



## OPEN Quality of basketballs of international level sports using an intelligent decision algorithm involving multiple experts

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In the competitive climate of international sports, quality assessment of basketballs assumes a critical function to guarantee performance at its best. The research proposes a new intelligent decision-making method developed to measure the global quality of professional competition basketballs. To efficiently address uncertainty and imprecision present in expert judgments, we utilize circular picture fuzzy values (CPFVs), which provide a versatile framework for representing vagueness in judgment. Addressing multi-attribute decision-making (MADM) issues involving interdependent criteria, we propose aggregation operators in the framework of Aczel–Alsina (A–A) operations under a circular prioritized picture fuzzy (CPPF) setting. In particular, we introduce two novel operators the Cr-PPF Aczel–Alsina geometric (Cr-PPFAAG) and the Cr-PPF Aczel–Alsina average (Cr-PPFAAA) as generalizations of A–A t-norm and t-conorm operators for ordered fuzzy data. We analyze the mathematical properties and relationships among these operators and embed them into a MADM model. An empirical case study investigates five prominent global basketball brands on six quality features, e.g., the durability of the material, consistency of the bounce, quality of grip, and air retention. The results of the evaluation place the Brand at the highest rank with a cumulative score of, followed by Brand ranks the lowest with 0.61. Comparative evaluation verifies the superior performance of the suggested model in dealing with ambiguity and generating strong, consistent rankings compared to other methods. The study provides real-world guidance for sports associations, manufacturers, and procurement decision-makers when choosing high-quality basketballs that comply with international performance standards and enhance professional play.

**Keywords** Aczel–Alsina operations, Prioritization, International sports assessments, Quality of basketballs, Multi-attribute decision making

### Abbreviations

AO	Aggregation operator
DM	Decision making
A–A TN	Aczel Alsina t-norm
A–A TCN	Aczel Alsina t-conorm
Cr-PFS	Circular picture fuzzy set
FS	Fuzzy set
IFS	Intuitionistic fuzzy set
PFS	Picture fuzzy set
CPPF	Circular prioritized picture fuzzy
CPVs	Circular picture fuzzy values
Cr-PPFAAA	Cr-PPF Aczel–Alsina averaging
Cr-PPFAAG	Cr-PPF Aczel–Alsina geometric
MADM	Multi-attribute decision making
DoM	Degree of membership
DoNM	Degree of non-membership

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MADM-related research is becoming more and more popular these days. Many difficult real-world issues have already been successfully solved using theories and concepts connected to MADM. MADM techniques have drawn a lot of interest nowadays because of their flexibility in handling challenging issues in a variety of fields. For instance, MADM techniques have been used to prioritize real estate companies using credit risk assessments within group decision support frameworks<sup>1</sup>, assess blockchain adoption barriers in medical supply chains with a fairness-aware paradigm<sup>2</sup>, and improve the sustainability of critical material supply in the electrical vehicle market through AI-powered approaches<sup>3</sup>. Furthermore, research has investigated the use of smart contracts to reduce the hazards of extreme weather in construction supply chains<sup>4,5</sup> and the synergy between urban rail systems and urban form using spatial modeling tools<sup>6</sup>. Building on previous developments, this study uses MADM techniques to efficiently handle ambiguity and uncertainty to create a strong framework for making decisions about basketball quality about international standards.

In global-scale sports, the quality of basketballs is important in guaranteeing equal competition and best performance. Current practices, though, rely extensively on subjective evaluation without an organized, objective, and measurable system. Fuzzy MADM approaches have been utilized in comparable scenarios, but prevailing models fail to effectively combine different expert opinions, deal with uncertainty, and prioritize conflicting evaluation factors. This shortfall is due to the difficulty of combining subjective opinions and the insufficiency of conventional fuzzy models, which tend to neglect the subtle significance of various criteria under diverse situations. The necessity then occurs for a more precise, expert-combined, and uncertainty-tolerant decision-making model able to explicitly evaluate and rank basketball options based on global standards. To address this deficiency, this research formulates a new Cr-PFPAA framework by integrating the attributes of probabilistic hesitant fuzzy sets, prioritized operators, and Aczel–Alsina aggregation methods to provide a more powerful and applicable solution.

In 1965, Zadeh<sup>7</sup> introduced the concept of fuzzy sets (FS), defining simply the degree of membership (DoM). One of the commonly recognized theories for treating MADM is FS because decisions are often made based on abstract facts, vulnerability, and ambiguity. Numerous direct and indirect extensions of the FS have been developed and successfully implemented in the majority of the current reality situation's issues. FS has been extended with fixed point theory to solve fuzzy ordinary equations which is proposed by Sarwar and Li<sup>8</sup>. FS has been also extending neural networks<sup>9</sup>, switching systems with deception attacks<sup>10</sup> and sampled-data stabilization<sup>11</sup>, other studies on FS have been developed and can be shown here<sup>12–14</sup>. Using two words, DoM and degree of non-membership (DoNM), Atanassov<sup>15,16</sup> introduced intuitionistic fuzzy sets (IFSs), which are a generalization of FS to understand ambiguity or imprecise information. The duplets' sum must fall between 0 and 1. References<sup>17–19</sup> talk about some current research on IFS theory and applications. The requirements for handling conclusions including several response forms, such as yes, no, abstention, and rejection, cannot be met by FSs and IFSs. IFS were chosen over probability-based preferences due to their superior ability to manage uncertainty and hesitation characteristics in complex decision-making processes. The combined representation of DoM, DoNM, and DoH is possible with IFS, in contrast to probability-based approaches that require exact numerical probabilities. Because it better conveys the uncertainty and vagueness that are inherent, this flexibility is especially helpful in situations where experts struggle to clear their opinions. An accurate and nuanced depiction of expert perspectives is ensured by this study's use of IFS, which is essential for producing trustworthy and significant results.

The concept of IFS has strict constraints and lacks independence when it comes to allocating the DoM and DoNM and limiting their sum lies between 0 and 1. Yager<sup>20</sup> first proposed Pythagorean FS (PyFS), which is a generalization of IFS, to address the case where the total of the duplet exceeds 1<sup>21–24</sup>. Contain the current work on the PyFS hypothesis. Yager<sup>25</sup> introduced the idea of q-rung ortho-pair FS (qROFS), in which q is the independent variable and the sum of the *q*th power of the duplet is constrained between 0 and 1. The current work is shown in<sup>26–30</sup>. To avoid this shortage of data, Cuong<sup>31</sup> proposed Picture fuzzy sets (PFSs), which are more useful than FSs or IFSs. Membership, abstention (DoA), and non-membership degrees are used to indicate PFSs; the sum of these degrees shouldn't be more than one. PFS appears to be a more accurate and logical method of explaining ambiguous data than FSs and IFSs. As soon as PFS was created, several researchers began working on it. Numerous studies on PFS are available in<sup>32–35</sup>. Menger<sup>36</sup> established the concept of TN. Norms like the Hamacher t-norm (HTN) and Hamacher t-conorm (HTCN) have been discovered to operate<sup>37</sup>, the spherical TN and spherical TCN<sup>38</sup>, the Einstein TN and Einstein TCN<sup>39</sup>, the Archimedean TN and TCN<sup>40</sup>, the Frankfurt TN and TCN, the Underlying TN and TCN<sup>41,42</sup>, and so forth, are crucial to fuzzy set theory. More effectively connected aspects of triangular norms and their relevant features have been researched in recent years by Klement et al.<sup>31</sup>. In 1982, Aczel Alsina proposed the idea of Aczel–Alsina norms and the Aczel–Alsina t-conorm<sup>43,44</sup>, which can be changed under certain restrictions. To overcome the MADM challenges, Senapati et al.<sup>28,29</sup> recently developed Interval-valued IFSs (IVIFSs) and an IFS formation using Aczel–Alsina (A–AAOs). Aczel–Alsina AOs based on PFSs were examined by Senapati<sup>45</sup> with MADM. Hussain et al.<sup>46,47</sup> introduced the idea of Pythagorean FS (PyFS) and T-spherical fuzzy Aczel Alsina (TSFAA) to address MADM issues and interval-valued T-spherical by<sup>48</sup> and<sup>49</sup>. Recent developments in fuzzy MADM techniques<sup>50,51</sup>, have proved enhancement in aggregation procedures as well as uncertainty management, yet the integration of prioritized expert judgments is still challenging<sup>52–54</sup>. This study paper's primary goals are to introduce the idea of Aczel Alsina within the context of PPFs, explain AAPPF values (AASFVs) and their operational laws, present their aggregation operators using the score function for comparison, and go over their characteristics.

In many real-world scenarios, It is essential to have a function in mathematics that can lessen a collection of information to a single value. MADM difficulties are significantly impacted by the aggregation operator (AO) study. Due to its widespread application across many industries, many researchers have focused on data aggregation in recent years. However, there are several circumstances in which there is a rigid relationship between the data that must be combined to prioritize. The MADM issue where the criteria have a priority connection is

the main topic of this investigation. The criteria have varying levels of priority. Consider the following scenario: we want to buy some land and build a house on it based on location ( $c_2$ ), utility access ( $c_1$ ), and cost ( $c_3$ ). Given the price and location, we have no interest in paying for utility access. As a result, the parameters in this case are strictly prioritized, with  $>$  signifying preferable to. To deal with the previously prioritized MADM issue. The prioritized scoring (PS) operator is one of the aggregation operators that Yager<sup>55</sup> introduced. There are three operators: prioritized averaging (PA), prioritized “and,” and prioritized “or.” Yager also suggested the prioritized OWA (POWA) operator, which is based on the BUM function<sup>56</sup>. Yan et al.<sup>57</sup> suggested a prioritized weighted aggregation operator based on triangle norms (t-norms) and the OWA operator. Recent works<sup>50,51,58</sup> have enhanced fuzzy MCDM models using improved aggregation techniques, motivating the need for further advancements such as our Cr-PFPAA approach.

Given the previously mentioned extensive analysis, it is crucial and advantageous to extend the prioritized aggregation operator to CPFVs based on the A–A t-norms and t-norms and use them to address MADM issues. To address more challenging MADM problems in this study, we develop a novel MADM technique within the Picture fuzzy environment and introduce some new Circular Picture fuzzy prioritized A–A (Cr-PFPAA) aggregation operator operators by combining the prioritized aggregation operator with CPFVs based on the A–A. Given the growing popularity of fuzzy logic in decision-making, this work develops the idea of C-IFS and comprehensively proposes new features to enhance its use in MADM methods. It explains new angles on radius computation and the incorporation of interior points into the mindset of defuzzification decision-makers. To make using C-IFS numbers in decision-making situations simple, MADM processes are suggested and demonstrated in the example. Sensitivity analysis is carried out based on changes in parameters, and the outcomes are contrasted with the IF MADM process. By introducing a new C-IFS MADM process and a C-IFS defuzzification function, this paper is a pioneer in the C-IFS literature. Over the past decade, a variety of neural network- and wavelet-based computational models have come to the fore to tackle sophisticated nonlinear systems, most of which are applicable to improve decision-making models in situations of uncertainty. These consist of trustworthy neural network methods that have been successfully implemented in pantograph differential models<sup>59</sup>, language learning systems<sup>60</sup>, and hepatitis B virus dynamics<sup>61</sup>. Morlet and Meyer wavelet-based neural computation processes have been utilized for delay differential models, transport networks, and predator–prey interactions<sup>62–65</sup>. Further works have considered functional nonlinear singular differential equations through Meyer wavelets<sup>66</sup>, in addition to heuristic designs for modeling biological and medical systems like the modeling of smoking behavior and SIRT COVID-19 models<sup>67–71</sup>. Higher-order neuro-swarm hybrid methods, fractional-order and time-delay-based Lane–Emden equations and Emden–Fowler differential models have also been investigated to enhance the accuracy and stability of solutions in intricate systems<sup>72–76</sup>.

This paper’s remainder is organized as follows: Section “**Introduction**” offers a concise synopsis of the current work’s goal. We present the idea of Cr-PPFs and their particular situations in section “**Literature review**”. The idea of PPAAFs and Cr-Picture prioritized fuzzy Aczel–Alsina Average Aggregation operators, such as the Cr-PPAAA operator and the Cr-PPAAG operator, are intended to be introduced in sections three and four. In section “**Multi-criteria group decision-making by using circular picture fuzzy valued Aczel–Alsina**”, we define a method of MADM that solves algebraic problems by utilizing the Cr-PPFA-A geometric operators and the Cr-PPFA-A averaging. In section “**Application (quality of basketballs of international level sports)**”, we use different  $\eta$  to examine the parameter’s impact. We wrap up this study work with a few remarks in the final part.

### Motivation for the proposed framework

While various aggregation methods have been designed in fuzzy environments, the evaluation of basketball quality requires a richer method to overcome conflicting expert opinions, vagueness, and priority among attributes. Cr-PPFs are effective in representing such sophisticated evaluations. In addition, prioritization processes are necessary to reflect expert influence realistically. Aczel–Alsina aggregation can be incorporated for a nonlinear, flexible combination of expert inputs with robustness against extreme evaluations. Such a methodological approach guarantees an accurate and thorough decision-making process for high-stakes sports analysis.

The following are the major reasons for choosing this methodology:

*Evaluation complexity problem:* Basketball quality evaluation has not only several criteria (like player performance, team cohesiveness, coaching effectiveness, etc.) but also different levels of uncertainty and subjectivity in judging these factors. The CPFV model is specifically tailored to address such complexities because it permits the modeling of fuzzy and uncertainty-involving aspects within the data.

*Prioritization of criteria:* In sports quality assessment, certain criteria are more important than others (e.g., individual player performance compared to team coordination). The Prioritized approach enables us to give different importance levels to each criterion, which is crucial to make sure that the most important aspects of basketball performance are assigned greater weight in the overall assessment.

*Aczel–Alsina aggregation:* Aczel–Alsina operator is especially useful to aggregate fuzzy information from several sources since it takes into account the weights as well as the connection between the criteria. The operator is more complex than other simpler aggregations because it guarantees that the global judgment captures not only the intensity of each criterion but also their relative importance and interdependencies.

*Advanced capabilities for realistic and stable decision-making:* Employing a less complicated substitute such as classic fuzzy sets or mere weighted averages would not grant the degree of complexity required for proper basketball quality evaluation. The CPFV + Prioritized + Aczel–Alsina approach enables a more stable decision-making process, particularly in situations where various assessors might have differing views or the data is incomplete or vague.

## Literature review

Over the past decade, many methods of MADM have been constructed to deal with uncertainty and vagueness in expert assessments. Classical methods, e.g., FS theory, IFSs, and PyFSs, have been extensively used. Most of these models are not good at dealing with scenarios where decision-makers are required to indicate hesitation, refusal, or prioritize competing criteria. To mitigate some of these problems, PFSs were proposed, and extensions like CPFSs have enabled better uncertainty and ambiguity modeling. Even with these developments, most of the aggregation methods either pay no attention to attribute prioritization or are not robust in case criteria have interdependencies. Also, recent applications involving conventional TN and TCN operators might experience a loss of information and very low flexibility in handling uncertain data. In application fields like sports equipment rating, where opinions of experts are diverse and attribute weighting is essential, these constraints diminish decision trustworthiness. To fill these gaps, we introduce new aggregation operators Cr PPFAAG and Cr-PPFAAA using CPPFVs and Aczel–Alsina operations. Our method successfully captures decision-makers' hesitancy, refusal, and attribute prioritization, thus providing a more reliable and precise MADM framework. This is especially useful for intricate evaluations like international basketball quality assessments, where subtle expert judgments are crucial.

## Enhanced methodology significance and literature contextualization

The suggested Fermatean fuzzy MADM strategy through the application of the FFSW-WPA operator has better uncertainty management, particularly under vigorously dynamic and vague conditions like foreign trade. In contrast with other recent research, e.g.,<sup>51</sup> applying complicated q-rung orthopair fuzzy Aczel–Alsina power aggregation operators, our model has better interpretability and adaptability in linguistic evaluations. Although Aczel–Alsina operators are effective in capturing hesitancy and intricate information (e.g., in<sup>77</sup> ranking of decision-making frameworks), Fermatean fuzzy sets have a wider range for membership and non-membership values, which is specially beneficial under high-risk, uncertain economic situations.

Similarly, compared to intuitionistic fuzzy Dombi models of aggregation by lower and upper approximations, our model does not suffer from the computational complexity of set approximation techniques but does capture deeper levels of uncertainty. While rough approximations (e.g., applied in assessing Dublin's bike-sharing safety system) are particularly suitable for imprecise spatial data, our approach is preferable for strategic and economic assessments because it involves finer-grained expert-led fuzzification. In addition, although the q-rung orthopair fuzzy logic with Aczel–Alsina operators<sup>78</sup> is a great enhancement for dealing with higher uncertainty in attribute assessments as demonstrated in current group decision literature our application of Fermatean fuzzy sets accommodates even more flexibility. Finally, as opposed to the smart multi-expert approach employed in testing international-level basketballs with a focus on accuracy under quality testing controlled conditions, our approach focuses on dynamic flexibility and policy-level decision-making support, hence optimal for real-world macroeconomic uncertainty.

Recent research has shown the applicability of advanced fuzzy MADM models similar to those discussed above in actual decision problems. For example, researchers have used q-rung orthopair fuzzy Aczel–Alsina aggregation operators in selecting construction materials and established the model's flexibility in handling conflicting attributes and vague ratings<sup>79</sup>. Similarly, other research has used power aggregation operators to improve the stability of MADM in fuzzy environments, such as in the choice of infrastructure materials and safety assessments<sup>80</sup>. These recent approaches authenticate the expanding applicability of MADM in uncertain, high-risk scenarios and support the validity of the proposed CPPF–Aczel–Alsina model in this research. In conclusion, although all these models offer useful tools for fuzzy decision-making, the suggested Fermatean fuzzy approach optimally reconciles uncertainty modeling, expert knowledge, and strategic usefulness in the context of international trade under uncertain economic dynamics.

## Preliminaries

In this section, some basic concepts are used to introduce the proposed study. These ideas can help us comprehend this content more fully. Here are definitions for PFS, Aczel–Alsina TN, and TCN.

**Definition 1** (Wei<sup>81</sup>) A PFS on a non-empty set  $z$  has the shape  $\zeta = (\mathfrak{X}, (\mathfrak{A}, \mathfrak{T}, \mathfrak{H}))$ :  $0 \leq \text{sum}(\mathfrak{A}(\mathfrak{X}), \mathfrak{T}(\mathfrak{X}), \mathfrak{H}(\mathfrak{X})) \leq 1$ . Here  $\mathfrak{A}, \mathfrak{T}, \mathfrak{H} : z \rightarrow [0, 1]$  denoting the MG, AG and NMG correspondingly. Additionally,  $\nu(\mathfrak{Z}) = 1 - \text{sum}(\mathfrak{A}(\mathfrak{Z}), \mathfrak{T}(\mathfrak{Z}), \mathfrak{H}(\mathfrak{Z}))$  stands for the RG of  $\mathfrak{Z} \in z$  and the trio  $(\mathfrak{A}, \mathfrak{T}, \mathfrak{H})$  is termed as a picture fuzzy value (PFV). The fundamental operations in set theory of union, intersection, inclusion and complement of PFVs were also proposed by Cuong<sup>31</sup> which are provided as follows.

**Definition 2** (Cuong and Kreinovich<sup>31</sup>) Suppose that  $\zeta = (\mathfrak{A}, \mathfrak{T}, \mathfrak{H})$ ,  $\zeta_1 = (\mathfrak{A}_1, \mathfrak{T}_1, \mathfrak{H}_1)$  and  $\zeta_2 = (\mathfrak{A}_2, \mathfrak{T}_2, \mathfrak{H}_2)$  are three PFVs. Then.

1.  $\zeta_1 \subseteq \zeta_2$  iff  $\mathfrak{A}_1 \leq \mathfrak{A}_2, \mathfrak{T}_1 \geq \mathfrak{T}_2, \mathfrak{H}_1 \geq \mathfrak{H}_2$
2.  $\zeta_1 = \zeta_2$  iff  $\zeta_1 \subseteq \zeta_2$  and  $\zeta_2 \subseteq \zeta_1$
3.  $\zeta_1 \cup \zeta_2 = (\bigvee(\mathfrak{A}_1, \mathfrak{A}_2), \bigwedge(\mathfrak{T}_1, \mathfrak{T}_2), \bigwedge(\mathfrak{H}_1, \mathfrak{H}_2))$
4.  $\zeta_1 \cap \zeta_2 = (\bigwedge(\mathfrak{A}_1, \mathfrak{A}_2), \bigvee(\mathfrak{T}_1, \mathfrak{T}_2), \bigvee(\mathfrak{H}_1, \mathfrak{H}_2))$
5.  $\zeta^c = (\mathfrak{H}, \mathfrak{T}, \mathfrak{A})$

**Definition 3** A CPFS on a non-empty set  $z$  has the shape.

$$\zeta = \left\{ (\mathfrak{Z}, (\mathfrak{A}, \mathfrak{T}, \mathfrak{H}); \mathfrak{C}\nu) : 0 \leq \text{sum} \left( \begin{array}{l} \mathfrak{A}(\mathfrak{Z}), \mathfrak{T}(\mathfrak{Z}), \\ \mathfrak{H}(\mathfrak{Z}), \nu(\mathfrak{Z}) \end{array} \right) \leq 1 \right\}.$$

Let,  $c_r \in [0, 1]$  be the radius of the circle around each  $\mathfrak{Z} \in z$ . Here  $\mathfrak{A}, \mathfrak{T}, \mathfrak{H} : z \rightarrow [0, 1]$  denoting the MG, AG, and NMG correspondingly. Additionally,  $v(\mathfrak{Z}) = 1 - \text{sum}(\mathfrak{A}(\mathfrak{Z}), \mathfrak{T}(\mathfrak{Z}), \mathfrak{H}(\mathfrak{Z}))$  represents the RG of  $\mathfrak{Z} \in z$  and the triplet  $(\mathfrak{A}, \mathfrak{T}, \mathfrak{H})$  It is recommended to a “picture fuzzy value” (PFV). Furthermore, the fundamental operations in the set theory of inclusion, intersection, union, and complement of PFV were proposed by Cuong<sup>31</sup> which are provided as follows.

**Definition 4** (Cuong and Kreinovich<sup>31</sup>) Suppose that  $\zeta = (\mathfrak{A}, \mathfrak{T}, \mathfrak{H}, v)$  be CPFVs. Next the score of  $\zeta$  is define as  $s_{\mathfrak{C}}(\zeta) = c_r * (\mathfrak{A}(\mathfrak{Z}) - \mathfrak{T}(\mathfrak{Z}) - \mathfrak{H}(\mathfrak{Z}))$  and  $s_{\mathfrak{C}}(\zeta) \in [-1, 1]$ .

These score functions mean that, for two CPFVs  $\zeta_1 = (\mathfrak{A}_1, \mathfrak{T}_1, \mathfrak{H}_1, v_1)$  and  $\zeta_2 = (\mathfrak{A}_2, \mathfrak{T}_2, \mathfrak{H}_2, v_2)$  we have,

- $\zeta_1$  is superior to  $\zeta_2$  if  $s_{\mathfrak{C}}(\zeta_1) > s_{\mathfrak{C}}(\zeta_2)$ .
- $\zeta_1$  is inferior to  $\zeta_2$  if  $s_{\mathfrak{C}}(\zeta_1) < s_{\mathfrak{C}}(\zeta_2)$ .

**Comparison method for CPFVs**

For comparing two CPFVs, we mainly employ the score function. If the score values of two CPFVs are unequal, the CPFV with the greater score is preferred. If the score values are the same, we employ the accuracy function. The CPFV with the greater accuracy value is preferred. If the score and accuracy values are the same, we compare the refusal degrees of the CPFV with the smaller refusal degree is the one that gets preference, showing a stronger evaluation.

*Significance of the proposed MADM approach*

The importance of creating sophisticated MADM methods stems from their capacity to address uncertainty, impressiveness, and the intricate interactions between attributes and decision-maker preferences. The introduced Cr-PFP-based MADM method advances with these necessities through an augmented structure that, at the same time, supports membership, non-membership, neutrality, and refusal information in decision settings. Current developments in MADM have illustrated the increasing significance of advanced aggregation operators in capturing real-world complexity. Especially, studies like Intuitionistic fuzzy Dombi aggregation with lower and upper approximations and Some Aczel–Alsina power aggregation operators for complex q-rung orthopair fuzzy sets have extended the use of decision theory to situations that demand more complex data interpretations. Likewise, Multi-attribute group decision-making with q-rung orthopair fuzzy Aczel–Alsina power aggregation operators and Multi-attribute decision-making with Archimedean aggregation operators in T-spherical fuzzy settings reflect the increasing trend of integrating mathematical rigor with practical flexibility. In addition, recent works like Multi-attribute decision-making based on complex T-spherical fuzzy Frank prioritized aggregation operators and Performance measures using a MADM approach based on complex T-spherical fuzzy power aggregation operators emphasize the need to create novel operators to handle the inherent vagueness of real-world decision-making. Our suggested approach directly addresses these research trends by overcoming the weaknesses of current theories, specifically by providing a more flexible, precise, and integrated decision-making process through the innovative Circular Prioritized Picture Fuzzy Aczel–Alsina aggregation operators. This makes our approach a major step forward that closes existing research gaps and adds value to the existing toolbox for dealing with complex, high-risk decision situations.

**Definition 5** (Senapati<sup>45</sup>) Suppose that  $\zeta = (\mathfrak{A}, \mathfrak{T}, \mathfrak{H}, v)$   $\zeta_1 = (\mathfrak{A}_1, \mathfrak{T}_1, \mathfrak{H}_1, v_1)$  and  $\zeta_2 = (\mathfrak{A}_2, \mathfrak{T}_2, \mathfrak{H}_2, v_2)$  be three CPFVs where  $\lambda \geq 1$  and  $\lambda > 0$ . Then A–A operations of CPFVs are defined by:

$$\zeta_1 \oplus \zeta_2 = \left( \begin{array}{c} 1 - e^{-((-\ln(1-\mathfrak{A}_1))^{\lambda} + (-\ln(1-\mathfrak{A}_2))^{\lambda})^{1/\lambda}}, \\ e^{-((-\ln \mathfrak{T}_1)^{\lambda} + (-\ln \mathfrak{T}_2)^{\lambda})^{1/\lambda}}, \\ e^{-((-\ln \mathfrak{H}_1)^{\lambda} + (-\ln \mathfrak{H}_2)^{\lambda})^{1/\lambda}}, \\ 1 - e^{-((-\ln(1-v_1))^{\lambda} + (-\ln(1-v_2))^{\lambda})^{1/\lambda}} \end{array} \right)$$

$$\zeta_1 \otimes \zeta_2 = \left( \begin{array}{c} e^{-((-\ln \mathfrak{A}_1)^{\lambda} + (-\ln \mathfrak{A}_2)^{\lambda})^{1/\lambda}}, \\ 1 - e^{-((-\ln(1-\mathfrak{T}_1))^{\lambda} + (-\ln(1-\mathfrak{T}_2))^{\lambda})^{1/\lambda}}, \\ 1 - e^{-((-\ln(1-\mathfrak{H}_1))^{\lambda} + (-\ln(1-\mathfrak{H}_2))^{\lambda})^{1/\lambda}}, \\ e^{-((-\ln v_1)^{\lambda} + (-\ln v_2)^{\lambda})^{1/\lambda}} \end{array} \right)$$

$$\lambda \zeta = \left( \begin{array}{c} 1 - e^{-\lambda(-\ln(1-\mathfrak{A}))^{1/\lambda}}, \\ e^{-\lambda(-\ln \mathfrak{T})^{1/\lambda}}, e^{-\lambda(-\ln \mathfrak{H})^{1/\lambda}}, \\ 1 - e^{-\lambda(-\ln(1-v))^{1/\lambda}} \end{array} \right)$$

$$\zeta^{\lambda} = \left( \begin{array}{c} e^{-\lambda(-\ln \mathfrak{A})^{1/\lambda}}, \\ 1 - e^{-\lambda(-\ln(1-\mathfrak{T}))^{1/\lambda}}, \\ 1 - e^{-\lambda(-\ln(1-\mathfrak{H}))^{1/\lambda}}, e^{-\lambda(-\ln v)^{1/\lambda}} \end{array} \right)$$

**Remark on Definition 5**

The operation expressions on CPFVs are built following the rules of Aczél–Alsina aggregation, with the behavior of each component being logical: DoM transformation employs a complement form  $1 - \mathfrak{A}$  to express acceptance. DoNM and DoA are handled directly by their respective values. DoR is structurally opposite to acceptance, so the operation is built with  $1 - \mathfrak{r}$  in a similar manner to  $1 - \mathfrak{A}$ . This balanced treatment allows all the parts to develop synchronously under aggregation and retain the overall constraint  $\mathfrak{A} + \mathfrak{T} + \mathfrak{H} \leq 1$ . Furthermore, the selection preserves decision rationality by ensuring refusal degrees grow according to aggregation whenever aggregation favors lower acceptance and lowers when there is greater acceptance.

### Cr-picture fuzzy prioritized Aczel–Alsina aggregation operators

In the third section, we define prioritized Aczel–Alsina averaging (Cr-PFPAAA) operators for circular pictures and explain their fundamental characteristics using Aczel–Alsina procedures. We examined the features of the Cr-PFPAAA operators and provided examples to further clarify them.

#### Geometric representation of the proposed work

A CPPFS can be geometrically represented inside a unit circle, with each decision element characterized by four elements: DOM, DoNM, DoA, and DoR. These elements are constrained by  $+\mathfrak{T} + \mathfrak{H} \leq 1$ . In this geometric representation, the horizontal axis is used to represent the DoM, whereas the vertical axis is used to represent the DoNM. The DoA is shown along a third axis pointing into the page (Z-axis), and the DoR can be viewed as the distance from the origin, with uncertainty or indecision in choice. Each option thus takes a point on or in the boundary of the constrained unit sphere, with the sum of the degrees indicating the general evaluation status. This geometric form enables CPPFSs to extract richer and more detailed information than traditional fuzzy or picture fuzzy sets, particularly in situations involving multiple dimensions of uncertainty and priority-based decision-making.

**Definition 6** Suppose that  $\zeta_n = (n = 1, 2, 3 \dots h)$  be several  $\mathfrak{C}_\nu$ - CPFVs. Then mapping is the Cr- PFPAAA operator.  $\zeta^n \rightarrow \zeta$  is described by:

$$\begin{aligned} C_\nu - PFPAAA(\zeta_1, \zeta_2, \dots, \zeta_n) &= \oplus_{q=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} \zeta_n \\ &= \frac{\mathbb{T}_1}{\sum_{n=1}^h \mathbb{T}_n} \zeta_1 \oplus \frac{\mathbb{T}_2}{\sum_{n=1}^h \mathbb{T}_n} \zeta_2 \\ &\oplus \frac{\mathbb{T}_3}{\sum_{n=1}^h \mathbb{T}_n} \zeta_3, \dots, \oplus \frac{\mathbb{T}_n}{\sum_{\mathcal{J}=1}^n \mathbb{T}_n} \zeta_h \end{aligned} \tag{1}$$

Thus, we proposed the following theorem utilizing the Aczel–Alsina operations on CPFVs. Here,  $\omega = \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n}$ ,  $\mathbb{T}_{1j} = 1$  is the priority degree, which works as a weight of CPFV  $\mathbb{T}_j$ .

#### Prioritized weight vectors

Prioritized weight vectors are employed in MADM problems in a situation where the relative importance of attributes (or criteria) is not the same and has a certain priority order. In contrast to standard weight assignments, where just the magnitude of importance is considered, prioritized weighting maintains respect for the relative ranking between attributes. Assume we have a collection of attributes  $\mathfrak{J} = \{\mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3 \dots \mathfrak{J}_m\}$  and the set of attributes is  $\mathfrak{C} = \{\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3 \dots \mathfrak{C}_z\}$ , and that the attributes are prioritized according to the linear ordering  $\mathfrak{C}_1 > \mathfrak{C}_2 > \mathfrak{C}_3 \dots > \mathfrak{C}_z$ .

To create prioritized weights, one may utilize techniques such as exponential decreasing weighting, linear prioritization, or Aczel–Alsina functions. A linear prioritization can simply be carried out as: Assign initial priority scores inversely proportional to the rank:

$$\mathfrak{C}_1 = 4, \mathfrak{C}_1 = 3, \mathfrak{C}_1 = 2 \text{ and } \mathfrak{C}_1 = 1$$

Now, normalize these scores to form a weight vector:

$$\begin{aligned} \omega &= \frac{\text{priority score of } \mathfrak{C}_z}{\text{sum of all priority scores}} \\ \text{sum of all priority scores} &= 10 \end{aligned}$$

Therefore, the prioritized weight vector is:

$$\omega_1 = \frac{4}{10} = 0.4, \omega_2 = \frac{3}{10} = 0.3, \omega_3 = \frac{2}{10} = 0.2$$

$$\begin{aligned} \text{And } \omega_4 &= \frac{1}{10} = 0.1 \\ \omega &= (0.4, 0.3, 0.2, 0.1). \end{aligned}$$

**Theorem 1** Let  $\zeta_n = (\mathfrak{A}_n, \mathfrak{T}_n, \mathfrak{H}_n, \mathfrak{r}_n)$  ( $n = 1, 2, 3 \dots h$ ) are a build-up of CPFV. In that case, the total value of their use of the Cr-PFPAAA procedure is a CPFV.

$$C_v - PFPAAA(\zeta_1, \zeta_2, \dots, \zeta_n) = \oplus_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} \zeta_n$$

$$\left( \begin{array}{c} - \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\text{Ln}(1-\mathfrak{A}_n))^l \right)^{1/l} \\ 1 - \mathfrak{e} \\ - \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\text{Ln}\mathfrak{T}_n)^l \right)^{1/l} \\ \mathfrak{e} \\ - \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\text{Ln}\mathfrak{H}_n)^l \right)^{1/l} \\ \mathfrak{e} \\ - \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\text{Ln}(1-\mathfrak{v}_n))^l \right)^{1/l} \\ 1 - \mathfrak{e} \end{array} \right)^{1/l} \tag{2}$$

*Proof* We are prepared to demonstrate theorem 1 in the following way using the mathematical induction method:

(1) In the relation of  $n = 2$  and the CPFV A-A operations, we acquire

$$\frac{\mathbb{T}_1}{\sum_{\mathcal{J}=1}^n \mathbb{T}_{\mathcal{J}}} \zeta_1 = \left( \begin{array}{c} - \left( \frac{\mathbb{T}_1}{\sum_{n=1}^h \mathbb{T}_n} (-\text{Ln}(1-\mathfrak{A}_1))^l \right)^{1/l} \\ 1 - \mathfrak{e} \\ - \left( \frac{\mathbb{T}_1}{\sum_{n=1}^h \mathbb{T}_n} (-\text{Ln}\mathfrak{T}_1)^l \right)^{1/l} \\ \mathfrak{e} \\ - \left( \frac{\mathbb{T}_1}{\sum_{n=1}^h \mathbb{T}_n} (-\text{Ln}\mathfrak{H}_1)^l \right)^{1/l} \\ \mathfrak{e} \\ - \left( \frac{\mathbb{T}_1}{\sum_{n=1}^h \mathbb{T}_n} (-\text{Ln}(1-\mathfrak{v}_1))^l \right)^{1/l} \\ 1 - \mathfrak{e} \end{array} \right)^{1/l}$$

$$\frac{\mathbb{T}_2}{\sum_{\mathcal{J}=1}^n \mathbb{T}_{\mathcal{J}}} \zeta_2 = \left( \begin{array}{c} - \left( \frac{\mathbb{T}_2}{\sum_{n=1}^h \mathbb{T}_n} (-\text{Ln}(1-\mathfrak{A}_2))^l \right)^{1/l} \\ 1 - \mathfrak{e} \\ - \left( \frac{\mathbb{T}_2}{\sum_{n=1}^h \mathbb{T}_n} (-\text{Ln}\mathfrak{T}_2)^l \right)^{1/l} \\ \mathfrak{e} \\ - \left( \frac{\mathbb{T}_2}{\sum_{n=1}^h \mathbb{T}_n} (-\text{Ln}\mathfrak{H}_2)^l \right)^{1/l} \\ \mathfrak{e} \\ - \left( \frac{\mathbb{T}_2}{\sum_{n=1}^h \mathbb{T}_n} (-\text{Ln}(1-\mathfrak{v}_2))^l \right)^{1/l} \\ 1 - \mathfrak{e} \end{array} \right)^{1/l}$$

We obtain

$$C_v - PFPAAA(\zeta_1 \zeta_2) = \frac{\mathbb{T}_1}{\sum_{\mathcal{J}=1}^n \mathbb{T}_{\mathcal{J}}} \zeta_1 \oplus \frac{\mathbb{T}_2}{\sum_{\mathcal{J}=1}^n \mathbb{T}_{\mathcal{J}}} \zeta_2$$

$$= \left( \begin{array}{c} - \left( \frac{\mathbb{T}_1}{\sum_{n=1}^2 \mathbb{T}_n} (-\text{Ln}(1-\mathfrak{A}_1))^l \right)^{1/l} \\ 1 - \mathfrak{e} \\ - \left( \frac{\mathbb{T}_1}{\sum_{n=1}^2 \mathbb{T}_n} (-\text{Ln}\mathfrak{T}_1)^l \right)^{1/l} \\ \mathfrak{e} \\ - \left( \frac{\mathbb{T}_1}{\sum_{n=1}^2 \mathbb{T}_n} (-\text{Ln}\mathfrak{H}_1)^l \right)^{1/l} \\ \mathfrak{e} \\ - \left( \frac{\mathbb{T}_1}{\sum_{n=1}^2 \mathbb{T}_n} (-\text{Ln}(1-\mathfrak{v}_1))^l \right)^{1/l} \\ 1 - \mathfrak{e} \end{array} \right)^{1/l}$$

$$\begin{aligned}
 & \oplus \begin{pmatrix} 1 - \mathbb{E} \left( \frac{\mathbb{T}_2}{\sum_{n=1}^2 \mathbb{T}_n} (-\text{Ln}(1-\mathfrak{A}_2))^l \right)^{1/l} \\ \mathbb{E} \left( \frac{\mathbb{T}_2}{\sum_{n=1}^2 \mathbb{T}_n} (-\text{Ln}\mathfrak{T}_2)^l \right)^{1/l} \\ \mathbb{E} \left( \frac{\mathbb{T}_2}{\sum_{n=1}^h \mathbb{T}_n} (-\text{Ln}\mathfrak{H}_2)^l \right)^{1/l} \\ 1 - \mathbb{E} \left( \frac{\mathbb{T}_2}{\sum_{n=1}^2 \mathbb{T}_n} (-\text{Ln}(1-\mathfrak{v}_2))^l \right)^{1/l} \end{pmatrix} \\
 = & \begin{pmatrix} 1 - \mathbb{E} \left( \frac{\mathbb{T}_1}{\sum_{n=1}^2 \mathbb{T}_n} (-\text{Ln}(1-\mathfrak{A}_1))^l + \frac{\mathbb{T}_2}{\sum_{n=1}^2 \mathbb{T}_n} (-\text{Ln}(1-\mathfrak{A}_2))^l \right)^{1/l} \\ \mathbb{E} \left( \frac{\mathbb{T}_1}{\sum_{n=1}^2 \mathbb{T}_n} (-\text{Ln}\mathfrak{T}_1)^l + \frac{\mathbb{T}_2}{\sum_{n=1}^2 \mathbb{T}_n} (-\text{Ln}\mathfrak{T}_2)^l \right)^{1/l} \\ \mathbb{E} \left( \frac{\mathbb{T}_1}{\sum_{n=1}^2 \mathbb{T}_n} (-\text{Ln}\mathfrak{H}_1)^l + \frac{\mathbb{T}_2}{\sum_{n=1}^2 \mathbb{T}_n} (-\text{Ln}\mathfrak{H}_2)^l \right)^{1/l} \\ 1 - \mathbb{E} \left( \frac{\mathbb{T}_1}{\sum_{n=1}^2 \mathbb{T}_n} (-\text{Ln}(1-\mathfrak{v}_1))^l + \frac{\mathbb{T}_2}{\sum_{n=1}^2 \mathbb{T}_n} (-\text{Ln}(1-\mathfrak{v}_2))^l \right)^{1/l} \end{pmatrix} \\
 = & \begin{pmatrix} 1 - \mathbb{E} \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\text{Ln}(1-\mathfrak{A}_n))^l \right)^{1/l} \\ \mathbb{E} \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\text{Ln}\mathfrak{T}_n)^l \right)^{1/l} \\ \mathbb{E} \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\text{Ln}\mathfrak{H}_n)^l \right)^{1/l} \\ 1 - \mathbb{E} \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\text{Ln}(1-\mathfrak{v}_n))^l \right)^{1/l} \end{pmatrix}
 \end{aligned}$$

This is true for  $n = 2$  (1) holds

(2) assuming that for  $n = k$  (1) holds, We obtain

$$\begin{aligned}
 Cr - PFPA\text{AA} (\zeta_1, \zeta_2, \dots, \zeta_k) &= \oplus_{n=1}^k \left( \frac{\mathbb{T}_n}{\sum_{n=1}^k \mathbb{T}_n} \zeta_n \right) \\
 &= \begin{pmatrix} 1 - \mathbb{E} \left( \sum_{n=1}^k \frac{\mathbb{T}_n}{\sum_{n=1}^k \mathbb{T}_n} (-\text{Ln}(1-\mathfrak{A}_n))^l \right)^{1/l} \\ \mathbb{E} \left( \sum_{n=1}^k \frac{\mathbb{T}_n}{\sum_{n=1}^k \mathbb{T}_n} (-\text{Ln}\mathfrak{T}_n)^l \right)^{1/l} \\ \mathbb{E} \left( \sum_{n=1}^k \frac{\mathbb{T}_n}{\sum_{n=1}^k \mathbb{T}_n} (-\text{Ln}\mathfrak{H}_n)^l \right)^{1/l} \\ 1 - \mathbb{E} \left( \sum_{n=1}^k \frac{\mathbb{T}_n}{\sum_{n=1}^k \mathbb{T}_n} (-\text{Ln}(1-\mathfrak{v}_n))^l \right)^{1/l} \end{pmatrix}
 \end{aligned}$$

Now for  $n = k + 1$  then

$$Cr - PFPA\text{AA} (\zeta_1, \zeta_2, \dots, \zeta_{k+1}) = \oplus_{n=1}^k \left( \frac{\mathbb{T}_n}{\sum_{n=1}^{k+1} \mathbb{T}_n} \zeta_n \right) \oplus \left( \frac{\mathbb{T}_{k+1}}{\sum_{n=1}^{k+1} \mathbb{T}_n} \zeta_{k+1} \right)$$

$$\begin{aligned}
 &= \left( \begin{array}{c} 1 - \mathbb{P} \left( \sum_{n=1}^k \frac{\mathbb{T}_n}{\sum_{n=1}^{k+1} \mathbb{T}_n} (-\text{Ln}(1-\mathfrak{A}_n))^l \right)^{1/l}, \\ \mathbb{P} \left( \sum_{n=1}^k \frac{\mathbb{T}_n}{\sum_{n=1}^{k+1} \mathbb{T}_n} (-\text{Ln}\mathfrak{T}_n)^l \right)^{1/l}, \\ \mathbb{P} \left( \sum_{n=1}^k \frac{\mathbb{T}_n}{\sum_{n=1}^{k+1} \mathbb{T}_n} (-\text{Ln}\mathfrak{H}_n)^l \right)^{1/l}, \\ 1 - \mathbb{P} \left( \sum_{n=1}^k \frac{\mathbb{T}_n}{\sum_{n=1}^{k+1} \mathbb{T}_n} (-\text{Ln}(1-\mathfrak{v}_n))^l \right)^{1/l} \end{array} \right) \\
 &\oplus \left( \begin{array}{c} 1 - \mathbb{P} \left( \sum_{n=1}^{\mathbb{T}_{k+1}} \frac{\mathbb{T}_n}{\sum_{n=1}^{\mathbb{T}_{k+1} + \mathbb{T}_n} \mathbb{T}_n} (-\text{Ln}(1-\mathfrak{A}_{k+1}))^l \right)^{\frac{1}{\mathbb{P}L}}, \\ \mathbb{P} \left( \sum_{n=1}^{\mathbb{T}_{k+1}} \frac{\mathbb{T}_n}{\sum_{n=1}^{\mathbb{T}_{k+1} + \mathbb{T}_n} \mathbb{T}_n} (-\text{Ln}\mathfrak{T}_{k+1})^l \right)^{\frac{1}{\mathbb{P}L}}, \\ \mathbb{P} \left( \sum_{n=1}^{\mathbb{T}_{k+1}} \frac{\mathbb{T}_n}{\sum_{n=1}^{\mathbb{T}_{k+1} + \mathbb{T}_n} \mathbb{T}_n} (-\text{Ln}\mathfrak{H}_{k+1})^l \right)^{\frac{1}{\mathbb{P}L}}, \\ 1 - \mathbb{P} \left( \sum_{n=1}^{\mathbb{T}_{k+1}} \frac{\mathbb{T}_n}{\sum_{n=1}^{\mathbb{T}_{k+1} + \mathbb{T}_n} \mathbb{T}_n} (-\text{Ln}(1-\mathfrak{v}_{k+1}))^l \right)^{\frac{1}{\mathbb{P}L}} \end{array} \right) \\
 &= \left( \begin{array}{c} 1 - \mathbb{P} \left( \sum_{n=1}^{k+1} \frac{\mathbb{T}_n}{\sum_{n=1}^{k+1} \mathbb{T}_n} (-\text{Ln}(1-\mathfrak{A}_n))^l \right)^{1/l}, \\ \mathbb{P} \left( \sum_{n=1}^{k+1} \frac{\mathbb{T}_n}{\sum_{n=1}^{k+1} \mathbb{T}_n} (-\text{Ln}\mathfrak{T}_n)^l \right)^{1/l}, \\ \mathbb{P} \left( \sum_{n=1}^{k+1} \frac{\mathbb{T}_n}{\sum_{n=1}^{k+1} \mathbb{T}_n} (-\text{Ln}\mathfrak{H}_n)^l \right)^{1/l}, \\ 1 - \mathbb{P} \left( \sum_{n=1}^{k+1} \frac{\mathbb{T}_n}{\sum_{n=1}^{k+1} \mathbb{T}_n} (-\text{Ln}(1-\mathfrak{v}_n))^l \right)^{1/l} \end{array} \right)
 \end{aligned}$$

**Theorem 2** (Idempotency). *If all  $\zeta_n = (\mathfrak{A}_n, \mathfrak{T}_n, \mathfrak{H}_n)$  ( $n = 1, 2, 3 \dots h$ ) are comparable, that is  $\zeta_n = \zeta$  for every  $n$ , then.*

$$Cr - PFPAAA(\zeta_1, \zeta_2, \dots, \zeta_h) = \zeta \tag{3}$$

*Proof* Since  $\zeta_n = (\mathfrak{A}_n, \mathfrak{T}_n, \mathfrak{H}_n) = \zeta$  ( $n = 1, 2, 3 \dots h$ ) then by Eq. (1) we have

$$\begin{aligned}
 Cr - PFPAAA(\zeta_1, \zeta_2, \dots, \zeta_h) &= \oplus_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} \zeta_n \\
 &= \left( \begin{array}{c} 1 - \mathbb{P} \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\text{Ln}(1-\mathfrak{A}_n))^l \right)^{1/l}, \\ \mathbb{P} \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\text{Ln}\mathfrak{T}_n)^l \right)^{1/l}, \\ \mathbb{P} \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\text{Ln}\mathfrak{H}_n)^l \right)^{1/l}, \\ 1 - \mathbb{P} \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\text{Ln}(1-\mathfrak{v}_n))^l \right)^{1/l} \end{array} \right) \\
 &= \left( \begin{array}{c} 1 - \mathbb{P} \left( (-\text{Ln}(1-\mathfrak{A}))^l \right)^{1/l}, \\ \mathbb{P} \left( (-\text{Ln}\mathfrak{T})^l \right)^{1/l}, \\ \mathbb{P} \left( (-\text{Ln}\mathfrak{H})^l \right)^{1/l}, \\ 1 - \mathbb{P} \left( (-\text{Ln}(1-\mathfrak{v}))^l \right)^{1/l} \end{array} \right)
 \end{aligned}$$

$$= \left( \begin{matrix} 1 - \mathbb{P}^{(\mathbb{L}n(1-\mathfrak{A}))}, \mathbb{P}^{(\mathbb{L}n\mathfrak{X})} \\ \mathbb{P}^{(\mathbb{L}n\mathfrak{Y})}, 1 - \mathbb{L}^{(\mathbb{L}n(1-\nu))} \end{matrix} \right) = (\mathfrak{A}, \mathfrak{X}, \mathfrak{Y}, \nu) = \zeta$$

Thus,  $C_\nu - PFPAAA(\zeta_1, \zeta_2, \dots, \zeta_h) = \zeta$  holds.

**Theorem 3** (Boundedness). *Suppose that  $\zeta_n = (\mathfrak{A}_n, \mathfrak{X}_n, \mathfrak{Y}_n, \nu_n)$  ( $n = 1, 2, 3 \dots h$ ) be an accumulation of CPFVs. Let  $\zeta^- = \min(\zeta_1, \zeta_2, \dots, \zeta_n)$  and  $\zeta^+ = \max(\zeta_1, \zeta_2, \dots, \zeta_h)$ . Then,*

$$\zeta^- \leq C_\nu - PFPAAA(\zeta_1, \zeta_2, \zeta_3 \dots \zeta_h) \leq \zeta^+ \tag{4}$$

*Proof* Let  $\zeta^- = \min(\zeta_1, \zeta_2, \dots, \zeta_n) = (\mathfrak{A}^-, \mathfrak{X}^-, \mathfrak{Y}^-, \nu^-)$  and  $\zeta^+ = \max(\zeta_1, \zeta_2, \dots, \zeta_n) = (\mathfrak{A}^+, \mathfrak{X}^+, \mathfrak{Y}^+, \nu^+)$ . Consequently, we have the following inequalities:

$$\begin{aligned} & 1 - \mathbb{P}^{-\left(\sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\mathbb{L}n(1-\mathfrak{A}^-))^l\right)^{1/l}} \\ & \leq 1 - \mathbb{P}^{-\left(\sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\mathbb{L}n(1-\mathfrak{A}_n))^l\right)^{1/l}} \\ & \leq 1 - \mathbb{P}^{-\left(\sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\mathbb{L}n(1-\mathfrak{A}^+))^l\right)^{1/l}} \\ & \mathbb{P}^{-\left(\sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\mathbb{L}n(\mathfrak{X}^-))^l\right)^{1/l}} \\ & \leq \mathbb{P}^{-\left(\sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\mathbb{L}n(1-\mathfrak{X}_n))^l\right)^{1/l}} \\ & \leq \mathbb{P}^{-\left(\sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\mathbb{L}n(1-\mathfrak{X}^+))^l\right)^{1/l}} \\ & \mathbb{P}^{-\left(\sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\mathbb{L}n(1-\mathfrak{Y}^-))^l\right)^{1/l}} \\ & \leq \mathbb{P}^{-\left(\sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\mathbb{L}n(1-\mathfrak{Y}_n))^l\right)^{1/l}} \\ & \leq \mathbb{P}^{-\left(\sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\mathbb{L}n(1-\mathfrak{Y}^+))^l\right)^{1/l}} \end{aligned}$$

Therefore,  $\zeta^- \leq Cr - PFPAAA(\zeta_1, \zeta_2, \zeta_3 \dots \zeta_h) \leq \zeta^+$ .

**Theorem 4** (Monotonicity). *Suppose that  $\zeta_n$  and  $\zeta_n'$  ( $n = 1, 2, 3 \dots h$ ) are two CPFV sets, if  $\zeta_n \leq \zeta_n'$  for every  $n$ , consequently  $Cr - PFPAAA(\zeta_1, \zeta_2, \zeta_3 \dots \zeta_h) \leq (\zeta_1', \zeta_2', \zeta_3' \dots \zeta_h')$ .*

**Theorem 5** *Suppose that  $\zeta_n = (\mathfrak{A}_n, \mathfrak{X}_n, \mathfrak{Y}_n, \nu_n)$  ( $n = 1, 2, 3 \dots h$ ) be an accumulation of CPFVs,  $\mathbb{T}_n = \prod_{n=1}^{h-1} \mathfrak{s}(\zeta_h)$*

*( $n = 2, 3, \dots, h$ ),  $\mathbb{T}_1 = 1$  and  $\mathfrak{s}(\zeta_h)$  is the CPFV score  $\zeta_h$  if  $\psi = (\mu, \phi, \nu)$  is a CPFV on  $z$ , afterward,*

$$C_\nu - PFPAAA(\varrho_1 \oplus \psi, \varrho_2 \oplus \psi, \dots, \varrho_h \oplus \psi) = Cr - PFPAAA(\zeta_1, \zeta_2, \zeta_3, \dots, \zeta_h) \oplus \psi \tag{5}$$

*Proof:*  $Cr - PFPAAA(\varrho_1, \varrho_2, \dots, \varrho_h)$

$$= \begin{pmatrix} 1 - \Phi \left( - \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(1-\mathfrak{A}_n))^{\iota} \right)^{1/\iota} \right), \\ \Phi \left( - \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(\mathfrak{X}_n))^{\iota} \right)^{1/\iota} \right), \\ \Phi \left( - \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(\mathfrak{H}_n))^{\iota} \right)^{1/\iota} \right), \\ 1 - \Phi \left( - \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(\mathfrak{e}_n))^{\iota} \right)^{1/\iota} \right) \end{pmatrix}$$

$$\varrho_n \oplus \psi = \begin{pmatrix} 1 - \Phi \left( - \left( (-\text{Ln}(1-\mathfrak{A}_n))^{\iota} + (-\text{Ln}(1-\mu))^{\iota} \right)^{\frac{1}{\sigma} L} \right), \\ \Phi \left( - \left( (-\text{Ln}(\mathfrak{X}_n))^{\iota} + (-\text{Ln}(\phi))^{\iota} \right)^{\frac{1}{\sigma} L} \right), \\ \Phi \left( - \left( (-\text{Ln}(\mathfrak{H}_n))^{\iota} + (-\text{Ln}(\nu))^{\iota} \right)^{\frac{1}{\sigma} L} \right), \\ 1 - \Phi \left( - \left( (-\text{Ln}(1-\mathfrak{L}_n))^{\iota} + (-\text{Ln}(1-\mu))^{\iota} \right)^{\frac{1}{\sigma} L} \right) \end{pmatrix}$$

Theorem 1 states that we have

$$C_{\nu} - PFPAAA(\varrho_1 \oplus \psi, \varrho_2 \oplus \psi, \dots, \varrho_h \oplus \psi)$$

$$\begin{pmatrix} 1 - \Phi \left( - \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left( -\text{Ln} \left( 1 - \left( 1 - \Phi \left( - \left( (-\text{Ln}(1-\mathfrak{A}_n))^{\iota} + (-\text{Ln}(1-\mu))^{\iota} \right)^{1/\iota} \right) \right) \right) \right)^{\iota} \right)^{1/\iota} \right), \\ \Phi \left( - \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left( -\text{Ln} \left( \Phi \left( - \left( (-\text{Ln}(\mathfrak{A}_n))^{\iota} + (-\text{Ln}(\phi))^{\iota} \right)^{1/\iota} \right) \right) \right)^{\iota} \right)^{1/\iota} \right), \\ \Phi \left( - \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left( -\text{Ln} \left( \Phi \left( - \left( (-\text{Ln}(\mathfrak{A}_n))^{\iota} + (-\text{Ln}(\nu))^{\iota} \right)^{1/\iota} \right) \right) \right)^{\iota} \right)^{1/\iota} \right), \\ 1 - \Phi \left( - \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left( -\text{Ln} \left( 1 - \left( 1 - \Phi \left( - \left( (-\text{Ln}(1-\mathfrak{L}_n))^{\iota} + (-\text{Ln}(1-\mu))^{\iota} \right)^{1/\iota} \right) \right) \right) \right)^{\iota} \right)^{1/\iota} \right) \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \Phi \left( - \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(1-\mathfrak{A}_n))^{\iota} + (-\text{Ln}(1-\mu))^{\iota} \right)^{1/\iota} \right), \\ \Phi \left( - \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(\mathfrak{X}_n))^{\iota} + (-\text{Ln}(\phi))^{\iota} \right)^{1/\iota} \right), \\ \Phi \left( - \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(\mathfrak{H}_n))^{\iota} + (-\text{Ln}(\nu))^{\iota} \right)^{1/\iota} \right), \\ 1 - \Phi \left( - \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(1-\mathfrak{L}_n))^{\iota} + (-\text{Ln}(1-\mu))^{\iota} \right)^{1/\iota} \right) \end{pmatrix}$$

Using the definition and the operational laws of CPFVs, we arrive at

$$C_{\nu} - PFPAAA(\varrho_1, \varrho_2, \dots, \varrho_h) \oplus \psi$$

$$\begin{aligned}
 &= \left( \begin{array}{c} 1 - e^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(1-\mathfrak{A}_n))^l\right)^{1/l}} \\ e^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(\mathfrak{X}_n))^l\right)^{1/l}} \\ e^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(\mathfrak{S}_n))^l\right)^{1/l}} \\ 1 - e^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(1-\mathfrak{v}_n))^l\right)^{1/l}} \end{array} \right) \oplus \psi \\
 &= \left( \begin{array}{c} 1 - e^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(1-\mathfrak{A}_n))^l\right)^{1/l}} \\ e^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(\mathfrak{X}_n))^l\right)^{1/l}} \\ e^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(\mathfrak{S}_n))^l\right)^{1/l}} \\ 1 - e^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(1-\mathfrak{v}_n))^l\right)^{1/l}} \end{array} \right) \oplus (\mu, \phi, \nu) \\
 &\left( \begin{array}{c} - \left( \left( -\text{Ln} \left( 1 - \left( 1 - e^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(1-\mathfrak{A}_n))^l\right)^{1/l}} \right) \right) \right)^l + \right)^{1/l} \\ e^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(\mathfrak{X}_n))^l\right)^{1/l}} \\ e^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(\mathfrak{S}_n))^l\right)^{1/l}} \\ - \left( \left( -\text{Ln} \left( 1 - \left( 1 - e^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(1-\mathfrak{v}_n))^l\right)^{1/l}} \right) \right) \right)^l + \right)^{1/l} \end{array} \right) \\
 &= \left( \begin{array}{c} 1 - e^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(1-\mathfrak{A}_n \oplus \mu))^l\right)^{1/l}} \\ e^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(\mathfrak{X}_n \oplus \phi))^l\right)^{1/l}} \\ e^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(\mathfrak{S}_n \oplus \nu))^l\right)^{1/l}} \\ 1 - e^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(1-\mathfrak{v}_n \oplus \mu))^l\right)^{1/l}} \end{array} \right)
 \end{aligned}$$

Thus

$$\begin{aligned}
 &C_v - PFPAAA(\zeta_1 \oplus \psi, \zeta_2 \oplus \psi, \dots, \zeta_h \oplus \psi) \\
 &= C_v - PFPAAA(\zeta_1, \zeta_2, \zeta_3, \dots, \zeta_h) \oplus \psi.
 \end{aligned}$$

**Theorem 6** Let  $\zeta_n = (\mathfrak{A}_n, \mathfrak{T}_n, \mathfrak{H}_n, \nu)$  ( $n = 1, 2, 3 \dots h$ ) be a collection of CPFVs.

$$T_n = \prod_{n=1}^{h-1} s(\zeta_h) \quad (n = 2, 3, \dots h),$$

$T_1 = 1$  along with  $s(\zeta_h)$  is the CPFV score  $\zeta_h$  if  $\nu > 0$  picture the fuzzy value on  $z$ . Afterward,

$$C\nu - PFPAAA(\nu\varrho_1, \nu\varrho_2, \dots, \nu\varrho_h) = C\nu - \nu PFPAAA(\zeta_1, \zeta_2, \zeta_3, \dots \zeta_h) \tag{6}$$

*Proof* Following the guidelines for operations outlined in “Literature review” section, we obtain

$$\lambda\zeta = \begin{pmatrix} 1 - e^{-(\lambda(-\ln(1-\mathfrak{A}))^t)^{1/t}}, \\ e^{-(\lambda(-\ln\mathfrak{T})^t)^{1/t}}, \\ e^{-(\lambda(-\ln\mathfrak{H})^t)^{1/t}}, \\ 1 - e^{-(\lambda(-\ln(1-\nu))^t)^{1/t}} \end{pmatrix} \tag{7}$$

According to Theorem 1, we have

$$\begin{aligned} & C\nu - PFPAAA(\nu\zeta_1, \nu\zeta_2, \dots, \nu\zeta_h) \\ &= \begin{pmatrix} 1 - e^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left(-\ln\left(1 - \left(1 - e^{-\left(\nu(-\ln(1-\mathfrak{A}))^t\right)^{1/t}}\right)\right)\right)^t} \right)^{1/t}}, \\ e^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left(-\ln\left(e^{-\left(\nu(-\ln\mathfrak{T})^t\right)^{1/t}}\right)\right)\right)^t} \right)^{1/t}}, \\ e^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left(-\ln\left(e^{-\left(\nu(-\ln\mathfrak{H})^t\right)^{1/t}}\right)\right)\right)^t} \right)^{1/t}}, \\ 1 - e^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left(-\ln\left(1 - \left(1 - e^{-\left(\nu(-\ln(1-\nu))^t\right)^{1/t}}\right)\right)\right)\right)^t} \right)^{1/t}} \end{pmatrix} \\ &= \begin{pmatrix} 1 - e^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \nu \left(-\ln(1-\mathfrak{A})\right)^t\right)^{1/t}}, \\ e^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \nu \left(-\ln\mathfrak{T}\right)^t\right)^{1/t}}, \\ e^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \nu \left(-\ln\mathfrak{H}\right)^t\right)^{1/t}}, \\ 1 - e^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \nu \left(-\ln(1-\nu)\right)^t\right)^{1/t}} \end{pmatrix} \\ & C\nu - \nu PFPAAA(\zeta_1, \zeta_2, \zeta_3, \dots \zeta_h) \\ &= \nu \begin{pmatrix} 1 - e^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left(-\ln\left(1 - \left(1 - e^{-\left((-ln(1-\mathfrak{A}))^t\right)^{1/t}}\right)\right)\right)\right)^t} \right)^{1/t}}, \\ e^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left(-\ln\left(e^{-\left((-ln\mathfrak{T})^t\right)^{1/t}}\right)\right)\right)^t} \right)^{1/t}}, \\ e^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left(-\ln\left(e^{-\left((-ln\mathfrak{H})^t\right)^{1/t}}\right)\right)\right)^t} \right)^{1/t}}, \\ 1 - e^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left(-\ln\left(1 - \left(1 - e^{-\left((-ln(1-\nu))^t\right)^{1/t}}\right)\right)\right)\right)^t} \right)^{1/t}} \end{pmatrix} \end{aligned}$$

$$= \nu \begin{pmatrix} 1 - \mathbb{P} \left( - \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\ln(1-\mathfrak{A}))^t \right)^{1/\iota} \right), \\ \mathbb{P} \left( - \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\ln \mathfrak{T})^t \right)^{1/\iota} \right), \\ \mathbb{P} \left( - \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\ln \mathfrak{H})^t \right)^{1/\iota} \right), \\ 1 - \mathbb{P} \left( - \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\ln(1-\nu))^t \right)^{1/\iota} \right) \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \mathbb{P} \left( - \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} \nu (-\ln(1-\mathfrak{A}))^t \right)^{1/\iota} \right), \\ \mathbb{P} \left( - \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} \nu (-\ln \mathfrak{T})^t \right)^{1/\iota} \right), \\ \mathbb{P} \left( - \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} \nu (-\ln \mathfrak{H})^t \right)^{1/\iota} \right), \\ 1 - \mathbb{P} \left( - \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} \nu (-\ln(1-\nu))^t \right)^{1/\iota} \right) \end{pmatrix}$$

Thus

$$C_\nu - PFPAAA(\nu\zeta_1, \nu\zeta_2, \dots, \nu\zeta_h) = C_\nu - \nu PFPAAA(\zeta_1, \zeta_2, \zeta_3, \dots, \zeta_h)$$

**Theorem 7** Let  $\zeta_n = (\mathfrak{A}_n, \mathfrak{T}_n, \mathfrak{H}_n, \nu)$  ( $n = 1, 2, 3 \dots h$ ) be an assortment of CP-FVs.  $\mathbb{T}_n = \prod_{n=1}^{h-1} \mathfrak{s}(\zeta_h)$  ( $n = 2, 3, \dots, h$ ),  $\mathbb{T}_1 = 1$  along with  $\mathfrak{s}(\zeta_h)$  is the CPFV score  $\zeta_h$  if  $\nu > 0$ ,  $\psi = (\mu, \phi, \nu)$  is a CPFV on  $z$ . Afterward,

$$C_\nu - PFPAAA(\nu\varrho_1 \oplus \psi, \nu\varrho_2 \oplus \psi, \dots, \nu\varrho_h \oplus \psi) = C_\nu - \nu PFPAAA(\varrho_1, \varrho_2, \varrho_3, \dots, \varrho_h) \oplus \psi \tag{8}$$

*Proof*  $C_\nu - PFPAAA(\nu\varrho_1 \oplus \psi, \nu\varrho_2 \oplus \psi, \dots, \nu\varrho_h \oplus \psi)$

$$\nu\varrho_n = \begin{pmatrix} 1 - \mathbb{P} \left( \nu(-\ln(1-\mathfrak{A}_n))^t \right)^{1/\iota}, \mathbb{P} \left( \nu(-\ln \mathfrak{T}_n)^t \right)^{1/\iota} \\ \mathbb{P} \left( \nu(-\ln \mathfrak{H}_n)^t \right)^{1/\iota}, 1 - \mathbb{P} \left( \nu(-\ln(1-\nu_n))^t \right)^{1/\iota} \end{pmatrix}$$

$$\nu\varrho_n \oplus \psi = \begin{pmatrix} 1 - \mathbb{P} \left( \nu(-\ln(1-\mathfrak{A}_n))^t \right)^{1/\iota}, \\ \mathbb{P} \left( \nu(-\ln \mathfrak{T}_n)^t \right)^{1/\iota}, \\ \mathbb{P} \left( \nu(-\ln \mathfrak{H}_n)^t \right)^{1/\iota}, \\ 1 - \mathbb{P} \left( \nu(-\ln(1-\nu_n))^t \right)^{1/\iota} \end{pmatrix} \oplus (\mu, \phi, \nu)$$

$$\nu\varrho_n \oplus \psi = \begin{pmatrix} 1 - \mathbb{P} \left( \left( -\ln \left( 1 - \left( 1 - e^{-\left( \nu(-\ln(1-\mathfrak{A}_n))^t \right)^{1/\iota}} \right) \right) \right)^2 + (-\ln(\mu_n))^t \right)^{1/\iota}, \\ \mathbb{P} \left( \left( -\ln \left( e^{-\left( \nu(-\ln \mathfrak{T}_n)^t \right)^{1/\iota}} \right) \right)^2 + (-\ln \phi_n)^t \right)^{1/\iota}, \\ \mathbb{P} \left( \left( -\ln e^{-\left( \nu(-\ln \mathfrak{H}_n)^t \right)^{1/\iota}} \right)^2 + (-\ln \nu_n)^t \right)^{1/\iota}, \\ 1 - \mathbb{P} \left( \left( -\ln \left( 1 - \left( 1 - e^{-\left( \nu(-\ln(1-\nu_n))^t \right)^{1/\iota}} \right) \right) \right)^2 + (-\ln(\mu_n))^t \right)^{1/\iota} \end{pmatrix}$$

$$\begin{pmatrix}
 1 - \mathbb{E} \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left( -\text{Ln} \left( 1 - \frac{1}{1 - e^{-\left( (-\text{Ln} \left( 1 - \left( 1 - e^{-j(-\ln(1-\alpha_n))^l} \right)^{1/l} \right) \right)^l} \right) \right)^{1/l} + \left( (-\text{Ln} (1 - \mu_n))^l \right)^{1/l} \right) \right)^{1/l} \\
 \mathbb{E} \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left( -\text{Ln} \left( e^{-\left( (-\text{Ln} \left( e^{-j(-\ln \mathfrak{I}_n)^l} \right)^{1/l} \right)^l} + (-\text{Ln} \phi_n)^l \right) \right)^{1/l} \right) \right)^{1/l} \\
 \mathbb{E} \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left( -\text{Ln} \left( e^{-\left( (-\text{Ln} e^{-j(-\ln \mathfrak{H}_n)^l} \right)^{1/l} \right)^l} + (-\text{Ln} \nu_n)^l \right) \right)^{1/l} \right) \right)^{1/l} \\
 1 - \mathbb{E} \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left( -\text{Ln} \left( 1 - \frac{1}{1 - e^{-\left( (-\text{Ln} \left( 1 - \left( 1 - e^{-j(-\ln(1-j_n))^l} \right)^{1/l} \right) \right)^l} \right) \right)^{1/l} + \left( (-\text{Ln} (1 - \mu_n))^l \right)^{1/l} \right) \right)^{1/l}
 \end{pmatrix}$$

$$\begin{pmatrix}
 1 - \mathbb{E} \left( \left( (-\text{Ln} \left( 1 - \left( 1 - e^{-\left( \nu(-\ln(1-\alpha_n))^l \right)^{1/l} \right) \right)^l \right) \right)^{1/l} + (-\text{Ln}(\mu_n))^l \right)^{1/l} \\
 \mathbb{E} \left( \left( (-\text{Ln} \left( e^{-\left( \nu(-\ln \mathfrak{I}_n)^l \right)^{1/l} \right) \right)^l \right)^{1/l} + (-\text{Ln} \phi_n)^l \right)^{1/l} \\
 \mathbb{E} \left( \left( (-\text{Ln} e^{-\left( \nu(-\ln \mathfrak{H}_n)^l \right)^{1/l} \right)^l \right)^{1/l} + (-\text{Ln} \nu_n)^l \right)^{1/l} \\
 1 - \mathbb{E} \left( \left( (-\text{Ln} \left( 1 - \left( 1 - e^{-\left( \nu(-\ln(1-\nu_n))^l \right)^{1/l} \right) \right)^l \right) \right)^{1/l} + (-\text{Ln}(\mu_n))^l \right)^{1/l}
 \end{pmatrix}$$

$$C_{\nu} - \nu PFPAAA(\zeta_1, \zeta_2, \zeta_3, \dots, \zeta_h) \oplus \psi$$

$$C_{\nu} - PFPAAA(\zeta_1, \zeta_2, \zeta_3, \dots, \zeta_h) = \begin{pmatrix}
 1 - \mathbb{E} \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(1 - \alpha_n))^l \right)^{1/l} \\
 \mathbb{E} \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln} \mathfrak{I}_n)^l \right)^{1/l} \\
 \mathbb{E} \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln} \mathfrak{H}_n)^l \right)^{1/l} \\
 1 - \mathbb{E} \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(1 - \nu_n))^l \right)^{1/l}
 \end{pmatrix}$$

$$C_{\nu} - \nu PFPAAA(\varrho_1, \varrho_2, \varrho_3, \dots, \varrho_h) = \nu \begin{pmatrix}
 1 - \mathbb{E} \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(1 - \alpha_n))^l \right)^{1/l} \\
 \mathbb{E} \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln} \mathfrak{I}_n)^l \right)^{1/l} \\
 \mathbb{E} \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln} \mathfrak{H}_n)^l \right)^{1/l} \\
 1 - \mathbb{E} \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(1 - \nu_n))^l \right)^{1/l}
 \end{pmatrix}$$

$$C_{\nu} - \nu PFPAAA(\varrho_1, \varrho_2, \varrho_3, \dots, \varrho_h) = \begin{pmatrix} 1 - \varrho - \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (\nu(-\text{Ln}(1-\mathfrak{A}_n))^{\iota}) \right)^{1/\iota} \\ \varrho - \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (\nu(-\text{Ln}\mathfrak{T}_n)^{\iota}) \right)^{1/\iota} \\ \varrho - \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (\nu(-\text{Ln}\mathfrak{H}_n)^{\iota}) \right)^{1/\iota} \\ 1 - \varrho - \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (\nu(-\text{Ln}(1-\nu_n))^{\iota}) \right)^{1/\iota} \end{pmatrix},$$

$$C_{\nu} - \nu PFPAAA(\zeta_1, \zeta_2, \zeta_3, \dots, \zeta_h) \oplus \psi = \begin{pmatrix} 1 - \varrho - \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (\nu(-\text{Ln}(1-\mathfrak{A}_n))^{\iota}) \right)^{1/\iota} \\ \varrho - \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (\nu(-\text{Ln}\mathfrak{T}_n)^{\iota}) \right)^{1/\iota} \\ \varrho - \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (\nu(-\text{Ln}\mathfrak{H}_n)^{\iota}) \right)^{1/\iota} \\ 1 - \varrho - \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (\nu(-\text{Ln}(1-\nu_n))^{\iota}) \right)^{1/\iota} \end{pmatrix} \oplus (\mu, \phi, \nu)$$

Thus

$$C_{\nu} - PFPAAA(\nu\zeta_1 \oplus \psi, \nu\zeta_2 \oplus \psi, \dots, \nu\zeta_h \oplus \psi) = C_{\nu} - \nu PFPAAA(\zeta_1, \zeta_2, \zeta_3, \dots, \zeta_h) \oplus \psi$$

**Theorem 8** Let  $\zeta_n = (\mathfrak{A}_n, \mathfrak{T}_n, \mathfrak{H}_n, \nu_n)$  and  $\psi = (\mu_n, \phi_n, \nu_n, \nu_n)$  ( $n = 1, 2, 3, \dots, h$ ) be a collection of CPFVs.  $\mathbb{T}_n = \prod_{n=1}^{h-1} \mathfrak{s}(\zeta_h)$  ( $n = 2, 3, \dots, h$ ),  $\mathbb{T}_1 = 1$  and  $\mathfrak{s}(\zeta_h)$  is the score of CPFVs  $\zeta_h$  if  $\nu > 0$ , is a CPFV on  $z$ . Then

$$C_{\nu} - PFPAAA(\zeta_1 \oplus \psi_1, \zeta_2 \oplus \psi_2, \dots, \zeta_h \oplus \psi_h) = C_{\nu} - PFPAAA(\zeta_1, \zeta_2, \zeta_3, \dots, \zeta_h) \oplus C_{\nu} - PFPAAA(\psi_1, \psi_2, \psi_3, \dots, \psi_h) \quad (9)$$

*Proof* According to Definition, we have

$$C_{\nu} - PFPAAA(\zeta_1 \oplus \psi_1, \zeta_2 \oplus \psi_2, \dots, \zeta_h \oplus \psi_h)$$

$$\zeta_n \oplus \psi_n = \begin{pmatrix} 1 - \varrho - ((-\text{Ln}(1-\mathfrak{A}_n))^{\iota} + (-\text{Ln}(1-\mu_n))^{\iota})^{1/\iota} \\ \varrho - ((-\text{Ln}\mathfrak{T}_n)^{\iota} + (-\text{Ln}\phi_n)^{\iota})^{1/\iota} \\ \varrho - ((-\text{Ln}\mathfrak{H}_n)^{\iota} + (-\text{Ln}\nu_n)^{\iota})^{1/\iota} \\ 1 - \varrho - ((-\text{Ln}(1-\nu_n))^{\iota} + (-\text{Ln}(1-\mu_n))^{\iota})^{1/\iota} \end{pmatrix}$$

$$\begin{pmatrix} 1 - \varrho - \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} \left( -\text{Ln} \left( 1 - \left( 1 - \varrho - \left( \frac{(-\text{Ln}(1-\mathfrak{A}_n))^{\iota} + (-\text{Ln}(1-\mu_n))^{\iota}}{(-\text{Ln}(1-\mu_n))^{\iota}} \right)^{1/\iota} \right) \right) \right)^{\iota} \right)^{1/\iota} \\ \varrho - \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} \left( -\text{Ln} \left( e^{-((-\text{Ln}\mathfrak{T}_n)^{\iota} + (-\text{Ln}\phi_n)^{\iota})^{1/\iota}} \right) \right)^{\iota} \right)^{1/\iota} \\ \varrho - \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} \left( -\text{Ln} \left( e^{-((-\text{Ln}\mathfrak{H}_n)^{\iota} + (-\text{Ln}\nu_n)^{\iota})^{1/\iota}} \right) \right)^{\iota} \right)^{1/\iota} \\ 1 - \varrho - \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} \left( -\text{Ln} \left( 1 - \left( 1 - \varrho - \left( \frac{(-\text{Ln}(1-\nu_n))^{\iota} + (-\text{Ln}(1-\mu_n))^{\iota}}{(-\text{Ln}(1-\mu_n))^{\iota}} \right)^{1/\iota} \right) \right) \right)^{\iota} \right)^{1/\iota} \end{pmatrix}$$

$$\begin{aligned}
 &= \left( \begin{array}{c} 1 - \mathbb{e}^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(1-\mathfrak{A}_n))^l + (-\text{Ln}(1-\mu_n))^l\right)^{1/l}} \\ \mathbb{e}^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(\mathfrak{I}_n))^l + (-\text{Ln}\phi_n)^l\right)^{1/l}} \\ \mathbb{e}^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(\mathfrak{S}_n))^l + (-\text{Ln}\nu_n)^l\right)^{1/l}} \\ 1 - \mathbb{e}^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(1-\mathfrak{v}_n))^l + (-\text{Ln}(1-\mu_n))^l\right)^{1/l}} \end{array} \right) \\
 &Cr - PFPAAA(\zeta_1, \zeta_2, \zeta_3, \dots, \zeta_h) \oplus Cr - PFPAAA(\psi_1, \psi_2, \psi_3, \dots, \psi_h) \\
 Cr - PFPAAA(\zeta_1, \zeta_2, \dots, \zeta_n) &= \left( \begin{array}{c} 1 - \mathbb{e}^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(1-\mathfrak{A}_n))^l\right)^{1/l}} \\ \mathbb{e}^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}\phi_n)^l\right)^{1/l}} \\ \mathbb{e}^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}\mathfrak{S}_n)^l\right)^{1/l}} \\ 1 - \mathbb{e}^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(1-\mathfrak{v}_n))^l\right)^{1/l}} \end{array} \right) \\
 Cr - PFPAAA(\psi_1, \psi_2, \psi_3, \dots, \psi_h) &= \left( \begin{array}{c} 1 - \mathbb{e}^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(1-\mu_n))^l\right)^{1/l}} \\ \mathbb{e}^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}\mathfrak{I}_n)^l\right)^{1/l}} \\ \mathbb{e}^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}\nu_n)^l\right)^{1/l}} \\ 1 - \mathbb{e}^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(1-\mathfrak{v}_n))^l\right)^{1/l}} \end{array} \right) \\
 &Cr - PFPAAA(\zeta_1, \zeta_2, \zeta_3, \dots, \zeta_h) \oplus Cr - PFPAAA(\psi_1, \psi_2, \psi_3, \dots, \psi_h) \\
 &= \left( \left( \begin{array}{c} 1 - \mathbb{e}^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(1-\mathfrak{A}_n))^l\right)^{1/l}} \\ \mathbb{e}^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}\phi_n)^l\right)^{1/l}} \\ \mathbb{e}^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}\mathfrak{S}_n)^l\right)^{1/l}} \\ 1 - \mathbb{e}^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(1-\mathfrak{v}_n))^l\right)^{1/l}} \end{array} \right) \oplus \left( \begin{array}{c} 1 - \mathbb{e}^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(1-\mu_n))^l\right)^{1/l}} \\ \mathbb{e}^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}\mathfrak{I}_n)^l\right)^{1/l}} \\ \mathbb{e}^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}\nu_n)^l\right)^{1/l}} \\ 1 - \mathbb{e}^{-\left(\sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} (-\text{Ln}(1-\mu_n))^l\right)^{1/l}} \end{array} \right) \right)
 \end{aligned}$$

$$\begin{aligned}
 & \left( \begin{array}{c} - \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left( -\text{Ln} \left( 1 - \left( 1 - e^{-((- \text{Ln}(1 - \mathfrak{A}_n))^{\lambda})^{1/\lambda}} \right) \right) \right)^{\lambda} \\ 1 - \mathfrak{e} \\ - \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left( -\text{Ln} \left( e^{-((- \text{Ln} \mathfrak{T}_n)^{\lambda})^{1/\lambda}} \right) \right)^{\lambda} \\ \mathfrak{e} \\ - \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left( -\text{Ln} \left( e^{-((- \text{Ln} \mathfrak{H}_n)^{\lambda})^{1/\lambda}} \right) \right)^{\lambda} \\ \mathfrak{e} \\ - \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left( -\text{Ln} \left( 1 - \left( 1 - e^{-((- \text{Ln}(1 - \nu_n))^{\lambda})^{1/\lambda}} \right) \right) \right)^{\lambda} \\ 1 - \mathfrak{e} \end{array} \right)^{1/\lambda} \\
 & \oplus \left( \begin{array}{c} - \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left( -\text{Ln} \left( 1 - \left( 1 - e^{-((- \text{Ln}(1 - \mu_n))^{\lambda})^{1/\lambda}} \right) \right) \right)^{\lambda} \\ 1 - \mathfrak{e} \\ - \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left( -\text{Ln} \left( e^{-((- \text{Ln} \phi_n)^{\lambda})^{1/\lambda}} \right) \right)^{\lambda} \\ \mathfrak{e} \\ - \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left( -\text{Ln} \left( e^{-((- \text{Ln} \nu_n)^{\lambda})^{1/\lambda}} \right) \right)^{\lambda} \\ \mathfrak{e} \\ - \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left( -\text{Ln} \left( 1 - \left( 1 - e^{-((- \text{Ln}(1 - \mu_n))^{\lambda})^{1/\lambda}} \right) \right) \right)^{\lambda} \\ 1 - \mathfrak{e} \end{array} \right)^{1/\lambda} \\
 & = \left( \begin{array}{c} - \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left( -\text{Ln}(1 - \mathfrak{A}_n)^{\lambda} \right) \right)^{1/\lambda} \\ 1 - \mathfrak{e} \\ - \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left( -\text{Ln} \mathfrak{T}_n^{\lambda} \right) \right)^{1/\lambda} \\ \mathfrak{e} \\ - \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left( -\text{Ln} \mathfrak{H}_n^{\lambda} \right) \right)^{1/\lambda} \\ \mathfrak{e} \\ - \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left( -\text{Ln}(1 - \nu_n)^{\lambda} \right) \right)^{1/\lambda} \\ 1 - \mathfrak{e} \\ \oplus \left( \begin{array}{c} - \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left( -\text{Ln}(1 - \mu_n)^{\lambda} \right) \right)^{1/\lambda} \\ 1 - \mathfrak{e} \\ - \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left( -\text{Ln} \phi_n^{\lambda} \right) \right)^{1/\lambda} \\ \mathfrak{e} \\ - \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left( -\text{Ln} \nu_n^{\lambda} \right) \right)^{1/\lambda} \\ \mathfrak{e} \\ - \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left( -\text{Ln}(1 - \mu_n)^{\lambda} \right) \right)^{1/\lambda} \\ 1 - \mathfrak{e} \end{array} \right) \\
 & = \left( \begin{array}{c} - \left( \left( -\text{Ln} \left( 1 - \left( 1 - e^{- \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left( -\text{Ln}(1 - \mathfrak{A}_n)^{\lambda} \right) \right)^{1/\lambda} \right) \right) \right) \right)^{\lambda} + \left( \text{Ln} \left( 1 - \left( 1 - e^{- \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left( -\text{Ln}(1 - \mu_n)^{\lambda} \right) \right)^{1/\lambda} \right) \right) \right)^{\lambda} \\ 1 - \mathfrak{e} \\ - \left( \left( -\text{Ln} \left( e^{- \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left( -\text{Ln} \mathfrak{T}_n^{\lambda} \right) \right)^{1/\lambda} \right) \right) \right)^{\lambda} + \left( -\text{Ln} \left( e^{- \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left( -\text{Ln} \phi_n^{\lambda} \right) \right)^{1/\lambda} \right) \right)^{\lambda} \\ \mathfrak{e} \\ - \left( \left( -\text{Ln} \left( e^{- \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left( -\text{Ln} \mathfrak{H}_n^{\lambda} \right) \right)^{1/\lambda} \right) \right) \right)^{\lambda} + \left( -\text{Ln} \left( e^{- \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left( -\text{Ln} \nu_n^{\lambda} \right) \right)^{1/\lambda} \right) \right)^{\lambda} \\ \mathfrak{e} \\ - \left( \left( -\text{Ln} \left( 1 - \left( 1 - e^{- \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left( -\text{Ln}(1 - \nu_n)^{\lambda} \right) \right)^{1/\lambda} \right) \right) \right)^{\lambda} + \left( -\text{Ln} \left( 1 - \left( 1 - e^{- \left( \sum_{n=1}^h \frac{T_n}{\sum_{n=1}^h T_n} \left( -\text{Ln}(1 - \mu_n)^{\lambda} \right) \right)^{1/\lambda} \right) \right) \right)^{\lambda} \\ 1 - \mathfrak{e} \end{array} \right)^{1/\lambda}
 \end{aligned}$$

$$\begin{aligned}
 & \left( \begin{array}{c} - \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\mathbb{L}n(1-\mathfrak{A}_n))^{\iota} + \right)^{1/\iota} \\ 1 - \mathfrak{e} \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\mathbb{L}n(1-\mu_n))^{\iota} \right)^{1/\iota} \\ - \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\mathbb{L}n\mathfrak{T}_n)^{\iota} + \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\mathbb{L}n\phi_n)^{\iota} \right)^{1/\iota} \\ \mathfrak{e} \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\mathbb{L}n\mathfrak{H}_n)^{\iota} + \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\mathbb{L}n\nu_n)^{\iota} \right)^{1/\iota} \\ - \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\mathbb{L}n(1-\mathfrak{v}_n))^{\iota} + \right)^{1/\iota} \\ 1 - \mathfrak{e} \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\mathbb{L}n(1-\mu_n))^{\iota} \right)^{1/\iota} \end{array} \right) \\
 & = \left( \begin{array}{c} - \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\mathbb{L}n(1-\mathfrak{A}_n))^{\iota} + (-\mathbb{L}n(1-\mu_n))^{\iota} \right)^{1/\iota} \\ 1 - \mathfrak{e} \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\mathbb{L}n(1-\mu_n))^{\iota} \right)^{1/\iota} \\ - \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\mathbb{L}n(\mathfrak{T}_n))^{\iota} + (-\mathbb{L}n\phi_n)^{\iota} \right)^{1/\iota} \\ \mathfrak{e} \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\mathbb{L}n(\mathfrak{H}_n))^{\iota} + (-\mathbb{L}n\nu_n)^{\iota} \right)^{1/\iota} \\ - \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\mathbb{L}n(1-\mathfrak{v}_n))^{\iota} + (-\mathbb{L}n(1-\mu_n))^{\iota} \right)^{1/\iota} \\ 1 - \mathfrak{e} \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\mathbb{L}n(1-\mu_n))^{\iota} \right)^{1/\iota} \end{array} \right)
 \end{aligned}$$

Thus

$$\begin{aligned}
 & C_{\mathfrak{v}} - PFPAAA(\zeta_1 \oplus \psi_1, \zeta_2 \oplus \psi_2, \dots, \zeta_h \oplus \psi_h) \\
 & = C_{\mathfrak{v}} - PFPAAA(\zeta_1, \zeta_2, \zeta_3, \dots, \zeta_h) \oplus Cr - PFPAAA(\psi_1, \psi_2, \psi_3, \dots, \psi_h)
 \end{aligned}$$

**Definition 8** Suppose that  $\zeta_n = (n = 1, 2, 3 \dots h)$  be an assortment of several CPFVs. Then Cr-PFPAAG operator is a mapping  $\zeta^n \rightarrow \zeta$  is defined.

$$\begin{aligned}
 & \mathbb{T}_n = \prod_{n=1}^{h-1} \mathfrak{s}(\zeta_h) \quad (n = 2, 3, \dots, h), \\
 & Cr - PFPAAAG(\zeta_1, \zeta_2, \dots, \zeta_n) = \zeta_1 \frac{\mathbb{T}_1}{\sum_{n=1}^h \mathbb{T}_n} \otimes \zeta_2 \frac{\mathbb{T}_2}{\sum_{n=1}^h \mathbb{T}_n} \otimes \zeta_3 \frac{\mathbb{T}_3}{\sum_{n=1}^h \mathbb{T}_n}, \dots, \otimes \zeta_h \frac{\mathbb{T}_n}{\sum_{\mathcal{J}=1}^n \mathbb{T}_n} \quad (10)
 \end{aligned}$$

Consequently, we derive the following theorem by applying the Aczel–Alsina procedures on CPFVs.

**Theorem 9** Let  $\zeta_n = (\mathfrak{A}_n, \mathfrak{T}_n, \mathfrak{H}_n, \mathfrak{v}_n) \quad (n = 1, 2, 3 \dots h)$  are a combination of CPFVs. Then the combined value that is true is that they are using the Cr-PFPAAG operation.

CPFVs,

$$Cr - PFPAAAG(\zeta_1, \zeta_2, \dots, \zeta_n) = \left( \begin{array}{c} - \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\mathbb{L}n(1-\mathfrak{A}_n))^{\iota} \right)^{1/\iota} \\ \mathfrak{e} \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\mathbb{L}n(1-\mu_n))^{\iota} \right)^{1/\iota} \\ 1 - \mathfrak{e} \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\mathbb{L}n\mathfrak{T}_n)^{\iota} \right)^{1/\iota} \\ - \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\mathbb{L}n\mathfrak{H}_n)^{\iota} \right)^{1/\iota} \\ 1 - \mathfrak{e} \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\mathbb{L}n(1-\mathfrak{v}_n))^{\iota} \right)^{1/\iota} \\ \mathfrak{e} \left( \sum_{n=1}^h \frac{\mathbb{T}_n}{\sum_{n=1}^h \mathbb{T}_n} (-\mathbb{L}n(1-\mu_n))^{\iota} \right)^{1/\iota} \end{array} \right) \quad (11)$$

*Proof* Theorem 9 can be proven by using Theorem 1.

**Theorem 10** (Idempotency). Suppose that  $\zeta_n = (\mathfrak{A}_n, \mathfrak{T}_n, \mathfrak{H}_n, \mathfrak{v}_n) \quad (n = 1, 2, 3 \dots h)$  are equal, that is  $\zeta_n = \zeta$  for every  $n$ , afterward,

$$Cr - PFPAAG(\zeta_1, \zeta_2, \dots, \zeta_h) = \zeta \tag{12}$$

*Proof* Theorem 10 can be proven by using Theorem 2.

**Theorem 11** (Boundedness). *Suppose that  $\zeta_n = (\mathfrak{A}_n, \mathfrak{T}_n, \mathfrak{H}_n, \nu_n)$  ( $n = 1, 2, 3 \dots h$ ) be an accumulation of CPFVs. Let  $\zeta^- = \min(\zeta_1, \zeta_2, \dots, \zeta_h)$  and  $\zeta^+ = \max(\zeta_1, \zeta_2, \dots, \zeta_h)$  afterward,*

$$\zeta^- \leq Cr - PFPAAG(\zeta_1, \zeta_2, \zeta_3 \dots \zeta_h) \leq \zeta^+ \tag{13}$$

*Proof* Theorem 11 can be proven by using Theorem 3.

**Theorem 12** *Suppose that  $\zeta_n = (\mathfrak{A}_n, \mathfrak{T}_n, \mathfrak{H}_n, \nu_n)$  ( $n = 1, 2, 3 \dots h$ ) be an accumulation of CPFVs,  $\mathbb{T}_n = \prod_{n=1}^{h-1} \mathfrak{s}(\zeta_h)$  ( $n = 2, 3, \dots h$ ),  $\mathbb{T}_1 = 1$  along with  $\mathfrak{s}(\zeta_h)$  is the CPFV score  $\zeta_h$  if  $\psi = (\mu, \phi, \nu)$  is a CPFV upon  $X$ , afterward.*

$$Cr - PFPAAG(\varrho_1 \otimes \psi, \varrho_2 \otimes \psi, \dots, \varrho_h \otimes \psi) = Cr - PFPAAG(\zeta_1, \zeta_2, \zeta_3, \dots, \zeta_h) \otimes \psi \tag{14}$$

*Proof* Theorem 12 can be proven by using Theorem 4.

**Theorem 13** *Suppose  $\zeta_n = (\mathfrak{A}_n, \mathfrak{T}_n, \mathfrak{H}_n, \nu_n)$  ( $n = 1, 2, 3 \dots h$ ) be a collection of CPFVs.  $\mathbb{T}_1 = 1$  and  $\mathfrak{s}(\zeta_h)$  is the score of CPFVs  $\zeta_h$  if  $\nu > 0$  CPFVs on  $z$ , then*

$$Cr - PFPAAG((\zeta_1)^\nu, (\zeta_2)^\nu, \dots, (\zeta_h)^\nu) = (Cr - PFPAAG(\zeta_1, \zeta_2, \zeta_3, \dots, \zeta_h))^\nu \tag{15}$$

*Proof* Theorem 13 can be proven by using Theorem 5.

**Theorem 14** *Let  $\zeta_n = (\mathfrak{A}_n, \mathfrak{T}_n, \mathfrak{H}_n, \nu_n)$  ( $n = 1, 2, 3 \dots h$ ) be a collection of CPFVs.  $\mathbb{T}_n = \prod_{n=1}^{h-1} \mathfrak{s}(\zeta_h)$  ( $n = 2, 3, \dots h$ ),  $\mathbb{T}_1 = 1$  along with  $\mathfrak{s}(\zeta_h)$  is the CPFV score  $\zeta_h$  if  $\nu > 0$ ,  $\psi = (\mu, \phi, \nu)$  is a CPFV upon  $z$  afterward,*

$$Cr - PFPAAG((\varrho_1)^\nu \otimes \psi, (\varrho_2)^\nu \otimes \psi, \dots, (\varrho_h)^\nu \otimes \psi) = (Cr - PFPAAG(\varrho_1, \varrho_2, \varrho_3, \dots, \varrho_h))^\nu \oplus \psi \tag{16}$$

*Proof* Theorem 14 can be proven by using Theorem 6.

**Theorem 15** *Let  $\zeta_n = (\mathfrak{A}_n, \mathfrak{T}_n, \mathfrak{H}_n, \nu_n)$  ( $n = 1, 2, 3 \dots h$ ) be a collection of CPFVs.  $\mathbb{T}_n = \prod_{n=1}^{h-1} \mathfrak{s}(\zeta_h)$  ( $n = 2, 3, \dots h$ ),  $\mathbb{T}_1 = 1$  along with  $\mathfrak{s}(\zeta_h)$  is the CPFV score  $\zeta_h$  if  $\nu > 0$ ,  $\psi = (\mu, \phi, \nu)$  is a CPFV upon  $z$ . Afterward,*

$$\begin{aligned} Cr - PFPAAG((\varrho_1)^\nu \otimes \psi, (\varrho_2)^\nu \otimes \psi, \dots, (\varrho_h)^\nu \otimes \psi) \\ = (Cr - PFPAAG(\varrho_1, \varrho_2, \varrho_3, \dots, \varrho_h))^\nu \oplus \psi \end{aligned} \tag{17}$$

*Proof* Theorem 15 can be proven by using Theorem 7.

**Theorem 16** *Suppose  $\zeta_n = (\mathfrak{A}_n, \mathfrak{T}_n, \mathfrak{H}_n, \nu_n)$  and  $\psi = (\mu_n, \phi_n, \nu_n)$  ( $n = 1, 2, 3 \dots h$ ) is a collection of CPFVs.  $\mathbb{T}_n = \prod_{n=1}^{h-1} \mathfrak{s}(\zeta_h)$  ( $n = 2, 3, \dots h$ ),  $\mathbb{T}_1 = 1$  along with  $\mathfrak{s}(\zeta_h)$  is the CPFV score  $\zeta_h$  if  $\nu > 0$ , is a CPFV upon  $z$ . Afterward,*

$$Cr - PFPAAG(\zeta_1 \otimes \psi_1, \zeta_2 \otimes \psi_2, \dots, \zeta_h \otimes \psi_h) = Cr - PFPAAG(\zeta_1, \zeta_2, \zeta_3, \dots, \zeta_h) \otimes Cr - PFPAAG(\psi_1, \psi_2, \psi_3, \dots, \psi_h)$$

*Proof* Theorem 8 can be used to prove Theorem 16.

### Multi-criteria group decision-making by using circular picture fuzzy valued Aczel–Alsina

In this section, we will create a MADM approach based on the fuzzy environment image to demonstrate dependability and efficiency. Assuming that the collection of options in this problem is  $\mathfrak{Z} = \{\mathfrak{Z}_1, \mathfrak{Z}_2, \mathfrak{Z}_3 \dots \mathfrak{Z}_m\}$  and the set of attributes is  $\mathfrak{C} = \{\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3 \dots \mathfrak{C}_z\}$ , and that the attributes are prioritized according to the linear ordering  $\mathfrak{C}_1 > \mathfrak{C}_2 > \mathfrak{C}_3 \dots > \mathfrak{C}_z$ . If the collection of decision makers is  $\mathfrak{E} = \{\mathfrak{E}_1, \mathfrak{E}_2, \mathfrak{E}_3 \dots \mathfrak{E}_p\}$  and  $\mathfrak{E}_1 > \mathfrak{E}_2 > \mathfrak{E}_3 \dots > \mathfrak{E}_z$ , which indicates that  $\mathfrak{E}_\tau$  is more important than  $\mathfrak{E}_\zeta$  if  $\zeta < \tau$ . Let  $\mathbb{k} = (\mathbb{k}_{i\mathcal{J}}^q)_{n \times m}$  correspond to the picture value matrix of Aczel–Alsina decisions, and  $\mathbb{k}_{i\mathcal{J}}^q = (\mathfrak{A}^q, \mathfrak{T}^q, \mathfrak{H}^q, \nu^q)$  is an attribute value that is stated by the person who makes the decision  $\mathfrak{E}_q$  which is supplied in a Cr-PFPAAG, where the degree is indicated by the  $(\zeta)$  range that the alternative  $y_j$  satisfies the attribute  $\mathfrak{C}_{\mathcal{J}}$  expressed by the decision-maker  $\mathfrak{E}_q$ , and  $(\mathfrak{T}^q, \mathfrak{H}^q)$  indicates the degree range that the alternative  $y_i$  does not satisfy the attribute  $\mathfrak{C}_{\mathcal{J}}$  expresses the decision-maker  $\mathfrak{E}_q$ , assuming that  $(\mathfrak{A}^q) \subset [0, 1]$ ,  $(\mathfrak{T}^q, \mathfrak{H}^q) \subset [0, 1]$ ,  $(\mathfrak{A}^q + \mathfrak{T}^q + \mathfrak{H}^q + \nu^q) \leq 1, \nu^q = 1, 2, 3 \dots m, \mathcal{J} = 1, 2, 3 \dots z$ . The

attribute value does not require normalization of all of the attributes.  $\mathbb{C}_{\mathcal{J}}(1, 2, 3 \dots m)$  are of the same type. The decision-maker matrix  $K^q = (\mathbb{k}_{i\mathcal{J}}^q)_{n \times 3m}$  is normalized to  $R^q = (\mathbb{k}_{i\mathcal{J}}^q)_{n \times 3m}$  otherwise.

$$v_{i\mathcal{J}}^q = \begin{cases} \frac{\mathbb{k}_{i\mathcal{J}}^q \text{ for benefit attribute } \mathbb{C}_{\mathcal{J}}}{\mathbb{k}_{i\mathcal{J}}^q \text{ for benefit attribute } \mathbb{C}_{\mathcal{J}}} & = 1, 2, 3 \dots, m \mathcal{J} = 1, 2, 3 \dots, z \end{cases}$$

where  $\bar{\mathbb{k}}_{i\mathcal{J}}^q$  is the complement of  $\mathbb{k}_{i\mathcal{J}}^q$  such that  $\bar{\mathbb{k}}_{i\mathcal{J}}^q = (\omega^q, \alpha^q, \beta^q)$  and  $v_{i\mathcal{J}}^q = (\alpha^q, \beta^q, \omega^q)$   $i = 1, 2, 3 \dots, m \mathcal{J} = 1, 2, 3 \dots, z$ .

**Proposed methodology**

Due to the inconsistency, vagueness, and prioritization involved in expert ratings of basketball quality, a methodology framework integrating Circular Picture Fuzzy Sets and prioritized Aczel–Alsina aggregation operators is suggested. Such a method ensures that uncertainty and relative importance in ratings are both systematically dealt with. The main phases are as follows:

**Step 1: Define the problem**

Identifying a set of alternatives is  $\mathfrak{Z} = \{\mathfrak{Z}_1, \mathfrak{Z}_2, \mathfrak{Z}_3 \dots \mathfrak{Z}_m\}$  and the set of attributes is  $\mathbb{C} = \{\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3 \dots \mathbb{C}_z\}$ , and that the attributes are prioritized according to the linear ordering  $\mathbb{C}_1 > \mathbb{C}_2 > \mathbb{C}_3 \dots > \mathbb{C}_z$ .

Determine the collection of decision makers is  $\mathbb{E} = \{\mathbb{E}_1, \mathbb{E}_2, \mathbb{E}_3 \dots \mathbb{E}_p\}$  and  $\mathbb{E}_1 > \mathbb{E}_2 > \mathbb{E}_3 \dots > \mathbb{E}_p$ , which indicates that  $\mathbb{E}_\zeta$  is more important than  $\mathbb{E}_\tau$  if  $\zeta < \tau$ .

By using the following equations, determine the values of  $\mathbb{T}_{i\mathcal{J}}^q$  ( $q = 1, 2, 3 \dots s$ ):

$$\mathbb{T}_{i\mathcal{J}}^q = \prod_{k=1}^{q-1} s(v_{i\mathcal{J}}^k) \quad (q = 2, 3 \dots p)$$

$$\mathbb{T}_{i\mathcal{J}}^1 = 1$$

**Step 2: Collect evaluation data**

Collect expert assessments using CPFVs and each assessment is provided as a quadruple. By using the Cr-PFPAAA operator:

$$v_{i\mathcal{J}} = \mathbb{C} \nu - PFPAAA(v_{i\mathcal{J}}^1, v_{i\mathcal{J}}^2, v_{i\mathcal{J}}^3 \dots v_{i\mathcal{J}}^s)$$

$$= \left( \begin{array}{c} - \left( \sum_{n=1}^h \frac{\mathbb{T}_{i\mathcal{J}}}{\sum_{n=1}^h \mathbb{T}_{i\mathcal{J}}} (-\text{Ln}(1-\alpha_{i\mathcal{J}}^q))^l \right)^{1/l} \\ 1 - \mathbb{E} \\ - \left( \sum_{n=1}^h \frac{\mathbb{T}_{i\mathcal{J}}}{\sum_{n=1}^h \mathbb{T}_{i\mathcal{J}}} (-\text{Ln}\beta_{i\mathcal{J}}^q)^l \right)^{1/l} \\ \mathbb{E} \\ - \left( \sum_{n=1}^h \frac{\mathbb{T}_{i\mathcal{J}}}{\sum_{n=1}^h \mathbb{T}_{i\mathcal{J}}} (-\text{Ln}\omega_{i\mathcal{J}}^q)^l \right)^{1/l} \\ \mathbb{E} \\ - \left( \sum_{n=1}^h \frac{\mathbb{T}_{i\mathcal{J}}}{\sum_{n=1}^h \mathbb{T}_{i\mathcal{J}}} (-\text{Ln}(1-v_{i\mathcal{J}}^q))^l \right)^{1/l} \\ 1 - \mathbb{E} \end{array} \right)^{1/l}$$

Or the Cr-PFPAAG operator

$$v_{i\mathcal{J}} = \mathbb{C} \nu - PFPAAA(v_{i\mathcal{J}}^1, v_{i\mathcal{J}}^2, v_{i\mathcal{J}}^3 \dots v_{i\mathcal{J}}^s)$$

$$= \left( \begin{array}{c} - \left( \sum_{n=1}^h \frac{\mathbb{T}_{i\mathcal{J}}}{\sum_{n=1}^h \mathbb{T}_{i\mathcal{J}}} (-\text{Ln}\alpha_{i\mathcal{J}}^q)^l \right)^{1/l} \\ \mathbb{E} \\ - \left( \sum_{n=1}^h \frac{\mathbb{T}_{i\mathcal{J}}}{\sum_{n=1}^h \mathbb{T}_{i\mathcal{J}}} (-\text{Ln}(1-\beta_{i\mathcal{J}}^q))^l \right)^{1/l} \\ 1 - \mathbb{E} \\ - \left( \sum_{n=1}^h \frac{\mathbb{T}_{i\mathcal{J}}}{\sum_{n=1}^h \mathbb{T}_{i\mathcal{J}}} (-\text{Ln}(1-\omega_{i\mathcal{J}}^q))^l \right)^{1/l} \\ 1 - \mathbb{E} \\ - \left( \sum_{n=1}^h \frac{\mathbb{T}_{i\mathcal{J}}}{\sum_{n=1}^h \mathbb{T}_{i\mathcal{J}}} (-\text{Ln}v_{i\mathcal{J}}^q)^l \right)^{1/l} \\ \mathbb{E} \end{array} \right)^{1/l}$$

To aggregate the decision-making of each CPFV  $R^q = (v_{i\mathcal{J}}^q)_{n \times 3m}$  ( $q = 1, 2, 3 \dots s$ ) into the decision-making of collective CPFVs  $R^q = (v_{i\mathcal{J}}^q)_{n \times 3m}$   $i = 1, 2, 3 \dots, m, \mathcal{J} = 1, 2, 3 \dots, z$ .

Step 3: Using the following equation, determine the value  $\mathbb{T}_{i\mathcal{J}}$  ( $i = 1, 2, 3 \dots, m \mathcal{J} = 2, 3 \dots, s$ ).

$$\mathbb{T}_{i\mathcal{J}} = 1 \quad i = 1, 2, 3, \dots, m \tag{20}$$

*Step 4: Aggregate Expert Opinions* Combine individual assessments using Cr-PPFAAG or Cr-PPFAAA operators to get a collective decision matrix. For every alternative  $\mathfrak{J}_j$  aggregate the CPFVs  $v_{i\mathcal{J}}$  by using the Cr-PPFAAA operator.

$$v_i = \text{Cr-PPFAAA}(v_{i1}, v_{i2}, v_{i3}, \dots, v_{ik})$$

$$= \left( \begin{array}{c} - \left( \sum_{n=1}^h \frac{\mathbb{T}_{i\mathcal{J}}}{\sum_{n=1}^h \mathbb{T}_{i\mathcal{J}}} (-\text{Ln}(1-\alpha_{\mathcal{J}}))^i \right)^{1/i} \\ 1 - e \\ - \left( \sum_{n=1}^h \frac{\mathbb{T}_{i\mathcal{J}}}{\sum_{n=1}^h \mathbb{T}_{i\mathcal{J}}} (-\text{Ln}\beta_{\mathcal{J}})^i \right)^{1/i} \\ e \\ - \left( \sum_{n=1}^h \frac{\mathbb{T}_{i\mathcal{J}}}{\sum_{n=1}^h \mathbb{T}_{i\mathcal{J}}} (-\text{Ln}\omega_{\mathcal{J}})^i \right)^{1/i} \\ e \\ - \left( \sum_{n=1}^h \frac{\mathbb{T}_{i\mathcal{J}}}{\sum_{n=1}^h \mathbb{T}_{i\mathcal{J}}} (-\text{Ln}(1-\nu_{\mathcal{J}}))^i \right)^{1/i} \\ 1 - e \end{array} \right)$$

Or

$$v_i = \text{Cr-PPFAAG}(v_{i1}, v_{i2}, v_{i3}, \dots, v_{ik})$$

$$= \left( \begin{array}{c} - \left( \sum_{n=1}^h \frac{\mathbb{T}_{i\mathcal{J}}}{\sum_{n=1}^h \mathbb{T}_{i\mathcal{J}}} (-\text{Ln}\alpha_{\mathcal{J}})^i \right)^{1/i} \\ e \\ - \left( \sum_{n=1}^h \frac{\mathbb{T}_{i\mathcal{J}}}{\sum_{n=1}^h \mathbb{T}_{i\mathcal{J}}} (-\text{Ln}(1-\beta_{\mathcal{J}}))^i \right)^{1/i} \\ 1 - e \\ - \left( \sum_{n=1}^h \frac{\mathbb{T}_{i\mathcal{J}}}{\sum_{n=1}^h \mathbb{T}_{i\mathcal{J}}} (-\text{Ln}(1-\omega_{\mathcal{J}}))^i \right)^{1/i} \\ 1 - e \\ - \left( \sum_{n=1}^h \frac{\mathbb{T}_{i\mathcal{J}}}{\sum_{n=1}^h \mathbb{T}_{i\mathcal{J}}} (-\text{Ln}\nu_{\mathcal{J}})^i \right)^{1/i} \\ e \end{array} \right)$$

*Step 5: Compute Score, Accuracy, and Refusal Values and Compute the score function for every alternative.* If they have the same score, apply the accuracy function. If still equal, compare the refusal degree. Using the scoring mechanism outlined in “Literature review” section, rank each option.

$$s(v_i) = \omega_i - \alpha_i, \quad i = 1, 2, 3, \dots, m$$

Next the greater value of  $s(v_i)$ , the greater the total CPFFAAA  $v_i$  and hence the substitute  $k_i$  ( $i = 1, 2, 3, \dots, m$ ).

*Step 5: Rank the Alternatives:* Rank the alternatives according to their final evaluated values.

*Step 6: Make the Decision.*

*Step 7: Choose the alternative(s) with the best (highest) ranking.*

**Significance of the selected environment**

The CPPF setting is especially appropriate for solving complex decision-making issues with uncertainty, vagueness, and hesitation. Classical fuzzy and intuitionistic fuzzy settings usually have difficulties dealing with situations where DoM, DoNM, DoA, and refusal all play a role in the evaluation process at the same time. The CPPF model further increases flexibility by adding another parameter of refusal while maintaining the prioritization of decision-makers and criteria, which reflects more realistic real-world decision situations. This setting facilitates decision-makers to articulate their judgments more fully and record subtle opinions more precisely. In addition, employing circular structures facilitates uniform aggregation and comparison of alternatives in the face of intricate uncertainty, which renders it extremely effective for high-stakes uses like quality evaluation of sports equipment at the global level. The chosen setting thus improves the strength, reliability, and interpretability of the presented MADM framework.

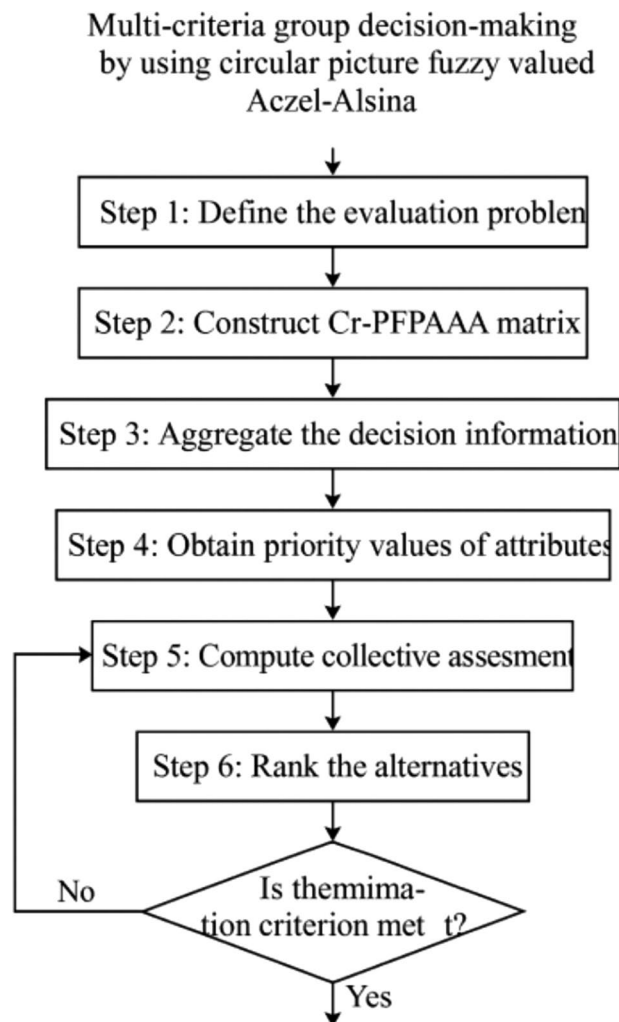
**Application (quality of basketballs of international level sports)**

Basketball<sup>82</sup> has become the primary sport of the contemporary day and can help people stay healthy and more physically active. Basketball, however, typically suffers from irrational sports plans and even degrades physical fitness. Finding a basketball program that works is therefore a pressing issue that needs to be resolved right now. The quality of international basketball is suggested in this research as a way to raise the standard of public health. The quality of basketballs and their attributes are examined using an intelligent algorithm that incorporates the methods of several specialists. Unsatisfactory analysis results were obtained from earlier basketball research that overlooked thorough analysis and lacked correlation analysis of basketball indices. Basketball’s technical approach, level of difficulty, and curriculum can all be modified using the correlation analysis method in conjunction with teens’ physical growth. A MATLAB simulation demonstrates that the correlation analysis

approach can 95% accurately assess how basketball affects teens' health. As a result, the correlation analysis approach developed in this work can help basketball players and raise teens' health levels. Professional basketball quantitative analysis has grown in popularity among seasoned data analysts, and the recent release of high-resolution datasets has advanced data-driven basketball analytics. We discuss professional basketball from a qualitative point of view. We suggest going over the quantitative basketball task kinds, granularity levels, and dimensions. We suggest going over the quantitative basketball task kinds, granularity levels, and dimensions. With a focus on current developments and an evolutionary viewpoint, we examine significant works from the last 20 years and map them into the suggested qualitative framework.

Future study directions are indicated by a list of topics about professional basketball that could be addressed with quantitative tools.

The purpose of this study was to compare the introduction of a new basketball program for healthy male students at Saigon University with the new basketball election classes in the physical education curriculum. The overview of the quality assessment stages for international standard basketballs is shown in Fig. 1. 64 healthy male students were chosen, and they were randomly assigned to one of two groups (the experimental group or the control group). Every Wednesday morning for 15 weeks, the participants trained on the basketball floor at Saigon University. The study's findings showed that, except for core strength, a new basketball program improved speed, agility, leg power, and maximum aerobic speed. Thus, a new basketball program at Saigon University might enable more improvements, offer more advantages than the current program, foster a healthy environment for studying physical education, and meet the growing training needs of male students. Future research should elucidate the program's effects on female students participating in other sports, taking into account participant classifications such as reduced activity, prolonged training, and advanced student training. International basketball teams' quality assessment procedure is represented geometrically in Fig. 2, which graphically illustrates the methodical methodology used to guarantee accordance with international standards. The evaluation phases include everything from the selection of raw materials, quality checks throughout manufacture, and extensive testing procedures such as size, weight, bounce, grip, and air retention, to durability



**Fig. 1.** MADM method with circular picture fuzzy Aczel–Alsina model.

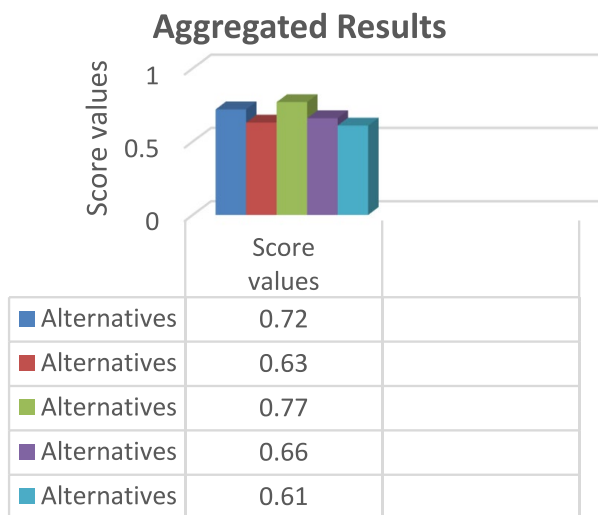


Fig. 2. Graphical presentation of score values.

Alternatives	$t_1$	$t_2$	$t_3$	$t_4$
$s_1$	(0.6, 0.5, 0.2, 0.3)	(0.5, 0.6, 0.1, 0.3)	(0.4, 0.5, 0.3, 0.1)	(0.6, 0.7, 0.2, 0.3)
$s_2$	(0.4, 0.5, 0.1, 0.3)	(0.3, 0.4, 0.2, 0.2)	(0.3, 0.4, 0.2, 0.2)	(0.4, 0.6, 0.2, 0.3)
$s_3$	(0.7, 0.6, 0.1, 0.2)	(0.4, 0.4, 0.2, 0.3)	(0.5, 0.5, 0.3, 0.4)	(0.5, 0.7, 0.2, 0.3)
$s_4$	(0.4, 0.5, 0.3, 0.1)	(0.4, 0.5, 0.1, 0.2)	(0.3, 0.5, 0.1, 0.4)	(0.5, 0.7, 0.1, 0.2)
$s_5$	(0.3, 0.5, 0.1, 0.2)	(0.4, 0.6, 0.1, 0.1)	(0.5, 0.5, 0.1, 0.3)	(0.4, 0.5, 0.2, 0.2)

Table 1. CR. Picture fuzzy prioritized decision matrix by expert 1.

and adherence to FIBA/NBA regulations. The geometric representation of each process highlights decision points such as “pass or rework,” which conclude in final approval, packing, and distribution. This methodical procedure guarantees premium basketballs fit for professional competitions and emphasizes the link between product standards improved player performance and public health.

Peace-building goals and objectives and sports were heavily used to reconstruct the nation. Therefore, among other things, themes of gender equality, peace and reconciliation, and illness prevention were disseminated through various sports. The International Basketball Foundation (IBF) recently supported a program that employed basketball, one of the most popular sports in Rwanda, to transform the lives of underprivileged adolescents and families. This project aims to use the abilities gained from basketball practice to enhance the living conditions of some poor areas. It is connected to bigger development plans of rising National Federations (NFs). This article attempts to show how basketball is used to improve living conditions for young players and their families from disadvantaged neighborhoods through an effort made feasible with support from the IBF. In two elementary schools in the districts of Rubavu and Muhanga, the Basketball for Good initiative had its start. We conclude by examining the prospects of Sport for Development (SFD) programs in Rwanda. The comparison of Saigon University’s basketball programs begins with participant selection (64 male students were split into experimental and control groups), followed by the introduction of a new basketball program, 15 weeks of training, and evaluation of physical fitness measurements. Improvements in speed, agility, leg power, and aerobic speed were demonstrated by the results, indicating potential avenues for further study, such as advanced training and testing on female students. The results of the study are analyzed based on the selection criteria established through a comprehensive literature search and feedback from potential players. Based on the data gathered, the following criteria are established: strength, power, agility, and excellent ball-handling abilities, where a group of decision-makers  $s = \{s_1, s_2, s_3, s_4, s_5\}$ . Evaluate the performance and chooses the best to get the most benefits. The stages of the algorithm in question are chosen based on the following criteria as shown in Tables 1, 2, and 3.

The attribute values do not require normalization if you edit the type, therefore  $R^q = D^q = (d_{i\mathcal{J}}^q)_{5 \times 4} = (r_{i\mathcal{J}}^q)_{5 \times 4}$ . The Cr-PPFAA operator has specified the following as the primary steps:  
 Step 1: Find the values of  $\mathbb{T}_{i\mathcal{J}}^1, \mathbb{T}_{i\mathcal{J}}^2, \mathbb{T}_{i\mathcal{J}}^3$

Alternatives	$t_1$	$t_2$	$t_3$	$t_4$
$s_1$	(0.2, 0.6, 0.1, 0.3)	(0.6, 0.7, 0.1, 0.2)	(0.4, 0.5, 0.2, 0.4)	(0.4, 0.7, 0.2, 0.1)
$s_2$	(0.3, 0.7, 0.1, 0.2)	(0.3, 0.4, 0.1, 0.3)	(0.3, 0.5, 0.2, 0.3)	(0.6, 0.4, 0.2, 0.3)
$s_3$	(0.4, 0.6, 0.1, 0.2)	(0.8, 0.3, 0.1, 0.1)	(0.4, 0.4, 0.3, 0.2)	(0.4, 0.6, 0.3, 0.2)
$s_4$	(0.5, 0.5, 0.3, 0.1)	(0.3, 0.3, 0.1, 0.2)	(0.6, 0.4, 0.1, 0.1)	(0.6, 0.4, 0.1, 0.2)
$s_5$	(0.6, 0.5, 0.2, 0.2)	(0.4, 0.6, 0.1, 0.3)	(0.3, 0.5, 0.1, 0.2)	(0.4, 0.3, 0.3, 0.1)

**Table 2.** CR. Picture fuzzy prioritized decision matrix by expert 2.

Alternatives	$t_1$	$t_2$	$t_3$	$t_4$
$s_1$	(0.6, 0.3, 0.2, 0.2)	(0.4, 0.5, 0.1, 0.3)	(0.6, 0.5, 0.2, 0.2)	(0.5, 0.6, 0.2, 0.3)
$s_2$	(0.4, 0.4, 0.1, 0.1)	(0.5, 0.5, 0.2, 0.4)	(0.5, 0.4, 0.3, 0.3)	(0.6, 0.5, 0.2, 0.3)
$s_3$	(0.6, 0.4, 0.1, 0.2)	(0.4, 0.4, 0.2, 0.3)	(0.4, 0.6, 0.3, 0.1)	(0.7, 0.4, 0.1, 0.3)
$s_4$	(0.4, 0.5, 0.3, 0.1)	(0.4, 0.5, 0.1, 0.2)	(0.3, 0.5, 0.1, 0.2)	(0.4, 0.3, 0.2, 0.2)
$s_5$	(0.3, 0.2, 0.1, 0.1)	(0.2, 0.4, 0.1, 0.1)	(0.2, 0.5, 0.1, 0.3)	(0.30.2, 0.1, 0.1)

**Table 3.** CR. picture fuzzy prioritized decision matrix by expert 3.

Alternatives	$t_1$	$t_2$	$t_3$	$t_4$
$s_1$	(0.50, 0.32, 0.16, 0.17)	(0.51, 0.39, 0.10, 0.16)	(0.44, 0.31, 0.24, 0.11)	(0.52, 0.43, 0.20, 0.13)
$s_2$	(0.37, 0.35, 0.10, 0.13)	(0.34, 0.26, 0.16, 0.16)	(0.34, 0.27, 0.21, 0.15)	(0.51, 0.33, 0.20, 0.19)
$s_3$	(0.59, 0.35, 0.10, 0.12)	(0.56, 0.23, 0.16, 0.13)	(0.45, 0.31, 0.30, 0.16)	(0.52, 0.39.0.19, 0.16)
$s_4$	(0.43, 0.31, 0.30, 0.06)	(0.37, 0.28, 0.10, 0.12)	(0.40, 0.30, 0.10, 0.14)	(0.51, 0.34, 0.11, 0.12)
$s_5$	(0.41, 0.28, 0.12, 0.11)	(0.36, 0.35, 0.10, 0.09)	(0.39, 0.31, 0.10, 0.16)	(0.38, 0.25, 0.20, 0.09)

**Table 4.** The matrix was combined, by using the data in Tables 1, 2 and 3.

$$\begin{aligned}
 \mathbb{T}_{i\mathcal{J}}^1 &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \\
 \mathbb{T}_{i\mathcal{J}}^2 &= \begin{pmatrix} 0.65 & 0.68 & 0.63 & 0.70 \\ 0.63 & 0.58 & 0.58 & 0.63 \\ 0.75 & 0.58 & 0.58 & 0.68 \\ 0.63 & 0.65 & 0.58 & 0.73 \\ 0.63 & 0.70 & 0.65 & 0.63 \end{pmatrix} \\
 \mathbb{T}_{i\mathcal{J}}^3 &= \begin{pmatrix} 0.39 & 0.51 & 0.36 & 0.05 \\ 0.42 & 0.33 & 0.33 & 0.04 \\ 0.51 & 0.42 & 0.33 & 0.04 \\ 0.41 & 0.37 & 0.40 & 0.05 \\ 0.42 & 0.46 & 0.41 & 0.04 \end{pmatrix}
 \end{aligned}$$

*Step 2:* By making use of the Cr-PFPAAA operator Eq. (8) to combine each CPFV decision-making  $R^q = (v_{i\mathcal{J}}^q)_{5 \times 4}$  ( $q = 1, 2, 3$ ) into overall picture fuzzy decision matrix  $\tilde{R} = (\tilde{v}_{i\mathcal{J}})_{5 \times 4}$  as shown in Table 4.

*Step 3:* By making use of the Eqs. (19) and (20), locate the values of  $\mathbb{T}_{i\mathcal{J}}$  ( $i = 1, 2, 3 \dots, m\mathcal{J} = 1, 2, 3 \dots, z$ ).

$$\mathbb{T}_{i\mathcal{J}}^1 = \begin{pmatrix} 10.62 & 0.40 & 0.24 \\ 10.62 & 0.35 & 0.19 \\ 10.68 & 0.42 & 0.24 \\ 10.59 & 0.36 & 0.22 \\ 10.61 & 0.38 & 0.23 \end{pmatrix}$$

*Step 4:* To obtain the overall values of preference  $v_{i\mathcal{J}}$  ( $i = 1, 2, 3, 4, 5$ ) the Cr-PFPAAA operator was used to add up all of the values of preference  $v_j$  on the line of  $R$ .

$$\begin{aligned}
 v_1 &= (0.49, 0.35, 0.15, 0.15) \\
 v_2 &= (0.37, 0.31, 0.13, 0.14) \\
 v_3 &= (0.55, 0.32, 0.15, 0.14) \\
 v_4 &= (0.42, 0.30, 0.16, 0.09) \\
 v_5 &= (0.39, 0.30, 0.11, 0.11)
 \end{aligned}$$

Step 5: Determine the score of  $v_j$  ( $i = 1, 2, 3, 4, 5$ ) respectively:

$$s_1 = 0.49 \quad s_2 = 0.37 \quad s_3 = 0.55 \quad s_4 = 0.42 \quad s_5 = 0.39$$

Since

$$s_3 > s_1 > s_4 > s_5 > s_2$$

We have

$$\mathfrak{J}_3 > \mathfrak{J}_1 > \mathfrak{J}_4 > \mathfrak{J}_5 > \mathfrak{J}_2$$

Consequently, the optimum choice is  $\mathfrak{J}_3$

The following are the main steps when utilizing Cr-PFPAAG operators:

Step 1: Examine step 1.

Step 2:  $\tilde{R} = (\tilde{v}_{i\mathcal{J}})_{5z4}$  ( $q = 1, 2, 3$ ) is transformed into a group picture fuzzy matrix of decisions.

$\tilde{R}' = (\tilde{v}'_{i\mathcal{J}})_{5z4}$  ( $q = 1, 2, 3$ ) by applying the Cr-PFPAAG operations to calculate all CPFVs decision-making.

Step 3:  $T'_{i\mathcal{J}}$  ( $i = 1, 2, 3 \dots, m$   $\mathcal{J} = 1, 2, 3 \dots, z$ ) should be aggregated based on Eqs. (19) and (20).

$$T'_{i\mathcal{J}} = \begin{pmatrix} 1 & 0.62 & 0.40 & 0.24, \\ 1 & 0.62 & 0.35 & 0.19 \\ 1 & 0.68 & 0.42 & 0.24 \\ 1 & 0.59 & 0.36 & 0.22 \\ 1 & 0.61 & 0.38 & 0.23 \end{pmatrix}$$

Step 4: All of the values of preference  $v'_i$  ( $i = 1, 2, 3, 4, 5$ ) on the line of  $\tilde{R}'$ , were aggregated using the Cr-PFPAAA operator to obtain the overall preference values  $v'_{i\mathcal{J}}$

$$\begin{aligned}
 \tilde{v}'_1 &= (0.72, 0.58, 0.03, 0.02) \\
 \tilde{v}'_2 &= (0.63, 0.52, 0.02, 0.02) \\
 \tilde{v}'_3 &= (0.77, 0.53, 0.03, 0.02) \\
 \tilde{v}'_4 &= (0.66, 0.51, 0.02, 0.01) \\
 \tilde{v}'_5 &= (0.61, 0.50, 0.01, 0.01)
 \end{aligned}$$

Step 5: Determine the score of  $v_i$  ( $i = 1, 2, 3, 4, 5$ ) in the following manner:

$$s_1 = 0.72 \quad s_2 = 0.63 \quad s_3 = 0.77 \quad s_4 = 0.66 \quad s_5 = 0.61$$

Since

$$s_3 > s_1 > s_4 > s_2 > s_5$$

We have  $\mathfrak{J}_3 > \mathfrak{J}_1 > \mathfrak{J}_4 > \mathfrak{J}_2 > \mathfrak{J}_5$ . Consequently, the optimum choice is  $\mathfrak{J}_3$ . As a result, the Cr-PFPAAA and Cr-PFPAAG operators receive different rankings of alternatives. The graphical representation of the aggregated result of the proposed work using the score function has been shown in Fig. 2.

### Results and discussion

The results of the study are analyzed based on the selection criteria established through a comprehensive literature search and feedback from potential players. The criteria were evaluated by a group of decision-makers  $s = \{s_1, s_2, s_3, s_4, s_5\}$  who assessed the performance of basketballs and selected the best alternative to maximize benefits. Each decision-maker assigned evaluations for each criterion, and the algorithm was applied to aggregate these evaluations. The algorithm stages, determined by the aforementioned criteria, ensured a systematic and objective assessment process.

The results indicated that basketballs with a higher balance of strength and agility scored the best overall, supporting the expectations of potential players. This analysis underscores the effectiveness of the proposed method in identifying high-quality basketballs while maximizing their suitability for players' needs.

By including this section, the study provides a healthy framework for evaluating performance data, supporting the validity of the proposed algorithm and its criteria-based decision-making process.

### Practical application of the suggested theory

The suggested Circular Prioritized Picture Fuzzy MADM can be efficiently applied to various real-world decision-making situations. For example, in the sports sector, it can assist in exhaustive quality checking of basketballs or other sporting goods for their adherence to global standards. In supplier selection issues, companies can employ this methodology to deal with conflicting factors such as cost, quality, and reliability in the context of the relative importance of experts' views. In project planning, the method can rank uncertain and incomplete alternative projects. It may also facilitate choosing treatment interventions in medical decision-making where expertise prioritization and uncertainty are imperative. The strategy can be of assistance in material selection for buildings with various qualities such as strength, sustainability, and cost having to be judged collectively under conditions of uncertainty. Therefore, the suggested approach has widespread practical applicability across various domains needing sophisticated decision analysis in fuzzy settings.

### Comparative analysis

In this section, we aim to contrast the newly developed aggregation operators with several existing methodologies to demonstrate their superiority and practical viability. Comparative analysis is not only essential for showcasing the advantages of a proposed theory but also a critical component of any valuable manuscript. It allows for a thorough understanding of how the new approach stands against established techniques in various scenarios. Finally, we focus on contrasting the presented procedures with several currently used techniques to evaluate the superiority and viability of the developed theory, because the comparative analysis is one of the best procedures and the main part of every valuable manuscript.

For this purpose, we utilize certain current theories to compare the presented operators with some existing operators.

1. The Picture fuzzy prioritized aczel–alsina weighted averaging (PFPAAWA) and Picture fuzzy prioritized aczel–alsina weighted geometric operator (PFPAAWG) which is proposed by Ijaz et al.<sup>83</sup>.
2. The Picture Fuzzy weighted averaging (PFWA) operator Wei<sup>84</sup> and Picture Fuzzy weighted geometric (PFWG) operator Wei<sup>84</sup>.
3. Picture fuzzy hybrid weighted averaging (PFHWA) operator Wei<sup>81</sup> and Picture fuzzy hybrid weighted geometric (PFHWG) operator Wei<sup>81</sup>.
4. Picture fuzzy frank weighted averaging (PFFWA) operator Seikh et al.<sup>41</sup> and Picture fuzzy frank weighted geometric (PFFWG) operator Seikh et al.<sup>41</sup>.
5. Picture fuzzy Dombi weighted averaging (PFDWA) operator Jana et al.<sup>85</sup> and Picture fuzzy Dombi weighted geometric (PFDWG) operator Jana et al.<sup>85</sup> are examples of existing methods that we compare with our suggested approach in the following section.

The comparison is summarized in Table 5. Using the data in Table 6, make a side-by-side comparison of the proposed and current operators.

### Advantages over traditional multi-criteria assessment tools in evaluating basketball quality

Compared with traditional multi-criteria assessment tools, the suggested CPPF technique has clear advantages for assessing basketball quality. Integrating DoM, DoNM, and hesitation through PFSSs, provides a more complex representation of the opinions of experts than traditional techniques. By carefully managing ambiguity and opposing needs, its advanced aggregation operators improve decision robustness, while its circular modeling method captures complex interactions among criteria like grip, bounce, and stability. Furthermore, the CPPF approach is appropriate for situations involving imprecise or incomplete information since it can adjust to different degrees of data quality. Because of these characteristics, CPPF offers a more comprehensive, accurate, and adaptable framework for evaluating basketball quality than other approaches.

Existing operators	Ranking results
Weighted averaging mean	0.72, 0.63, 0.77, 0.66, 0.61
Weighted geometric mean	0.49, 0.37, 0.55, 0.42, 0.39
PFPAWA <sup>83</sup>	Cannot handle prioritization among CPFVs
PFPAAG <sup>83</sup>	Not adaptable for circular picture fuzzy data
PFWA <sup>84</sup>	Designed for classical PFS, not CPFV
PFWG <sup>84</sup>	Geometric aggregation; no prioritization handling
PFHWA <sup>81</sup>	Hesitant fuzzy set focused; incompatible
PFHWG <sup>81</sup>	Hesitant framework mismatch
PFFWA	Fermatean fuzzy environment, not CPFV
PFFWG	Fermatean fuzzy environment, not CPFV
PFDWA	Designed for different divergence measures

**Table 5.** Comparative analysis of the proposed aggregation operators with current operators.

Proposed method	Cocoso method	EDAS method	Promethee method
3	2	1	2
1	1	2	1
2	3	3	3
4	5	4	5
5	4	5	4

**Table 6.** Comparative analysis of the proposed methods with different methods.

The findings with Cr-PFWA and Cr-PFWG reflect crucial differences in collective values as per the suggested procedure. Notably, the old procedures like PFWA, PFWG, PFHWA, PFHWG, PFFWA, PFFWG, and PFDWA are marked as “Not Applicable” as they do not support prioritized aggregation.

These prevailing methodologies do not account for attribute prioritization an essential need for professional basketball quality assessment, where, for instance, “durability” and “performance consistency” need to take precedence over aesthetic attributes such as “appearance.”

Therefore, the suggested CPFV-based operators with priority (Cr-PPFAAA and Cr-PPFAAG) capture real-world assessment requirements more accurately by including the different weights of the criteria in the decision-making. From the comparison, it is apparent that prioritization plays a crucial role in affecting the outcomes of decisions, as non-prioritized operators cannot identify.

### Advantages of the proposed method

The Cr-PPFAA-based decision-making model exhibits several strengths compared to current approaches. Firstly, the model is highly robust, as sensitivity analysis indicates little variation in alternative rankings under changes in major parameters. This suggests that the proposed method is less susceptible to subjective oscillations and ensures decision stability. Second, with the integration of probabilistic hesitant fuzzy data and prioritized aggregation and Aczel–Alsina operations, the model better captures expert uncertainty and preference systems, so there are more realistic and reliable outcomes. The Cr-PPFAA method yields more stable rankings and copes with conflicting criteria more effectively compared to the IF-MCDM approach, and thus it is highly applicable to complex decision-making contexts like international sports equipment evaluation. These strengths together emphasize the pragmatic utility and dependability of the suggested framework.

### Comparison with fuzzy madm models

To confirm the performance of the proposed method, we compared its results with fuzzy CoCoSo, fuzzy EDAS, and fuzzy PROMETHEE methods. Table 6 provides the ranking outcomes of every method. The proposed method generated rankings that were more in line with expert opinions than the other methods.

#### Observation

The suggested approach yields rankings very close to those of experts. Small differences were noted in fuzzy CoCoSo and PROMETHEE, especially for alternatives  $s_4$  and  $s_5$ .

### Conclusion

This study suggested an intelligent decision-making algorithm to systematically assess the quality of basketballs for international-level sports competitions. In consideration of the requirement for objective, transparent, and consistent assessment approaches, the study presented a new framework that combines expert judgments and multi-criteria evaluations into a single ranking system.

The central contribution of this paper is the establishment of the Cr-PPFAA model, which aggregates the merits of probabilistic hesitant fuzzy sets (PF), priority operators, and Aczel–Alsina (A–A) t-norm and t-conorm functions. This aggregation brings added flexibility and solidity, solving the unavoidable uncertainty and priority issues commonly found in real decision-making contexts.

Numerical results from the case study validated the performance and applicability of the new model. The Cr-PPFAA methodology indicated greater accuracy of decision-making compared to standard fuzzy MCDM. Primary quality attributes for basketballs e.g., durability, grip, consistency in bounce, consistency of weight, and quality of materials were analyzed in-depth, and the method effectively characterized the best of international standard matching basketball alternatives. Also, the model proved superior under sensitivity analysis for stability, solidifying its feasibility for actual practice.

In general, this research provides a valuable contribution to sports equipment evaluation with a scientifically based, expert-combined decision-making instrument. The presented Cr-PPFAA method is highly promising for more widespread applications where multi-criteria evaluation under uncertainty is needed, and follow-up studies could apply it to other equipment used in sports and real-time decision-support systems.

*Contribution:* We introduced two aggregation operators:

1. The Cr-PPFAA geometric operator with the priority degrees and the Cr-PPFAA averaging operator with the priority degrees. The principal feature of the proposed operators is their prioritization of the criteria; various desirable properties of these prioritized aggregation operators are explored.
2. The proposed Cr-PPFAA is explored to demonstrate that it is a generalization of the existing sets. Using the Cr-PPFAA features and numerical examples, the multi-criteria group decision-making approach that is pro-

vided takes into account the relationship between the decision-makers and the prioritization of these criteria, allowing for a wider range of practical applications.

3. We used the benefits, sensitive analysis, and geometrical expressions of the proposed work to prove the supremacy of the developed methods to determine the superiority and efficacy of the investigated operators.

In the future, we'll attempt to apply the aforementioned technique and broaden its scope to encompass more hazy scenarios. In the future, we aim to extend our. We can use it in PF<sup>86</sup>, SF and TSF<sup>87–89</sup> and interval-valued TSF<sup>90</sup> environments. We can also use it in Interval-valued complex single-valued neutrosophic<sup>91</sup>. Work to develop methods in several other fuzzy frameworks. We also aim for a T-spherical fuzzy hesitant fuzzy set. We can use it in PF<sup>34</sup> environments. We can also use it in Interval-valued complex single-valued neutrosophic<sup>35</sup> and interval 2-type fuzzy environments<sup>92</sup>. To identify gaps or propose extending the semi-Markov jump T-S fuzzy systems to new domains such as heritage preservation.

### Limitations and future work

Even though the suggested method proved robust performance, some limitations must be considered. Firstly, the method heavily depends on expert judgments for weight determination of criteria, and thus, subjectivity and possible bias can be introduced. Secondly, although the fuzzy approach efficiently handles uncertainty, it remains contingent on eliciting accurate linguistic judgments, which might differ between decision-makers. Third, the present validation was conducted on a limited number of case studies, which could limit the generalizability of the results. In future work, we plan to create hybrid models that minimize expert dependency by utilizing automated learning methods and validate the approach on broader and more diverse application domains.

### Data availability

The datasets used and/or analyzed during the current study are available from the corresponding author on reasonable request.

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