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Construction of improved comprehensive classes of estimators for population distribution function

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The primary purpose of this article is to examine the issue of estimating the finite population distribution function from auxiliary information, such as population mean and rank of the auxiliary variables, that are already known. In order to better estimate the distribution function (DF) of a finite population, two improved estimators are developed. The bias and mean squared error of the suggested and existing estimators are derived up to the first order of approximation. To improve the efficiency of an estimators, we compare the suggested estimators with existing counterpart. Based on the numerical outcomes, it is to be noted that the suggested classes of estimators perform well using six actual data sets. The strength and generalization of the suggested estimators are also verified using a simulation analysis. Based on the result of actual data sets and a simulation study, we observe that the suggested estimator outperforms as compared to all existing estimators which is compared in this study.

Keywords Distribution function (DF), Simple random sampling, MSE, PRE, Simulation study, Visualization

In the literature on survey sampling, the use of auxiliary information progresses the precision of an estimators. The finest possible estimates of population metrics like mean, median, variance, standard deviation, etc. have previously been discovered by researchers. To achieve this goal, it is necessary to draw sample from the population; when the target population is uniform, a simple random sampling provide better result. When the study variable and the auxiliary variables have a high degree of association, then the rank of the auxiliary information is also associated to the study variable. The ratio and product estimators can enhance the accuracy of estimators when there is either a positive or negative association between the studied variable and the extra information. By consulting[1–7], the researcher can further investigate these findings using auxiliary variables.

There is a substantial amount of literature available on the topic of population parameter estimate using different sampling approaches. But research based on distribution function (DF) has received less attention compared to the many estimators for estimating distinct finite population parameters under diverse sampling procedures in the literature. In order to determine what percentage of values are less than or equal to the threshold value, it is necessary to estimate a finite population DF. As an example, a doctor would wonder what percentage of the population get at least 20% of their caloric intake from cholesterol in their food. A soil scientist is interested in

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learning the poverty rate in a developing nation. Initially the technique for estimating the population DF was proposed by[8]. Some essential resources for learning how to estimate population DF using auxiliary information are given in[9–16].

There is a substantial amount of literature available on the topic of population parameter estimate using different sampling approaches. But research based on distribution function (DF) has received less attention compared to the many estimators for estimating population parameters. In this paper we suggested improved classes of estimators for estimation of population DF using dual use of an auxiliary variables. Estimation of population DF is required when the percentage of particular values are less than or equal to the specific threshold. To check the robustness and generalizability we have utilized six real data sets and a simulation study.

The remaining of the article is designed as follows. In "Notation and symbols" section, the notations and symbols for the said work is given. The existing estimators were analyzed in "Existing estimators" section. In "Suggested estimators" section, we suggested two improved classes estimators for determining the DF. In "Numerical study" section, the empirical study are given. In "Simulation study" section, we also comportment a simulation study to test the efficacy of our proposed families of estimators using a simple random sample. In "Discussion" section, the numerical results are discussed. "Conclusion" section, provides conclusion of the article.

Notation and symbols

Let a population $\Omega = \{1, 2, \ldots, N\}$ consist of N separate and identifiable units, we take a sample of size n from Ω using a $SRSWOR$. Let Y and X be the study variable and auxiliary variable. Consider Z is used for the rank of X. Let $I(Y \le y)$ signify the indicator variable for Y, and $I(X \le y)$ signify the display variable for X.

$$\mathcal{F}(y) = \sum_{i=1}^{N} I(Y_i \le y)/N, \widehat{\mathcal{F}}(y) = \sum_{i=1}^{n} I(Y_i \le y)/n, \mathcal{F}(x) = \sum_{i=1}^{N} I(X_i \le y)/N,$$

$\widehat{\mathcal{F}}(x) = \sum_{i=1}^{n} I(X_i \le y)/n$, are the DF functions of Y and X for population and sample, respectively. Similarly,

$$\overline{\mathcal{X}} = \sum_{i=1}^{N} X_i/N, \widehat{\overline{\mathcal{X}}} = \sum_{i=1}^{n} X_i/n, \overline{\mathcal{Z}} = \sum_{i=1}^{N} Z_i/N, \widehat{\overline{\mathcal{Z}}} = \sum_{i=1}^{n} Z_i/n.$$

$$\xi_0 = \frac{\widehat{\mathcal{F}}(y) - \mathcal{F}(y)}{\mathcal{F}(y)}, \xi_1 = \frac{\widehat{\mathcal{F}}(x) - \mathcal{F}(x)}{\mathcal{F}(x)}, \xi_2 = \frac{\widehat{\overline{\mathcal{X}}} - \overline{\mathcal{X}}}{\overline{\mathcal{X}}} \text{and} \xi_3 = \frac{\widehat{\overline{\mathcal{Z}}} - \overline{\mathcal{Z}}}{\overline{\mathcal{Z}}},$$

$$E(\xi_0^2) = \lambda C_{F_y}^2, E(\xi_1^2) = \lambda C_{F_x}^2, E(\xi_2^2) = \lambda C_x^2, E(\xi_3^2) = \lambda C_{rx}^2, E(\xi_0\xi_1) = \lambda\rho_{12} C_{F_y} C_{F_x},$$

$$E(\xi_0\xi_2) = \lambda\rho_{13} C_{F_y} C_x, E(\xi_0\xi_3) = \lambda\rho_{14} C_{F_y} C_{rx}, E(\xi_1\xi_2) = \lambda\rho_{23} C_{F_x} C_x, E(\xi_1\xi_3) = \lambda\rho_{24} C_{F_x} C_{rx},$$

$$\rho_1^2 = \sum_{i=1}^{N} \left(I(Y_i \le y) - \mathcal{F}(y)\right)^2/(N-1), \rho_2^2 = \sum_{i=1}^{N} (I(X_i \le x) - \mathcal{F}(x))^2/(N-1),$$

$$\rho_3^2 = \sum_{i=1}^{N} (X_i - \overline{\mathcal{X}})^2/(N-1), \rho_4^2 = \sum_{i=1}^{N} (Z_i - \overline{\mathcal{Z}})^2/(N-1),$$

$$C_{F_y} = \rho_1/\mathcal{F}(y), C_{F_x} = \rho_2/\mathcal{F}(x), C_x = \rho_3/\overline{\mathcal{X}}, C_{rx} = \rho_4/\overline{\mathcal{Z}},$$

$\rho_{12} = \sigma_{12}/(\sigma_1\sigma_2), \rho_{13} = \sigma_{13}/(\sigma_1\sigma_3), \rho_{23} = \sigma_{23}/(\sigma_2\sigma_3), \rho_{14} = \sigma_{14}/(\sigma_1\sigma_4), \rho_{24} = \sigma_{24}/(\sigma_2\sigma_4)$.

$\sigma_{12} = \sum_{i=1}^{N} \left\{(I(Y_i \le y) - \mathcal{F}(y))(I(X_i \le x) - \mathcal{F}(x))\right\}/(N-1), \sigma_{13} = \sum_{i=1}^{N} \left\{(I(Y_i \le y) - \mathcal{F}(y)) (X_i - \overline{\mathcal{X}})\right\}/(N-1), \sigma_{23} = \sum_{i=1}^{N} \left\{(I(X_i \le x) - \mathcal{F}(x))(X_i - \overline{\mathcal{X}})\right\}/(N-1), \sigma_{14} = \sum_{i=1}^{N} \left\{(I(Y_i \le y) -\mathcal{F}(y))(Z_i - \overline{\mathcal{Z}})\right\}/(N-1), \sigma_{24} = \sum_{i=1}^{N} \left\{(I(X_i \le x) - \mathcal{F}(x))(Z_i - \overline{\mathcal{Z}})\right\}/(N-1)$,

where $\lambda = (1/n - 1/N)$.

let $R_{1.23}^2 = \rho_{12}^2 + \rho_{13}^2 - 2\rho_{12}\rho_{13}\rho_{23}/\left(1 - \rho_{23}^2\right)$. Similarly, $R_{1.24}^2 = \rho_{12}^2 + \rho_{14}^2 - 2\rho_{12}\rho_{14}\rho_{24}/\left(1 - \rho_{24}^2\right)$.

Existing estimators

Here, we take some adopted existing for population DF, which is given by

1. The usual estimator for DF, is given by:

$$\widehat{\mathcal{F}}(y) = \frac{1}{n}\sum_{i=1}^{n} Y_i. \tag{1}$$

The variance of $\widehat{\mathcal{F}}(y)$:

$$\mathrm{Var}(\widehat{\mathcal{F}}(y)) = \lambda \mathcal{F}^2(y) C_{Fy}^2. \tag{2}$$

2. Reference[17] give a ratio estimator for estimating $\mathcal{F}(y)$:

$$\widehat{\mathcal{F}}_R(\mathcal{Y}) = \widehat{\mathcal{F}}(y)\left(\frac{\mathcal{F}(x)}{\widehat{\mathcal{F}}(x)}\right). \tag{3}$$

$$\mathrm{Bias}(\widehat{\mathcal{F}}_R(\mathcal{Y})) \cong \lambda \mathcal{F}(y)(C_{Fx}^2 - \rho_{12} C_{F_y} C_{F_x}),$$

and

$$\mathrm{MSE}(\widehat{\mathcal{F}}_R(\mathcal{Y})) \cong \lambda \mathcal{F}^2(y)(C_{Fy}^2 + C_{Fx}^2 - 2\rho_{12} C_{F_y} C_{F_x}). \tag{4}$$

3. Reference[18] suggested a product estimator for $\mathcal{F}(y)$:

$$\widehat{\mathcal{F}}_P(\mathcal{Y}) = \widehat{\mathcal{F}}(y)\left(\frac{\widehat{\mathcal{F}}(x)}{\mathcal{F}(x)}\right). \tag{5}$$

$$\mathrm{Bias}(\widehat{\mathcal{F}}_P(\mathcal{Y})) = \lambda \mathcal{F}(y) \rho_{12} C_{F_y} C_{F_x},$$

and

$$\mathrm{MSE}(\widehat{\mathcal{F}}_P(\mathcal{Y})) \cong \lambda \mathcal{F}^2(y)(C_{Fy}^2 + C_{Fx}^2 + 2\rho_{12} C_{F_y} C_{F_x}). \tag{6}$$

4. The regression estimator of $\mathcal{F}(y)$:

$$\widehat{\mathcal{F}}_{Reg}(\mathcal{Y}) = \left[\widehat{\mathcal{F}}(y) + w(\mathcal{F}(x) - \widehat{\mathcal{F}}(x))\right] \tag{7}$$

where w is constant.

$$w_{(\mathrm{opt})} = \rho_{12}(\rho_Y/\rho_X),$$

$$\mathrm{Var}_{\min}(\widehat{\mathcal{F}}_{Reg}(\mathcal{Y})) = \lambda \mathcal{F}^2(y) C_{Fy}^2 (1 - \rho_{12}^2). \tag{8}$$

5. Reference[19] suggested a difference estimator, given by:

$$\widehat{\mathcal{F}}_{R,D}(\mathcal{Y}) = \left[w_1 \widehat{\mathcal{F}}(y) + w_2(\mathcal{F}(x) - \widehat{\mathcal{F}}(x))\right] \tag{9}$$

$$\mathrm{Bias}(\widehat{\mathcal{F}}_{R,D}(\mathcal{Y})) = \left[\mathcal{F}(y)(w_1 - 1)\right]$$

and

$$\mathrm{MSE}(\widehat{\mathcal{F}}_{R,D}(\mathcal{Y})) \cong \mathcal{F}^2(y)(w_1 - 1)^2 + \lambda \mathcal{F}^2(y) C_{Fy}^2 w_1^2 + \lambda \mathcal{F}^2(x) C_{F_x}^2 w_2^2$$

$$-2\lambda \mathcal{F}(y)\mathcal{F}(x)\rho_{12} C_{F_y} C_{F_x} w_1 w_2. \tag{10}$$

where

$$w_{1(\mathrm{opt})} = \frac{1}{1 + \lambda C_{Fy}^2(1 - \rho_{12}^2)},$$

$$w_{2(\mathrm{opt})} = \frac{\mathcal{F}(y)\rho_{12} C_{F_y}}{\mathcal{F}(x) C_{F_x}\{1 + \lambda C_{Fy}^2(1 - \rho_{12}^2)\}},$$

Using $w_{1(\mathrm{opt})}$, $w_{1(\mathrm{opt})}$ we got:

$$\mathrm{MSE}_{\min}(\widehat{\mathcal{F}}_{R,D}(\mathcal{Y})) \cong \frac{\lambda \mathcal{F}^2(y) C_{Fy}^2(1 - \rho_{12}^2)}{1 + \lambda C_{Fy}^2(1 - \rho_{12}^2)}. \tag{11}$$

6. Reference[20] suggested exponential type estimators, given by:

$$\widehat{\mathcal{F}}_{BT,R}(\mathcal{Y}) = \widehat{\mathcal{F}}(y)\exp\left(\frac{\mathcal{F}(x) - \widehat{\mathcal{F}}(x)}{\widehat{\mathcal{F}}(x) + \mathcal{F}(x)}\right), \tag{12}$$

$$\widehat{\mathcal{F}}_{BT,P}(\mathcal{Y}) = \widehat{\mathcal{F}}(y)\exp\left(\frac{\widehat{\mathcal{F}}(x) - \mathcal{F}(x)}{\widehat{\mathcal{F}}(x) + \mathcal{F}(x)}\right). \tag{13}$$

$$\text{Bias}(\widehat{\mathcal{F}}_{BT,R}(\mathcal{Y})) \cong \lambda \mathcal{F}(y)\left(\frac{3C_{Fx}^2}{8} - \frac{\rho_{12} C_{F_y} C_{F_x}}{2}\right),$$

$$\text{MSE}(\widehat{\mathcal{F}}_{BT,R}(\mathcal{Y})) \cong \frac{\lambda \mathcal{F}(y)^2}{4}(4C_{Fy}^2 + C_{Fx}^2 - 4\rho_{12} C_{F_y} C_{F_x}), \tag{14}$$

and

$$\text{Bias}(\widehat{\mathcal{F}}_{BT,P}(\mathcal{Y})) \cong \lambda \mathcal{F}(y)\left(\frac{\rho_{12} C_{F_y} C_{F_x}}{2} - \frac{C_{Fx}^2}{8}\right),$$

$$\text{MSE}(\widehat{\mathcal{F}}_{BT,P}(\mathcal{Y})) \cong \frac{\lambda \mathcal{F}^2(y)}{4}(4C_{Fy}^2 + C_{Fx}^2 + 4\rho_{12} C_{F_y} C_{F_x}). \tag{15}$$

7. Reference[21] suggested the following estimator, given by:

$$\widehat{\mathcal{F}}_S(\mathcal{Y}) = \widehat{\mathcal{F}}(y)\exp\left[\frac{\alpha(\mathcal{F}(x) - \widehat{\mathcal{F}}(x))}{\alpha(\mathcal{F}(x) + \widehat{\mathcal{F}}(x)) + 2\beta}\right] \tag{16}$$

The estimator $\widehat{\mathcal{F}}_S(\mathcal{Y})$ reduces to $\widehat{\mathcal{F}}_{BT,R}(\mathcal{Y})$ and $\widehat{\mathcal{F}}_{BT,P}(\mathcal{Y})$ when $(\alpha = 1, \beta = 0)$ and $(\alpha = -1, \beta = 0)$, respectively.

$$\text{Bias}(\widehat{\mathcal{F}}_S(\mathcal{Y})) \cong \lambda \mathcal{F}(y)\left(\frac{3\Theta^2 C_{Fx}^2}{8} - \frac{\Theta \rho_{12} C_{F_y} C_{F_x}}{2}\right),$$

and

$$\text{MSE}(\widehat{\mathcal{F}}_S(\mathcal{Y})) \cong \frac{\lambda \mathcal{F}^2(y)}{4}(4C_{Fy}^2 + \Theta^2 C_{Fx}^2 - 4\Theta \rho_{12} C_{F_y} C_{F_x}), \tag{17}$$

where $\Theta = \alpha \mathcal{F}(x)/(\alpha \mathcal{F}(x) + \beta)$.

8. Reference[22] suggested a generalized ratio-type exponential estimator, given by:

$$\widehat{\mathcal{F}}_{GK}(\mathcal{Y}) = \left\{w_3 \widehat{\mathcal{F}}(y) + w_4(\mathcal{F}(x) - \widehat{\mathcal{F}}(x))\right\}\exp\left[\frac{\alpha(\mathcal{F}(x) - \widehat{\mathcal{F}}(x))}{\alpha(\mathcal{F}(x) + \widehat{\mathcal{F}}(x)) + 2\beta}\right], \tag{18}$$

$$\text{Bias}(\widehat{\mathcal{F}}_{GK}(\mathcal{Y})) \cong \mathcal{F}(y) - w_3 \mathcal{F}(y) + \frac{3}{8} w_3 \Theta^2 \mathcal{F}(y) \lambda C_{Fy}^2 + \frac{1}{2} w_4 \Theta \mathcal{F}(x) \lambda C_{Fx}^2$$

$$-\frac{1}{2} w_3 \Theta \mathcal{F}(y) \lambda \rho_{12} C_{F_y} C_{F_x},$$

and

$$\begin{aligned}\text{MSE}(\widehat{\mathcal{F}}_{GK}(\mathcal{Y})) \cong\ & \mathcal{F}^2(y)(w_3 - 1)^2 + w_3^2 \mathcal{F}^2(y) \lambda C_{Fy}^2 + w_4^2 \mathcal{F}^2(x) \lambda C_{Fx}^2 + \Theta^2 \mathcal{F}^2(y) \lambda C_{Fx}^2 w_3^2 \\ & + 2 w_3 w_4 \Theta \mathcal{F}(y) \mathcal{F}(x) \lambda C_{Fx}^2 - \frac{3}{4} w_3 \Theta^2 \mathcal{F}^2(y) \lambda C_{Fx}^2 - w_4 \Theta \mathcal{F}(y) \mathcal{F}(x) \lambda C_{Fx}^2 \\ & + w_3 \Theta \mathcal{F}^2(y) \lambda \rho_{12} C_{F_y} C_{F_x} - 2 w_3^2 \Theta \mathcal{F}^2(y) \lambda \rho_{12} C_{F_y} C_{F_x} \\ & - 2 w_3 w_4 \mathcal{F}(y) \mathcal{F}(x) \lambda \rho_{12} C_{F_y} C_{F_x}.\end{aligned}$$

$$w_{3(\text{opt})} = \frac{8 - \lambda \Theta^2 C_{Fx}^2}{8\{1 + \lambda C_{Fy}^2 (1 - \rho_{12}^2)\}}$$

$$w_{4(\text{opt})} = \frac{\mathcal{F}(y)\left[\lambda \Theta^3 C_{Fx}^3 + 8\rho_{12} C_{F_y} - \lambda \Theta^2 \rho_{12} C_{F_y} C_{Fx}^2 - 4\Theta C_{F_x}\{1 - \lambda C_{Fy}^2 (1 - \rho_{12}^2)\}\right]}{8\mathcal{F}(x) C_{F_x}\{1 + \lambda C_{Fy}^2 (1 - \rho_{12}^2)\}},$$

$$\mathrm{MSE}_{\min}\left(\widehat{\mathcal{F}}_{GK}(\mathcal{Y})\right) \cong \frac{\lambda\mathcal{F}^2(y)\{64C_{Fy}^2(1-\rho_{12}^2) - \lambda\Theta^4C_{Fx}^4 - 16\lambda\Theta^2C_{Fy}^2C_{Fx}^2(1-\rho_{12}^2)\}}{64\{1+\lambda C_{Fy}^2(1-\rho_{12}^2)\}}. \tag{19}$$

Here, (19) may be written as

$$\mathrm{MSE}_{\min}\left(\widehat{\mathcal{F}}_{GK}(\mathcal{Y})\right) \cong \mathrm{Var}_{\min}(\widehat{\mathcal{F}}^*_{st\,Reg}(\mathcal{Y})) - \frac{\lambda^2\mathcal{F}^2(y)\left\{\Theta^2C_{Fx}^2 + 8C_{Fy}^2\left(1-\rho_{12}^2\right)\right\}^2}{64\{1+\lambda C_{Fy}^2(1-\rho_{12}^2)\}}, \tag{20}$$

Suggested estimators

By incorporating the auxiliary variables, the design and estimation stages of an estimator can take benefit. When the study variable is associated with the auxiliary variable, then rank of the auxiliary variable is also correlated with each other. Therefore, the rank of the auxiliary variable can be considered as an additional information, it helps to improve the estimator accuracy. To calculate an approximation of the population distribution function, we use more information regarding the sample means and the rank of the auxiliary variable, along with the sample distribution functions of $F(y)$ and $F(x)$.

First improved class of estimator

Taking motivation from $\widehat{\mathcal{F}}_{R,D}(y)$, $\widehat{\mathcal{F}}_S(y)$ and average of $\widehat{\mathcal{F}}_{BT,R}(y)$ and $\widehat{\mathcal{F}}_{BT,P}(y)$, our first proposed class of the estimator, is given by:

$$\widehat{\mathcal{F}}_{Pr_1}(\mathcal{Y}) = \left[\begin{array}{c} \frac{1}{2}\widehat{\mathcal{F}}(y)\left\{\exp\left[\frac{\mathcal{F}(x)-\widehat{\mathcal{F}}(x)}{\widehat{\mathcal{F}}(x)+\mathcal{F}(x)}\right] + \exp\left[\frac{\widehat{\mathcal{F}}(x)-\mathcal{F}(x)}{\widehat{\mathcal{F}}(x)+\mathcal{F}(x)}\right]\right\} \\ +w_5\left\{\mathcal{F}(x)-\widehat{\mathcal{F}}(x)\right\} + w_6\widehat{\mathcal{F}}(y) + w_7\left(\overline{\mathcal{X}}-\widehat{\overline{\mathcal{X}}}\right)\end{array}\right]\exp\left[\frac{\alpha\left\{\mathcal{F}(x)-\widehat{\mathcal{F}}(x)\right\}}{\alpha(\mathcal{F}(x)+\widehat{\mathcal{F}}(x))+2\beta}\right].$$

The estimator $\widehat{\mathcal{F}}_{Pr_1}(\mathcal{Y})$, is expressed as:

$$\widehat{\mathcal{F}}_{Pr_1}(\mathcal{Y}) = \left\{\mathcal{F}\left(y\right)(1+\xi_0)(1+w_6) - w_5\xi_1 - w_7\xi_2 + \frac{1}{8}\Theta^2\mathcal{F}\left(y\right)\xi_1^2\right\}\left(1 - \frac{1}{2}\Theta\xi_1 + \frac{3}{8}\Theta^2\xi_1^2 + \cdots\right). \tag{21}$$

By simplifying (21), we have

$$\begin{aligned}\widehat{\mathcal{F}}_{Pr_1}(\mathcal{Y}) - \mathcal{F}(y) \cong \Big[& w_6\mathcal{F}(y) + \mathcal{F}(y)\xi_0 + w_6\mathcal{F}(y)\xi_0 - \frac{1}{2}\Theta\mathcal{F}(y)\xi_1 + \Theta^2\mathcal{F}(y)\xi_1^2 - \frac{1}{2}\Theta\mathcal{F}(y)\xi_0\xi_1 - w_5\xi_1 \\ &+\frac{1}{2}\Theta w_5\xi_1^2 - \frac{1}{2}\Theta w_6\mathcal{F}(y)\xi_1 + \frac{3}{8}\Theta^2w_6\mathcal{F}(y)\xi_1^2 - \frac{1}{2}\Theta w_6\mathcal{F}(y)\xi_0\xi_1 - w_7\xi_2 + \frac{1}{2}\Theta w_7\xi_1\xi_2\Big]\end{aligned} \tag{22}$$

The bias and MSE of $\widehat{\mathcal{F}}_{Pr_1}(\mathcal{Y})$, are given as

$$\begin{aligned}\mathrm{Bias}(\widehat{\mathcal{F}}_{Pr_1}(\mathcal{Y})) \cong{}& \frac{1}{2}\Theta^2\mathcal{F}(y)\lambda C_{F_x}^2 - \frac{1}{2}\Theta\mathcal{F}(y)\lambda\rho_{12}C_{F_y}C_{F_x} + \frac{1}{2}w_5\Theta\lambda C_{F_x}^2 + w_6\mathcal{F}(y) \\ &+ \frac{3}{8}w_6\Theta^2\mathcal{F}(y)\lambda C_{F_x}^2 - \frac{1}{2}w_6\Theta\mathcal{F}(y)\lambda\rho_{12}C_{F_y}C_{F_x} + \frac{1}{2}w_7\Theta\lambda\rho_{23}C_{F_x}C_x,\end{aligned}$$

and

$$\begin{aligned}\mathrm{MSE}(\widehat{\mathcal{F}}_{Pr_1}(\mathcal{Y})) \cong{}& -\Theta\mathcal{F}^2(y)\lambda\rho_{12}C_{F_y}C_{F_x} + \frac{3}{2}w_6\Theta^2\mathcal{F}^2(y)\lambda C_{F_x}^2 + w_6^2\Theta^2\mathcal{F}^2(y)\lambda C_{F_x}^2 + w_5\Theta\mathcal{F}(y)\lambda C_{F_x}^2 \\ &- 2w_6^2\Theta\mathcal{F}^2(y)\lambda\rho_{12}C_{F_y}C_{F_x} + \mathcal{F}^2(y)\lambda C_{F_y}^2 + w_6^2\mathcal{F}^2(y) + w_7\Theta\mathcal{F}(y)\lambda\rho_{23}C_{F_x}C_x \\ &- 3w_6\Theta\mathcal{F}^2(y)\lambda\rho_{12}C_{F_y}C_{F_x} - 2w_5\mathcal{F}(y)\lambda\rho_{12}C_{F_y}C_{F_x} - 2w_7\mathcal{F}(y)\lambda\rho_{13}C_{F_y}C_x \\ &- 2w_6w_7\mathcal{F}(y)\lambda\rho_{13}C_{F_y}C_x + \frac{1}{4}\Theta^2\mathcal{F}^2(y)\lambda C_{F_x}^2 + 2w_6\mathcal{F}^2(y)\lambda C_{F_y}^2 + w_5^2\lambda C_{F_x}^2 \\ &+ 2w_5w_6\Theta\mathcal{F}(y)\lambda C_{F_x}^2 - 2w_5w_6\mathcal{F}(y)\lambda\rho_{12}C_{F_y}C_{F_x} + 2w_5w_7\lambda\rho_{23}C_{F_x}C_x \\ &+ w_6^2\mathcal{F}^2\left(y\right)\lambda C_{F_y}^2 + m_7^2\lambda C_x^2 + 2w_6w_7\Theta\mathcal{F}\left(y\right)\lambda\rho_{23}C_{F_x}C_x.\end{aligned} \tag{23}$$

The optimum values for w_5, w_6 and w_7, determined (23) are given as:

$$w_{5(\mathrm{opt})} = \frac{\mathcal{F}(y)\left[\begin{array}{c}\lambda\Theta^3C_{F_x}^3\rho_{23}^2 - \lambda\Theta^2C_{F_y}C_{F_x}^2\rho_{13}\rho_{23} - 4\lambda\Theta C_{F_y}^2C_{F_x}\rho_{12}\rho_{13}\rho_{23} - \lambda\Theta^3C_{F_x}^3 \\ +\lambda\Theta^2C_{F_y}C_{F_x}^2\rho_{12} + 2\lambda\Theta C_{F_y}^2C_{F_x}\rho_{12}^2 + 2\lambda\Theta C_{F_y}^2C_{F_x}\rho_{13}^2 + 2\lambda\Theta C_{F_y}^2C_{F_x}\rho_{23}^2 \\ -2\lambda\Theta C_{F_y}^2C_{F_x} - 2\Theta C_{F_x}\rho_{23}^2 + 4C_{F_y}\rho_{13}\rho_{23} + 2\Theta C_{F_x} - 4C_{F_y}\rho_{12}\end{array}\right]}{4C_{F_x}\left(-2\lambda C_{F_y}^2\rho_{12}\rho_{13}\rho_{23} + \lambda C_{F_y}^2\rho_{12}^2 + \lambda C_{F_y}^2\rho_{13}^2 + \lambda C_{F_y}^2\rho_{23}^2 - \lambda C_{F_y}^2 + \rho_{23}^2 - 1\right)}$$

$$w_{6(\text{opt})} = -\frac{(\Theta^2 C_{F_x}^2 \rho_{23}^2 - 8C_{F_y}^2 \rho_{12}\rho_{13}\rho_{23} - (\Theta^2 C_{F_x}^2 + 4C_{F_y}^2 \rho_{12}^2 + 4C_{F_y}^2 \rho_{13}^2 + 4C_{F_y}^2 \rho_{23}^2 - 4C_{F_y}^2)\lambda}{4(-2\lambda C_{F_y}^2 \rho_{12}\rho_{13}\rho_{23} + \lambda C_{F_y}^2 \rho_{12}^2 + \lambda C_{F_y}^2 \rho_{13}^2 + \lambda C_{F_y}^2 \rho_{23}^2 - \lambda C_{F_y}^2 + \rho_{23}^2 - 1)}$$

and

$$w_{7(\text{opt})} = \frac{-\mathcal{F}(y) C_{F_y}(\lambda\Theta^2 C_{F_x}^2 \rho_{12}\rho_{23} - \lambda\Theta^2 C_{F_x}^2 \rho_{13} - 4\rho_{12}\rho_{23} + 4\rho_{13})}{4C_x(-2\lambda C_{F_y}^2 \rho_{12}\rho_{13}\rho_{23} + \lambda C_{F_y}^2 \rho_{12}^2 + \lambda C_{F_y}^2 \rho_{13}^2 + \lambda C_{F_y}^2 \rho_{23}^2 - \lambda C_{F_y}^2 + \rho_{23}^2 - 1)},$$

$$\text{MSE}_{\min}\left(\widehat{\mathcal{F}}_{Pr_1}(\mathcal{Y})\right) \cong \frac{\mathcal{F}^2(y)\lambda\{16C_{Fy}^2(1 - R_{1.23}^2) - \lambda\Theta^4 C_{Fx}^4 - 8\lambda\Theta^2 C_{Fx}^2 C_{Fy}^2 (1 - R_{1.23}^2)\}}{16\{1 + \lambda C_{Fy}^2 (1 - R_{1.23}^2)\}} \tag{24}$$

where

$$R_{1.23}^2 = \rho_{12}^2 + \rho_{13}^2 - 2\rho_{12}\rho_{13}\rho_{23}/\left(1 - \rho_{23}^2\right).$$

Second modified class of estimator

Both the design and estimating stages of an estimator can benefit by incorporating of additional information. When the study variable is highly connected with the auxiliary variable, the rank of the auxiliary variable will also be connected with the study variable. That's why the rank of the auxiliary variable can serve as yet another piece of supplementary data. Using the idea of rank, we proposed a second modified class of estimator, given by:

$$\widehat{\mathcal{F}}_{Pr_2}(\mathcal{Y}) = \begin{bmatrix} \frac{1}{2}\widehat{\mathcal{F}}(y)\left\{\exp\left[\frac{\mathcal{F}(x)-\widehat{\mathcal{F}}(x)}{\widehat{\mathcal{F}}(x)+\mathcal{F}(x)}\right] + \exp\left[\frac{\widehat{\mathcal{F}}(x)-\mathcal{F}(x)}{\widehat{\mathcal{F}}(x)+\mathcal{F}(x)}\right]\right\} \\ +w_8\left(\mathcal{F}(x) - \widehat{\mathcal{F}}(x)\right) + w_9\widehat{\mathcal{F}}(y) + w_{10}\left(\overline{\mathcal{Z}} - \widehat{\overline{\mathcal{Z}}}\right) \end{bmatrix} \exp\left\{\frac{\alpha(\mathcal{F}(x) - \widehat{\mathcal{F}}(x))}{\alpha(\mathcal{F}(x) + \widehat{\mathcal{F}}(x)) + 2\beta}\right\}.$$

The estimator $\widehat{\mathcal{F}}_{Pr_2}(\mathcal{Y})$, can also be written as

$$\widehat{\mathcal{F}}_{Pr_2}(\mathcal{Y}) = \left\{\mathcal{F}\left(y\right)(1+\xi_0)(1+w_9) - w_8\xi_1 - w_{10}\xi_3 + \frac{1}{8}\Theta^2\mathcal{F}\left(y\right)\xi_1^2\right\}\left(1 - \frac{1}{2}\Theta\xi_1 + \frac{3}{8}\Theta^2\xi_1^2 + \cdots\right). \tag{25}$$

$$\begin{aligned}\left(\widehat{\mathcal{F}}_{Pr_2}(\mathcal{Y}) - \mathcal{F}(y)\right) \cong{}& w_9\mathcal{F}\left(y\right) + \mathcal{F}\left(y\right)\xi_0 + w_9\mathcal{F}\left(y\right)\xi_0 - \frac{1}{2}\Theta\mathcal{F}\left(y\right)\xi_1 + \frac{1}{2}\Theta^2\mathcal{F}\left(y\right)\xi_1^2 \\ & - \frac{1}{2}\Theta\mathcal{F}\left(y\right)\xi_0\xi_1 - w_8\xi_1 + \frac{1}{2}\Theta w_8\xi_1^2 - \frac{1}{2}\Theta w_9\mathcal{F}\left(y\right)\xi_1 + \frac{3}{8}\Theta^2 w_9\mathcal{F}\left(y\right)\xi_1^2 \\ & - \frac{1}{2}\Theta w_9\mathcal{F}\left(y\right)\xi_0\xi_1 - w_{10}\xi_3 + \frac{1}{2}\Theta w_{10}\xi_1\xi_3.\end{aligned} \tag{26}$$

$$\begin{aligned}\text{Bias}(\widehat{\mathcal{F}}_{Pr_2}(\mathcal{Y})) \cong{}& \frac{1}{2}\Theta^2\mathcal{F}(y)\lambda C_{F_x}^2 - \frac{1}{2}\Theta\mathcal{F}(y)\lambda\rho_{12}C_{F_y}C_{F_x} + \frac{1}{2}w_8\Theta\lambda C_{F_x}^2 + w_9\mathcal{F}(y) \\ & + \frac{3}{8}w_9\Theta^2\mathcal{F}(y)\lambda C_{F_x}^2 - \frac{1}{2}w_9\Theta\mathcal{F}(y)\lambda\rho_{12}C_{F_y}C_{F_x} + \frac{1}{2}w_{10}\Theta\lambda\rho_{24}C_{F_x}C_{rx},\end{aligned}$$

$$\begin{aligned}\text{MSE}(\widehat{\mathcal{F}}_{Pr_2}(\mathcal{Y}) \cong{}& -\Theta\mathcal{F}^2(y)\lambda\rho_{12}C_{F_y}C_{F_x} + \frac{3}{2}w_9\Theta^2\mathcal{F}^2(y)\lambda C_{F_x}^2 + w_9^2\Theta^2\mathcal{F}^2(y)\lambda C_{F_x}^2 \\ & + w_8\Theta\mathcal{F}(y)\lambda C_{F_x}^2 - 2w_9^2\Theta\mathcal{F}^2(y)\lambda\rho_{12}C_{F_y}C_{F_x} + \mathcal{F}^2(y)\lambda C_{F_y}^2 + w_9^2\mathcal{F}^2(y) \\ & + w_{10}\Theta\mathcal{F}(y)\lambda\rho_{24}C_{F_x}C_{rx} - 3w_9\Theta\mathcal{F}^2(y)\lambda\rho_{12}C_{F_y}C_{F_x} - 2w_8\mathcal{F}(y)\lambda\rho_{12}C_{F_y}C_{F_x} \\ & - 2w_{10}\mathcal{F}(y)\lambda\rho_{14}C_{F_y}C_{rx} - 2w_9w_{10}\mathcal{F}(y)\lambda\rho_{14}C_{F_y}C_{rx} + \frac{1}{4}\Theta^2\mathcal{F}^2(y)\lambda C_{F_x}^2 \\ & + 2w_9\mathcal{F}^2(y)\lambda C_{F_y}^2 + w_8^2\lambda C_{F_x}^2 + 2w_8w_9\Theta\mathcal{F}(y)\lambda C_{F_x}^2 - 2w_8w_9\mathcal{F}(y)\lambda\rho_{12}C_{F_y}C_{F_x} \\ & + 2w_8w_{10}\lambda\rho_{24}C_{F_x}C_{rx} + w_9^2\mathcal{F}^2(y)\lambda C_{F_y}^2 + m_7^2\lambda C_{rx}^2 + 2w_9w_{10}\Theta\mathcal{F}(y)\lambda\rho_{24}C_{F_x}C_{rx}.\end{aligned} \tag{27}$$

The values of w_8, w_9 and w_{10}, are given by:

$$w_{8(\text{opt})} = \frac{\mathcal{F}(y)\begin{bmatrix} \lambda\Theta^3 C_{F_x}^3\rho_{24}^2 - \lambda\Theta^2 C_{F_y}C_{F_x}^2\rho_{14}\rho_{24} - 4\lambda\Theta C_{F_y}^2 C_{F_x}\rho_{12}\rho_{14}\rho_{24} - \lambda\Theta^3 C_{F_x}^3 \\ +\lambda\Theta^2 C_{F_y}C_{F_x}^2\rho_{12} + 2\lambda\Theta C_{F_y}^2 C_{F_x}\rho_{12}^2 + 2\lambda\Theta C_{F_y}^2 C_{F_x}\rho_{14}^2 + 2\lambda\Theta C_{F_y}^2 C_{F_x}\rho_{24}^2 \\ -2\lambda\Theta C_{F_y}^2 C_{F_x} - 2\Theta C_{F_x}\rho_{24}^2 + 4C_{F_y}\rho_{14}\rho_{24} + 2\Theta C_{F_x} - 4C_{F_y}\rho_{12} \end{bmatrix}}{4C_{F_x}\left(-2\lambda C_{F_y}^2\rho_{12}\rho_{14}\rho_{24} + \lambda C_{F_y}^2\rho_{12}^2 + \lambda C_{F_y}^2\rho_{14}^2 + \lambda C_{F_y}^2\rho_{24}^2 - \lambda C_{F_y}^2 + \rho_{24}^2 - 1\right)}$$

$$w_{9(\text{opt})} = -\frac{(\Theta^2 C_{F_x}^2 \rho_{24}^2 - 8C_{F_y}^2 \rho_{12}\rho_{14}\rho_{24} - (\Theta^2 C_{F_x}^2 + 4C_{F_y}^2 \rho_{12}^2 + 4C_{F_y}^2 \rho_{14}^2 + 4C_{F_y}^2 \rho_{24}^2 - 4C_{F_y}^2)\lambda}{4\left(-2\lambda C_{F_y}^2 \rho_{12}\rho_{14}\rho_{24} + \lambda C_{F_y}^2 \rho_{12}^2 + \lambda C_{F_y}^2 \rho_{14}^2 + \lambda C_{F_y}^2 \rho_{24}^2 - \lambda C_{F_y}^2 + \rho_{24}^2 - 1\right)},$$

and

$$w_{10(\text{opt})} = \frac{-\mathcal{F}(y) C_{F_y}(\lambda\Theta^2 C_{F_x}^2 \rho_{12}\rho_{24} - \lambda\Theta^2 C_{F_x}^2 \rho_{14} - 4\rho_{12}\rho_{24} + 4\rho_{14})}{4C_{rx}(-2\lambda C_{F_y}^2 \rho_{12}\rho_{14}\rho_{24} + \lambda C_{F_y}^2 \rho_{12}^2 + \lambda C_{F_y}^2 \rho_{14}^2 + \lambda C_{F_y}^2 \rho_{24}^2 - \lambda C_{F_y}^2 + \rho_{24}^2 - 1)},$$

The MSE of $\widehat{\mathcal{F}}_{Pr_2}(\mathcal{Y})$ at the values of w_8, w_9, and w_{10}, is given by:

$$\text{MSE}_{\min}\left(\widehat{\mathcal{F}}_{Pr_2}(\mathcal{Y})\right) \cong \left(\frac{\mathcal{F}^2(y)\lambda\{16C_{Fy}^2(1 - R_{1.24}^2) - \lambda\Theta^4 C_{Fx}^4 - 8\lambda\Theta^2 C_{Fx}^2 C_{Fy}^2(1 - R_{1.24}^2)\}}{16\{1 + \lambda C_{Fy}^2(1 - R_{1.24}^2)\}}\right) \tag{28}$$

where $R_{1.24}^2 = \left[\rho_{12}^2 + \rho_{14}^2 - 2\rho_{12}\rho_{14}\rho_{24}/\left(1 - \rho_{24}^2\right)\right]$ (Table 1).

Numerical study

We take a numerical analysis to compare the existing and the suggested classes of estimators. Six actual data sets are used for this purpose. Tables 2, 3, 4, 5, 6 and 7 present aggregate statistics for the provided data. PRE of an estimator $\widehat{\mathcal{F}}_i(\mathcal{Y})$ concerning $\widehat{\mathcal{F}}(y)$ is

$$\text{PRE}\left(\widehat{\mathcal{F}}_i(\mathcal{Y}), \widehat{\mathcal{F}}(y)\right) = \frac{\text{Var}\left(\widehat{\mathcal{F}}(y)\right)}{\text{MSE}\left(\widehat{\mathcal{F}}_i(\mathcal{Y})\right)} \times 100,$$

Population-I: [Source:[23]]:
Y: Number of instructors,

α	β	$\widehat{\mathcal{F}}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}(\mathcal{Y})$
1	C_{F_x}	$\widehat{\mathcal{F}}_{GK}^{(1)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}^{(1)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}^{(1)}(\mathcal{Y})$
1	β_2	$\widehat{\mathcal{F}}_{GK}^{(2)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}^{(2)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}^{(2)}(\mathcal{Y})$
β_2	C_{F_x}	$\widehat{\mathcal{F}}_{GK}^{(3)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}^{(3)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}^{(3)}(\mathcal{Y})$
C_{F_x}	β_2	$\widehat{\mathcal{F}}_{GK}^{(4)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}^{(4)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}^{(4)}(\mathcal{Y})$
1	ρ_{12}	$\widehat{\mathcal{F}}_{GK}^{(5)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}^{(5)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}^{(5)}(\mathcal{Y})$
C_{F_x}	ρ_{12}	$\widehat{\mathcal{F}}_{GK}^{(6)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}^{(6)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}^{(6)}(\mathcal{Y})$
ρ_{12}	C_{F_x}	$\widehat{\mathcal{F}}_{GK}^{(7)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}^{(7)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}^{(7)}(\mathcal{Y})$
β_2	ρ_{12}	$\widehat{\mathcal{F}}_{GK}^{(8)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}^{(8)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}^{(8)}(\mathcal{Y})$
ρ_{12}	β_2	$\widehat{\mathcal{F}}_{GK}^{(9)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}^{(9)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}^{(9)}(\mathcal{Y})$
1	$N\mathcal{F}(x)$	$\widehat{\mathcal{F}}_{GK}^{(10)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}^{(10)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}^{(10)}(\mathcal{Y})$

Table 1. Some elements of existing and suggested estimators.

Parameter	Value	Parameter	*x* and *y* (value)			
			$\overline{\mathcal{X}}, \overline{\mathcal{Y}}$	$Q_1(x), Q_1(y)$	$\widetilde{\mathcal{X}}, \widetilde{\mathcal{Y}}$	$Q_3(x), Q_3(y)$
N	923	$\mathcal{F}(y)$	0.7703	0.2556	0.5016	0.7497
n	180	C_{F_y}	0.5463	1.7070	0.9973	0.5780
λ	0.00447	$\mathcal{F}(x)$	0.7693	0.2503	0.5005	0.7508
$\overline{\mathcal{X}}$	11,440.5	C_{F_x}	0.5480	1.7317	0.9995	0.5764
C_x	1.86453	ρ_{12}	0.8930	0.8711	0.8462	0.8930
$\overline{\mathcal{Z}}$	461.000	ρ_{13}	−0.6640	−0.2861	−0.4416	−0.6465
C_{rx}	0.57703	ρ_{23}	−0.6753	−0.2838	0.4480	−0.6563
		ρ_{14}	−0.7169	−0.7424	−0.8286	−0.7402
		ρ_{24}	−0.7298	−0.7503	−0.8660	−0.7492
		β_2	1.6333	1.3295	1.0000	1.3449

Table 2. Data description using Population 1.

Parameter	Value	Parameter	x and y (value)			
			$\overline{\mathcal{X}}, \overline{\mathcal{Y}}$	$Q_1(x), Q_1(y)$	$\widetilde{\mathcal{X}}, \widetilde{\mathcal{Y}}$	$Q_3(x), Q_3(y)$
N	923	$\mathcal{F}(y)$	0.7703	0.2556	0.5016	0.7497
n	180	C_{F_y}	0.5463	1.7070	0.9973	0.5780
λ	0.00447	$\mathcal{F}(x)$	0.7291	0.2524	0.5016	0.7508
$\overline{\mathcal{X}}$	333.165	C_{F_x}	0.6098	1.7218	0.9973	0.5764
C_x	1.32809	ρ_{12}	0.8727	0.1910	0.8917	0.9162
$\overline{\mathcal{Z}}$	461.000	ρ_{13}	−0.7385	−0.3656	−0.5366	−0.7221
C_{rx}	0.57703	ρ_{23}	−0.7120	−0.1046	−0.5419	−0.7298
		ρ_{14}	−0.7223	−0.7424	−0.8486	−0.7430
		ρ_{24}	−0.7697	−0.7503	−0.8660	−0.7491
		β_2	1.0634	1.2990	1.0000	1.3449

Table 3. Data description using Population 2.

Parameter	Value	Parameter	x and y (value)			
			$\overline{\mathcal{X}}, \overline{\mathcal{Y}}$	$Q_1(x), Q_1(y)$	$\widetilde{\mathcal{X}}, \widetilde{\mathcal{Y}}$	$Q_3(x), Q_3(y)$
N	69	$\mathcal{F}(y)$	0.7246	0.2464	0.5072	0.7536
n	10	C_{F_y}	0.6209	1.7618	0.9928	0.5759
λ	0.08550	$\mathcal{F}(x)$	0.7681	0.2464	0.5072	0.7536
$\overline{\mathcal{X}}$	4954.44	C_{F_x}	0.5535	1.7618	0.9928	0.5759
C_x	1.42478	ρ_{12}	0.6607	0.7658	0.9420	0.7658
$\overline{\mathcal{Z}}$	35.0000	ρ_{13}	−0.7129	−0.3612	−0.5709	−0.7424
C_{rx}	0.57321	ρ_{23}	−0.7745	−0.3650	−0.5427	−0.7584
		ρ_{14}	−0.7168	−0.7109	−0.8558	−0.7008
		ρ_{24}	−0.7310	−0.7464	−0.8660	−0.7464
		β_2	1.6144	1.3857	1.0000	1.3857

Table 4. Data description using Population 3.

Parameter	Value	Parameter	x and y (value)			
			$\overline{\mathcal{X}}, \overline{\mathcal{Y}}$	$Q_1(x), Q_1(y)$	$\widetilde{\mathcal{X}}, \widetilde{\mathcal{Y}}$	$Q_3(x), Q_3(y)$
N	69	$\mathcal{F}(y)$	0.7246	0.2464	0.5072	0.7536
n	10	C_{F_y}	0.6209	1.7618	0.9928	0.5759
λ	0.08550	$\mathcal{F}(x)$	0.7391	0.2464	0.5072	0.7536
$\overline{\mathcal{X}}$	4591.72	C_{F_x}	0.5984	1.7618	0.9928	0.5759
C_x	1.37554	ρ_{12}	0.8159	0.6878	0.7100	0.7658
$\overline{\mathcal{Z}}$	35.0000	ρ_{13}	−0.7102	−0.3677	−0.5519	−0.7354
C_{rx}	0.57321	ρ_{23}	−0.7556	−0.3717	−0.7689	−0.7584
		ρ_{14}	−0.7282	−0.7424	−0.7903	−0.7008
		ρ_{24}	−0.7606	−0.7464	−0.8660	−0.7464
		β_2	1.1863	1.385	1.0000	1.3857

Table 5. Data description using Population 4.

X: number of pupils,
Z: order of X.
Population-II: [Source:[23]].
Y: number of an instructors,
X: number of classes,
Z: order of X.
Population III: [Source:[24]].
Y: the number of fish caught in 1995,
X: The number of fish caught in 1994,
Z: order of X.

Parameter	Value	Parameter	x and y (value)			
			$\overline{\mathcal{X}},\overline{\mathcal{Y}}$	$Q_1(x), Q_1(y)$	$\widetilde{\mathcal{X}},\widetilde{\mathcal{Y}}$	$Q_3(x), Q_3(y)$
N	50	$\mathcal{F}(y)$	0.6800	0.2400	0.5000	0.7600
n	5	C_{F_y}	0.6929	1.7975	1.0102	0.5677
λ	0.18000	$\mathcal{F}(x)$	0.5800	0.2600	0.5000	0.7600
$\overline{\mathcal{X}}$	78.2900	C_{F_x}	0.8596	1.7042	1.0102	0.5677
C_x	0.27229	ρ_{12}	−0.1494	−0.0128	−0.120	−0.2061
$\overline{\mathcal{Z}}$	25.5000	ρ_{13}	0.2841	0.3322	0.2292	0.2174
C_{rx}	0.57159	ρ_{23}	−0.8094	−0.6202	−0.7894	−0.8215
		ρ_{14}	0.2526	0.2402	0.1843	0.1882
		ρ_{24}	−0.8551	−0.7599	−0.8663	−0.7399
		β_2	1.1050	1.1975	1.0000	1.4824

Table 6. Data description using Population 5.

Parameter	Value	Parameter	x and y (value)			
			$\overline{\mathcal{X}},\overline{\mathcal{Y}}$	$Q_1(x), Q_1(y)$	$\widetilde{\mathcal{X}},\widetilde{\mathcal{Y}}$	$Q_3(x), Q_3(y)$
N	854	$\mathcal{F}(y)$	0.8934	0.2494	0.5012	0.7506
n	140	C_{F_y}	0.3456	1.7358	0.9982	0.5768
λ	0.18000	$\mathcal{F}(x)$	0.8279	0.2494	0.5012	0.7506
$\overline{\mathcal{X}}$	37,600.1	C_{F_x}	0.4563	1.7358	0.9982	0.5768
C_x	3.85089	ρ_{12}	0.6870	0.7623	0.7658	0.7498
$\overline{\mathcal{Z}}$	427.500	ρ_{13}	−0.4550	−0.1413	−0.2260	−0.3351
C_{rx}	0.57700	ρ_{23}	−0.4118	−0.1448	−0.2345	−0.3522
		ρ_{14}	−0.5137	−0.6914	−0.7834	−0.7003
		ρ_{24}	−0.6538	−0.7494	−0.8660	−0.7490
		β_2	1.1050	1.1975	1.0000	1.4824

Table 7. Data description using Population 6.

Population IV: [Source:[24]]:
Y: the number of fish caught in 1995,
X: The number of fish caught in 1993,
Z: order of X.
Population V: [Source:[25]].
Y: The eggs formed in 1990,
X: The amount of per dozen eggs in 1990,
Z: order of X.
Population VI: [Source:[26]].
Y: The production of apple in 1999,
X: The number of apple plants in 1999,
Z: order of X.

Simulation study

We have generated three populations of size 1000 from a multivariate normal distribution with different covariance matrices. All the populations have different correlations i.e., the auxiliary variable (X) and study variable (Y) are negatively correlated in Population I, but the same variables are positively correlated in Population II, and strongly positive association in case of Population III correlation.

Population-I:

$$\mu_1 = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

and

$$\sum_1 = \begin{bmatrix} 4 & -9.0 \\ -9.0 & 64 \end{bmatrix}$$

$$\rho_{XY} = -0.590220$$

Population-II:

$$\mu_2 = \begin{bmatrix} 5 \\ 5 \end{bmatrix},$$

$$\sum_2 = \begin{bmatrix} 4 & 9.5 \\ 9.5 & 63 \end{bmatrix}$$

$$\rho_{XY} = 0.612254$$

Population-III:

$$\mu_3 = \begin{bmatrix} 5 \\ 5 \end{bmatrix},$$

$$\sum_3 = \begin{bmatrix} 2 & 4 \\ 6 & 10 \end{bmatrix}$$

$\rho_{XY} = 0.902645$.

The Percentage Relative Efficiency (PRE) is calculated as follows:

$$\text{PRE}\left(\widehat{\mathcal{F}}_i(\mathcal{Y}), \widehat{\mathcal{F}}(y)\right) = \frac{\text{Var}\left(\widehat{\mathcal{F}}(y)\right)}{\text{MSE}\left(\widehat{\mathcal{F}}_i(\mathcal{Y})\right)} \times 100,$$

The results of MSE and PRE are given in Tables 16 and 17. Here we can only point out the best results of MSEs and PREs in these tables when $\Theta = \frac{\alpha \mathcal{F}(x)}{\alpha \mathcal{F}(x) + \beta}$ if $\alpha = C_{F_x}$ and $\beta = \beta_2$.

Discussion

Table 1, include some elements of the existing and suggested classes of estimators. From the numerical results, which are presented in Tables 8, 9, 10, 11, 12, 13, 14 and 15, We want to bring back the fact that PRE varies for the different choices of a and b. For the data sets, if we use ($\alpha = 1$ and $\beta = \rho_{12}$), ($\alpha = C_{F_x}$ and $\beta = \rho_{12}$) and ($\alpha = \beta_2$ and $\beta = \rho_{12}$) we get the largest values of PRE of all families among different classes. Consequently, the

Estimators			Population 1			Population 2			Population 3		
			$\widehat{\mathcal{F}}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}(\mathcal{Y})$
$\widehat{\mathcal{F}}^{(1)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(1)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(1)}_{Pr_2}(\mathcal{Y})$	493.99	511.31	517.30	419.78	475.32	431.20	181.17	220.67	228.54
$\widehat{\mathcal{F}}^{(2)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(2)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(2)}_{Pr_2}(\mathcal{Y})$	493.94	511.19	517.18	419.74	475.20	431.10	180.82	219.78	227.61
$\widehat{\mathcal{F}}^{(3)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(3)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(3)}_{Pr_2}(\mathcal{Y})$	494.05	511.47	517.46	419.83	475.45	431.31	181.53	221.65	229.55
$\widehat{\mathcal{F}}^{(4)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(4)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(4)}_{Pr_2}(\mathcal{Y})$	493.93	511.17	517.17	419.73	475.19	431.09	180.78	219.68	227.51
$\widehat{\mathcal{F}}^{(5)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(5)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(5)}_{Pr_2}(\mathcal{Y})$	493.97	511.25	517.25	419.77	475.27	431.16	181.11	220.51	228.37
$\widehat{\mathcal{F}}^{(6)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(6)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(6)}_{Pr_2}(\mathcal{Y})$	493.94	511.20	517.19	419.75	475.23	431.12	180.95	220.08	227.93
$\widehat{\mathcal{F}}^{(7)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(7)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(7)}_{Pr_2}(\mathcal{Y})$	493.98	511.29	517.29	419.78	475.30	431.18	181.04	220.32	228.17
$\widehat{\mathcal{F}}^{(8)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(8)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(8)}_{Pr_2}(\mathcal{Y})$	494.02	511.38	517.38	419.80	475.38	431.26	181.47	221.47	229.37
$\widehat{\mathcal{F}}^{(9)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(9)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(9)}_{Pr_2}(\mathcal{Y})$	493.93	511.18	517.18	419.74	475.20	431.09	180.79	219.70	227.53
$\widehat{\mathcal{F}}^{(10)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(10)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(10)}_{Pr_2}(\mathcal{Y})$	493.93	511.17	517.16	419.73	475.18	431.08	180.76	219.63	227.46
	$\widehat{\mathcal{F}}(y)$			100.00			100.00				100.00
	$\widehat{\mathcal{F}}_R(\mathcal{Y})$			465.57			336.04				161.16
	$\widehat{\mathcal{F}}_P(\mathcal{Y})$			25.470			23.310				33.340
	$\widehat{\mathcal{F}}_{BT,R}(\mathcal{Y})$			281.07			296.44				164.00
	$\widehat{\mathcal{F}}_{BT,P}(\mathcal{Y})$			46.570			43.750				55.940
	$\widehat{\mathcal{F}}_{Reg}(\mathcal{Y})$			493.79			419.59				177.46
	$\widehat{\mathcal{F}}_{R,D}(\mathcal{Y})$			493.93			419.73				180.76

Table 8. Percentage relative efficiency using Population 1, 2, 3 when $\{x = \overline{\mathcal{X}}, y = \overline{\mathcal{Y}}\}$.

Estimators			Population 1			Population 2			Population 3		
			$\widehat{\mathcal{F}}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}(\mathcal{Y})$
$\widehat{\mathcal{F}}^{(1)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(1)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(1)}_{Pr_2}(\mathcal{Y})$	415.95	418.83	449.59	105.10	119.99	227.16	268.65	273.50	297.65
$\widehat{\mathcal{F}}^{(2)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(2)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(2)}_{Pr_2}(\mathcal{Y})$	415.94	418.80	449.57	105.09	119.98	227.15	268.54	273.28	297.40
$\widehat{\mathcal{F}}^{(3)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(3)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(3)}_{Pr_2}(\mathcal{Y})$	416.02	418.96	449.74	105.12	120.03	227.23	269.51	275.29	299.59
$\widehat{\mathcal{F}}^{(4)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(4)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(4)}_{Pr_2}(\mathcal{Y})$	415.96	418.85	449.62	105.10	119.99	227.17	268.60	273.83	298.00
$\widehat{\mathcal{F}}^{(5)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(5)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(5)}_{Pr_2}(\mathcal{Y})$	416.00	418.92	449.69	105.20	120.24	227.67	269.45	275.16	299.46
$\widehat{\mathcal{F}}^{(6)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(6)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(6)}_{Pr_2}(\mathcal{Y})$	416.09	419.10	449.89	105.27	120.39	227.96	270.78	277.99	301.54
$\widehat{\mathcal{F}}^{(7)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(7)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(7)}_{Pr_2}(\mathcal{Y})$	415.95	418.82	449.58	105.09	119.97	227.13	268.55	273.3)	297.43
$\widehat{\mathcal{F}}^{(8)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(8)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(8)}_{Pr_2}(\mathcal{Y})$	416.16	419.27	450.07	105.29	120.46	228.13	271.88	280.38	305.18
$\widehat{\mathcal{F}}^{(9)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(9)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(9)}_{Pr_2}(\mathcal{Y})$	415.94	418.80	449.57	105.93	119.97	227.13	268.48	273.15	297.27
$\widehat{\mathcal{F}}^{(10)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(10)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(10)}_{Pr_2}(\mathcal{Y})$	415.93	418.78	449.54	105.09	119.97	227.13	268.39	271.97	297.06
	$\widehat{\mathcal{F}}(y)$			100.00			100.00				100.00
	$\widehat{\mathcal{F}}_{R}(\mathcal{Y})$			381.08			61.280				213.53
	$\widehat{\mathcal{F}}_{P}(\mathcal{Y})$			18.950			25.700				19.220
	$\widehat{\mathcal{F}}_{BT,R}(\mathcal{Y})$			267.67			94.190				206.54
	$\widehat{\mathcal{F}}_{BT,P}(\mathcal{Y})$			46.700			69.100				49.600
	$\widehat{\mathcal{F}}_{Reg}(\mathcal{Y})$			414.63			103.79				241.84
	$\widehat{\mathcal{F}}_{R,D}(\mathcal{Y})$			415.93			105.09				268.39

Table 9. Percentage relative efficiency using Population 1, 2, 3 when $\left\{x = Q_1(x), y = Q_1(y)\right\}$.

Estimators			Population 1			Population 2			Population 3		
			$\widehat{\mathcal{F}}^{(1)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(1)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}(\mathcal{Y})$
$\widehat{\mathcal{F}}^{(1)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(1)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(1)}_{Pr_2}(\mathcal{Y})$	351.58	358.79	404.92	488.43	498.33	551.21	140.84	898.79	913.16
$\widehat{\mathcal{F}}^{(2)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(2)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(2)}_{Pr_2}(\mathcal{Y})$	351.58	358.79	404.92	488.43	498.33	551.21	140.56	898.77	913.10
$\widehat{\mathcal{F}}^{(3)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(3)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(3)}_{Pr_2}(\mathcal{Y})$	351.78	358.79	404.92	488.43	498.33	551.21	140.86	898.79	913.17
$\widehat{\mathcal{F}}^{(4)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(4)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(4)}_{Pr_2}(\mathcal{Y})$	351.58	358.79	404.92	488.43	498.33	551.21	140.32	898.75	913.05
$\widehat{\mathcal{F}}^{(5)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(5)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(5)}_{Pr_2}(\mathcal{Y})$	351.59	358.81	404.94	488.45	498.35	551.24	141.67	898.97	913.56
$\widehat{\mathcal{F}}^{(6)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(6)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(6)}_{Pr_2}(\mathcal{Y})$	351.59	358.81	404.94	488.44	498.35	551.23	141.41	898.94	913.50
$\widehat{\mathcal{F}}^{(7)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(7)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(7)}_{Pr_2}(\mathcal{Y})$	351.57	358.77	404.89	488.43	498.30	551.19	138.82	898.60	911.74
$\widehat{\mathcal{F}}^{(8)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(8)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(8)}_{Pr_2}(\mathcal{Y})$	351.59	358.81	404.94	488.45	498.35	551.24	141.70	898.98	913.57
$\widehat{\mathcal{F}}^{(9)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(9)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(9)}_{Pr_2}(\mathcal{Y})$	351.57	358.77	404.89	488.43	498.30	551.19	138.56	989.59	911.69
$\widehat{\mathcal{F}}^{(10)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(10)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(10)}_{Pr_2}(\mathcal{Y})$	351.53	358.69	404.80	488.37	489.19	551.07	101.36	896.49	908.20
	$\widehat{\mathcal{F}}(y)$			100.00			100.00				100.00
	$\widehat{\mathcal{F}}_{R}(\mathcal{Y})$			324.29			461.49				861.32
	$\widehat{\mathcal{F}}_{P}(\mathcal{Y})$			27.060			26.450				25.790
	$\widehat{\mathcal{F}}_{BT,R}(\mathcal{Y})$			248.08			279.06				324.69
	$\widehat{\mathcal{F}}_{BT,P}(\mathcal{Y})$			47.640			46.690				45.620
	$\widehat{\mathcal{F}}_{Reg}(\mathcal{Y})$			351.08			487.23				888.06
	$\widehat{\mathcal{F}}_{R,D}(\mathcal{Y})$			351.53			488.37				896.49

Table 10. Percentage relative efficiency using Population 1, 2, 3 when $\left\{x = \widetilde{\mathcal{X}}, y = \widetilde{\mathcal{Y}}\right\}$.

ideal results from the families of estimators are attained by choosing and as the coefficients of variation, kurtosis, and correlation, respectively. It is also found that the second proposed class of estimators $\widehat{\mathcal{F}}_{Pr_2}(\mathcal{Y})$ behaves slightly better than the first proposed family of estimators $\widehat{\mathcal{F}}_{Pr_1}(\mathcal{Y})$, shown in Tables 8, 9, 10, 11, 12, 13, 14 and 15 which demonstrate the average gain inadequacies for the six populations, respectively, while the first suggested class of estimator $\widehat{\mathcal{F}}_{Pr_1}(\mathcal{Y})$ performs better over the second suggetsed class of estimator $\widehat{\mathcal{F}}_{Pr_2}(\mathcal{Y})$ with substantial normal

Estimators			Population 1			Population 2			Population 3		
			$\widehat{\mathcal{F}}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}(\mathcal{Y})$
$\widehat{\mathcal{F}}_{GK}^{(1)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}^{(1)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}^{(1)}(\mathcal{Y})$	494.04	510.24	523.86	621.69	647.49	651.53	245.26	288.33	270.29
$\widehat{\mathcal{F}}_{GK}^{(2)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}^{(2)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}^{(2)}(\mathcal{Y})$	493.99	510.12	523.81	621.62	647.33	651.36	244.78	287.10	269.15
$\widehat{\mathcal{F}}_{GK}^{(3)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}^{(3)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}^{(3)}(\mathcal{Y})$	494.10	510.39	524.09	621.77	647.70	651.74	245.76	289.70	271.56
$\widehat{\mathcal{F}}_{GK}^{(4)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}^{(4)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}^{(4)}(\mathcal{Y})$	493.98	510.10	523.79	621.61	647.32	651.35	244.72	286.97	269.02
$\widehat{\mathcal{F}}_{GK}^{(5)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}^{(5)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}^{(5)}(\mathcal{Y})$	494.02	210.18	523.88	621.66	647.42	651.45	245.12	287.96	269.95
$\widehat{\mathcal{F}}_{GK}^{(6)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}^{(6)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}^{(6)}(\mathcal{Y})$	493.99	510.14	523.83	621.63	647.36	651.39	244.91	287.43	269.45
$\widehat{\mathcal{F}}_{GK}^{(7)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}^{(7)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}^{(7)}(\mathcal{Y})$	494.04	510.22	523.92	621.68	647.48	651.51	245.13	287.98	269.97
$\widehat{\mathcal{F}}_{GK}^{(8)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}^{(8)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}^{(8)}(\mathcal{Y})$	494.07	510.30	524.00	621.72	647.58	651.62	245.59	289.24	271.13
$\widehat{\mathcal{F}}_{GK}^{(9)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}^{(9)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}^{(9)}(\mathcal{Y})$	493.99	510.11	523.80	621.62	647.33	651.36	244.74	287.03	269.08
$\widehat{\mathcal{F}}_{GK}^{(10)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}^{(10)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}^{(10)}(\mathcal{Y})$	493.98	510.09	523.79	621.6	647.30	651.33	244.68	286.87	268.93
	$\widehat{\mathcal{F}}(y)$			100.00			100.00				100.00
	$\widehat{\mathcal{F}}_{R}(\mathcal{Y})$			468.75			598.06				213.53
	$\widehat{\mathcal{F}}_{P}(\mathcal{Y})$			26.04			25.730				27.780
	$\widehat{\mathcal{F}}_{BT,R}(\mathcal{Y})$			279.25			298.47				206.54
	$\widehat{\mathcal{F}}_{BT,P}(\mathcal{Y})$			46.750			46.250				49.600
	$\widehat{\mathcal{F}}_{Reg}(\mathcal{Y})$			493.83			621.46				241.84
	$\widehat{\mathcal{F}}_{R,D}(\mathcal{Y})$			493.98			621.60				244.68

Table 11. Percentage relative efficiency using Population 1, 2, 3 when $\{x = Q_3(x), y = Q_3(y)\}$.

Estimators			Population 4			Population 5			Population 6		
			$\widehat{\mathcal{F}}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}(\mathcal{Y})$
$\widehat{\mathcal{F}}_{GK}^{(1)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}^{(1)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}^{(1)}(\mathcal{Y})$	303.27	323.67	330.99	111.54	121.05	118.69	189.48	203.23	191.16
$\widehat{\mathcal{F}}_{GK}^{(2)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}^{(2)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}^{(2)}(\mathcal{Y})$	301.67	321.31	329.60	111.37	120.67	118.33	189.46	203.17	191.10
$\widehat{\mathcal{F}}_{GK}^{(3)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}^{(3)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}^{(3)}(\mathcal{Y})$	303.87	325.16	331.53	111.62	121.22	118.87	189.51	203.30	191.22
$\widehat{\mathcal{F}}_{GK}^{(4)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}^{(4)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}^{(4)}(\mathcal{Y})$	301.58	321.13	329.42	111.29	120.49	118.15	189.45	203.17	191.10
$\widehat{\mathcal{F}}_{GK}^{(5)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}^{(5)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}^{(5)}(\mathcal{Y})$	303.06	323.19	330.51	119.42	143.76	140.81	189.47	203.21	191.14
$\widehat{\mathcal{F}}_{GK}^{(6)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}^{(6)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}^{(6)}(\mathcal{Y})$	301.81	321.63	329.93	120.76	148.63	145.52	189.46	203.19	191.12
$\widehat{\mathcal{F}}_{GK}^{(7)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}^{(7)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}^{(7)}(\mathcal{Y})$	303.13	323.35	330.67	110.97	119.80	117.47	189.47	203.21	191.14
$\widehat{\mathcal{F}}_{GK}^{(8)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}^{(8)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}^{(8)}(\mathcal{Y})$	303.62	324.54	331.89	118.76	141.50	138.62	189.50	203.28	191.21
$\widehat{\mathcal{F}}_{GK}^{(9)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}^{(9)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}^{(9)}(\mathcal{Y})$	301.63	321.22	329.51	110.95	119.75	117.63	189.45	203.17	191.10
$\widehat{\mathcal{F}}_{GK}^{(10)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}^{(10)}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}^{(10)}(\mathcal{Y})$	301.52	321.99	329.74	110.93	119.70	117.38	189.45	203.17	191.10
	$\widehat{\mathcal{F}}(y)$			100.00			100.00				100.00
	$\widehat{\mathcal{F}}_{R}(\mathcal{Y})$			280.87			34.370				17.380
	$\widehat{\mathcal{F}}_{P}(\mathcal{Y})$			28.490			51.640				107.62
	$\widehat{\mathcal{F}}_{BT,R}(\mathcal{Y})$			224.30			63.690				189.38
	$\widehat{\mathcal{F}}_{BT,P}(\mathcal{Y})$			49.400			83.380				189.45
	$\widehat{\mathcal{F}}_{Reg}(\mathcal{Y})$			299.22			101.28				189.13
	$\widehat{\mathcal{F}}_{R,D}(\mathcal{Y})$			301.52			110.93				41.680

Table 12. Percentage relative efficiency using Population 4, 5, 6 when $\{x = \overline{\mathcal{X}}, y = \overline{\mathcal{Y}}\}$.

improvement inadequacies for the second population. While from Tables 8, 9, 10, 11, 12, 13, 14 and 15 the PRE of all families are diminishing diagonally the values of $(\alpha = 1$ and $\beta = N\mathcal{F}(x))$.

For visualization, we take population 1 and 4 respectively, in descriptions of these graphs we mention that what kind of trash holed we used for finding distribution function. The comparison of numerous estimators in terms of PRE for six populations is depicted in Figs. 1, 2, 3, 4, 5, 6, 7 and 8. The length of a bar is directly associated with the efficiency of an estimator. However, it can be conditional that the suggested estimators, in our

Estimators			Population 4			Population 5			Population 6		
			$\widehat{\mathcal{F}}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}(\mathcal{Y})$
$\widehat{\mathcal{F}}^{(1)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(1)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(1)}_{Pr_2}(\mathcal{Y})$	216.53	221.15	251.90	158.54	179.63	173.36	240.55	241.12	260.94
$\widehat{\mathcal{F}}^{(2)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(2)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(2)}_{Pr_2}(\mathcal{Y})$	216.44	221.97	251.69	158.41	179.33	173.07	240.54	241.11	260.93
$\widehat{\mathcal{F}}^{(3)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(3)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(3)}_{Pr_2}(\mathcal{Y})$	217.22	223.59	253.54	159.50	181.82	175.48	240.60	241.23	261.06
$\widehat{\mathcal{F}}^{(4)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(4)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(4)}_{Pr_2}(\mathcal{Y})$	216.65	221.41	251.19	158.76	180.13	173.85	240.56	241.14	260.96
$\widehat{\mathcal{F}}^{(5)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(5)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(5)}_{Pr_2}(\mathcal{Y})$	217.32	223.81	253.78	186.88	267.89	257.67	240.60	241.22	261.05
$\widehat{\mathcal{F}}^{(6)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(6)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(6)}_{Pr_2}(\mathcal{Y})$	218.52	226.37	256.71	185.40	261.80	251.90	240.68	241.39	261.23
$\widehat{\mathcal{F}}^{(7)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(7)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(7)}_{Pr_2}(\mathcal{Y})$	216.42	221.94	251.65	158.18	178.80	171.56	240.54	241.11	260.93
$\widehat{\mathcal{F}}^{(8)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(8)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(8)}_{Pr_2}(\mathcal{Y})$	219.48	228.48	259.13	184.96	260.00	250.20	240.74	241.52	261.37
$\widehat{\mathcal{F}}^{(9)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(9)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(9)}_{Pr_2}(\mathcal{Y})$	216.37	221.84	251.54	158.18	178.80	171.56	240.54	241.10	260.92
$\widehat{\mathcal{F}}^{(10)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(10)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(10)}_{Pr_2}(\mathcal{Y})$	216.31	221.72	251.40	158.19	178.80	171.58	240.53	241.09	260.91
	$\widehat{\mathcal{F}}(y)$			100.00			100.00				100.00
	$\widehat{\mathcal{F}}_R(\mathcal{Y})$			160.14			51.000				19.560
	$\widehat{\mathcal{F}}_P(\mathcal{Y})$			19.810			26.780				210.36
	$\widehat{\mathcal{F}}_{BT,R}(\mathcal{Y})$			177.87			80.850				238.73
	$\widehat{\mathcal{F}}_{BT,P}(\mathcal{Y})$			51.60			81.470				240.53
	$\widehat{\mathcal{F}}_{Reg}(\mathcal{Y})$			189.77			100.02				205.05
	$\widehat{\mathcal{F}}_{R,D}(\mathcal{Y})$			216.30			158.18				49.690

Table 13. Percentage relative efficiency using Population 4, 5, 6 when $\left\{x = Q_1(x), y = Q_1(y)\right\}$.

Estimators			Population 4			Population 5			Population 6		
			$\widehat{\mathcal{F}}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}_{Pr_2}(\mathcal{Y})$
$\widehat{\mathcal{F}}^{(1)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(1)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(1)}_{Pr_2}(\mathcal{Y})$	210.65	225.70	301.93	120.44	126.31	123.83	241.45	243.84	281.75
$\widehat{\mathcal{F}}^{(2)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(2)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(2)}_{Pr_2}(\mathcal{Y})$	21.64	225.69	301.91	120.45	126.33	123.85	241.45	243.84	281.75
$\widehat{\mathcal{F}}^{(3)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(3)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(3)}_{Pr_2}(\mathcal{Y})$	210.65	225.70	301.93	120.44	126.31	123.83	241.45	243.84	281.75
$\widehat{\mathcal{F}}^{(4)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(4)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(4)}_{Pr_2}(\mathcal{Y})$	210.64	225.68	301.89	120.46	126.35	123.87	241.45	243.84	281.75
$\widehat{\mathcal{F}}^{(5)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(5)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(5)}_{Pr_2}(\mathcal{Y})$	210.92	226.31	301.77	131.42	155.44	151.23	241.47	243.87	281.79
$\widehat{\mathcal{F}}^{(6)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(6)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(6)}_{Pr_2}(\mathcal{Y})$	210.91	226.30	301.75	131.34	155.16	151.95	241.47	243.87	281.79
$\widehat{\mathcal{F}}^{(7)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(7)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(7)}_{Pr_2}(\mathcal{Y})$	210.45	225.26	301.34	119.85	125.07	121.61	241.44	243.81	281.72
$\widehat{\mathcal{F}}^{(8)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(8)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(8)}_{Pr_2}(\mathcal{Y})$	210.92	226.31	301.77	131.42	155.44	151.23	241.47	243.87	281.79
$\widehat{\mathcal{F}}^{(9)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(9)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(9)}_{Pr_2}(\mathcal{Y})$	210.44	225.26	301.32	119.85	125.07	121.61	241.44	243.81	281.72
$\widehat{\mathcal{F}}^{(10)}_{GK}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(10)}_{Pr_1}(\mathcal{Y})$	$\widehat{\mathcal{F}}^{(10)}_{Pr_2}(\mathcal{Y})$	210.13	224.58	300.42	119.83	125.03	111.57	241.41	243.75	281.65
	$\widehat{\mathcal{F}}(y)$			100.00			100.00				100.00
	$\widehat{\mathcal{F}}_R(\mathcal{Y})$			171.46			44.640				28.330
	$\widehat{\mathcal{F}}_P(\mathcal{Y})$			29.290			56.480				213.50
	$\widehat{\mathcal{F}}_{BT,R}(\mathcal{Y})$			185.21			71.990				241.81
	$\widehat{\mathcal{F}}_{BT,P}(\mathcal{Y})$			51.02			88.500				241.41
	$\widehat{\mathcal{F}}_{Reg}(\mathcal{Y})$			201.70			101.46				206.53
	$\widehat{\mathcal{F}}_{R,D}(\mathcal{Y})$			210.13			119.83				49.610

Table 14. Percentage relative efficiency using Population 4, 5, 6 when $\left\{x = \widetilde{\mathcal{X}}, y = \widetilde{\mathcal{Y}}\right\}$.

case shown by $\widehat{\mathcal{F}}_{Pr_1}(\mathcal{Y})$ and $\widehat{\mathcal{F}}_{Pr_2}(\mathcal{Y})$, have outperformed the other competitive estimators. Across the suggested class, it is observed that the second proposed class of estimator is more robust than the first proposed class of estimators, because of higher efficiency. Tables 16 and 17 show that the proposed estimators outperform all other estimators currently in use. When X and Y are highly positively correlated, the PRE demonstrates that the second family of estimators proposed in SRS provides a reliable estimate.

Estimators			Population 4			Population 5			Population 6		
			$\hat{\mathcal{F}}_{GK}(\mathcal{Y})$	$\hat{\mathcal{F}}_{Pr_1}(\mathcal{Y})$	$\hat{\mathcal{F}}_{Pr_2}(\mathcal{Y})$	$\hat{\mathcal{F}}_{GK}(\mathcal{Y})$	$\hat{\mathcal{F}}_{Pr_1}(\mathcal{Y})$	$\hat{\mathcal{F}}_{Pr_2}(\mathcal{Y})$	$\hat{\mathcal{F}}_{GK}(\mathcal{Y})$	$\hat{\mathcal{F}}_{Pr_1}(\mathcal{Y})$	$\hat{\mathcal{F}}_{Pr_2}(\mathcal{Y})$
$\hat{\mathcal{F}}^{(1)}_{GK}(\mathcal{Y})$	$\hat{\mathcal{F}}^{(1)}_{Pr_1}(\mathcal{Y})$	$\hat{\mathcal{F}}^{(1)}_{Pr_2}(\mathcal{Y})$	245.26	281.27	270.29	110.78	111.13	111.65	228.65	231.74	254.02
$\hat{\mathcal{F}}^{(2)}_{GK}(\mathcal{Y})$	$\hat{\mathcal{F}}^{(2)}_{Pr_1}(\mathcal{Y})$	$\hat{\mathcal{F}}^{(2)}_{Pr_2}(\mathcal{Y})$	244.78	280.08	269.15	110.33	111.20	110.72	228.62	231.67	253.94
$\hat{\mathcal{F}}^{(3)}_{GK}(\mathcal{Y})$	$\hat{\mathcal{F}}^{(3)}_{Pr_1}(\mathcal{Y})$	$\hat{\mathcal{F}}^{(3)}_{Pr_2}(\mathcal{Y})$	245.76	281.60	271.56	111.23	113.10	111.63	228.69	231.81	254.10
$\hat{\mathcal{F}}^{(4)}_{GK}(\mathcal{Y})$	$\hat{\mathcal{F}}^{(4)}_{Pr_1}(\mathcal{Y})$	$\hat{\mathcal{F}}^{(4)}_{Pr_2}(\mathcal{Y})$	244.72	279.94	269.02	110.27	111.09	110.62	228.62	231.66	253.93
$\hat{\mathcal{F}}^{(5)}_{GK}(\mathcal{Y})$	$\hat{\mathcal{F}}^{(5)}_{Pr_1}(\mathcal{Y})$	$\hat{\mathcal{F}}^{(5)}_{Pr_2}(\mathcal{Y})$	245.12	280.91	269.95	113.72	119.14	118.62	228.65	231.72	254.00
$\hat{\mathcal{F}}^{(6)}_{GK}(\mathcal{Y})$	$\hat{\mathcal{F}}^{(6)}_{Pr_1}(\mathcal{Y})$	$\hat{\mathcal{F}}^{(6)}_{Pr_2}(\mathcal{Y})$	244.91	280.39	269.45	118.02	131.80	131.20	228.63	231.69	253.96
$\hat{\mathcal{F}}^{(7)}_{GK}(\mathcal{Y})$	$\hat{\mathcal{F}}^{(7)}_{Pr_1}(\mathcal{Y})$	$\hat{\mathcal{F}}^{(7)}_{Pr_2}(\mathcal{Y})$	245.13	280.93	269.97	110.47	111.50	111.02	228.65	231.72	253.99
$\hat{\mathcal{F}}^{(8)}_{GK}(\mathcal{Y})$	$\hat{\mathcal{F}}^{(8)}_{Pr_1}(\mathcal{Y})$	$\hat{\mathcal{F}}^{(8)}_{Pr_2}(\mathcal{Y})$	245.59	281.15	271.13	111.46	115.96	115.46	228.68	231.79	254.07
$\hat{\mathcal{F}}^{(9)}_{GK}(\mathcal{Y})$	$\hat{\mathcal{F}}^{(9)}_{Pr_1}(\mathcal{Y})$	$\hat{\mathcal{F}}^{(9)}_{Pr_2}(\mathcal{Y})$	244.74	280.00	269.08	110.25	111.03	11,056	228.62	231.67	253.94
$\hat{\mathcal{F}}^{(10)}_{GK}(\mathcal{Y})$	$\hat{\mathcal{F}}^{(10)}_{Pr_1}(\mathcal{Y})$	$\hat{\mathcal{F}}^{(10)}_{Pr_2}(\mathcal{Y})$	244.68	279.68	268.93	110.240	111.02	110.55	228.62	231.66	253.93
	$\hat{\mathcal{F}}(y)$			100.00			100.00				100.00
	$\hat{\mathcal{F}}_R(\mathcal{Y})$			213.53			41.450				28.040
	$\hat{\mathcal{F}}_P(\mathcal{Y})$			27.780			59.820				199.84
	$\hat{\mathcal{F}}_{BT,R}(\mathcal{Y})$			206.54			68.670				228.42
	$\hat{\mathcal{F}}_{BT,P}(\mathcal{Y})$			49.600			95.800				228.62
	$\hat{\mathcal{F}}_{Reg}(\mathcal{Y})$			241.84			104.44				199.92
	$\hat{\mathcal{F}}_{R,D}(\mathcal{Y})$			244.68			110.24				50.000

Table 15. Percentage relative efficiency using Population 4, 5, 6 when $\{x = Q_3(x), y = Q_3(y)\}$.

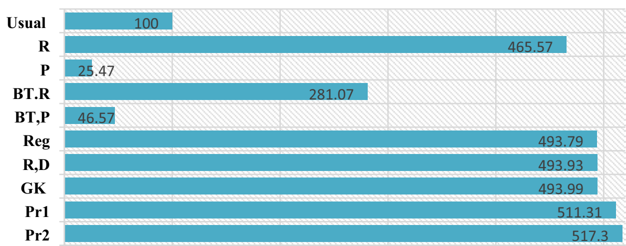

Figure 1. Percentage of relative efficiencies of existing and proposed estimators when $\{x = \overline{\mathcal{X}}, y = \overline{\mathcal{Y}}\}$, using Population 1.

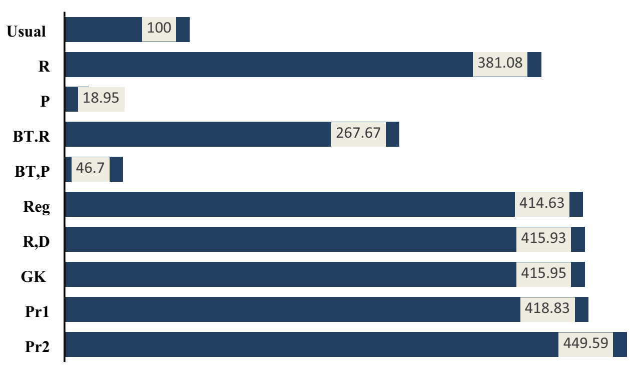

Figure 2. Percentage of relative efficiencies of existing and proposed estimators when $\{x = Q_1(x), y = Q_1(y)\}$, using Population 1.

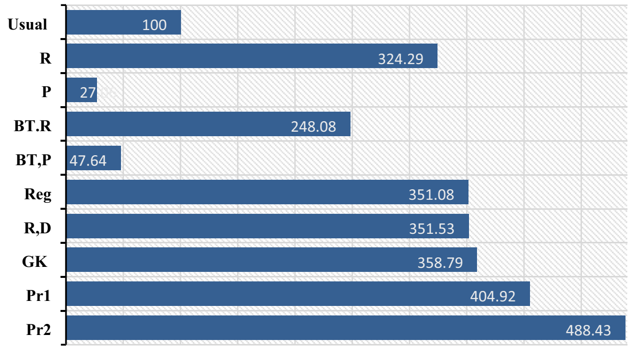

Figure 3. Percentage of relative efficiencies of existing and proposed estimators when $\left\{x = \widetilde{\mathcal{X}}, y = \widetilde{\mathcal{Y}}\right\}$, using Population 1.

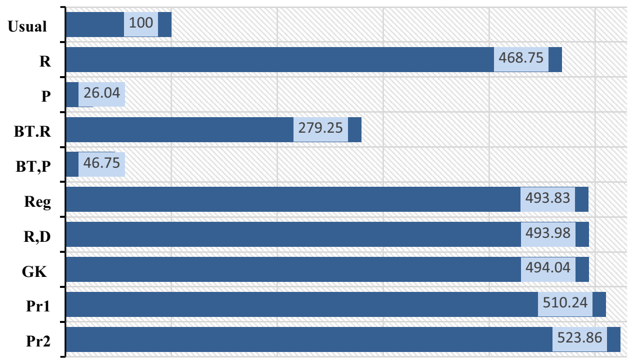

Figure 4. Percentage of relative efficiencies of existing and proposed estimators when $\left\{x = Q_3(x), y = Q_3(y)\right\}$, using Population 1.

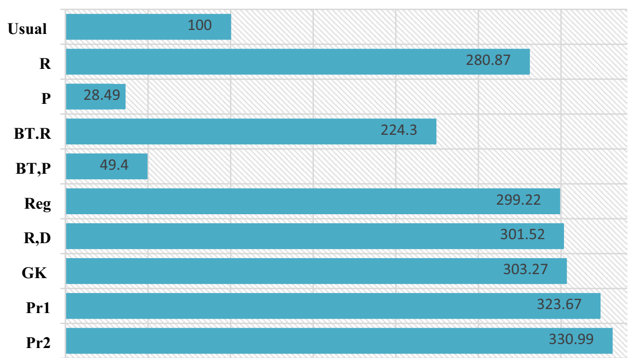

Figure 5. Percentage relative efficiencies of existing and proposed estimators when $\{x = \overline{\mathcal{X}}, y = \overline{\mathcal{Y}}\}$, using Population 4.

Conclusion

In this article, we have suggested two improved classes of estimators to estimate the finite population DF using dual auxiliary varaible. The bias and MSE of the suggested classes of estimators are derived up to the first order of approximmation. To observe the efficiency of estimators, six real data sets are used. Also To check the uniqueness and generalizability of the suggested classes of estimaators, we also employ a simulation study. Based on the numerical outcomes, it is observed that the suggested classes of estimators are more efficient than the exisitng estimators, for all the considered populations. The suggested modified classes of estimators $\widehat{\mathcal{F}}_{Pr_1}(\mathcal{Y})$ and $\widehat{\mathcal{F}}_{Pr_2}(\mathcal{Y})$ perform better as compared to all other considered estimators, although $\widehat{\mathcal{F}}_{Pr_2}(\mathcal{Y})$ is the best. The current work can be extended to estimate population mean using calibration approach under stratified random sampling.

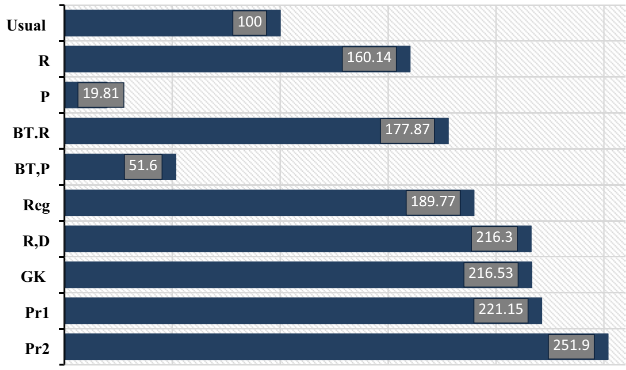

Figure 6. Percentage of relative efficiencies of existing and proposed estimators when $\{x = Q_1(x), y = Q_1(y)\}$, using Population.

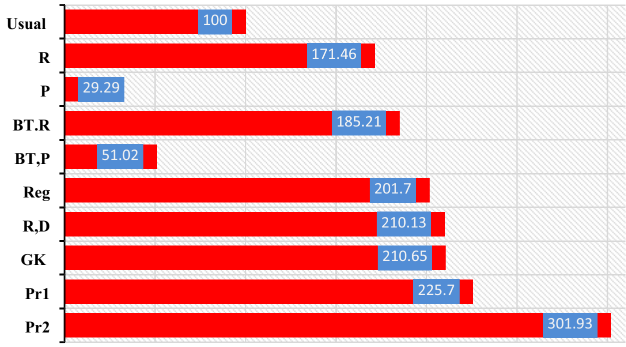

Figure 7. Percentage relative efficiencies of existing and proposed estimators when $\left\{x = \widetilde{\mathcal{X}}, y = \widetilde{\mathcal{Y}}\right\}$, using Population 4.

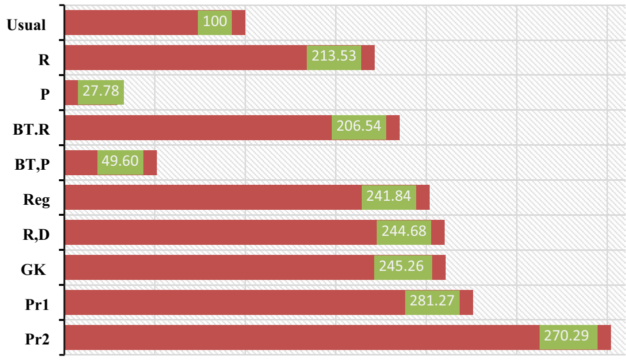

Figure 8. Percentage relative efficiencies of existing and proposed estimators when $\{x = Q_3(x), y = Q_3(y)\}$, using Population 4.

Estimator	Population I	Population II	Population III
$\widehat{\mathcal{F}}(y)$	0.0022520	0.0022500	0.0022520
$\widehat{\mathcal{F}}_R(\mathcal{Y})$	0.0064510	0.0026960	0.0012260
$\widehat{\mathcal{F}}_P(\mathcal{Y})$	0.0026130	0.0064800	0.0077600
$\widehat{\mathcal{F}}_{BT,R}(\mathcal{Y})$	0.0037770	0.0018730	0.0011830
$\widehat{\mathcal{F}}_{BT,P}(\mathcal{Y})$	0.0018750	0.0038270	0.0044330
$\widehat{\mathcal{F}}_{Reg}(\mathcal{Y})$	0.0018580	0.0018530	0.0010650
$\widehat{\mathcal{F}}_{R,D}(\mathcal{Y})$	0.0018450	0.0018400	0.0010600
$\widehat{\mathcal{F}}_{G,K}(\mathcal{Y})$	0.0018440	0.0018390	0.0010600
$\widehat{\mathcal{F}}_{Pr_1}(\mathcal{Y})$	0.0018390	0.0018370	0.0010580
$\widehat{\mathcal{F}}_{Pr_2}(\mathcal{Y})$	0.0017370	0.0016740	0.0008970

Table 16. MSEs of population DF estimators using simulation.

Estimator	Population I	Population II	Population III
$\widehat{\mathcal{F}}(y)$	100	100	100
$\widehat{\mathcal{F}}_R(\mathcal{Y})$	34.90917	83.46483	183.5737
$\widehat{\mathcal{F}}_P(\mathcal{Y})$	86.18418	34.73246	29.01847
$\widehat{\mathcal{F}}_{BT,R}(\mathcal{Y})$	59.62183	120.1159	190.3484
$\widehat{\mathcal{F}}_{BT,P}(\mathcal{Y})$	120.0678	58.80547	50.79537
$\widehat{\mathcal{F}}_{Reg}(\mathcal{Y})$	121.1700	121.4586	211.4561
$\widehat{\mathcal{F}}_{R,D}(\mathcal{Y})$	122.0601	122.3173	212.3715
$\widehat{\mathcal{F}}_{G,K}(\mathcal{Y})$	122.0912	122.3487	212.4257
$\widehat{\mathcal{F}}_{Pr_1}(\mathcal{Y})$	122.4285	122.4696	212.8426
$\widehat{\mathcal{F}}_{Pr_2}(\mathcal{Y})$	129.6135	134.4366	250.9312

Table 17. PREs of population DF estimators using simulation.

Data availability
The datasets generated and/or analysed during the current study are available in the current study are available from the corresponding author on reasonable request.

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Author contributions
M.S.M. interpretation of the results; wrote the main manuscript S.A. wrote the main manuscript H.M.A. Analysis; wrote the main manuscript F.M.A. Conceptualizations; wrote the main manuscript R.A. helped us to improve the language of the paper; wrote the main manuscript M.E. supervision; wrote the main manuscript S.M.A. helped us in the revised manuscript S.N. helped us to answer all the questions arises during revision.

Competing interests
The authors declare no competing interests.

Additional information
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