, A. & Sababheh, M. A new definition of fractional derivative. *J. Comput. Appl. Math.* **264**, 65–70 (2014).
- Hilfer, R. (ed.) *Applications of fractional calculus in physics* (World scientific, 2000).
- Kilbas, A. A. Hadamard-type fractional calculus. *J. Korean Math. Soc.* **38**(6), 1191–1204 (2001).
- Zhang, T. & Li, Y. Global exponential stability of discrete-time almost automorphic Caputo-Fabrizio BAM fuzzy neural networks via exponential Euler technique. *Knowl.-Based Syst.* **246**, 108675 (2022).
- Khan, M. *et al.* Dynamics of two-step reversible enzymatic reaction under fractional derivative with Mittag-Leffler Kernel. *PLoS One* **18**(3), e0277806 (2023).
- Caputo, M. & Fabrizio, M. Applications of new time and spatial fractional derivatives with exponential kernels. *Progress Fract. Differ. Appl.* **2**(1), 1–11 (2016).
- Caputo, M. & Fabrizio, M. A new definition of fractional derivative without singular kernel. *Progress Fract. Differ. Appl.* **1**(2), 73–85 (2015).
- Losada, J. & Nieto, J. J. Properties of a new fractional derivative without singular kernel. *Progr. Fract. Differ. Appl.* **1**(2), 87–92 (2015).
- Gul, R., Sarwar, M., Shah, K., Abdeljawad, T. & Jarad, F. Qualitative analysis of implicit Dirichlet boundary value problem for Caputo-Fabrizio fractional differential equations. *J. Function Spaces* **2020**, 1–9 (2020).
- Atangana, A. & Baleanu, D. New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model. (2016). arXiv preprint [arXiv:1602.03408](https://arxiv.org/abs/1602.03408).
- Xu, C., Liu, Z., Pang, Y., Saifullah, S. & Inc, M. Oscillatory, crossover behavior and chaos analysis of HIV-1 infection model using piece-wise Atangana-Baleanu fractional operator: Real data approach. *Chaos, Solitons Fractals* **164**, 112662 (2022).

24. Saifullah, S., Ali, A., Irfan, M. & Shah, K. Time-fractional Klein-Gordon equation with solitary/shock waves solutions. *Math. Probl. Eng.* **2021**, 1–15 (2021).
25. Alomari, A. K., Abdeljawad, T., Baleanu, D., Saad, K. M. & Al-Mdallal, Q. M. Numerical solutions of fractional parabolic equations with generalized Mittag-Leffler kernels. *Numer. Methods Partial Differ. Equ.* **40**(1), e22699 (2024).
26. Saad Alshehry, A., Imran, M., Shah, R. & Weera, W. Fractional-view analysis of fokker-planck equations by ZZ transform with mittag-leffler kernel. *Symmetry* **14**(8), 1513 (2022).
27. Atangana, A. Fractal-fractional differentiation and integration: connecting fractal calculus and fractional calculus to predict complex system. *Chaos, Solitons Fractals* **102**, 396–406 (2017).
28. He, J. H. Fractal calculus and its geometrical explanation. *Res. Phys.* **10**, 272–276 (2018).
29. Hu, Y. & He, J. H. On fractal space-time and fractional calculus. *Therm. Sci.* **20**(3), 773–777 (2016).
30. Qureshi, S. & Atangana, A. Fractal-fractional differentiation for the modeling and mathematical analysis of nonlinear diarrhea transmission dynamics under the use of real data. *Chaos Solitons Fractals* **136**, 109812 (2020).
31. Srivastava, H. M. & Saad, K. M. Numerical simulation of the fractal-fractional Ebola virus. *Fractal Fract.* **4**(4), 49 (2020).
32. Xiao, B., Huang, Q., Chen, H., Chen, X. & Long, G. A fractal model for capillary flow through a single tortuous capillary with roughened surfaces in fibrous porous media. *Fractals* **29**(01), 2150017 (2021).
33. Liang, M. *et al.* An analytical model for the transverse permeability of gas diffusion layer with electrical double layer effects in proton exchange membrane fuel cells. *Int. J. Hydrogen Energy* **43**(37), 17880–17888 (2018).
34. Yu, X. *et al.* Characterization of water migration behavior during spontaneous imbibition in coal: From the perspective of fractal theory and NMR. *Fuel* **355**, 129499 (2024).
35. Ahmad, I., Ahmad, N., Shah, K. & Abdeljawad, T. Some appropriate results for the existence theory and numerical solutions of fractals-fractional order malaria disease mathematical model. *Res. Control Optim.* **14**, 100386 (2024).
36. Ur Rahman, M. Generalized fractal-fractional order problems under non-singular Mittag-Leffler kernel. *Res. Phys.* **35**, 105346 (2022).
37. Eiman, Shah K., Sarwar, M. & Abdeljawad, T. A comprehensive mathematical analysis of fractal-fractional order nonlinear re-infection model. *Alex. Eng. J.* **103**, 353–365 (2024).
38. Khan, S. Existence theory and stability analysis to a class of hybrid differential equations using confirmable fractal fractional derivative. *J. Frac. Calc. Nonlinear Sys.* **5**(1), 1–11 (2024).
39. El-Dessoky, M. M. & Khan, M. A. Modeling and analysis of an epidemic model with fractal-fractional Atangana-Baleanu derivative. *Alex. Eng. J.* **61**(1), 729–746 (2022).
40. Smith, H. *An Introduction to Delay Differential Equations with Applications to the Life Sciences* 119–130 (Springer, 2011).
41. Balachandran, B., Kalmár-Nagy, T. & Gilsinn, D. E. *Delay differential equations* (Springer, 2009).
42. Balachandran, K., Kiruthika, S. & Trujillo, J. Existence of solutions of nonlinear fractional pantograph equations. *Acta Mathe. Sci.* **33**(3), 712–720 (2013).
43. Basim, M., Ahmadian, A., Senu, N. & Ibrahim, Z. B. Numerical simulation of variable-order fractal-fractional delay differential equations with nonsingular derivative. *Eng. Sci. Technol. Int. J.* **42**, 101412 (2023).
44. Shafiullah, Shah K., Sarwar, M. & Abdeljawad, T. On theoretical and numerical analysis of fractal-fractional non-linear hybrid differential equations. *Nonlinear Eng.* **13**(1), 20220372 (2024).
45. Abbas, M. I. & Ragusa, M. A. On the hybrid fractional differential equations with fractional proportional derivatives of a function with respect to a certain function. *Symmetry* **13**(2), 264 (2021).
46. Guo, C., Hu, J., Hao, J., Celikovskiy, S. & Hu, X. Fixed-time safe tracking control of uncertain high-order nonlinear pure-feedback systems via unified transformation functions. *Kybernetika* **59**(3), 342–364 (2023).
47. Ulam, S. M. *Problems in modern mathematics* (Courier Corporation, 2004).
48. Hyers, D. H. On the stability of the linear functional equation. *Proc. Natl. Acad. Sci.* **27**(4), 222–224 (1941).
49. Rassias, T. M. On the stability of the linear mapping in Banach spaces. *Proc. Am. Mathe. Soc.* **72**(2), 297–300 (1978).
50. Rhaima, M., Mchiri, L., Makhlof, A. B. & Ahmed, H. Ulam type stability for mixed Hadamard and Riemann-Liouville fractional stochastic differential equations. *Chaos Solitons Fractals* **178**, 114356 (2024).
51. Huang, J. & Luo, D. Ulam-Hyers stability of fuzzy fractional non-instantaneous impulsive switched differential equations under generalized Hukuhara differentiability. *Int. J. Fuzzy Syst.* **2024**, 1–12 (2024).
52. Guo, C., Hu, J., Wu, Y. & Celikovskiy, S. Non-Singular Fixed-Time Tracking Control of Uncertain Nonlinear Pure-Feedback Systems With Practical State Constraints. *IEEE Trans. Circuits Syst. I Regul. Pap.* **70**(9), 3746–3758 (2023).
53. Peng, Y., Zhao, Y. & Hu, J. On The Role of Community Structure in Evolution of Opinion Formation: A New Bounded Confidence Opinion Dynamics. *Inf. Sci.* **621**, 672–690 (2023).
54. Khan, S., Shah, K., Debbouche, A., Zeb, S. & Antonov, V. Solvability and Ulam-Hyers stability analysis for nonlinear piecewise fractional cancer dynamic systems. *Phys. Scr.* **99**(2), 025225 (2024).
55. Khan, M. A. & Atangana, A. *Numerical Methods for Fractal-fractional Differential Equations and Engineering: Simulations and Modeling* (CRC Press, 2023).
56. Chen, X., Shi, C. & Wang, D. Dynamic behaviors for a delay Lasota-Ważewska model with feedback control on time scales. *Adv. Differ. Equ.* **2020**(1), 1–13 (2020).
57. Xu, G., Huang, M., Hu, J., Liu, S. & Yang, M. Bisphenol A and its structural analogues exhibit differential potential to induce mitochondrial dysfunction and apoptosis in human granulosa cells. *Food Chem. Toxicol.* **188**, 114713 (2024).
58. Luo, W. *et al.* Update: Innate lymphoid cells in inflammatory bowel disease. *Dig. Dis. Sci.* **67**(1), 56–66 (2022).

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Author contributions

T.A. edited the work. M.S. has written the paper. K.S. has designed the problem and done numerical. M.S. included theoretical results. I.A. included literature review. M.A. has supervised the work and contributed in the revision. A. A. has edited the final version and done significant work in the revised version.

Competing interests

The authors declare no competing interests.

Additional information

Correspondence and requests for materials should be addressed to I.A.

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