scientific reports

OPEN

Diverse solitons wave structures for coupled NLSEs in birefringent fibers with higher nonlinearities using the modified extended mapping algorithm

Karim K. Ahmed[1], Njah A. Alsahafi[2], Hamdy M. Ahmed[3], Salah Boulaaras[4]✉ & M. S. Osman[2,5,6]✉

This work is a thorough investigation of mathematical modeling with an emphasis on efficiency and performance optimization. Our research is centered on the cubic–quartic nonlinear Schrödinger equation, specifically concerning birefringent fibers exhibiting nonlinearity in the cubic–quintic–septic–nonic continuum. This work makes a unique and significant addition to the field of science. We have obtained a wide range of soliton solutions for cubic–quintic optical solitons in birefringent fibers by using sophisticated mathematical techniques, most notably the modified extended mapping technique. The solitons that fall under these obtained solutions are dark, singular, bright, and combo bright-dark. Besides, we get other exact wave solutions such as singular periodic, exponential, rational, and Weierstrass elliptic doubly periodic solutions. The study presented in this publication is novel and creative, shedding light on how mathematical techniques might improve the functionality and architecture of fiber communication networks. These results are crucial for understanding pulse propagation in birefringent optical fibers governed by the cubic–quartic nonlinear Schrödinger equation, particularly when nonlinear effects extend into the cubic–quintic–septic–nonic continuum. It highlights the innovative nature of our work and highlights the relevance of our results in furthering the science of nonlinear optics and its possible applications in the real world. Graphical depictions of some of the extracted solutions are included to aid readers in physically understanding the obtained solutions' behavior and characteristics.

The visualization of basic physical phenomena in fields like fluid mechanics, chemical reactions, electromagnetism, magneto-hydrodynamics, quantum mechanics, thermodynamics, optics, neuroscience, and elasticity is often achieved through the use of nonlinear partial differential equations, or NPDEs. Understanding the exact solutions to these equations is essential to comprehending the propagation of waves[1,2]. Chou et al.[3] analyze heat conduction using Lie symmetry, solitons, and modulation instability. Kudryashov et al.[4] studied cubic–quartic optical solitons under higher-order self-phase modulation. Gonzalez-Gaxiola et al.[5] investigate highly dispersive solitons with a quadratic–cubic refractive index using the variational iteration method. Rehan Akber et al.[6] explore Brownian motion in the stochastic Schrödinger equation via the Sardar sub-equation method. These works enhance understanding in nonlinear optics, stochastic waves, and thermal transport.

[1]Department of Mathematics, Faculty of Engineering, German International University (GIU), New Administrative Capital, Cairo, Egypt. [2]Mathematics Department, Faculty of Sciences, Umm Al-Qura University, Makkah 21955, Saudi Arabia. [3]Department of Physics and Engineering Mathematics, Higher Institute of Engineering, El Shorouk Academy, P.O. Box 11837, Cairo, Egypt. [4]Department of Mathematics, College of Science, Qassim University, Buraydah 51452, Saudi Arabia. [5]Department of Mathematics, Faculty of Science, Cairo University, Giza 12613, Egypt. [6]Research Center of Mathematical and Physical Sciences, Khazar University, Baku, Azerbaijan. ✉email: s.boularas@qu.edu.sa; msosman@uqu.edu.sa; mofatzi@sci.cu.edu.eg

Thanks to its ability to carry massive amounts of data over long distances in a reliable and fast manner, fiber optic communications have become the backbone of the contemporary telecommunications network. The persistent advancement of fiber communication networks is propelled by the growing need for data-intensive applications, including cloud computing algorithms[7], Internet of Things devices[8], and video streaming programs[9]. Given this, understanding the complexities of these systems and optimising their performance are crucial tasks for which mathematical modelling and analysis must be used. Many models that involve NPDEs are investigated by many researchers for the seek of finding solitons by implementing several approaches and techniques such as the improved modified extended Tanh-Function method[10], the sub ordinary differential equation[11], the extended F-expansion method[12] and many others (see[13–15]).

Ever since their discovery in the nineteenth century, solidions have been the subject of considerable investigation, revolutionizing a number of scientific and technological domains and resulting in groundbreaking discoveries in nonlinear dynamics, optics, communications, and other sectors[16]. Soliton solutions are one topic of special interest in mathematical fiber communications, especially in birefringent fibers. Because they can maintain their form and integrity throughout transmission, self-sustaining, non-dispersive optical pulses called soliton pulses are very attractive for long-distance communication. Birefringent fibers come with special benefits when it comes to controlling and preserving polarization because of their capacity to keep light signals polarized. Many types of soliton solutions, including single, combo-bright-dark, brilliant, and combo-singular solitons, may be studied by scientists in cubic–quintic (CQ) optical systems by using mathematical techniques[17,18]. Through the use of mathematical tools, scientists and engineers may develop a deeper understanding of the intricate dynamics related to light transmission, signal deterioration, interference, and other elements that affect the system's overall performance. More dependable and effective fiber communication systems may be developed, thanks to mathematical models, which offer a quantitative foundation for system parameter design and optimization. Some valuable contributions to the study of nonlinear wave phenomena, soliton dynamics, mathematical modeling of complex physical systems, and fractional differential equations across various scientific disciplines are described briefly. Ismael et al.[19] investigate autonomous multiple wave solutions and hybrid behaviors in a (3+1)-dimensional Boussinesq-type equation, which is significant for fluid mechanics. Muhammad et al.[20] examine optical wave propagation in fiber optics using a nonlinear fractional Schrödinger equation, highlighting the influence of fractional derivatives on optical communication systems. Younas et al.[21] focus on the solitary wave dynamics and interaction mechanisms in ultrasound imaging, modeled through a fractional nonlinear system, providing insights into medical wave applications. In[22], Muhammad et al. analyze soliton solutions and their interaction properties in the Estevez-Mansfield-Clarkson equation, which appears in multiple physical fields. Their research in[23] explores the propagation of fractional solitary waves in the modified Korteweg-de Vries-Kadomtsev-Petviashvili equation, emphasizing parametric effects and qualitative wave characteristics. Furthermore, Muhammad et al.[24] delve into the fractional impact on optical wave transmission in the extended Kairat-II equation, demonstrating how fractional-order derivatives affect nonlinear optical models. Raza et al.[25] investigate complexiton and resonant multi-soliton solutions for a $(4 + 1)$-dimensional Boiti-Leon-Manna-Pempinelli equation, providing insights into high-dimensional nonlinear wave interactions in optical and quantum systems. Javid et al.[26] explore dual-wave resonant solutions of the nonlinear Schrödinger equation with varying nonlinearities, shedding light on wave dynamics in dispersive and nonlinear media. Kazmi et al.[27] conduct a bifurcation analysis of soliton and quasi-periodic patterns in a generalized q-deformed Sinh-Gordon equation, emphasizing the role of symmetry and integrability in nonlinear systems. Finally, Raza et al.[28] derive new exact periodic elliptic wave solutions for the extended quantum Zakharov-Kuznetsov equation, enhancing the understanding of nonlinear wave behavior in quantum plasmas. Together, these studies enhance the theoretical and applied understanding of nonlinear waves in diverse fields such as fluid dynamics, optics, and medical imaging.

Utilizing a powerful mathematical tool, the modified extended mapping technique (MEMT)[13], these soliton solutions are found in this study. In this study, the dynamics and properties of solitons in birefringent fibers are investigated by using this approach on the cubic–quintic–septic–nonic nonlinear Schrödinger's equation (CQSN-NLSE). For enhanced performance and efficiency, fiber communication system design and optimization can be guided by the soliton solutions generated, which offer insightful information on the behavior of the system. In addition to offering a thorough examination of the system under study and its results, the paper includes novel and unique research. This paper demonstrates the potential of mathematical modeling in fiber communications, laying the groundwork for further development and research in the design and optimization of fiber communication systems. It also contributes to the field's ongoing development. Focusing on the CQ-NLSE in birefringent fibers with CQSN nonlinearity[17,18], this study explores the intricate realm of nonlinear optical systems and is considered significant. In fields like optical signal processing, laser technology, and telecommunications, understanding this complex nonlinear behavior is crucial from a practical standpoint.

An important new concept is presented in this work. Literature that is currently accessible does not offer a thorough examination of soliton solutions inside the birefringent fibers CQSN nonlinearity model. This opens up new areas in the field of nonlinear optics. Here, we examine for the first time the effect of CQSN terms on the dynamic behavior of solitons in optical systems using the widely recognized MEMT.

A dimensionless variation of the CQ-NLSE with CQSN nonlinearity has recently been written as[4]:

$$i\mathcal{H}_t + ia\mathcal{H}_{zzz} + b\mathcal{H}_{zzzz} + \left(c_1 |\mathcal{H}|^2 + c_2 |\mathcal{H}|^4 + c_3 |\mathcal{H}|^6 + c_4 |\mathcal{H}|^8\right)\mathcal{H} = 0, \qquad i = \sqrt{-1}, \qquad (1)$$

where $\mathcal{H}(z, t)$ defines the complex-valued wave packet's amplitude and z, t symbolizes the the spatial and temporal variables, respectively. The first term is the evolution term, while the a and b are the 3rd and 4th order dispersion coefficients, in the given sequence. Additionally, the self-phase modulation (SPM) influences are represented by c_j, $(1 \leq j \leq 4)$.

In this article, since we treat birefringent fibers, we can split Eq. (1) into the following coupled NLSEs (CNLSEs)[17]:

$$
\begin{aligned}
& i\phi_t + i\alpha_1\phi_{zzz} + \beta_1\phi_{zzzz} + \left(a_{11}\,|\phi|^2 + a_{12}\,|\psi|^2\right)\phi + \left(b_{11}\,|\phi|^4 + b_{12}\,|\psi|^2\,|\phi|^2 + b_{13}\,|\psi|^4\right)\phi \\
& + \left(c_{11}\,|\phi|^6 + c_{12}\,|\phi|^4\,|\psi|^2 + c_{13}\,|\phi|^2\,|\psi|^4 + c_{14}\,|\psi|^6\right)\phi \\
& + \left(d_{11}\,|\phi|^8 + d_{12}\,|\phi|^6\,|\psi|^2 + d_{13}\,|\phi|^4\,|\psi|^4 + d_{14}\,|\phi|^2\,|\psi|^6 + d_{15}\,|\psi|^8\right)\phi = 0,
\end{aligned}
\tag{2}
$$

$$
\begin{aligned}
& i\psi_t + i\alpha_2\psi_{zzz} + \beta_2\psi_{zzzz} + \left(a_{21}\,|\psi|^2 + a_{22}\,|\phi|^2\right)\psi + \left(b_{21}\,|\psi|^4 + b_{22}\,|\phi|^2\,|\psi|^2 + b_{23}\,|\phi|^4\right)\psi \\
& + \left(c_{21}\,|\psi|^6 + c_{22}\,|\psi|^4\,|\phi|^2 + c_{23}\,|\psi|^2\,|\phi|^4 + c_{24}\,|\phi|^6\right)\psi \\
& + \left(d_{21}\,|\psi|^8 + d_{22}\,|\psi|^6\,|\phi|^2 + d_{23}\,|\psi|^4\,|\phi|^4 + d_{24}\,|\psi|^2\,|\phi|^6 + d_{25}\,|\phi|^8\right)\psi = 0,
\end{aligned}
\tag{3}
$$

where the functions $\phi = \phi(z,t)$ and $\psi = \psi(z,t)$ describe wave profiles, whereas the first terms relate to linear temporal evolutions. The coefficients α_i and β_i, $(i = 1,2)$ reflect the 3rd and 4th order dispersions, respectively. The constants a_{i2}, b_{i2}, b_{i3}, c_{i2}, c_{i3}, c_{i4}, d_{i2}, d_{i3}, d_{i4}, and d_{i5}, $(i = 1,2)$ represent the coefficients of the cross-phase modulation (XPM) terms. The coefficients of the SPM terms are a_{i1}, b_{i1}, c_{i1}, and d_{i1}, $(i = 1,2)$.

The novelty of this work lies in deriving a broader class of solutions than previously reported, including singular, periodic, and elliptic function solutions, expanding the known solution space. Unlike earlier studies limited to cubic–quintic or cubic–septic models, this research incorporates ninth-order nonlinearities, providing a more complete understanding of extreme nonlinear regimes in birefringent fibers, which is essential for advanced optical communication and fiber laser technologies. Each solution is substituted back into the original cubic–quartic nonlinear Schrödinger equation to confirm that it satisfies the equation exactly.

Below is the structure of this paper: The exact solutions for the suggested coupled system in Eqs. (2)–(3) are derived after a brief discussion of the concept's mathematical underpinnings and recent application. "Mathematical preliminaries for the proposed method" section provides a summary of the main elements of the suggested strategy. In addition to displaying all of the data, "Novel solutions extraction for the proposed coupled system" section explains the many dynamic wave shapes that make up the novel soliton solutions when implementing the MEMT. In "Graphical simulation of some retrieved solutions" section, graphical 2D, 3D, and contour drawings of several obtained solutions are shown. "Conclusion" section presents some inferences drawn from the data acquired and some conclusive remarks.

Mathematical preliminaries for the proposed method

This section is divided into two parts. General detailed steps of the proposed technique are presented in the first part. Finally, we mention the advantages of the applied technique compared to some other methods in the literature in the last subsection.

General algorithm of the MEMT

The key components of the MEMT that will be used in section 3 are presented in this section[13].

If we assume to have an NPDE of the following form :

$$
\mathfrak{Z}\left(\mathfrak{R}, \mathfrak{R}_t, \mathfrak{R}_x, \mathfrak{R}_{xx}, \mathfrak{R}_{xt}, \mathfrak{R}_{xxt}, \ldots\right) = 0,
\tag{4}
$$

where $\mathfrak{Z}$ denotes a polynomial function of $\mathfrak{R}(x,t)$ and its corresponding partial derivatives with respect to the two-dimensional time and space.

Algorithm-(I): The next travelling wave transformation will be used to solve Eq. (4):

$$
\mathfrak{R}(x,t) = \mathfrak{K}(\zeta), \qquad \zeta = x - \kappa t, \quad \kappa \neq 0,
\tag{5}
$$

where κ denotes the speed of the wave.

The transformation of Eq. (5) into Eq. (4) yields the following nonlinear ordinary differential equation (NODE):

$$
\mathcal{Z}(\mathfrak{K}, \mathfrak{K}', \mathfrak{K}'', \mathfrak{K}''', \ldots) = 0.
\tag{6}
$$

Algorithm-(II): We suggest that the general solution of Eq. (6) is:

$$
\mathfrak{K}(\zeta) = \sum_{j=0}^{\mathbb{M}} \mathbb{A}_j \mathcal{W}^j(\zeta) + \sum_{j=-1}^{-\mathbb{M}} \mathbb{B}_{-j} \mathcal{W}^j(\zeta) + \sum_{j=2}^{\mathbb{M}} \theta_j \mathcal{W}^{j-2}(\zeta)\mathcal{W}'(\zeta) + \sum_{j=-1}^{-\mathbb{M}} \mathbb{C}_{-j}\mathcal{W}^j(\zeta)\mathcal{W}'(\zeta),
\tag{7}
$$

where $\mathbb{A}_j, \mathbb{B}_{-j}, \theta_j, \mathbb{C}_{-j}$ represent real-valued constants that will be estimated and the function $\mathcal{W}(\zeta)$ obeys the following constraint:

$$
\mathcal{W}'(\zeta) = \sqrt{\tau_0 + \tau_1 \mathcal{W}(\zeta) + \tau_2 \mathcal{W}^2(\zeta) + \tau_3 \mathcal{W}^3(\zeta) + \tau_4 \mathcal{W}^4(\zeta) + \tau_6 \mathcal{W}^6(\zeta)},
\tag{8}
$$

where τ_i, $(i = 0,1,2,3,4,6)$ are constants. Many types of solutions can be extracted from Eq. (8) by putting $\tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \tau_6$ with differnt values as follows:

Case 1: When $\tau_0 = \tau_1 = \tau_3 = \tau_6 = 0$, the following solutions are raised:

$$\mathcal{W}(\zeta) = \sqrt{-\frac{\tau_2}{\tau_4}} \operatorname{sech}\left(\sqrt{\tau_2}\zeta\right), \quad \tau_2 > 0, \tau_4 < 0.$$

$$\mathcal{W}(\zeta) = \sqrt{-\frac{\tau_2}{\tau_4}} \sec\left(\sqrt{-\tau_2}\zeta\right), \quad \tau_2 < 0, \tau_4 > 0.$$

$$\mathcal{W}(\zeta) = \sqrt{-\frac{\tau_2}{\tau_4}} \csc\left(\sqrt{-\tau_2}\zeta\right), \quad \tau_2 < 0, \tau_4 > 0.$$

Case 2: When $\tau_1 = \tau_3 = \tau_6 = 0, \tau_0 = \frac{\tau_2^2}{4\tau_4}$, the following solutions are raised:

$$\mathcal{W}(\zeta) = \sqrt{-\frac{\tau_2}{2\tau_4}} \tanh\left(\sqrt{-\frac{\tau_2}{2}}\zeta\right), \quad \tau_2 < 0, \tau_4 > 0.$$

$$\mathcal{W}(\zeta) = \sqrt{\frac{\tau_2}{2\tau_4}} \tan\left(\sqrt{\frac{\tau_2}{2}}\zeta\right), \quad \tau_2 > 0, \tau_4 > 0.$$

Case 3: When $\tau_3 = \tau_4 = \tau_6 = 0$, the following solutions are raised:

$$\mathcal{W}(\zeta) = \frac{\tau_1 \sinh\left(2\sqrt{\tau_2}\zeta\right)}{2\tau_2} - \frac{\tau_1}{2\tau_2}, \quad \tau_2 > 0, \tau_0 = 0.$$

$$\mathcal{W}(\zeta) = \frac{\tau_1 \sin\left(\sqrt{-\tau_2}\zeta\right)}{2\tau_2} - \frac{\tau_1}{2\tau_2}, \quad \tau_2 < 0, \tau_0 = 0.$$

$$\mathcal{W}(\zeta) = \sqrt{\frac{\lambda_0}{\lambda_2}} \sinh\left(\sqrt{\lambda_2}\zeta\right), \quad \tau_0 > 0, \tau_2 > 0, \tau_1 = 0.$$

$$\mathcal{W}(\zeta) = \sqrt{-\frac{\lambda_0}{\lambda_2}} \sin\left(\sqrt{-\lambda_2}\zeta\right), \quad \tau_0 > 0, \tau_2 < 0, \tau_1 = 0.$$

$$\mathcal{W}(\zeta) = \exp\left(\sqrt{\tau_2}\zeta\right) - \frac{\tau_1}{2\tau_2}, \quad \tau_2 > 0, \tau_0 = \frac{\tau_1^2}{4\tau_2}.$$

Case 4: When $\tau_0 = \tau_1 = \tau_2 = \tau_6 = 0$, the following solution is raised:

$$\mathcal{W}(\zeta) = \frac{4\tau_3}{\tau_3^2\zeta^2 - 4\tau_4}.$$

Case 5: When $\tau_0 = \tau_1 = \tau_6 = 0$, the following solutions are raised:

$$\mathcal{W}(\zeta) = -\frac{\tau_2\left(\tanh\left(\frac{1}{2}\sqrt{\tau_2}\zeta\right)+1\right)}{\tau_3}, \quad \tau_3^2 = 4\tau_2\tau_4, \tau_2 > 0.$$

$$\mathcal{W}(\zeta) = -\frac{\tau_2\left(\coth\left(\frac{1}{2}\sqrt{\tau_2}\zeta\right)+1\right)}{\tau_3}, \quad \tau_3^2 = 4\tau_2\tau_4, \tau_2 > 0.$$

$$\mathcal{W}(\zeta) = \frac{\tau_2\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\tau_2}\zeta\right)}{2\sqrt{\tau_2\tau_4}\tanh\left(\frac{1}{2}\sqrt{\tau_2}\zeta\right)-\tau_3}, \quad \tau_3^2 \neq 4\tau_2\tau_4, \tau_2 > 0, \tau_4 > 0.$$

$$\mathcal{W}(\zeta) = -\frac{\tau_2\sec^2\left(\frac{1}{2}\sqrt{-\tau_2}\zeta\right)}{2\sqrt{-\tau_2\tau_4}\tan\left(\frac{1}{2}\sqrt{-\tau_2}\zeta\right)+\tau_3}, \quad \tau_3^2 \neq 4\tau_2\tau_4, \tau_2 < 0, \tau_4 > 0.$$

Case 6: When $\tau_2 = \tau_4 = \tau_6 = 0$, the following solution is raised:

$$\mathcal{W}(\zeta) = \wp\left(\frac{1}{2}\sqrt{\tau_3}\zeta; -\frac{4\tau_1}{\tau_3}, -\frac{4\tau_0}{\tau_3}\right), \quad \tau_3 > 0.$$

Case 7: When $\tau_1 = \tau_3 = \tau_6 = 0$, the following solutions are raised:

No.	τ_0	τ_2	τ_4	$\mathcal{W}(\zeta)$
1	1	$-(1+m^2)$	m^2	$\operatorname{cd}(\zeta, m)$ or $\operatorname{sn}(\zeta, m)$
2	$m^2 - 1$	$-m^2 + 2$	-1	$\operatorname{dn}(\zeta, m)$
3	$-m^2$	$2m^2 - 1$	$-m^2 + 1$	$\operatorname{nc}(\zeta, m)$

No.	τ_0	τ_2	τ_4	$\mathcal{W}(\zeta)$
4	-1	$-m^2 + 2$	$m^2 - 1$	$\mathrm{nd}\,(\zeta, m)$
5	$m^2 - 2m^3 + m^4$	$-\dfrac{4}{m}$	$-1 + 6m - m^2$	$\dfrac{m\,\mathrm{dn}(\zeta|m)\mathrm{cn}(\zeta|m)}{1 + m\,\mathrm{sn}(\zeta|m)^2}$
6	$\dfrac{1}{4}$	$\dfrac{1}{2}m^2 - 1$	$\dfrac{m^4}{4}$	$\dfrac{\mathrm{sn}(\zeta|m)}{1 + \mathrm{dn}(\zeta|m)}$ or $\dfrac{\mathrm{cn}(\zeta|m)}{\sqrt{1 - m^2} + \mathrm{dn}(\zeta|m)}$

Algorithm-(III): Equation (6) may be utilised to compute the integer $\mathbb{M}$, utilising the idea of homogeneous balancing the nonlinearity and the dispersion in the resultant ODE.

Algorithm-(IV): After plugging the suggested solution into Eq. (5) using Eq. (8) into Eq. (6), we equalize the coefficients of $\mathcal{W}'^{\,j}(\zeta)\mathcal{W}^i(\zeta)(\,j = 0, 1; \, i = 0, \pm 1, \pm 2, ...)$ to zero and obtain a set of nonlinear algebraic equations (NLAEs) for $\mathbb{A}_j, \mathbb{B}_{-j}, \theta_j, \mathbb{C}_{-j}$ and κ, which will be solved using the Wolfram Mathematica program. From there, we can estimate the used unknowns $\mathbb{A}_j, \mathbb{B}_{-j}, \theta_j,$ and $\mathbb{C}_{-j}$. Following that, it is possible to derive many precise solutions for Eq. (4).

Advantages of the applied technique

The MEMT has several advantages over traditional methods for solving nonlinear differential equations. Unlike the Inverse Scattering Transform[29], which is limited to integrable equations and requires intricate spectral analysis, MEMT provides a more direct and adaptable approach. Compared to Hirota's Bilinear Method[30], which involves tedious bilinear transformations and perturbative expansions, MEMT simplifies the derivation of exact solutions. Unlike Lie Symmetry Analysis[31], which demands extensive group-theoretic computations and is often limited to finding symmetry-invariant solutions, MEMT systematically generates multiple types of solutions, including solitons, periodic, and elliptic function solutions. Its efficiency and versatility make it a valuable tool in nonlinear wave analysis.

Novel solutions extraction for the proposed coupled system

The solutions of Eqs. (2)–(3) may be obtained by assuming the following wave transformation:

$$\phi(z, t) = \mathcal{P}_1(\zeta)\, e^{i\Upsilon(z,t)}, \tag{9}$$

$$\psi(z, t) = \mathcal{P}_2(\zeta)\, e^{i\Upsilon(z,t)}, \tag{10}$$

and

$$\zeta = z - \rho t, \quad \rho \neq 0, \quad \Upsilon(z, t) = -\mathcal{K}z + \omega t + \Delta, \tag{11}$$

where $\mathcal{P}_1(\zeta)$ and $\mathcal{P}_2(\zeta)$ denote the wave's solution amplitude, and $\mathcal{K}$, ρ, Δ and ω denote the frequency, velocity of the soliton, phase shift constant and wave number in the mentioned sequence.

Using the transformation described by Eqs. (9)–(11), after that, Eqs. (2)–(3) will be transformed into the subsequent system of NODEs, then separating the real and imaginary parts yield:

Real parts:

$$\begin{aligned}
&\beta_1 \mathcal{P}_1^{(4)} + 3\mathcal{K}\left(\alpha_1 - 2\beta_1\mathcal{K}\right)\mathcal{P}_1'' + \left(a_{12}\mathcal{P}_2^2 + b_{13}\mathcal{P}_2^4 + c_{14}\mathcal{P}_2^6 + d_{15}\mathcal{P}_2^8 + \beta_1\mathcal{K}^4 - \alpha_1\mathcal{K}^3 - \omega\right)\mathcal{P}_1 \\
&+ \left(a_{11} + b_{12}\mathcal{P}_2^2 + c_{13}\mathcal{P}_2^4 + d_{14}\mathcal{P}_2^6\right)\mathcal{P}_1^3 + \left(b_{11} + c_{12}\mathcal{P}_2^2 + d_{13}\mathcal{P}_2^4\right)\mathcal{P}_1^5 + \left(c_{11} + d_{12}\mathcal{P}_2^2\right)\mathcal{P}_1^7 \\
&+ d_{11}\mathcal{P}_1^9 = 0,
\end{aligned} \tag{12}$$

$$\begin{aligned}
&\beta_2 \mathcal{P}_2^{(4)} + 3\mathcal{K}\left(\alpha_2 - 2\beta_2\mathcal{K}\right)\mathcal{P}_2'' + \left(a_{22}\mathcal{P}_1^2 + b_{23}\mathcal{P}_1^4 + c_{24}\mathcal{P}_1^6 + d_{25}\mathcal{P}_1^8 + \beta_2\mathcal{K}^4 - \alpha_2\mathcal{K}^3 - \omega\right)\mathcal{P}_2 \\
&+ \left(a_{21} + b_{22}\mathcal{P}_1^2 + c_{23}\mathcal{P}_1^4 + d_{24}\mathcal{P}_1^6\right)\mathcal{P}_2^3 + \left(b_{21} + c_{22}\mathcal{P}_1^2 + d_{23}\mathcal{P}_1^4\right)\mathcal{P}_2^5 + \left(c_{21} + d_{22}\mathcal{P}_1^2\right)\mathcal{P}_2^7 \\
&+ d_{21}\mathcal{P}_2^9 = 0,
\end{aligned} \tag{13}$$

and the imaginary parts are:

$$\left(\alpha_1 - 4\beta_1\mathcal{K}\right)\mathcal{P}_1^{(3)} + \left(4\beta_1\mathcal{K}^3 - 3\alpha_1\mathcal{K}^2 - \rho\right)\mathcal{P}_1' = 0, \tag{14}$$

$$\left(\alpha_2 - 4\beta_2\mathcal{K}\right)\mathcal{P}_2^{(3)} + \left(4\beta_2\mathcal{K}^3 - 3\alpha_2\mathcal{K}^2 - \rho\right)\mathcal{P}_2' = 0, \tag{15}$$

By integrating Eqs. (14)–(15) with respect to ζ once with neglecting the constant of integration, and then inserting them in Eqs. (12)–(13), respectively, we get:

$$
\begin{aligned}
&\beta_1\left(\alpha_1 - 4\beta_1\mathcal{K}\right)\mathcal{P}_1^{(4)} + \left(-\beta_1\mathcal{K}\left(4\mathcal{P}_2^2\left(a_{12} + b_{13}\mathcal{P}_2^2 + c_{14}\mathcal{P}_2^4 + d_{15}\mathcal{P}_2^6\right) + 25\alpha_1\mathcal{K}^3 + 6\mathcal{K}\rho - 4\omega\right)\right. \\
&+\alpha_1\left(a_{12}\mathcal{P}_2^2 + b_{13}\mathcal{P}_2^4 + c_{14}\mathcal{P}_2^6 + d_{15}\mathcal{P}_2^8 + 8\alpha_1\mathcal{K}^3 + 3\mathcal{K}\rho - \omega\right) + 20\beta_1^2\mathcal{K}^5\right)\mathcal{P}_1 \\
&+ \left(\alpha_1 - 4\beta_1\mathcal{K}\right)\left(a_{11} + b_{12}\mathcal{P}_2^2 + c_{13}\mathcal{P}_2^4 + d_{14}\mathcal{P}_2^6\right)\mathcal{P}_1^3 + \left(\alpha_1 - 4\beta_1\mathcal{K}\right)\left(b_{11} + c_{12}\mathcal{P}_2^2 + d_{13}\mathcal{P}_2^4\right)\mathcal{P}_1^5 \\
&+ \left(c_{11} + d_{12}\mathcal{P}_2^2\right)\left(\alpha_1 - 4\beta_1\mathcal{K}\right)\mathcal{P}_1^7 + d_{11}\left(\alpha_1 - 4\beta_1\mathcal{K}\right)\mathcal{P}_1^9 = 0,
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
&\beta_2\left(\alpha_2 - 4\beta_2\mathcal{K}\right)\mathcal{P}_2^{(4)} + \left(-\beta_2\mathcal{K}\left(4\mathcal{P}_1^2\left(a_{22} + b_{23}\mathcal{P}_1^2 + c_{24}\mathcal{P}_1^4 + d_{25}\mathcal{P}_1^6\right) + 25\alpha_2\mathcal{K}^3 + 6\mathcal{K}\rho - 4\omega\right)\right. \\
&+\alpha_2\left(a_{22}\mathcal{P}_1^2 + b_{23}\mathcal{P}_1^4 + c_{24}\mathcal{P}_1^6 + d_{25}\mathcal{P}_1^8 + 8\alpha_2\mathcal{K}^3 + 3\mathcal{K}\rho - \omega\right) + 20\beta_2^2\mathcal{K}^5\right)\mathcal{P}_2 \\
&+ \left(\alpha_2 - 4\beta_2\mathcal{K}\right)\left(a_{21} + b_{22}\mathcal{P}_1^2 + c_{23}\mathcal{P}_1^4 + d_{24}\mathcal{P}_1^6\right)\mathcal{P}_2^3 + \left(\alpha_2 - 4\beta_2\mathcal{K}\right)\left(b_{21} + c_{22}\mathcal{P}_1^2 + d_{23}\mathcal{P}_1^4\right)\mathcal{P}_2^5 \\
&+ \left(c_{21} + d_{22}\mathcal{P}_1^2\right)\left(\alpha_2 - 4\beta_2\mathcal{K}\right)\mathcal{P}_2^7 + d_{21}\left(\alpha_2 - 4\beta_2\mathcal{K}\right)\mathcal{P}_2^9 = 0.
\end{aligned}
\tag{17}
$$

Now, let's set

$$
\mathcal{P}_2(\zeta) = \Omega\,\mathcal{P}_1(\zeta), \quad \Omega \neq 0, 1.
\tag{18}
$$

Then, Eqs. (16)–(17) can be written as:

$$
\begin{aligned}
\mathcal{P}_1^{(4)} &+ \left(\frac{\alpha_1\omega - 20\beta_1^2\mathcal{K}^5 + 25\alpha_1\beta_1\mathcal{K}^4 - 8\alpha_1^2\mathcal{K}^3 + 6\beta_1\mathcal{K}^2\rho - 3\alpha_1\mathcal{K}\rho - 4\beta_1\mathcal{K}\omega}{\beta_1\left(4\beta_1\mathcal{K} - \alpha_1\right)}\right)\mathcal{P}_1 \\
&+ \left(\frac{a_{12}\Omega^2 + a_{11}}{\beta_1}\right)\mathcal{P}_1^3 + \left(\frac{b_{13}\Omega^4 + b_{12}\Omega^2 + b_{11}}{\beta_1}\right)\mathcal{P}_1^5 + \left(\frac{c_{14}\Omega^6 + c_{13}\Omega^4 + c_{12}\Omega^2 + c_{11}}{\beta_1}\right)\mathcal{P}_1^7 \\
&+ \left(\frac{d_{15}\Omega^8 + d_{14}\Omega^6 + d_{13}\Omega^4 + d_{12}\Omega^2 + d_{11}}{\beta_1}\right)\mathcal{P}_1^9 = 0,
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
\mathcal{P}_1^{(4)} &+ \left(\frac{\alpha_2\omega - 20\beta_2^2\mathcal{K}^5 + 25\alpha_2\beta_2\mathcal{K}^4 - 8\alpha_2^2\mathcal{K}^3 + 6\beta_2\mathcal{K}^2\rho - 3\alpha_2\mathcal{K}\rho - 4\beta_2\mathcal{K}\omega}{\beta_2\left(4\beta_2\mathcal{K} - \alpha_2\right)}\right)\mathcal{P}_1 \\
&+ \left(\frac{a_{21}\Omega^2 + a_{22}}{\beta_2}\right)\mathcal{P}_1^3 + \left(\frac{b_{21}\Omega^4 + b_{22}\Omega^2 + b_{23}}{\beta_2}\right)\mathcal{P}_1^5 + \left(\frac{c_{21}\Omega^6 + c_{22}\Omega^4 + c_{23}\Omega^2 + c_{24}}{\beta_2}\right)\mathcal{P}_1^7 \\
&+ \left(\frac{d_{21}\Omega^8 + d_{22}\Omega^6 + d_{23}\Omega^4 + d_{24}\Omega^2 + d_{25}}{\beta_2}\right)\mathcal{P}_1^9 = 0.
\end{aligned}
\tag{20}
$$

So, by comparing the coefficients of Eqs. (19)–(20), they are equivalent by providing the following constraints:

$$
\begin{aligned}
&\frac{\alpha_1\omega - 20\beta_1^2\mathcal{K}^5 + 25\alpha_1\beta_1\mathcal{K}^4 - 8\alpha_1^2\mathcal{K}^3 + 6\beta_1\mathcal{K}^2\rho - 3\alpha_1\mathcal{K}\rho - 4\beta_1\mathcal{K}\omega}{\beta_1\left(4\beta_1\mathcal{K} - \alpha_1\right)} \\
&= \frac{\alpha_2\omega - 20\beta_2^2\mathcal{K}^5 + 25\alpha_2\beta_2\mathcal{K}^4 - 8\alpha_2^2\mathcal{K}^3 + 6\beta_2\mathcal{K}^2\rho - 3\alpha_2\mathcal{K}\rho - 4\beta_2\mathcal{K}\omega}{\beta_2\left(4\beta_2\mathcal{K} - \alpha_2\right)},
\end{aligned}
\tag{21}
$$

$$
\frac{a_{12}\Omega^2 + a_{11}}{\beta_1} = \frac{a_{21}\Omega^2 + a_{22}}{\beta_2},
\tag{22}
$$

$$
\frac{b_{13}\Omega^4 + b_{12}\Omega^2 + b_{11}}{\beta_1} = \frac{b_{21}\Omega^4 + b_{22}\Omega^2 + b_{23}}{\beta_2},
\tag{23}
$$

$$
\frac{c_{14}\Omega^6 + c_{13}\Omega^4 + c_{12}\Omega^2 + c_{11}}{\beta_1} = \frac{c_{21}\Omega^6 + c_{22}\Omega^4 + c_{23}\Omega^2 + c_{24}}{\beta_2},
\tag{24}
$$

$$
\frac{d_{15}\Omega^8 + d_{14}\Omega^6 + d_{13}\Omega^4 + d_{12}\Omega^2 + d_{11}}{\beta_1} = \frac{d_{21}\Omega^8 + d_{22}\Omega^6 + d_{23}\Omega^4 + d_{24}\Omega^2 + d_{25}}{\beta_2}.
\tag{25}
$$

Then, we can find the wave number from Eq. (21) to be

$$
\begin{aligned}
\omega = \frac{1}{\left(\beta_1 - \beta_2\right)\left(\alpha_1 - 4\beta_1\mathcal{K}\right)\left(4\beta_2\mathcal{K} - \alpha_2\right)}&[\mathcal{K}\left(\beta_2\left(-8\beta_1^2\mathcal{K}^2\left(10\alpha_2\mathcal{K}^2 + 3\rho\right) + \alpha_1\alpha_2\left(8\alpha_1\mathcal{K}^2 + 3\rho\right)\right.\right. \\
&+6\left(\alpha_1 - \alpha_2\right)\beta_1\mathcal{K}\rho\right) + \alpha_2\beta_1\left(4\beta_1\mathcal{K} - \alpha_1\right)\left(8\alpha_2\mathcal{K}^2 + 3\rho\right) + 4\beta_2^2\mathcal{K}\left(\alpha_1\left(20\beta_1\mathcal{K}^3 - 3\rho\right)\right. \\
&\left.\left.-8\alpha_1^2\mathcal{K}^2 + 6\beta_1\mathcal{K}\rho\right)\right)].
\end{aligned}
\tag{26}
$$

Then Eq. (19) can be expressed as follows:

$$
\mathcal{P}_1^{(4)} + \mathcal{L}_1\mathcal{P}_1 + \mathcal{L}_3\mathcal{P}_1^3 + \mathcal{L}_5\mathcal{P}_1^5 + \mathcal{L}_7\mathcal{P}_1^7 + \mathcal{L}_9\mathcal{P}_1^9 = 0,
\tag{27}
$$

where $\mathcal{L}_i$, $(i = 1, 3, 5, 7, 9)$ represent constants and expressed as:

$$\mathcal{L}_1 = \frac{\alpha_1\omega - 20\beta_1^2\mathcal{K}^5 + 25\alpha_1\beta_1\mathcal{K}^4 - 8\alpha_1^2\mathcal{K}^3 + 6\beta_1\mathcal{K}^2\rho - 3\alpha_1\mathcal{K}\rho - 4\beta_1\mathcal{K}\omega}{\beta_1\left(4\beta_1\mathcal{K} - \alpha_1\right)},$$

$$\mathcal{L}_3 = \frac{a_{12}\Omega^2 + a_{11}}{\beta_1},$$

$$\mathcal{L}_5 = \frac{b_{13}\Omega^4 + b_{12}\Omega^2 + b_{11}}{\beta_1}, \tag{28}$$

$$\mathcal{L}_7 = \frac{c_{14}\Omega^6 + c_{13}\Omega^4 + c_{12}\Omega^2 + c_{11}}{\beta_1},$$

$$\mathcal{L}_9 = \frac{d_{15}\Omega^8 + d_{14}\Omega^6 + d_{13}\Omega^4 + d_{12}\Omega^2 + d_{11}}{\beta_1}.$$

At this stage, we have to assume that

$$\mathcal{P}_1(\zeta) = \sqrt{\mathcal{X}(\zeta)}, \tag{29}$$

then, we can write Eq. (27) as:

$$8\mathcal{X}^3\mathcal{X}^{(4)} - 16\mathcal{X}^2\mathcal{X}^{(3)}\mathcal{X}' - 12\mathcal{X}^2\left(\mathcal{X}''\right)^2 - 15\left(\mathcal{X}'\right)^4 + 36\mathcal{X}\left(\mathcal{X}'\right)^2\mathcal{X}''$$
$$+ 16\left(\mathcal{L}_9\mathcal{X}^8 + \mathcal{L}_7\mathcal{X}^7 + \mathcal{L}_5\mathcal{X}^6 + \mathcal{L}_3\mathcal{X}^5 + \mathcal{L}_1\mathcal{X}^4\right) = 0, \tag{30}$$

According to the proposed technique discussed in section 2, by applying the balance between $\mathcal{X}^3\mathcal{X}^{(4)}$ and $\mathcal{X}^8$, we find that $\mathbb{M} = 1$. From Eq. (7), the solution of Eq. (30) can be formulated as:

$$\mathcal{X}(\zeta) = \mathbb{A}_0 + \mathbb{A}_1\mathcal{W}(\zeta) + \frac{\mathbb{B}_1}{\mathcal{W}(\zeta)} + \mathbb{C}_1\left(\frac{\mathcal{W}'(\zeta)}{\mathcal{W}(\zeta)}\right), \tag{31}$$

assuming that $\mathbb{A}_1$, $\mathbb{B}_1$, and $\mathbb{C}_1$ are all unknown constants, they may be estimated by concurrently applying the restrictions $\mathbb{A}_1$, $\mathbb{B}_1$, and $\mathbb{C}_1 \neq 0$.

The following results may be obtained by solving the system of NLAEs created by inserting Eqs. (31) and (8) into Eq. (30) with the help of the Wolfram Mathematica program, grouping coefficients of comparable powers and setting them all to zero:

Result-(1): When $\tau_0 = \tau_1 = \tau_3 = \tau_6 = \mathbb{A}_0 = \mathbb{C}_1 = 0$, these are the sets of solutions that we discovered:

Set (1.1): $\mathbb{B}_1 = \mathcal{L}_7 = \mathcal{L}_3 = 0$, $\mathbb{A}_1 = \frac{\sqrt[4]{-105}\sqrt{-\tau_4}}{\sqrt[4]{2}\sqrt[4]{\mathcal{L}_9}}$, $\mathcal{L}_5 = \frac{-13\tau_2}{2}\sqrt{-\frac{3\mathcal{L}_9}{35}}$, $\mathcal{L}_1 = -\frac{\tau_2^2}{16}$.

Set (1.2): $\mathbb{B}_1 = \mathcal{L}_7 = \mathcal{L}_3 = 0$, $\mathbb{A}_1 = \frac{\sqrt[4]{-105}\sqrt{\tau_4}}{2\sqrt[4]{\mathcal{L}_9}}$, $\mathcal{L}_5 = \frac{13\tau_2}{2}\sqrt{-\frac{3\mathcal{L}_9}{35}}$, $\mathcal{L}_1 = -\frac{\tau_2^2}{16}$.

According to set (1.1), the solutions of Eqs. (2)–(3) can be stated as:

(1.1) If $\tau_2 > 0$, $\tau_4 < 0$ and $\mathcal{L}_9 < 0$, then:

$$\phi_{1.1}(z, t) = \left(\frac{1}{2}\sqrt{\tau_2}\sqrt[4]{-\frac{105}{\mathcal{L}_9}}\operatorname{sech}\left[(z - \rho t)\sqrt{\tau_2}\right]\right)^{\frac{1}{2}} e^{i(-\mathcal{K}z + \omega t + \Delta)}, \tag{32}$$

$$\psi_{1.1}(z, t) = \Omega\left(\frac{1}{2}\sqrt{\tau_2}\sqrt[4]{-\frac{105}{\mathcal{L}_9}}\operatorname{sech}\left[(z - \rho t)\sqrt{\tau_2}\right]\right)^{\frac{1}{2}} e^{i(-\mathcal{K}z + \omega t + \Delta)}, \tag{33}$$

these solutions represent bright soliton solutions.

According to set (1.2), the solutions of Eqs. (2)–(3) can be stated as:

(1.2) If $\tau_2 < 0$, $\tau_4 > 0$ and $\mathcal{L}_9 < 0$, the solutions will be:

$$\phi_{1.2.1}(z, t) = \left(\frac{1}{2}\sqrt{-\tau_2}\sqrt[4]{-\frac{105}{\mathcal{L}_9}}\sec\left[(z - \rho t)\sqrt{-\tau_2}\right]\right)^{\frac{1}{2}} e^{i(-\mathcal{K}z + \omega t + \Delta)}, \tag{34}$$

$$\psi_{1.2,1}(z,t) = \Omega \left(\frac{1}{2}\sqrt{-\tau_2} \sqrt[4]{-\frac{105}{\mathcal{L}_9}} \sec\left[(z-\rho t)\sqrt{-\tau_2}\right] \right)^{\frac{1}{2}} e^{i(-\mathcal{K}z+\omega t+\Delta)}, \tag{35}$$

or

$$\phi_{1.2,2}(z,t) = \left(\frac{1}{2}\sqrt{-\tau_2} \sqrt[4]{-\frac{105}{\mathcal{L}_9}} \csc\left[(z-\rho t)\sqrt{-\tau_2}\right] \right)^{\frac{1}{2}} e^{i(-\mathcal{K}z+\omega t+\Delta)}, \tag{36}$$

$$\psi_{1.2,2}(z,t) = \Omega \left(\frac{1}{2}\sqrt{-\tau_2} \sqrt[4]{-\frac{105}{\mathcal{L}_9}} \csc\left[(z-\rho t)\sqrt{-\tau_2}\right] \right)^{\frac{1}{2}} e^{i(-\mathcal{K}z+\omega t+\Delta)}, \tag{37}$$

these solutions represent the singular periodic solutions.

Result-(2): When $\tau_1 = \tau_3 = \tau_6 = \mathbb{A}_0 = \mathbb{C}_1 = 0$ and $\tau_0 = \frac{\tau_2^2}{4\tau_4}$, this is the set of solutions that we gained:

$$\mathcal{L}_7 = \mathcal{L}_3 = 0, \ \ \mathbb{A}_1 = \frac{\sqrt[4]{-105}\tau_2}{4\sqrt{\tau_0}\sqrt[4]{\mathcal{L}_9}}, \ \ \mathbb{B}_1 = \frac{\sqrt[4]{-105}\sqrt{\tau_0}}{2\sqrt[4]{\mathcal{L}_9}}, \ \ \tau_4 = \frac{\tau_2^2}{4\tau_0}, \ \ \mathcal{L}_5 = -13\tau_2\sqrt{-\frac{3\mathcal{L}_9}{35}}, \ \ \mathcal{L}_1 = -\frac{\tau_2^2}{4}$$

Based on the recovered solutions' set, we may state the following solutions:
If $\tau_2 > 0, \ \tau_4 > 0$ and $\mathcal{L}_9 < 0$, we get singular periodic solutions:

$$\phi_2(z,t) = \left(\sqrt{\tau_2} \sqrt[4]{-\frac{105}{2\mathcal{L}_9}} \csc\left[(z-\rho t)\sqrt{2\tau_2}\right] \right)^{\frac{1}{2}} e^{i(-\mathcal{K}z+\omega t+\Delta)}, \tag{38}$$

$$\psi_2(z,t) = \Omega \left(\sqrt{\tau_2} \sqrt[4]{-\frac{105}{2\mathcal{L}_9}} \csc\left[(z-\rho t)\sqrt{2\tau_2}\right] \right)^{\frac{1}{2}} e^{i(-\mathcal{K}z+\omega t+\Delta)}. \tag{39}$$

Result-(3): When $\tau_3 = \tau_4 = \tau_6 = \mathbb{A}_0 = \mathbb{C}_1 = 0$, this is the set of solutions that we gained:

$$\mathbb{A}_1 = 0, \mathbb{B}_1 = \frac{2\tau_1\sqrt{-\frac{1}{7}(15\mathcal{L}_9)}}{\mathcal{L}_7}, \ \ \mathcal{L}_1 = -\frac{\tau_2^2}{16}, \ \ \mathcal{L}_3 = \frac{1}{8}\tau_2\sqrt{-\frac{35}{3\mathcal{L}_9}}\mathcal{L}_7,$$

$$\mathcal{L}_5 = \frac{13}{2}\tau_2\sqrt{-\frac{1}{35}(3\mathcal{L}_9)} + \frac{7\mathcal{L}_7^2}{40\mathcal{L}_9}, \ \ \tau_0 = \frac{16\sqrt{-\frac{15}{7}\mathcal{L}_9^3}\,\tau_1^2}{7\mathcal{L}_7^2}.$$

If $\tau_2 > 0, \ \tau_0 = \frac{\tau_1^2}{4\tau_2}, \ \mathcal{L}_9 < 0, \ \tau_1\tau_2\mathcal{L}_7 < 0$ and $\tau_1 \neq 2\tau_2 e^{\sqrt{\tau_2}(z-\rho t)}$, then, the solutions are:

$$\phi_3(z,t) = 2\sqrt[4]{\frac{15}{7}} \left(-\frac{\tau_1\tau_2\sqrt{-\mathcal{L}_9}}{\mathcal{L}_7\left(\tau_1 - 2\tau_2 e^{\sqrt{\tau_2}(z-\rho t)}\right)} \right)^{\frac{1}{2}} e^{i(-\mathcal{K}z+\omega t+\Delta)}, \tag{40}$$

$$\psi_3(z,t) = 2\Omega \sqrt[4]{\frac{15}{7}} \left(-\frac{\tau_1\tau_2\sqrt{-\mathcal{L}_9}}{\mathcal{L}_7\left(\tau_1 - 2\tau_2 e^{\sqrt{\tau_2}(z-\rho t)}\right)} \right)^{\frac{1}{2}} e^{i(-\mathcal{K}z+\omega t+\Delta)}, \tag{41}$$

these solutions are exponential ones.

Result-(4): When $\tau_0 = \tau_1 = \tau_2 = \tau_6 = \mathbb{A}_0 = \mathbb{C}_1 = 0$, then, we obtain:

$$\mathbb{B}_1 = \mathcal{L}_1 = \mathcal{L}_3 = 0, \ \ \mathbb{A}_1 = \frac{\sqrt[4]{-105}\sqrt{-\tau_4}}{2\sqrt[4]{\mathcal{L}_9}}, \ \ \mathcal{L}_5 = -\frac{2\tau_3^2\sqrt{-\frac{1}{35}(3\mathcal{L}_9)}}{\tau_4}, \ \ \mathcal{L}_7 = \frac{4\sqrt[4]{-15\mathcal{L}_9^3}\tau_3}{7^{3/4}\sqrt{-\tau_4}}.$$

Then, the solutions are:

$$\phi_4(z,t) = \left(\frac{\sqrt[4]{\frac{-105}{\mathcal{L}_9}}\tau_3\sqrt{-2\tau_4}}{(\tau_3^2(z-\rho t)^2 - 4\tau_4)} \right)^{\frac{1}{2}} e^{i(-\mathcal{K}z+\omega t+\Delta)}, \tag{42}$$

$$\psi_4(z,t) = \Omega \left(\frac{\sqrt[4]{\frac{-105}{\mathcal{L}_9}} \tau_3 \sqrt{-2\tau_4}}{(\tau_3^2(z-\rho t)^2 - 4\tau_4)} \right)^{\frac{1}{2}} e^{i(-\mathcal{K}z+\omega t+\Delta)}, \tag{43}$$

which provide rational wave solutions in the constraints that $\mathcal{L}_9 < 0$, $\tau_4 < 0$ and $\tau_3^2(z-\rho t)^2 - 4\tau_4 \neq 0$.

Result-(5): When $\tau_0 = \tau_1 = \tau_6 = \mathbb{A}_0 = \mathbb{C}_1 = 0$, then, we discover:

$$\mathbb{B}_1 = 0, \ \mathbb{A}_1 = -\frac{5\tau_2\tau_3}{4\mathcal{L}_3}, \ \mathcal{L}_1 = -\frac{\tau_2^2}{16}, \ \mathcal{L}_7 = \frac{96\tau_4\mathcal{L}_3^3}{25\tau_2^3\tau_3^2}, \ \mathcal{L}_5 = -\frac{6\left(4\tau_3^2 + 13\tau_2\tau_4\right)\mathcal{L}_3^2}{25\tau_2^2\tau_3^2}, \ \mathcal{L}_9 = -\frac{336\tau_4^2\mathcal{L}_3^4}{125\tau_2^4\tau_3^4}.$$

Then, the solutions will be:

(5.1) If $\tau_2 > 0$, $\tau_3^2 = 4\tau_2\tau_4$ and $\mathcal{L}_3 > 0$, then, the solutions are:

$$\phi_{5.1}(z,t) = \frac{1}{2}\tau_2 \left(\frac{5\left(\tanh\left[\frac{1}{2}(z-\rho t)\sqrt{\tau_2}\right] + 1\right)}{\mathcal{L}_3} \right)^{\frac{1}{2}} e^{i(-\mathcal{K}z+\omega t+\Delta)}, \tag{44}$$

$$\psi_{5.1}(z,t) = \frac{\Omega}{2}\tau_2 \left(\frac{5\left(\tanh\left[\frac{1}{2}(z-\rho t)\sqrt{\tau_2}\right] + 1\right)}{\mathcal{L}_3} \right)^{\frac{1}{2}} e^{i(-\mathcal{K}z+\omega t+\Delta)}, \tag{45}$$

these solutions are the dark solitons, or we can obtain singular soliton solutions as:

$$\phi_{5.2}(z,t) = \frac{1}{2}\tau_2 \left(\frac{5\left(\coth\left[\frac{1}{2}(z-\rho t)\sqrt{\tau_2}\right] + 1\right)}{\mathcal{L}_3} \right)^{\frac{1}{2}} e^{i(-\mathcal{K}z+\omega t+\Delta)}, \tag{46}$$

$$\psi_{5.2}(z,t) = \frac{\Omega}{2}\tau_2 \left(\frac{5\left(\coth\left[\frac{1}{2}(z-\rho t)\sqrt{\tau_2}\right] + 1\right)}{\mathcal{L}_3} \right)^{\frac{1}{2}} e^{i(-\mathcal{K}z+\omega t+\Delta)}. \tag{47}$$

(5.2) If $\tau_2 > 0$, $\tau_4 > 0$, $\tau_3^2 \neq 4\tau_2\tau_4$, $\tau_3\mathcal{L}_3 > 0$ and $\tau_3 - 2\sqrt{\tau_2\tau_4}\tanh\left(\frac{1}{2}\sqrt{\tau_2}(z-\rho t)\right) \neq 0$, then, the solutions are:

$$\phi_{5.3}(z,t) = \frac{1}{2}\tau_2 \sqrt{\frac{5\tau_3\text{sech}^2\left[\frac{1}{2}(z-\rho t)\sqrt{\tau_2}\right]}{\mathcal{L}_3\left(\tau_3 - 2\sqrt{\tau_2\tau_4}\tanh\left[\frac{1}{2}(z-\rho t)\sqrt{\tau_2}\right]\right)}} e^{i(-\mathcal{K}z+\omega t+\Delta)}, \tag{48}$$

$$\psi_{5.3}(z,t) = \frac{\Omega}{2}\tau_2 \sqrt{\frac{5\tau_3\text{sech}^2\left[\frac{1}{2}(z-\rho t)\sqrt{\tau_2}\right]}{\mathcal{L}_3\left(\tau_3 - 2\sqrt{\tau_2\tau_4}\tanh\left[\frac{1}{2}(z-\rho t)\sqrt{\tau_2}\right]\right)}} e^{i(-\mathcal{K}z+\omega t+\Delta)}, \tag{49}$$

these solutions represent combo bright-dark soliton solutions.

(5.3) If $\tau_2 < 0$, $\tau_4 > 0$, $\tau_3^2 \neq 4\tau_2\tau_4$, $\tau_3\mathcal{L}_3 > 0$ and $\tau_3 - 2\sqrt{\tau_2\tau_4}\tan\left(\frac{1}{2}\sqrt{-\tau_2}(z-\rho t)\right) \neq 0$, then:

$$\phi_{5.4}(z,t) = \frac{1}{2}\tau_2 \sqrt{\frac{5\tau_3\sec^2\left[\frac{1}{2}(z-\rho t)\sqrt{-\tau_2}\right]}{\mathcal{L}_3\left(\tau_3 - 2\sqrt{\tau_2\tau_4}\tan\left[\frac{1}{2}(z-\rho t)\sqrt{-\tau_2}\right]\right)}} e^{i(-\mathcal{K}z+\omega t+\Delta)}, \tag{50}$$

$$\psi_{5.4}(z,t) = \frac{\Omega}{2}\tau_2 \sqrt{\frac{5\tau_3\sec^2\left[\frac{1}{2}(z-\rho t)\sqrt{-\tau_2}\right]}{\mathcal{L}_3\left(\tau_3 - 2\sqrt{\tau_2\tau_4}\tan\left[\frac{1}{2}(z-\rho t)\sqrt{-\tau_2}\right]\right)}} e^{i(-\mathcal{K}z+\omega t+\Delta)}, \tag{51}$$

these solutions represent singular periodic solutions.

Result-(6): When $\tau_2 = \tau_4 = \tau_6 = \mathbb{A}_0 = \mathbb{C}_1 = 0$, then, we gain:

$$\mathbb{A}_1 = 0, \ \mathbb{B}_1 = \frac{\mathcal{L}_3\sqrt{-\frac{7}{15\mathcal{L}_9}}}{\tau_3}, \ \mathcal{L}_1 = \frac{-3}{4}\tau_1\tau_3, \ \mathcal{L}_7 = -\frac{30\tau_1\tau_3\mathcal{L}_9}{7\mathcal{L}_3}, \ \tau_0 = \frac{4\sqrt{-\frac{7}{15\mathcal{L}_9}}\mathcal{L}_3^2}{15\tau_3^2}, \ \mathcal{L}_5 = \frac{45\tau_1^2\tau_3^2\mathcal{L}_9}{14\mathcal{L}_3^2}.$$

So, we can have the following Weierstrass elliptic doubly periodic solutions:

$$\phi_6(z,t) = \sqrt{\frac{\sqrt{-\frac{7}{15\mathcal{L}_9}}\mathcal{L}_3}{\tau_3 \wp\left(\frac{1}{2}(z-t\rho)\sqrt{\tau_3}; -\frac{4\tau_1}{\tau_3}, -\frac{4\tau_0}{\tau_3}\right)}} \, e^{i(-\mathcal{K}z+\omega t+\Delta)}, \tag{52}$$

$$\psi_6(z,t) = \Omega\sqrt{\frac{\sqrt{-\frac{7}{15\mathcal{L}_9}}\mathcal{L}_3}{\tau_3 \wp\left(\frac{1}{2}(z-t\rho)\sqrt{\tau_3}; -\frac{4\tau_1}{\tau_3}, -\frac{4\tau_0}{\tau_3}\right)}} \, e^{i(-\mathcal{K}z+\omega t+\Delta)}, \tag{53}$$

where $\mathcal{L}_9 < 0$ and $\mathcal{L}_3\tau_3 > 0$.

Result-(7): When $\tau_1 = \tau_3 = \tau_6 = \mathbb{A}_0 = \mathbb{C}_1 = 0$, then, these are the sets of solutions that we discovered:

Set (7.1): $\mathbb{B}_1 = \mathcal{L}_3 = \mathcal{L}_7 = \tau_0 = 0, \ \mathbb{A}_1 = \sqrt{-\frac{39\tau_2\tau_4}{8\mathcal{L}_5}}, \ \mathcal{L}_1 = -\frac{\tau_2^2}{16}, \ \mathcal{L}_9 = -\frac{140\mathcal{L}_5^2}{507\tau_2^2}.$

S $\mathcal{L}_3^e = \mathcal{L}_7 = 0$, $\mathbb{A}_1 = -\sqrt{\frac{39\tau_2\tau_4}{4\mathcal{L}_5}}, \ \mathbb{B}_1 = -\tau_2\sqrt{\frac{39\tau_2}{16\tau_4\mathcal{L}_5}}, \ \mathcal{L}_1 = -\frac{\tau_2^2}{4}, 2\mathcal{L}_9 = -\frac{35\mathcal{L}_5^2}{507\tau_2^2}, , \tau_0 = \frac{.\tau_2^2}{4\tau_4}.$

As per the solution set (7.1), the solutions can be expressed as follows:

(7.1,1) If $\tau_0 = m^2-1, \ \tau_2 = 2-m^2, \ \tau_4 = -1,$ and $\mathcal{L}_5 > 0$, then, the solutions are:

$$\phi_{7.1,1}(z,t) = \left(\frac{1}{2}\sqrt{\frac{39}{2\mathcal{L}_5}}\mathrm{sech}[z-\rho t]\right)^{\frac{1}{2}} e^{i(-\mathcal{K}z+\omega t+\Delta)}, \tag{54}$$

$$\psi_{7.1,1}(z,t) = \Omega\left(\frac{1}{2}\sqrt{\frac{39}{2\mathcal{L}_5}}\mathrm{sech}[z-\rho t]\right)^{\frac{1}{2}} e^{i(-\mathcal{K}z+\omega t+\Delta)}, \tag{55}$$

these solutions represent bright soliton solutions.

(7.1,2) If $\tau_0 = -m^2, \ \tau_2 = 2m^2-1, \ \tau_4 = 1-m^2,$ and $\mathcal{L}_5 > 0$, then, the solutions are:

$$\phi_{7.1,2}(z,t) = \left(\frac{1}{2}\sqrt{\frac{39}{2\mathcal{L}_5}}\sec[z-\rho t]\right)^{\frac{1}{2}} e^{i(-\mathcal{K}z+\omega t+\Delta)}, \tag{56}$$

$$\psi_{7.1,2}(z,t) = \Omega\left(\frac{1}{2}\sqrt{\frac{39}{2\mathcal{L}_5}}\sec[z-\rho t]\right)^{\frac{1}{2}} e^{i(-\mathcal{K}z+\omega t+\Delta)}, \tag{57}$$

these solutions are singular periodic solutions.

As per the solution set (7.2), the solutions can be expressed as follows:

(7.2,1) If $\tau_0 = 1, \ \tau_2 = -m^2-1, \ \tau_4 = m^2,$ and $\mathcal{L}_5 < 0$, then, we raise singular soliton solutions as:

$$\phi_{7.2,1}(z,t) = \sqrt[4]{78}\left(\sqrt{-\frac{1}{\mathcal{L}_5}}\mathrm{csch}[2(z-\rho t)]\right)^{\frac{1}{2}} e^{i(-\mathcal{K}z+\omega t+\Delta)}, \tag{58}$$

$$\psi_{7.2,1}(z,t) = \Omega\sqrt[4]{78}\left(\sqrt{-\frac{1}{\mathcal{L}_5}}\mathrm{csch}[2(z-\rho t)]\right)^{\frac{1}{2}} e^{i(-\mathcal{K}z+\omega t+\Delta)}. \tag{59}$$

Graphical simulation of some retrieved solutions

Equations (2)–(3) yielded many families of solutions when specific values were assigned to the parameters. This investigation has, therefore, led to several previously unpublished original and revised results. The 3D, contour, and 2D figures of certain specific solutions are displayed in order to let the reader fully understand the physical structures of some extracted solutions that will be provided. The bright soliton solution of Eq. (32) is plotted in Fig. 1 where the parameter's values are $\alpha_1 = 0.7, \ \mathcal{K} = 0.5, \ \beta_1 = -0.6, \ \beta_2 = 0.8, \ \alpha_2 = 0.7, \ \rho = 0.5, \ \Omega = 0.6, \ d_{11} = 0.5, \ d_{12} = 0.75, \ d_{13} = 0.6, \ d_{14} = 0.7, \ d_{15} = 0.8, \ \Delta = 0.7,$ and $-15 < x < 15$. This solution depicts a bright soliton propagating over a nonlinear medium while keeping its localised form over time. The absence of oscillatory tails points to a basic soliton rather than a higher-order or periodic structure. This behaviour has applications in nonlinear optics, optical fibre communications, and wave dynamics. Eq. (34) shows a singular periodic solution which is drawn in Fig. 2 with $\alpha_1 = 0.7, \ \mathcal{K} = 0.6, \ \beta_1 = -0.7, \ \beta_2 = 0.9, \ \alpha_2 = 0.8, \ \rho = 0.6, \ \Omega = 0.7, \ d_{11} = 0.85, \ d_{12} = 0.7, \ d_{13} = 0.7,$
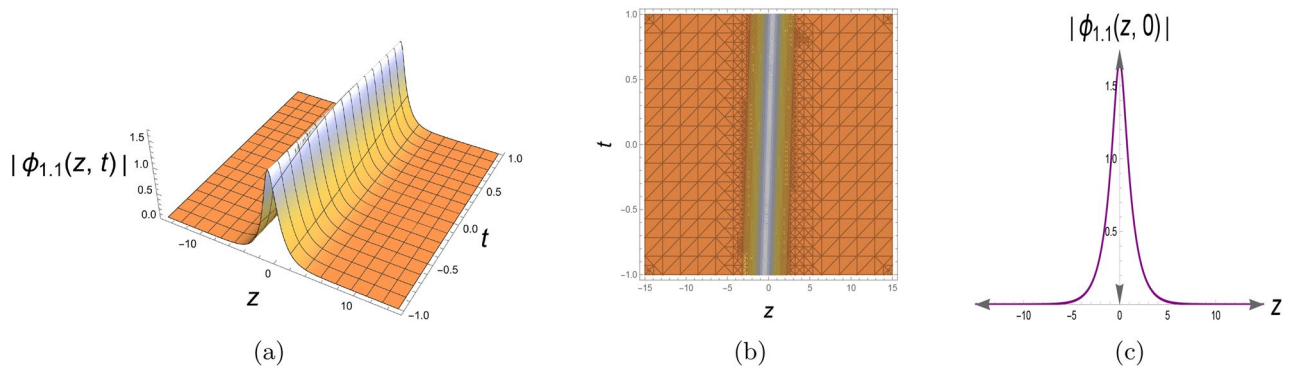
10

(a) (b) (c)

Fig. 1. Simulation of the solution of Eq. (32).
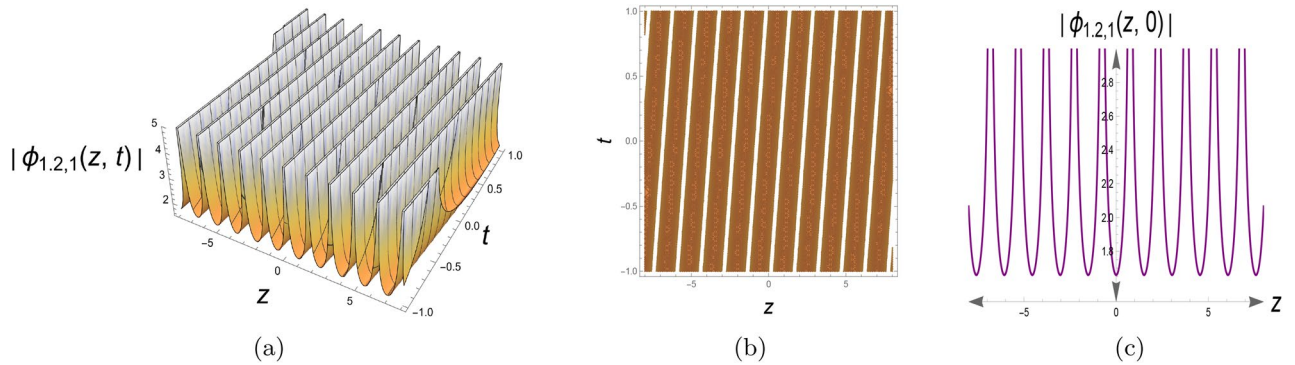

(a) (b) (c)

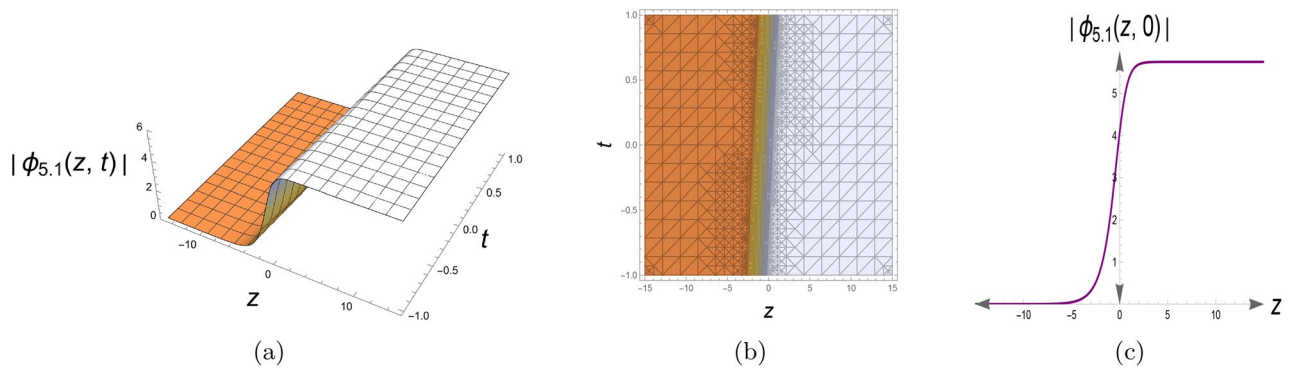**Fig. 2.** Simulation of the solution of Eq. (34).

(a) (b) (c)

Fig. 3. Simulation of the solution of Eq. (44).

$d_{14} = 0.6$, $d_{15} = 0.7$, $\Delta = 0.8$, and $-8 < x < 8$. This solution is a periodic wave train rather than a single soliton, with steady and repeating oscillations. Such solutions occur in birefringent fibres and nonlinear media where energy distribution is periodic rather than localised, making them useful for optical signal processing, waveguides, and nonlinear wave interactions. Figure 3 displays a dark soliton solution of Eq. (44) with $\alpha_1 = -0.5$, $\mathcal{K} = 0.7$, $\beta_1 = 0.8$, $\beta_2 = -0.7$, $\alpha_2 = -0.8$, $\rho = 0.7$, $\Omega = 0.7$, $a_{11} = 0.5$, $a_{12} = 0.7$, $\Delta = 0.8$, and $-15 < x < 15$. This solution is a dark soliton solution, which is an important structure in nonlinear optics, fluid mechanics, and phase transition models. The sudden shift in amplitude indicates shock-wave dynamics, which are common in fibre optics, plasma physics, and condensed matter systems. Additionally, Eq.(48) is a combo bright-dark soliton solution that is plotted in Fig. 4 when $\alpha_1 = -0.7$, $\mathcal{K} = 0.7$, $\beta_1 = 0.6$, $\beta_2 = -0.8$, $\alpha_2 = 0.6$, $\rho = 0.7$, $\Omega = 0.8$, $a_{11} = 0.5$, $a_{12} = 0.7$, $\rho = 0.7$, $\Omega = 0.8$, $a_{11} = 0.5$, $a_{12} = 0.7$, $\Delta = 0.85$, $\tau_4 = 0.7$, $\tau_3 = 0.9$, and $-15 < x < 15$.
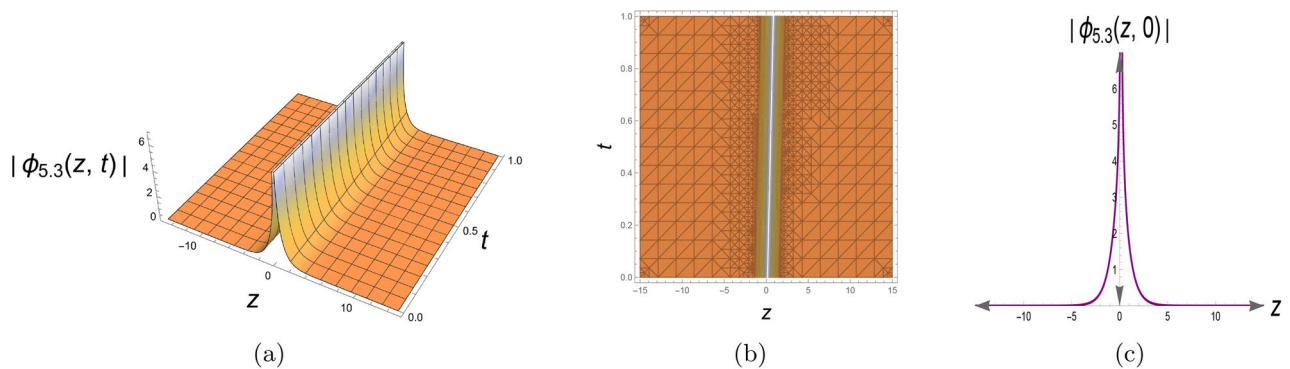
Fig. 4. Simulation of the solution of Eq. (48).

Conclusion

We take on a leading position in the field of nonlinear optics with our study. Our main goal is to thoroughly investigate soliton solutions as they relate to birefringent fibers with CQSN nonlinearity, using the framework of the CQ-NLSE model. This unexplored field adds something unique and important to the body of knowledge in science. In order to understand the complexities of fiber communication networks and optimize their performance, we emphasize repeatedly in this study how crucial mathematical modeling and analysis are. Researchers and engineers may comprehend the dynamics and properties of optical systems by skillfully applying mathematical tools. These discoveries promise improved performance and efficiency and have practical implications for the development of fiber communication networks. Key conclusions include the impact of higher-order self-phase modulation on soliton structure, the role of non-Kerr nonlinearities in shaping pulse dynamics, and the conditions for stable propagation. We found a large number of new soliton solutions. The retrieved solutions include singular, dark, bright, and combo bright-dark solitons. Also, we get numerous exact wave solutions, including rational, exponential, singular periodic, and Weierstrass elliptic double periodic solutions. With the fiber optic communications business developing so quickly, it is impossible to exaggerate the value of mathematical modelling and analysis. This dynamic interaction fosters creativity, maximizes system efficiency, and effectively handles the ever-increasing demands of contemporary telecommunications. More sophisticated fiber communication technologies are made possible by the combination of practical technical solutions and mathematical methodologies, which effectively meet the demands of our data-driven society. In order to clarify the physical nature of some solutions, graphic representations were also added. Since this model has never been examined using the suggested method, the answers we retrieved for our study report are unique. The method's aptitude for handling NPDEs and its ease of use, efficacy, and success rate are further indications of its trustworthiness.

Data availability

No data was used for the research described in the article.

References

1. Kukkar, A. et al. Optical solitons for the concatenation model with Kurdryashov's approaches. *Ukrain. J. Phys. Opt.* **24**(2), 56 (2023).
2. Ahmed, K. K., Badra, N. M., Ahmed, H. M. & Rabie, W. B. Unveiling optical solitons and other solutions for fourth-order (2+ 1)-dimensional nonlinear Schrödinger equation by modified extended direct algebraic method. *J. Opt.* **7**, 1–13 (2024).
3. Chou, D., Ifrah, I., Rehman, H. U., Khalil, O. H. & Osman, M. S. *Heat Conduction Dynamics: A Study of Lie Symmetry, Solitons, and Modulation Instability* (Rendiconti Lincei, Scienze Fisiche e Naturali, 2025).
4. Kudryashov, N. A., Biswas, A., Kara, A. H. & Yıldırım, Y. Cubic-quartic optical solitons and conservation laws having cubic–quintic–septic–nonic self-phase modulation. *Optik* **269**, 169834 (2022).
5. Gonzalez-Gaxiola, O., Biswas, A., Ekici, M. & Khan, S. Highly dispersive optical solitons with quadratic-cubic law of refractive index by the variational iteration method. *J. Opt.* **7**, 1–8 (2022).
6. Rehan Akber, H. U., Wazwaz, A. M., Alshehri, H. M. & Osman, M. S. Analysis of Brownian motion in stochastic Schrödinger wave equation using Sardar sub-equation method. *Optik* **289**, 171305 (2023).
7. Ogundipe, D. O. Conceptualizing cloud computing in financial services: Opportunities and challenges in Africa-US contexts. *Comput. Sci. IT Res. J.* **5**(4), 757–767 (2024).
8. Hasan, M. K., Weichen, Z., Safie, N., Ahmed, F. R. A. & Ghazal, T. M. A survey on key agreement and authentication protocol for Internet of Things application. *IEEE Access* (2024).
9. Evens, T., Henderickx, A. & Conradie, P. Technological affordances of video streaming platforms: Why people prefer video streaming platforms over television. *Eur. J. Commun.* **39**(1), 3–21 (2024).
10. Khalifa, A. S., Badra, N. M., Ahmed, H. M. & Rabie, W. B. Retrieval of optical solitons in fiber Bragg gratings for high-order coupled system with arbitrary refractive index. *Optik* **287**, 171116 (2023).
11. Osman, M. S., Machado, J. & Baleanu, D. On nonautonomous complex wave solutions described by the coupled Schrödinger–Boussinesq equation with variable-coefficients. *Opt. Quant. Electron.* **50**, 73 (2018).

12. Khalifa, A. S., Ahmed, H. M., Badra, N. M. & Rabie, W. B. Exploring solitons in optical twin-core couplers with Kerr law of nonlinear refractive index using the modified extended direct algebraic method. *Opt. Quant. Electron.* **56**(6), 1060 (2024).
13. Ahmed, K. K. et al. Diverse exact solutions to Davey–Stewartson model using modified extended mapping method. *Nonlinear Anal. Model. Control* **29**(5), 983–1002 (2024).
14. Nithyanandan, K. Soliton physics in India: A tribute to the late K. Porsezian. *Opt. Commun.* **553**, 130078 (2024).
15. Martel, Y. Asymptotic stability of small standing solitary waves of the one-dimensional cubic–quintic Schrödinger equation. *Inventiones Mathematicae* **8**, 1–76 (2024).
16. Zhong, Y., Triki, H. & Zhou, Q. Bright and kink solitons of time-modulated cubic–quintic–septic–nonic nonlinear Schrödinger equation under space-time rotated pt-symmetric potentials. *Nonlinear Dyn.* **112**(2), 1349–1364 (2024).
17. AlQahtani, S. A., Alngar, M. E., Shohib, R. & Alawwad, A. M. Enhancing the performance and efficiency of optical communications through soliton solutions in birefringent fibers. *J. Opt.* **6**, 1–11 (2024).
18. Yıldırım, Y., Biswas, A., Khan, S., Mahmood, M. F. & Alshehri, H. M. Highly dispersive optical soliton perturbation with Kudryashov's sextic-power law of nonlinear refractive index. *Ukr. J. Phys. Opt.* **23**(1), 24–29 (2022).
19. Ismael, H. F., Sulaiman, T. A., Younas, U. & Nabi, H. R. On the autonomous multiple wave solutions and hybrid phenomena to a (3+1)-dimensional Boussinesq-type equation in fluid mediums. *Chaos Solitons Fractals* **187**, 115374 (2024).
20. Muhammad, J., Younas, U., Nasreen, N., Khan, A. & Abdeljawad, T. Multicomponent nonlinear fractional Schrödinger equation: On the study of optical wave propagation in the fiber optics. *Part. Differ. Equ. Appl. Math.* **11**, 100805 (2024).
21. Younas, U. et al. On the study of solitary wave dynamics and interaction phenomena in the ultrasound imaging modelled by the fractional nonlinear system. *Sci. Rep.* **14**(1), 26080 (2024).
22. Muhammad, J., Nasreen, N., Hussain, E., Younas, U. & Alsubaie, A. S. On the study of analytical soliton solutions and interaction aspects to the Estevez–Mansfield–Clarkson equation arising in diversity of fields. *Phys. Scr.* **99**(11), 115221 (2024).
23. Muhammad, J. et al. Analysis of fractional solitary wave propagation with parametric effects and qualitative analysis of the modified Korteweg-de Vries-Kadomtsev-Petviashvili equation. *Sci. Rep.* **14**(1), 19736 (2024).
24. Muhammad, J., Rehman, S. U., Nasreen, N., Bilal, M. & Younas, U. Exploring the fractional effect to the optical wave propagation for the extended Kairat-II equation. *Nonlinear Dyn.* **6**, 1–12 (2024).
25. Raza, N., Kaplan, M., Javid, A. & Inc, M. Complexiton and resonant multi-solitons of a (4+ 1)-dimensional Boiti–Leon–Manna–Pempinelli equation. *Opt. Quant. Electron.* **54**, 1–16 (2022).
26. Javid, A., Seadawy, A. R. & Raza, N. Dual-wave of resonant nonlinear Schrödinger's dynamical equation with different nonlinearities. *Phys. Lett. A* **407**, 127446 (2021).
27. Kazmi, S. S. et al. The analysis of bifurcation, quasi-periodic and solitons patterns to the new form of the generalized *q*-deformed Sinh–Gordon equation. *Symmetry* **15**(7), 1324 (2023).
28. Raza, N., Abdullah, M., Butt, A. R., Murtaza, I. G. & Sial, S. New exact periodic elliptic wave solutions for extended quantum Zakharov–Kuznetsov equation. *Opt. Quant. Electron.* **50**, 1–17 (2018).
29. Xu, C., Xu, T., Li, M. & Huang, Y. Inverse scattering transform for the focusing PT-symmetric nonlinear Schrödinger equation with nonzero boundary conditions: Higher-order poles and multi-solition solutions. *Physica D* **472**, 134466 (2025).
30. Yang, L. & Gao, B. Multiple solitons solutions, lump solutions and rogue wave solutions of the complex cubic Ginzburg–Landau equation with the Hirota bilinear method. *Indian J. Phys.* **99**(1), 221–228 (2025).
31. Mandal, S., Sil, S. & Ghosh, S. On the Lie symmetry analysis of three-dimensional perturbed shear flows. *Chaos Solitons Fractals* **191**, 115875 (2025).

Acknowledgements

Author contributions
K.K and N.A: Writing-review & editing, Methodology, Investigation. H.M.: Writing-original draft, Software, Conceptualization. S.B.: Writing-review & editing, Funding acquisition. M.S.: Supervision, Writing-review & editing.

Declarations

Competing interests
The authors declare no competing interests.

Additional information
Correspondence and requests for materials should be addressed to S.B. or M.S.O.

Reprints and permissions information is available at www.nature.com/reprints.

Publisher's note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.