



OPEN Investigation of soliton solutions to the (2 + 1)-dimensional stochastic chiral nonlinear Schrödinger equation with bifurcation, sensitivity and chaotic analysis

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The stochastic chiral nonlinear Schrödinger equation has real life applications in developing advanced optical communication systems, involving description of wave propagation in noisy, chiral fiber networks. In the present study, the (2 + 1)-dimensional stochastic chiral nonlinear Schrödinger equation is investigated using two different formats of the generalized Kudryashov method. A variety of soliton solutions, such as kink, anti-kink, periodic, M-shaped, W-shaped, and V-shaped patterns, are derived, showing the graphical behavior of the system. Achieved solutions are verified with the use of Mathematica software. For further investigation to these solutions, 2D, 3D, and contour graphs are shown to graphically represent the corresponding solutions. Moreover, Bifurcation analysis is performed to investigate the qualitative changes in the dynamics of the system. Chaotic behaviour and sensitivity analysis are also investigated, highlighting the stochastic system's complexity. Additional determination of chaotic paths is carried out by 2D and 3D graphs and time series analysis. The findings provide valuable theoretical insights into chiral nonlinear systems under unexpected causes and provide useful analytical methods and visual models for future studies in nonlinear wave propagation, optical physics, and complex dynamical systems.

Keywords Stochastic Chiral nonlinear Schrödinger equation, Generalized Kudryashov Method, Bifurcation analysis, Chaotic & Sensitivity Analysis, Solitons.

The chaotic models of nonlinear partial differential equations (NLPDEs) can explain the mathematical prototypes of number of experiences in the family of physical sciences, including biology, nonlinear optics, mechanics, economy and current physics^{1–5}. In many different fields, the study of natural phenomena depends on exact solutions of NLPDEs. Therefore, the modern development of physical science, due to importance, intensified the search for solutions to nonlinear equations through various fields of sciences. There are lot of methods proposed to get exact solutions of NLPDEs, like Hirota method, $\frac{G'}{G}$ -expansion method⁶, sine-cosine method, the homotopy perturbation method⁷, the tanh-sech method, Jacobi elliptical function method, perturbation method, inverse scattering method, Petrov-Kudrin-Xiong method, variational iteration method⁸, Ricatti Bernoulli sub-ODE method, the $\exp(-\phi)$ -expansion method, the first integral method⁹, the Sardar sub-equation method¹⁰, the modified Khater method¹¹ and numerous others^{12–14}.

In order to retain the vast variety of rules in the nonlinear society, the nonlinear Schrödinger Equation (NSE)^{15–19} is essential for identifying the wave characteristics and exhibiting the solitary type solutions. In systems where nonlinearity can not be disregarded, the NLSE is an essential tool for simulating wave dynamics. For the purpose of comprehending wave propagation in nonlinear media, particularly in domains such as fluid dynamics, plasma physics, biophysics, quantum mechanics and nonlinear optics, it captures complex procedures including soliton generation, pulse distortion and interactions. NLPDEs are ideal mathematical

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models of complex dynamical systems when random effects are present. Compared to predictable equations, solving stochastic equations is more challenging because of unexpected arbitrary errors.

A Stochastic differential equation (SDE) is a type of differential equation where one or more of components of SDE are a kind of stochastic activities that result in solution or outcome that can be defined as stochastic process. SDEs can frequently generate a variety of phenomena, such as fluctuating market prices or somatic systems associated with thermal fluctuations.

The $(2 + 1)$ -dimensional stochastic chiral nonlinear Schrödinger equation (SCNLSE) is an extended model which governs the dynamics of complex wave fields in higher spatial dimension systems with nonlinear coupling, chiral dispersion, and stochastic fluctuations. It generalizes the NLSE by adding spatial anisotropy and chiral terms that are commonly employed to study wave propagation in nonlinear optics, quantum fluids, and Bose-Einstein condensates. The stochastic part, commonly represented as Gaussian noise, addresses randomness due to the environment, like thermal or quantum fluctuations. The equation is typically non-integrable and presents serious analytical difficulties, with research currently aiming to study it numerically, using perturbation theory, or statistical approaches. Since despite its importance in contemporary physics, analytical approaches such as the generalized kudryashov method have not yet been used to study this equation, a possible research venue with new findings remains.

In this paper, we consider the $(2 + 1)$ -dimensional Stochastic Chiral nonlinear Schrödinger (SCNS) equation, which is given in the following form²⁰

$$i\xi_t + a(\xi_{xx} + \xi_{yy}) + i(b_1(\xi\xi_x^* - \xi^*\xi_x) + b_2(\xi\xi_y^* - \xi^*\xi_y))\xi + \sigma\xi\Psi_t = 0, \quad (1.1)$$

where $\xi = \xi(x, y, t)$ is a complex function, * represents the complex conjugate, a is the second-order dispersion coefficients, b_1 and b_2 are self-steepening coefficients, σ is the noise strength and $\Psi(t) = \frac{d\Psi}{dt}$ is the time derivative of Brownian motion $\Psi(t)$.

Numerous applications of the above equation can be found in quantum optics, which studies prototypes that are limited by certain measurement results. The extended direct algebraic, extended trial equation methods, the $\exp(-\phi)$ -expansion method²¹, improved modified extended tanh-function method²², the trial solution method, the first integral method and functional variable method²³, the modified Jacobi elliptic expansion method²⁴, the extended Fan sub-equation technique²⁵ and the extended rational sine-cosine/sinh-cosh methods²⁶ are some of the methods that have been used by various researchers to construct the exact solutions to Eq. (1.1) with $\sigma = 0$ (without stochastic term). However, the stochastic factor in Eq. (1.1) has never been examined before.

The standard generalized Kudryashov method is an analytical approach developed to determine the exact traveling wave solutions to nonlinear equations. It accomplishes this through transforming the original nonlinear PDE to the form of an ODE through a wave variable, and then assuming the solution in the form of a polynomial function of a function that solves a simpler auxiliary equation (usually a Bernoulli or Riccati equation). By balancing the highest order derivatives and nonlinear terms, the approach simplifies the problem into solving algebraic equations for unspecified coefficients. The method is appreciated for its effectiveness and versatility in application across many nonlinear models in physics and engineering, allowing one to derive soliton, periodic, and rational solutions. Moreover, the modified Kudryashov method advances on this basis by improving the ansatz for the solution and auxiliary equation employed, commonly using more symmetrical or general functions. It also usually involves more systematic balancing process and can handle higher-order nonlinear PDEs. This approach transforms the nonlinear PDE directly into an algebraic system and is particularly useful in building new exact traveling wave solutions of complicated nonlinear equations. It is efficient in computation and simple to implement with symbolic software, thus proving to be a useful tool for studying nonlinear wave phenomena in mathematical physics. In short, both forms of this method works to translate complicated nonlinear PDEs into solvable algebraic problems, with second format of the method is providing increased flexibility and applicability to a broader category of nonlinear equations, thereby widening the scope of exact solutions to be used in theoretical and practical research.

In this present research, we intended to find some important soliton solutions to $(2 + 1)$ -dimensional SCNS equation by using two different formats of generalized kudryashov method (gKM)²⁷. Due to its nonlinearity, unpredictability and two spatial dimensions, the $(2 + 1)$ -dimensional SCNS equation is difficult to solve directly. It is reduced to a manageable ODE using the Kudryashov approach, which greatly facilitates the search for solutions. This method has never been applied on the mentioned model yet. This model is highly significant to the $(2 + 1)$ -dimensional SCNS equation because it gives a systematic, exact, and computationally effective method to obtain traveling wave solutions like kink, anti-kink, and multi-solitons of complex, nonlinear and stochastic models, which are challenging to solve analytically otherwise. Its novelty is that it can transform difficult multi-dimensional nonlinear PDEs into solvable algebraic systems by a systematic ansatz, making it adaptable to both deterministic and stochastic situations, as well as providing means of understanding wave stability and propagation under the influence of noise. Its flexibility and stability render gKM a useful tool for ongoing as well as future research, revealing new possibilities toward examining and anticipating nonlinear phenomena in quantum optics, chiral media and other high-level physical systems.

This is the format used for the rest of the paper. Our mathematical study of suggested $(2 + 1)$ -dimensional SCNS equation using the logarithmic transformation technique is presented in Sect. [Mathematical modeling](#). The SNS equation has been examined by using different variations of gKM in Sects. [Description of the methods](#) and [Application of Generalized Kudryashov methods](#). In Sect. [Qualitative dynamical analysis](#), qualitative analysis is conducted applying bifurcation analysis, studying chaotic behavior and sensitivity analysis to SNS equation, also we represent graphs of 3D, 2D and time series to demonstrate their physical interpretation. In Sect. [Physical significance of proposed study](#), physical interpretation of graphs, solutions and model is studied and limitations

and comparison of different formats of gKM methods are discussed in Sect. [Limitation of methods](#). Finally, in the last Sect. [Conclusion](#), conclusion is presented.

Mathematical modeling

The generalized Kudryashov method is implemented effectively to obtain unique but precise form of Eq. (1.1). In this section, some new results of solitary wave are obtained. Consider the wave transformation as:

$$\begin{aligned}\xi(x, y, t) &= \Omega(\rho) \exp(\iota(\theta_1 x + \theta_2 y + \theta_3 t - \sigma \Psi(t)), \\ \rho &= c_1 x + c_2 y + c_3 t.\end{aligned}\quad (2.1)$$

Here, $c_1, c_2, c_3, \theta_1, \theta_2$ and θ_3 are nonzero constants. The proposed model (1.1) is remodeled into ODE by Eq. (2.1). Now put (2.1) into (1.1) yields the imaginary and real part respectively as follows

$$\begin{aligned}c_3 + 2a(c_1\theta_1 + c_2\theta_2) &= 0, \\ a(c_1^2 + c_2^2)\Omega'' - 2(b_1\theta_1 + b_2\theta_2)\Omega^3 - (\theta_3 + a(\theta_1^2 + \theta_2^2))\Omega &= 0.\end{aligned}$$

Divide above equation by $a(c_1^2 + c_2^2)$ gives

$$\Omega'' - M_1\Omega^3 - M_2\Omega = 0. \quad (2.2)$$

In this equation, $M_1 = \frac{2(b_1\theta_1 + b_2\theta_2)}{a(c_1^2 + c_2^2)}$ and $M_2 = \frac{\theta_3 + a(\theta_1^2 + \theta_2^2)}{a(c_1^2 + c_2^2)}$. By applying mentioned methods, the solutions of (2 + 1)-dimensional SCNS Eq. (1.1) are obtained. The general discussion of proposed methods is given below:

Description of the methods

There are several formats or versions of the Generalized Kudryashov Method depending on the structure and selection of the auxiliary function and the solution ansatz. Here, two different formats are presented to obtain solutions.

Generalized Kudryashov method-1

Lets consider nonlinear partial differential equation

$$\chi_1(\xi, \xi_x, \xi_t, \xi_{xt}, \xi_{xx}, \xi_{tt} \dots) = 0, \quad (3.1)$$

where χ is a Polynomial in $\xi = \xi(x, t)$ and its partial derivatives.

The main steps of method are listed below²⁸,

Step-1 Assume the transformation

$$\xi(x, t) = \Omega(\rho), \rho = kx + vt + d,$$

where v is the wave speed and k and d are constants. Now, reduce (3.1) to a nonlinear ODE as

$$\zeta_1(\Omega, \Omega', \Omega'', \Omega''' \dots) = 0. \quad (3.2)$$

Step-2 The solution to Eq. (3.2) can be expressed by specific form that we take

$$\Omega(\rho) = \frac{\sum_{j=0}^S \alpha_j \phi^j(\rho)}{\sum_{i=0}^T \beta_i \phi^i(\rho)}, \quad (3.3)$$

where α_j and β_i are constants to be determined later such that $\alpha_S \neq 0, \beta_T \neq 0$ and $\phi(\rho) = \frac{1}{1+e^{\pm\rho}}$. The first-order Bernoulli differential equation must therefore be satisfied by the function ϕ :

$$\phi'(\rho) = \phi^2(\rho) - \phi(\rho). \quad (3.4)$$

Step-3 By balancing the nonlinear terms in Eq. (3.2) and the highest order derivatives, find the positive integer values S and T in Eq. (3.3).

Step-4 Substituting Eqs. (3.3) and (3.4) into Eq. (3.2), the polynomial in ϕ^{i-j} is obtained. Now set all the coefficients of this polynomial to zero, a system is left with algebraic equations that may be solved using the software programs Mathematica or Maple to determine the unknown parameters $\alpha_j (j = 0, 1, 2, 3, \dots, S)$ and $\beta_i (i = 0, 1, 2, 3, \dots, T)$. Hence, solutions of Eq. (3.1) are obtained.

Generalized Kudryashov method-2

Consider the following nonlinear partial differential equation

$$\chi_1(\xi, \xi_x, \xi_t, \xi_{xt}, \xi_{xx}, \xi_{tt} \dots) = 0,$$

where χ is a Polynomial in $\xi = \xi(x, t)$ and its partial derivatives.

The main steps of method are listed below²⁹,

Step-1 Assume the transformation

$$\xi(x, t) = \Omega(\rho), \rho = kx - vt,$$

where v is the speed of wave and k is constant. Now, reduce (3.1) to a nonlinear ODE as

$$\zeta_1(\Omega, \Omega', \Omega'', \Omega''' \dots) = 0.$$

Step-2 The solution to Eq. (3.2) can be expressed by specific form that is

$$\Omega(\rho) = \alpha_0 + \sum_{j=1}^S \frac{\alpha_j}{(1 + \psi(\rho))^j}, \quad (3.5)$$

where α_0 and α_j are constants to be determined later and ψ is the function of ρ which is the solution to general Riccati equation as follows:

$$\psi'(\rho) = z_1 + z_2\psi(\rho) + z_3\psi^2(\rho), \quad (3.6)$$

here z_1, z_2 and z_3 are constants. Following cases shows the solutions for Eq. (3.6):

Case 1:

If z_1, z_2 and z_3 are nonzero, then $\psi(\rho)$ is given by

$$\begin{aligned} \psi(\rho) &= \frac{1}{2z_3} (\sqrt{4z_1z_3 - z_2^2} \tan(\frac{1}{2}\sqrt{4z_1z_3 - z_2^2}(d_0 + \mu)) - z_2), 4z_1z_3 > z_2^2, \\ \psi(\rho) &= \frac{-1}{2z_3} (\sqrt{4z_1z_3 - z_2^2} \cot(\frac{1}{2}\sqrt{4z_1z_3 - z_2^2}(d_0 + \mu)) + z_2), 4z_1z_3 > z_2^2, \\ \psi(\rho) &= \frac{-1}{2z_3} (\sqrt{z_2^2 - 4z_1z_3} \tanh(\frac{1}{2}\sqrt{z_2^2 - 4z_1z_3}(d_0 + \mu)) + z_2), 4z_1z_3 < z_2^2, \\ \psi(\rho) &= \frac{-1}{2z_3} (\sqrt{z_2^2 - 4z_1z_3} \coth(\frac{1}{2}\sqrt{z_2^2 - 4z_1z_3}(d_0 + \mu)) + z_2), 4z_1z_3 < z_2^2, \\ \psi(\rho) &= \frac{-1}{z_3} (\frac{1}{d_0 + \mu} + \frac{z_2}{2}), 4z_1z_3 = z_2^2. \end{aligned}$$

Case 2:

If $z_1 = 0$ and $z_3 \neq 0$, then

$$\begin{aligned} \psi(\rho) &= \frac{-1}{2z_3} (z_2 \tanh(\frac{z_2}{2}(d_0 + \mu) + z_2), z_2^2 > 0, \\ \psi(\rho) &= \frac{-1}{2z_3} (z_2 \coth(\frac{z_2}{2}(d_0 + \mu) + z_2), z_2^2 > 0, \\ \psi(\rho) &= \frac{z_2}{z_2 \exp(-z_2(d_0 + \mu)) - z_3}, z_2^2 \neq 0, \\ \psi(\rho) &= \frac{-1}{z_3\mu}, z_2^2 = 0. \end{aligned}$$

Case 3:

If $z_2 = 0$ and $z_3 \neq 0$, then

$$\begin{aligned} \psi(\rho) &= \frac{\sqrt{z_1z_3}}{z_3} \tan(\sqrt{z_1z_3}(d_0 + \mu)), z_1z_3 > 0, \\ \psi(\rho) &= -\frac{\sqrt{z_1z_3}}{z_3} \cot(\sqrt{z_1z_3}(d_0 + \mu)), z_1z_3 > 0, \\ \psi(\rho) &= \frac{\sqrt{-z_1z_3}}{z_3} \tanh(\sqrt{-z_1z_3}(d_0 + \mu)), z_1z_3 < 0, \\ \psi(\rho) &= \frac{\sqrt{-z_1z_3}}{z_3} \coth(\sqrt{-z_1z_3}(d_0 + \mu)), z_1z_3 < 0, \\ \psi(\rho) &= \frac{-1}{z_3(d_0 + \mu)}, z_1 = 0. \end{aligned}$$

Case 4:

If $z_3 = 0$ and $z_2 \neq 0$, then

$$\psi(\rho) = \frac{-1}{z_2} (\exp(z_2(d_0 + \mu)) - z_1).$$

Step-3

Substitute Eq. (3.2) into Eq. (3.5), and add the coefficients of same power of $\psi(\rho)$. Now assuming the coefficient of every power to be zero, a system of algebraic equations involving α_0 and $\alpha_j, (j = 1, 2, 3 \dots S)$ and other parameters is obtained. Solving this system with the help of Mathematica software gives the unknown's values.

Step-4

Now by substituting the values of α_0 and $\alpha_j, (j = 1, 2, 3 \dots S)$ into Eq. (3.5) and utilizing the appropriate cases of general Riccati equation provided from above equations, the solutions of the Eq. (3.1) are obtained.

Application of Generalized Kudryashov methods

Generalized Kudryashov Method-1

We have particular form of solution:

$$\Omega(\rho) = \frac{\sum_{j=0}^S \alpha_j \phi^j(\rho)}{\sum_{i=0}^T \beta_i \phi^i(\rho)}$$

Firstly, applying homogeneous balance method on Eq. (2.2). $S = T + 1$ comes out³⁰. Let $T = 1$, then S becomes 2.

Put $S = 2$ in this solution yields

$$\Omega(\rho) = \frac{\alpha_0 + \alpha_1 \phi(\rho) + \alpha_2 \phi(\rho)^2}{\beta_0 + \beta_1 \phi(\rho)} \tag{4.1}$$

Putting (4.1) into (2.2) gives the coefficients

$$\begin{aligned} \phi^6(\rho) &= 2\alpha_2\beta_1^2 - \alpha_2^3 M_1 = 0, \\ \phi^5(\rho) &= -3\alpha_2(-2\beta_0\beta_1 + \beta_1^2 + \alpha_1\alpha_2 M_1) = 0, \\ \phi^4(\rho) &= \alpha_2(6\beta_0^2 - 9\beta_0\beta_1 - 3(\alpha_1^2 + \alpha_0\alpha_2)M_1 - \beta_1^2(-1 + M_2)) = 0, \\ \phi^3(\rho) &= -\alpha_0\beta_1(2\beta_0 + \beta_1) - \alpha_1^3 M_1 + \alpha_2\beta_0(-10\beta_0 + \beta_1(3 - 2M_2)) + \alpha_1(\beta_0(2\beta_0 + \beta_1) - 6\alpha_0\alpha_2 M_1 - \beta_1^2 M_2) = 0, \\ \phi^2(\rho) &= -3\alpha_0\alpha_1^2 M_1 - \alpha_2(3\alpha_0^2 M_1 + \beta_0^2(-4 + M_2)) + \alpha_0\beta_1(3\beta_0 + \beta_1 - \beta_1 M_2) - \alpha_1\beta_0(3\beta_0 + \beta_1(1 + 2M_2)) = 0, \\ \phi(\rho) &= \alpha_1(-3\alpha_0^2 M_1 - \beta_0^2(-1 + M_2)) - \alpha_0\beta_0\beta_1(1 + 2M_2) = 0. \end{aligned}$$

Solving the above mentioned set of equations yields following solutions:

Set-1

$$\alpha_0 = \pm \frac{i\beta_0\sqrt{M_2}}{\sqrt{M_1}}, \alpha_1 = \pm \frac{\sqrt{2}\beta_0}{\sqrt{M_1}}, \alpha_2 = \pm \frac{\sqrt{2}\beta_1}{\sqrt{M_1}}, \beta_0 = \beta_0, \beta_1 = \beta_1.$$

Set-2

$$\alpha_0 = \pm \frac{i\beta_0\sqrt{M_2}}{\sqrt{M_1}}, \alpha_1 = \alpha_1, \alpha_2 = \alpha_2, \beta_0 = \pm \frac{1}{2}(\beta_1 - \sqrt{2}\alpha_1\sqrt{M_1}), \beta_1 = \pm \frac{\alpha_2\sqrt{M_1}}{\sqrt{2}}.$$

Put above set of solutions into (4.1) with $\phi(\rho) = \frac{1}{1+e^{\pm\rho}}$, Solutions of Eq. (2.1) are:

$$\xi_1(x, y, t) = \left(\frac{\left(\frac{\sqrt{2}\beta_0}{(1+e^{\pm\rho})\sqrt{M_1}} \right) + \left(\frac{\sqrt{2}\beta_1}{(1+e^{\pm\rho})^2\sqrt{M_1}} \right) + \frac{i\beta_0\sqrt{M_2}}{\sqrt{M_1}}}{\beta_0 + \frac{\beta_1}{1+e^{\pm\rho}}} \right) e^{i(\theta_1 x + \theta_2 y + \theta_3 t - \sigma\Psi(t))}, \tag{4.2}$$

Where $\rho = c_1 x + c_2 y + c_3 t, c_3 = -2a(c_1\theta_1 + c_2\theta_2), M_1 = \frac{2(b_1\theta_1 + b_2\theta_2)}{a(c_1^2 + c_2^2)}$ and $M_2 = \frac{\theta_3 + a(\theta_1^2 + \theta_2^2)}{a(c_1^2 + c_2^2)}$.

$$\xi_2(x, y, t) = \left(\frac{\left(\frac{\alpha_1}{1+e^{\pm\rho}} \right) + \left(\frac{\alpha_2}{(1+e^{\pm\rho})^2} \right) + \frac{i(-\sqrt{2}\alpha_1\sqrt{M_1} + \frac{\alpha_2\sqrt{M_1}}{\sqrt{2}})\sqrt{M_2}}{2\sqrt{M_1}}}{\frac{1}{2}(-\sqrt{2}\alpha_1\sqrt{M_1} + \frac{\alpha_2\sqrt{M_1}}{\sqrt{2}}) + \frac{\alpha_2\sqrt{M_1}}{\sqrt{2}(1+e^{\pm\rho})}} \right) e^{i(\theta_1 x + \theta_2 y + \theta_3 t - \sigma\Psi(t))}, \tag{4.3}$$

Where $\rho = c_1 x + c_2 y + c_3 t, c_3 = -2a(c_1\theta_1 + c_2\theta_2), M_1 = \frac{2(b_1\theta_1 + b_2\theta_2)}{a(c_1^2 + c_2^2)}$ and $M_2 = \frac{\theta_3 + a(\theta_1^2 + \theta_2^2)}{a(c_1^2 + c_2^2)}$.

Generalized Kudryashov method-2

Applying homogeneous balance method on Eq. (2.2), $S = 1$ comes out. For $S = 1$, Eq. (3.5) becomes

$$\Omega(\rho) = \alpha_0 + \frac{\alpha_1}{(1 + \psi(\rho))}, \tag{4.4}$$

here, α_0 and α_1 are unknown constants. Now put Eq. (4.4) into Eq. (2.2) with Eq. (3.6) and collect all coefficients of same power of $\psi(\rho)$, the algebraic equation involving α_0, α_1 , and more parameters is obtained. Now, with Wolfram Mathematica, following sets of solution arises:

Set A:

$$\left\{ \alpha_0 = \frac{\sqrt{\theta_3}}{\sqrt{2}\sqrt{-b_1\theta_1 - b_2\theta_2}}, \alpha_1 = \frac{\alpha_0 (2b_1\alpha_0^2\theta_1 + a\theta_1^2 + 2b_2\alpha_0^2\theta_2 + a\theta_2^2 + \theta_3)}{a(c_1^2 + c_2^2)(z_2 - 2z_3)z_3}, a = \frac{-2b_1\alpha_0^2\theta_1 - 2b_2\alpha_0^2\theta_2 - \theta_3}{\theta_1^2 + \theta_2^2} \right\}.$$

By using above set of solution, following cases for solutions of Eq. (2.1) arises:

Case-1

$$\begin{aligned} \xi_{A11}(x, y, t) = & \frac{e^{i(-\gamma(xp_1+yp_2+tp_3)+x\theta_1+y\theta_2+t\theta_3)}\sqrt{\theta_3}}{\sqrt{2}\sqrt{-b_1\theta_1 - b_2\theta_2}} \\ & + \frac{e^{i(-\gamma(xp_1+yp_2+tp_3)+x\theta_1+y\theta_2+t\theta_3)}}{(\sqrt{2}(c_1^2 + c_2^2)(z_2 - 2z_3)z_3\sqrt{-b_1\theta_1 - b_2\theta_2}(-\theta_3 - \frac{b_1\theta_1\theta_3}{-b_1\theta_1 - b_2\theta_2} - \frac{b_2\theta_2\theta_3}{-b_1\theta_1 - b_2\theta_2}))} \\ & \times \frac{(\theta_1^2 + \theta_2^2)\sqrt{\theta_3}}{\left(1 + \frac{1}{2z_3} \left[-z_2 + \sqrt{-z_2^2 + 4z_1z_3} \tan\left(\mu + \frac{1}{2} \left(\sqrt{-z_2^2 + 4z_1z_3}\right) d_0\right)\right]\right)} \\ & \times \theta_3 + \frac{b_1\theta_1\theta_3}{-b_1\theta_1 - b_2\theta_2} + \frac{b_2\theta_2\theta_3}{-b_1\theta_1 - b_2\theta_2} \\ & + \frac{\theta_1^2 \left(-\theta_3 - \frac{b_1\theta_1\theta_3}{-b_1\theta_1 - b_2\theta_2} - \frac{b_2\theta_2\theta_3}{-b_1\theta_1 - b_2\theta_2}\right)}{\theta_1^2 + \theta_2^2} + \frac{\theta_2^2 \left(-\theta_3 - \frac{b_1\theta_1\theta_3}{-b_1\theta_1 - b_2\theta_2} - \frac{b_2\theta_2\theta_3}{-b_1\theta_1 - b_2\theta_2}\right)}{\theta_1^2 + \theta_2^2}, \end{aligned} \tag{4.5}$$

$$\begin{aligned} \xi_{A12}(x, y, t) = & \frac{e^{i(-\gamma(xp_1+yp_2+tp_3)+x\theta_1+y\theta_2+t\theta_3)}\sqrt{\theta_3}}{\sqrt{2}\sqrt{-b_1\theta_1 - b_2\theta_2}} \\ & + \frac{e^{i(-\gamma(xp_1+yp_2+tp_3)+x\theta_1+y\theta_2+t\theta_3)}}{(\sqrt{2}(c_1^2 + c_2^2)(z_2 - 2z_3)z_3\sqrt{-b_1\theta_1 - b_2\theta_2}(-\theta_3 - \frac{b_1\theta_1\theta_3}{-b_1\theta_1 - b_2\theta_2} - \frac{b_2\theta_2\theta_3}{-b_1\theta_1 - b_2\theta_2}))} \\ & \times \frac{(\theta_1^2 + \theta_2^2)\sqrt{\theta_3}}{\left(1 + \frac{1}{2z_3} \left[-z_2 + \sqrt{-z_2^2 + 4z_1z_3} \cot\left(\mu + \frac{1}{2} \left(\sqrt{-z_2^2 + 4z_1z_3}\right) d_0\right)\right]\right)} \\ & \times \theta_3 + \frac{b_1\theta_1\theta_3}{-b_1\theta_1 - b_2\theta_2} + \frac{b_2\theta_2\theta_3}{-b_1\theta_1 - b_2\theta_2} \\ & + \frac{\theta_1^2 \left(-\theta_3 - \frac{b_1\theta_1\theta_3}{-b_1\theta_1 - b_2\theta_2} - \frac{b_2\theta_2\theta_3}{-b_1\theta_1 - b_2\theta_2}\right)}{\theta_1^2 + \theta_2^2} + \frac{\theta_2^2 \left(-\theta_3 - \frac{b_1\theta_1\theta_3}{-b_1\theta_1 - b_2\theta_2} - \frac{b_2\theta_2\theta_3}{-b_1\theta_1 - b_2\theta_2}\right)}{\theta_1^2 + \theta_2^2}, \end{aligned} \tag{4.6}$$

$$\begin{aligned} \xi_{A13}(x, y, t) = & \frac{e^{i(x\theta_1+y\theta_2-\gamma(xp_1+yp_2-2t\lambda(p_1\theta_1+p_2\theta_2))+t\theta_3)}\sqrt{\theta_3}}{\sqrt{2}\sqrt{-b_1\theta_1 - b_2\theta_2}} \\ & + \frac{e^{i(x\theta_1+y\theta_2-\gamma(xp_1+yp_2-2t\lambda(p_1\theta_1+p_2\theta_2))+t\theta_3)}}{(\sqrt{2}(c_1^2 + c_2^2)(z_2 - 2z_3)z_3\sqrt{-b_1\theta_1 - b_2\theta_2}(-\theta_3 - \frac{b_1\theta_1\theta_3}{-b_1\theta_1 - b_2\theta_2} - \frac{b_2\theta_2\theta_3}{-b_1\theta_1 - b_2\theta_2}))} \\ & \times \frac{(\theta_1^2 + \theta_2^2)\sqrt{\theta_3}}{\left(1 - \frac{1}{2z_3} \left[z_2 + \sqrt{z_2^2 - 4z_1z_3} \tanh\left(\mu + \frac{1}{2} \left(\sqrt{z_2^2 - 4z_1z_3}\right) d_0\right)\right]\right)} \\ & \times \theta_3 + \frac{b_1\theta_1\theta_3}{-b_1\theta_1 - b_2\theta_2} + \frac{b_2\theta_2\theta_3}{-b_1\theta_1 - b_2\theta_2} \\ & + \frac{\theta_1^2 \left(-\theta_3 - \frac{b_1\theta_1\theta_3}{-b_1\theta_1 - b_2\theta_2} - \frac{b_2\theta_2\theta_3}{-b_1\theta_1 - b_2\theta_2}\right)}{\theta_1^2 + \theta_2^2} + \frac{\theta_2^2 \left(-\theta_3 - \frac{b_1\theta_1\theta_3}{-b_1\theta_1 - b_2\theta_2} - \frac{b_2\theta_2\theta_3}{-b_1\theta_1 - b_2\theta_2}\right)}{\theta_1^2 + \theta_2^2}, \end{aligned} \tag{4.7}$$

$$\begin{aligned} \xi_{A14}(x, y, t) = & \frac{e^{i(x\theta_1+y\theta_2-\gamma(xp_1+yp_2-2t\lambda(p_1\theta_1+p_2\theta_2))+t\theta_3)}\sqrt{\theta_3}}{\sqrt{2}\sqrt{-b_1\theta_1-b_2\theta_2}} \\ & + \frac{e^{i(x\theta_1+y\theta_2-\gamma(xp_1+yp_2-2t\lambda(p_1\theta_1+p_2\theta_2))+t\theta_3)}}{(\sqrt{2}(c_1^2+c_2^2)(z_2-2z_3)z_3\sqrt{-b_1\theta_1-b_2\theta_2}\left(-\theta_3-\frac{b_1\theta_1\theta_3}{-b_1\theta_1-b_2\theta_2}-\frac{b_2\theta_2\theta_3}{-b_1\theta_1-b_2\theta_2}\right))} \\ & \times \frac{(\theta_1^2+\theta_2^2)\sqrt{\theta_3}}{\left(1-\frac{1}{2z_3}\left[z_2+\sqrt{z_2^2-4z_1z_3}\coth\left(\mu+\frac{1}{2}\left(\sqrt{z_2^2-4z_1z_3}\right)d_0\right)\right]\right)} \\ & \times \theta_3 + \frac{b_1\theta_1\theta_3}{-b_1\theta_1-b_2\theta_2} + \frac{b_2\theta_2\theta_3}{-b_1\theta_1-b_2\theta_2} \\ & + \frac{\theta_1^2\left(-\theta_3-\frac{b_1\theta_1\theta_3}{-b_1\theta_1-b_2\theta_2}-\frac{b_2\theta_2\theta_3}{-b_1\theta_1-b_2\theta_2}\right)}{\theta_1^2+\theta_2^2} + \frac{\theta_2^2\left(-\theta_3-\frac{b_1\theta_1\theta_3}{-b_1\theta_1-b_2\theta_2}-\frac{b_2\theta_2\theta_3}{-b_1\theta_1-b_2\theta_2}\right)}{\theta_1^2+\theta_2^2}, \end{aligned} \tag{4.8}$$

$$\begin{aligned} \xi_{A15}(x, y, t) = & \frac{e^{i(-\gamma(xp_1+yp_2+tp_3)+x\theta_1+y\theta_2+t\theta_3)}\sqrt{\theta_3}}{\sqrt{2}\sqrt{-b_1\theta_1-b_2\theta_2}} \\ & + \frac{e^{i(-\gamma(xp_1+yp_2+tp_3)+x\theta_1+y\theta_2+t\theta_3)}}{(\sqrt{2}(c_1^2+c_2^2)(z_2-2z_3)z_3\sqrt{-b_1\theta_1-b_2\theta_2}\left(-\theta_3-\frac{b_1\theta_1\theta_3}{-b_1\theta_1-b_2\theta_2}-\frac{b_2\theta_2\theta_3}{-b_1\theta_1-b_2\theta_2}\right))} \\ & \times \frac{(\theta_1^2+\theta_2^2)\sqrt{\theta_3}}{\left(1+\frac{z_2}{2}-\frac{1}{z_3}\left(\frac{1}{\mu+d_0}\right)\right)} \times \theta_3 + \frac{b_1\theta_1\theta_3}{-b_1\theta_1-b_2\theta_2} + \frac{b_2\theta_2\theta_3}{-b_1\theta_1-b_2\theta_2} \\ & + \frac{\theta_1^2\left(-\theta_3-\frac{b_1\theta_1\theta_3}{-b_1\theta_1-b_2\theta_2}-\frac{b_2\theta_2\theta_3}{-b_1\theta_1-b_2\theta_2}\right)}{\theta_1^2+\theta_2^2} + \frac{\theta_2^2\left(-\theta_3-\frac{b_1\theta_1\theta_3}{-b_1\theta_1-b_2\theta_2}-\frac{b_2\theta_2\theta_3}{-b_1\theta_1-b_2\theta_2}\right)}{\theta_1^2+\theta_2^2}. \end{aligned} \tag{4.9}$$

Case-2

$$\begin{aligned} \xi_{A21}(x, y, t) = & \frac{e^{i(x\theta_1+y\theta_2-\gamma(xp_1+yp_2-2t\lambda(p_1\theta_1+p_2\theta_2))+t\theta_3)}\sqrt{\theta_3}}{\sqrt{2}\sqrt{-b_1\theta_1-b_2\theta_2}} \\ & + \frac{e^{i(x\theta_1+y\theta_2-\gamma(xp_1+yp_2-2t\lambda(p_1\theta_1+p_2\theta_2))+t\theta_3)}}{\sqrt{2}(c_1^2+c_2^2)(z_2-2z_3)z_3\sqrt{-b_1\theta_1-b_2\theta_2}\left(-\theta_3-\frac{b_1\theta_1\theta_3}{-b_1\theta_1-b_2\theta_2}-\frac{b_2\theta_2\theta_3}{-b_1\theta_1-b_2\theta_2}\right)} \\ & \times \frac{(\theta_1^2+\theta_2^2)\sqrt{\theta_3}}{\left(1-\frac{1}{2z_3}\left[z_2\tanh\left(z_2+\frac{z_2}{2}(\mu+d_0)\right)\right]\right)} \times \theta_3 + \frac{b_1\theta_1\theta_3}{-b_1\theta_1-b_2\theta_2} + \frac{b_2\theta_2\theta_3}{-b_1\theta_1-b_2\theta_2} \\ & + \frac{\theta_1^2\left(-\theta_3-\frac{b_1\theta_1\theta_3}{-b_1\theta_1-b_2\theta_2}-\frac{b_2\theta_2\theta_3}{-b_1\theta_1-b_2\theta_2}\right)}{\theta_1^2+\theta_2^2} + \frac{\theta_2^2\left(-\theta_3-\frac{b_1\theta_1\theta_3}{-b_1\theta_1-b_2\theta_2}-\frac{b_2\theta_2\theta_3}{-b_1\theta_1-b_2\theta_2}\right)}{\theta_1^2+\theta_2^2}, \end{aligned} \tag{4.10}$$

$$\begin{aligned} \xi_{A22}(x, y, t) = & \frac{e^{i(x\theta_1+y\theta_2-\gamma(xp_1+yp_2-2t\lambda(p_1\theta_1+p_2\theta_2))+t\theta_3)}\sqrt{\theta_3}}{\sqrt{2}\sqrt{-b_1\theta_1-b_2\theta_2}} \\ & + \frac{e^{i(x\theta_1+y\theta_2-\gamma(xp_1+yp_2-2t\lambda(p_1\theta_1+p_2\theta_2))+t\theta_3)}}{\sqrt{2}(c_1^2+c_2^2)(z_2-2z_3)z_3\sqrt{-b_1\theta_1-b_2\theta_2}\left(-\theta_3-\frac{b_1\theta_1\theta_3}{-b_1\theta_1-b_2\theta_2}-\frac{b_2\theta_2\theta_3}{-b_1\theta_1-b_2\theta_2}\right)} \\ & \times \frac{(\theta_1^2+\theta_2^2)\sqrt{\theta_3}}{\left(1-\frac{1}{2z_3}\left[z_2\coth\left(z_2+\frac{z_2}{2}(\mu+d_0)\right)\right]\right)} \times \theta_3 + \frac{b_1\theta_1\theta_3}{-b_1\theta_1-b_2\theta_2} + \frac{b_2\theta_2\theta_3}{-b_1\theta_1-b_2\theta_2} \\ & + \frac{\theta_1^2\left(-\theta_3-\frac{b_1\theta_1\theta_3}{-b_1\theta_1-b_2\theta_2}-\frac{b_2\theta_2\theta_3}{-b_1\theta_1-b_2\theta_2}\right)}{\theta_1^2+\theta_2^2} + \frac{\theta_2^2\left(-\theta_3-\frac{b_1\theta_1\theta_3}{-b_1\theta_1-b_2\theta_2}-\frac{b_2\theta_2\theta_3}{-b_1\theta_1-b_2\theta_2}\right)}{\theta_1^2+\theta_2^2}, \end{aligned} \tag{4.11}$$

$$\begin{aligned} \xi_{A23}(x, y, t) = & \frac{e^{i(x\theta_1+y\theta_2-\gamma(xp_1+yp_2-2t\lambda(p_1\theta_1+p_2\theta_2))+t\theta_3)}\sqrt{\theta_3}}{\sqrt{2}\sqrt{-b_1\theta_1-b_2\theta_2}} \\ & + \frac{e^{i(x\theta_1+y\theta_2-\gamma(xp_1+yp_2-2t\lambda(p_1\theta_1+p_2\theta_2))+t\theta_3)}}{\sqrt{2}(c_1^2+c_2^2)(z_2-2z_3)z_3\sqrt{-b_1\theta_1-b_2\theta_2}\left(-\theta_3-\frac{b_1\theta_1\theta_3}{-b_1\theta_1-b_2\theta_2}-\frac{b_2\theta_2\theta_3}{-b_1\theta_1-b_2\theta_2}\right)} \\ & \times \frac{(\theta_1^2+\theta_2^2)\sqrt{\theta_3}}{\left(1+\frac{z_2}{e^{-z_2[\mu+d_0]}z_2-z_3}\right)} \times \theta_3 + \frac{b_1\theta_1\theta_3}{-b_1\theta_1-b_2\theta_2} + \frac{b_2\theta_2\theta_3}{-b_1\theta_1-b_2\theta_2} \\ & + \frac{\theta_1^2\left(-\theta_3-\frac{b_1\theta_1\theta_3}{-b_1\theta_1-b_2\theta_2}-\frac{b_2\theta_2\theta_3}{-b_1\theta_1-b_2\theta_2}\right)}{\theta_1^2+\theta_2^2} + \frac{\theta_2^2\left(-\theta_3-\frac{b_1\theta_1\theta_3}{-b_1\theta_1-b_2\theta_2}-\frac{b_2\theta_2\theta_3}{-b_1\theta_1-b_2\theta_2}\right)}{\theta_1^2+\theta_2^2}, \end{aligned} \tag{4.12}$$

$$\begin{aligned} \xi_{A35}(x, y, t) = & \frac{e^{i(x\theta_1+y\theta_2-\gamma(xp_1+yp_2-2t\lambda(p_1\theta_1+p_2\theta_2))+t\theta_3)}\sqrt{\theta_3}}{\sqrt{2}\sqrt{-b_1\theta_1-b_2\theta_2}} \\ & + \frac{e^{i(x\theta_1+y\theta_2-\gamma(xp_1+yp_2-2t\lambda(p_1\theta_1+p_2\theta_2))+t\theta_3)}}{\sqrt{2}(c_1^2+c_2^2)(z_2-2z_3)z_3\sqrt{-b_1\theta_1-b_2\theta_2}\left(-\theta_3-\frac{b_1\theta_1\theta_3}{-b_1\theta_1-b_2\theta_2}-\frac{b_2\theta_2\theta_3}{-b_1\theta_1-b_2\theta_2}\right)} \\ & \times \frac{(\theta_1^2+\theta_2^2)\sqrt{\theta_3}}{\left(1-\frac{1}{z_3[\mu+d_0]}\right)} \times \theta_3 + \frac{b_1\theta_1\theta_3}{-b_1\theta_1-b_2\theta_2} + \frac{b_2\theta_2\theta_3}{-b_1\theta_1-b_2\theta_2} \\ & + \frac{\theta_1^2\left(-\theta_3-\frac{b_1\theta_1\theta_3}{-b_1\theta_1-b_2\theta_2}-\frac{b_2\theta_2\theta_3}{-b_1\theta_1-b_2\theta_2}\right)}{\theta_1^2+\theta_2^2} + \frac{\theta_2^2\left(-\theta_3-\frac{b_1\theta_1\theta_3}{-b_1\theta_1-b_2\theta_2}-\frac{b_2\theta_2\theta_3}{-b_1\theta_1-b_2\theta_2}\right)}{\theta_1^2+\theta_2^2}. \end{aligned} \tag{4.18}$$

Case-4

$$\begin{aligned} \xi_{A41}(x, y, t) = & \frac{e^{i(x\theta_1+y\theta_2-\gamma(xp_1+yp_2-2t\lambda(p_1\theta_1+p_2\theta_2))+t\theta_3)}\sqrt{\theta_3}}{\sqrt{2}\sqrt{-b_1\theta_1-b_2\theta_2}} \\ & + \frac{e^{i(x\theta_1+y\theta_2-\gamma(xp_1+yp_2-2t\lambda(p_1\theta_1+p_2\theta_2))+t\theta_3)}}{\sqrt{2}(c_1^2+c_2^2)(z_2-2z_3)z_3\sqrt{-b_1\theta_1-b_2\theta_2}\left(-\theta_3-\frac{b_1\theta_1\theta_3}{-b_1\theta_1-b_2\theta_2}-\frac{b_2\theta_2\theta_3}{-b_1\theta_1-b_2\theta_2}\right)} \\ & \times \frac{(\theta_1^2+\theta_2^2)\sqrt{\theta_3}}{\left(1+\frac{1}{z_2}[e^{z_2[\mu+d_0]}-z_1]\right)} \times \theta_3 + \frac{b_1\theta_1\theta_3}{-b_1\theta_1-b_2\theta_2} + \frac{b_2\theta_2\theta_3}{-b_1\theta_1-b_2\theta_2} \\ & + \frac{\theta_1^2\left(-\theta_3-\frac{b_1\theta_1\theta_3}{-b_1\theta_1-b_2\theta_2}-\frac{b_2\theta_2\theta_3}{-b_1\theta_1-b_2\theta_2}\right)}{\theta_1^2+\theta_2^2} + \frac{\theta_2^2\left(-\theta_3-\frac{b_1\theta_1\theta_3}{-b_1\theta_1-b_2\theta_2}-\frac{b_2\theta_2\theta_3}{-b_1\theta_1-b_2\theta_2}\right)}{\theta_1^2+\theta_2^2}. \end{aligned} \tag{4.19}$$

Set B:

$$\left\{ a = a, \alpha_0 = -\frac{\sqrt{-a\theta_1^2 - a\theta_2^2 - \theta_3}}{\sqrt{2}\sqrt{b_1\theta_1 + b_2\theta_2}}, \alpha_1 = \frac{\alpha_0(2b_1\alpha_0^2\theta_1 + a\theta_1^2 + 2b_2\alpha_0^2\theta_2 + a\theta_2^2 + \theta_3)}{a(z_2 - 2z_3)z_3(c_1^2 + c_2^2)} \right\}.$$

By using above set of solution, following cases for solutions of Eq (2.1) arises:

Case-1

$$\begin{aligned} \xi_{B11}(x, y, t) = & -\frac{e^{i(-\gamma(xp_1+yp_2+tp_3)+x\theta_1+y\theta_2+t\theta_3)}\sqrt{-a\theta_1^2 - a\theta_2^2 - \theta_3}}{\sqrt{2}\sqrt{b_1\theta_1 + b_2\theta_2}} \\ & - \frac{e^{i(-\gamma(xp_1+yp_2+tp_3)+x\theta_1+y\theta_2+t\theta_3)}}{\sqrt{2}a(c_1^2+c_2^2)(z_2-2z_3)z_3\sqrt{b_1\theta_1+b_2\theta_2}} \\ & \times \frac{\sqrt{-a\theta_1^2 - a\theta_2^2 - \theta_3}}{\left(1+\frac{1}{2z_3}\left[-z_2+\sqrt{-z_2^2+4z_1z_3}\tan\left(\mu+\frac{1}{2}\left(\sqrt{-z_2^2+4z_1z_3}\right)d_0\right)\right]\right)} \\ & \times a\theta_1^2 + a\theta_2^2 + \frac{b_1\theta_1(-a\theta_1^2 - a\theta_2^2 - \theta_3)}{b_1\theta_1 + b_2\theta_2} + \frac{b_2\theta_2(-a\theta_1^2 - a\theta_2^2 - \theta_3)}{b_1\theta_1 + b_2\theta_2} + \theta_3, \end{aligned} \tag{4.20}$$

$$\begin{aligned} \xi_{B12}(x, y, t) = & -\frac{e^{i(x\theta_1+y\theta_2-\gamma(xp_1+yp_2-2t\lambda(p_1\theta_1+p_2\theta_2))+t\theta_3)}\sqrt{-a\theta_1^2 - a\theta_2^2 - \theta_3}}{\sqrt{2}\sqrt{b_1\theta_1 + b_2\theta_2}} \\ & - \frac{e^{i(x\theta_1+y\theta_2-\gamma(xp_1+yp_2-2t\lambda(p_1\theta_1+p_2\theta_2))+t\theta_3)}}{\sqrt{2}a(c_1^2+c_2^2)(z_2-2z_3)z_3\sqrt{b_1\theta_1+b_2\theta_2}} \\ & \times \frac{\sqrt{-a\theta_1^2 - a\theta_2^2 - \theta_3}}{\left(1-\frac{1}{2z_3}\left[z_2+\cot\left(\mu+\frac{1}{2}\left(\sqrt{-z_2^2+4z_1z_3}\right)d_0\right)\right]\sqrt{-z_2^2+4z_1z_3}\right)} \\ & \times a\theta_1^2 + a\theta_2^2 + \frac{b_1\theta_1(-a\theta_1^2 - a\theta_2^2 - \theta_3)}{b_1\theta_1 + b_2\theta_2} + \frac{b_2\theta_2(-a\theta_1^2 - a\theta_2^2 - \theta_3)}{b_1\theta_1 + b_2\theta_2} + \theta_3, \end{aligned} \tag{4.21}$$

$$\begin{aligned} \xi_{B13}(x, y, t) = & - \frac{e^{i(x\theta_1+y\theta_2-\gamma(xp_1+yp_2-2t\lambda(p_1\theta_1+p_2\theta_2))+t\theta_3)} \sqrt{-a\theta_1^2 - a\theta_2^2 - \theta_3}}{\sqrt{2}\sqrt{b_1\theta_1 + b_2\theta_2}} \\ & - \frac{e^{i(x\theta_1+y\theta_2-\gamma(xp_1+yp_2-2t\lambda(p_1\theta_1+p_2\theta_2))+t\theta_3)}}{\sqrt{2}a(c_1^2 + c_2^2)(z_2 - 2z_3)z_3\sqrt{b_1\theta_1 + b_2\theta_2}} \\ & \times \frac{\sqrt{-a\theta_1^2 - a\theta_2^2 - \theta_3}}{\left(1 - \frac{1}{2z_3} \left[z_2 + \sqrt{z_2^2 - 4z_1z_3} \tanh\left(\mu + \frac{1}{2} \left(\sqrt{z_2^2 - 4z_1z_3}\right) d_0\right)\right]\right)} \\ & \times a\theta_1^2 + a\theta_2^2 + \frac{b_1\theta_1(-a\theta_1^2 - a\theta_2^2 - \theta_3)}{b_1\theta_1 + b_2\theta_2} + \frac{b_2\theta_2(-a\theta_1^2 - a\theta_2^2 - \theta_3)}{b_1\theta_1 + b_2\theta_2} + \theta_3, \end{aligned} \tag{4.22}$$

$$\begin{aligned} \xi_{B14}(x, y, t) = & - \frac{e^{i(x\theta_1+y\theta_2-\gamma(xp_1+yp_2-2t\lambda(p_1\theta_1+p_2\theta_2))+t\theta_3)} \sqrt{-a\theta_1^2 - a\theta_2^2 - \theta_3}}{\sqrt{2}\sqrt{b_1\theta_1 + b_2\theta_2}} \\ & - \frac{e^{i(x\theta_1+y\theta_2-\gamma(xp_1+yp_2-2t\lambda(p_1\theta_1+p_2\theta_2))+t\theta_3)}}{\sqrt{2}a(c_1^2 + c_2^2)(z_2 - 2z_3)z_3\sqrt{b_1\theta_1 + b_2\theta_2}} \\ & \times \frac{\sqrt{-a\theta_1^2 - a\theta_2^2 - \theta_3}}{\left(1 - \frac{1}{2z_3} \left[z_2 + \coth\left(\mu + \frac{1}{2} \left(\sqrt{z_2^2 - 4z_1z_3}\right) d_0\right)\right] \sqrt{z_2^2 - 4z_1z_3}\right)} \\ & \times a\theta_1^2 + a\theta_2^2 + \frac{b_1\theta_1(-a\theta_1^2 - a\theta_2^2 - \theta_3)}{b_1\theta_1 + b_2\theta_2} + \frac{b_2\theta_2(-a\theta_1^2 - a\theta_2^2 - \theta_3)}{b_1\theta_1 + b_2\theta_2} + \theta_3, \end{aligned} \tag{4.23}$$

$$\begin{aligned} \xi_{B15}(x, y, t) = & - \frac{e^{i(x\theta_1+y\theta_2-\gamma(xp_1+yp_2-2t\lambda(p_1\theta_1+p_2\theta_2))+t\theta_3)} \sqrt{-a\theta_1^2 - a\theta_2^2 - \theta_3}}{\sqrt{2}\sqrt{b_1\theta_1 + b_2\theta_2}} \\ & - \frac{e^{i(x\theta_1+y\theta_2-\gamma(xp_1+yp_2-2t\lambda(p_1\theta_1+p_2\theta_2))+t\theta_3)}}{\sqrt{2}a(c_1^2 + c_2^2)(z_2 - 2z_3)z_3\sqrt{b_1\theta_1 + b_2\theta_2}} \times \frac{\sqrt{-a\theta_1^2 - a\theta_2^2 - \theta_3}}{\left(1 + \frac{z_2}{2} - \frac{1}{z_3} \left[\frac{1}{\mu+d_0}\right]\right)} \\ & \times a\theta_1^2 + a\theta_2^2 + \frac{b_1\theta_1(-a\theta_1^2 - a\theta_2^2 - \theta_3)}{b_1\theta_1 + b_2\theta_2} + \frac{b_2\theta_2(-a\theta_1^2 - a\theta_2^2 - \theta_3)}{b_1\theta_1 + b_2\theta_2} + \theta_3. \end{aligned} \tag{4.24}$$

Case-2

$$\begin{aligned} \xi_{B21}(x, y, t) & = e^{i(x\theta_1+y\theta_2-\gamma(xp_1+yp_2-2t\lambda(p_1\theta_1+p_2\theta_2))+t\theta_3)} \times \left(- \frac{\sqrt{-a\theta_1^2 - a\theta_2^2 - \theta_3}}{\sqrt{2}\sqrt{b_1\theta_1 + b_2\theta_2}} \right. \\ & \left. - \frac{\sqrt{-a\theta_1^2 - a\theta_2^2 - \theta_3} \left(a\theta_1^2 + a\theta_2^2 + \frac{b_1\theta_1(-a\theta_1^2 - a\theta_2^2 - \theta_3)}{b_1\theta_1 + b_2\theta_2} + \frac{b_2\theta_2(-a\theta_1^2 - a\theta_2^2 - \theta_3)}{b_1\theta_1 + b_2\theta_2} + \theta_3 \right)}{\sqrt{2}a(c_1^2 + c_2^2)(z_2 - 2z_3)z_3\sqrt{b_1\theta_1 + b_2\theta_2} \left(1 - \frac{1}{2z_3} [z_2 \tanh [z_2 + \frac{z_2}{2} [\mu + d_0]]]\right)} \right) \end{aligned} \tag{4.25}$$

$$\begin{aligned} \xi_{B22}(x, y, t) & = e^{i(x\theta_1+y\theta_2-\gamma(xp_1+yp_2-2t\lambda(p_1\theta_1+p_2\theta_2))+t\theta_3)} \times \left(- \frac{\sqrt{-a\theta_1^2 - a\theta_2^2 - \theta_3}}{\sqrt{2}\sqrt{b_1\theta_1 + b_2\theta_2}} \right. \\ & \left. - \frac{\sqrt{-a\theta_1^2 - a\theta_2^2 - \theta_3} \left(a\theta_1^2 + a\theta_2^2 + \frac{b_1\theta_1(-a\theta_1^2 - a\theta_2^2 - \theta_3)}{b_1\theta_1 + b_2\theta_2} + \frac{b_2\theta_2(-a\theta_1^2 - a\theta_2^2 - \theta_3)}{b_1\theta_1 + b_2\theta_2} + \theta_3 \right)}{\sqrt{2}a(c_1^2 + c_2^2)(z_2 - 2z_3)z_3\sqrt{b_1\theta_1 + b_2\theta_2} \left(1 - \frac{1}{2z_3} [\coth [z_2 + \frac{z_2}{2} [\mu + d_0]] z_2]\right)} \right) \end{aligned} \tag{4.26}$$

$$\begin{aligned} \xi_{B23}(x, y, t) & = e^{i(x\theta_1+y\theta_2-\gamma(xp_1+yp_2-2t\lambda(p_1\theta_1+p_2\theta_2))+t\theta_3)} \times \left(- \frac{\sqrt{-a\theta_1^2 - a\theta_2^2 - \theta_3}}{\sqrt{2}\sqrt{b_1\theta_1 + b_2\theta_2}} \right. \\ & \left. - \frac{\sqrt{-a\theta_1^2 - a\theta_2^2 - \theta_3} \left(a\theta_1^2 + a\theta_2^2 + \frac{b_1\theta_1(-a\theta_1^2 - a\theta_2^2 - \theta_3)}{b_1\theta_1 + b_2\theta_2} + \frac{b_2\theta_2(-a\theta_1^2 - a\theta_2^2 - \theta_3)}{b_1\theta_1 + b_2\theta_2} + \theta_3 \right)}{\sqrt{2}a(c_1^2 + c_2^2)(z_2 - 2z_3)z_3\sqrt{b_1\theta_1 + b_2\theta_2} \left(1 + \frac{z_2}{e^{-z_2[\mu+d_0]} z_2 - z_3}\right)} \right) \end{aligned} \tag{4.27}$$

$$\begin{aligned} \xi_{B24}(x, y, t) &= e^{i(x\theta_1+y\theta_2-\gamma(xp_1+yp_2-2t\lambda(p_1\theta_1+p_2\theta_2))+t\theta_3)} \\ &\times \left(-\frac{\sqrt{-a\theta_1^2-a\theta_2^2-\theta_3}}{\sqrt{2}\sqrt{b_1\theta_1+b_2\theta_2}} \right. \\ &\left. - \frac{\sqrt{-a\theta_1^2-a\theta_2^2-\theta_3} \left(a\theta_1^2+a\theta_2^2+\frac{b_1\theta_1(-a\theta_1^2-a\theta_2^2-\theta_3)}{b_1\theta_1+b_2\theta_2}+\frac{b_2\theta_2(-a\theta_1^2-a\theta_2^2-\theta_3)}{b_1\theta_1+b_2\theta_2}+\theta_3 \right)}{\sqrt{2}a(c_1^2+c_2^2)\left(1-\frac{1}{\mu z_3}\right)(z_2-2z_3)z_3\sqrt{b_1\theta_1+b_2\theta_2}} \right). \end{aligned} \tag{4.28}$$

Case-3

$$\begin{aligned} \xi_{B31}(x, y, t) &= e^{i(x\theta_1+y\theta_2-\gamma(xp_1+yp_2-2t\lambda(p_1\theta_1+p_2\theta_2))+t\theta_3)} \times \left(-\frac{\sqrt{-a\theta_1^2-a\theta_2^2-\theta_3}}{\sqrt{2}\sqrt{b_1\theta_1+b_2\theta_2}} \right. \\ &\left. - \frac{\sqrt{-a\theta_1^2-a\theta_2^2-\theta_3} \left(a\theta_1^2+a\theta_2^2+\frac{b_1\theta_1(-a\theta_1^2-a\theta_2^2-\theta_3)}{b_1\theta_1+b_2\theta_2}+\frac{b_2\theta_2(-a\theta_1^2-a\theta_2^2-\theta_3)}{b_1\theta_1+b_2\theta_2}+\theta_3 \right)}{\sqrt{2}a(c_1^2+c_2^2)\left(1-\frac{\sqrt{z_1z_3}}{z_3}[\tan[\sqrt{z_1z_3}[\mu+d_0]]]\right)(z_2-2z_3)z_3\sqrt{b_1\theta_1+b_2\theta_2}} \right), \end{aligned} \tag{4.29}$$

$$\begin{aligned} \xi_{B32}(x, y, t) &= e^{i(x\theta_1+y\theta_2-\gamma(xp_1+yp_2-2t\lambda(p_1\theta_1+p_2\theta_2))+t\theta_3)} \times \left(-\frac{\sqrt{-a\theta_1^2-a\theta_2^2-\theta_3}}{\sqrt{2}\sqrt{b_1\theta_1+b_2\theta_2}} \right. \\ &\left. - \frac{\sqrt{-a\theta_1^2-a\theta_2^2-\theta_3} \left(a\theta_1^2+a\theta_2^2+\frac{b_1\theta_1(-a\theta_1^2-a\theta_2^2-\theta_3)}{b_1\theta_1+b_2\theta_2}+\frac{b_2\theta_2(-a\theta_1^2-a\theta_2^2-\theta_3)}{b_1\theta_1+b_2\theta_2}+\theta_3 \right)}{\sqrt{2}a(c_1^2+c_2^2)\left(1-\frac{\sqrt{z_1z_3}}{z_3}[\cot[\sqrt{z_1z_3}[\mu+d_0]]]\right)(z_2-2z_3)z_3\sqrt{b_1\theta_1+b_2\theta_2}} \right), \end{aligned} \tag{4.30}$$

$$\begin{aligned} \xi_{B33}(x, y, t) &= e^{i(x\theta_1+y\theta_2-\gamma(xp_1+yp_2-2t\lambda(p_1\theta_1+p_2\theta_2))+t\theta_3)} \times \left(-\frac{\sqrt{-a\theta_1^2-a\theta_2^2-\theta_3}}{\sqrt{2}\sqrt{b_1\theta_1+b_2\theta_2}} \right. \\ &\left. - \frac{\sqrt{-a\theta_1^2-a\theta_2^2-\theta_3} \left(a\theta_1^2+a\theta_2^2+\frac{b_1\theta_1(-a\theta_1^2-a\theta_2^2-\theta_3)}{b_1\theta_1+b_2\theta_2}+\frac{b_2\theta_2(-a\theta_1^2-a\theta_2^2-\theta_3)}{b_1\theta_1+b_2\theta_2}+\theta_3 \right)}{\sqrt{2}a(c_1^2+c_2^2)\left(1-\frac{\sqrt{-z_1z_3}}{z_3}[\tanh[\sqrt{-z_1z_3}[\mu+d_0]]]\right)(z_2-2z_3)z_3\sqrt{b_1\theta_1+b_2\theta_2}} \right), \end{aligned} \tag{4.31}$$

$$\begin{aligned} \xi_{B34}(x, y, t) &= e^{i(x\theta_1+y\theta_2-\gamma(xp_1+yp_2-2t\lambda(p_1\theta_1+p_2\theta_2))+t\theta_3)} \times \left(-\frac{\sqrt{-a\theta_1^2-a\theta_2^2-\theta_3}}{\sqrt{2}\sqrt{b_1\theta_1+b_2\theta_2}} \right. \\ &\left. - \frac{\sqrt{-a\theta_1^2-a\theta_2^2-\theta_3} \left(a\theta_1^2+a\theta_2^2+\frac{b_1\theta_1(-a\theta_1^2-a\theta_2^2-\theta_3)}{b_1\theta_1+b_2\theta_2}+\frac{b_2\theta_2(-a\theta_1^2-a\theta_2^2-\theta_3)}{b_1\theta_1+b_2\theta_2}+\theta_3 \right)}{\sqrt{2}a(c_1^2+c_2^2)\left(1-\frac{\sqrt{-z_1z_3}}{z_3}[\coth[\sqrt{-z_1z_3}[\mu+d_0]]]\right)(z_2-2z_3)z_3\sqrt{b_1\theta_1+b_2\theta_2}} \right), \end{aligned} \tag{4.32}$$

$$\begin{aligned} \xi_{B35}(x, y, t) &= e^{i(x\theta_1+y\theta_2-\gamma(xp_1+yp_2-2t\lambda(p_1\theta_1+p_2\theta_2))+t\theta_3)} \times \left(-\frac{\sqrt{-a\theta_1^2-a\theta_2^2-\theta_3}}{\sqrt{2}\sqrt{b_1\theta_1+b_2\theta_2}} \right. \\ &\left. - \frac{\sqrt{-a\theta_1^2-a\theta_2^2-\theta_3} \left(a\theta_1^2+a\theta_2^2+\frac{b_1\theta_1(-a\theta_1^2-a\theta_2^2-\theta_3)}{b_1\theta_1+b_2\theta_2}+\frac{b_2\theta_2(-a\theta_1^2-a\theta_2^2-\theta_3)}{b_1\theta_1+b_2\theta_2}+\theta_3 \right)}{\sqrt{2}a(c_1^2+c_2^2)\left(1-\frac{1}{z_3[\mu+d_0]}\right)(z_2-2z_3)z_3\sqrt{b_1\theta_1+b_2\theta_2}} \right). \end{aligned} \tag{4.33}$$

Case-4

$$\begin{aligned} \xi_{B41}(x, y, t) &= e^{i(x\theta_1 + y\theta_2 - \gamma(xp_1 + yp_2 - 2t\lambda(p_1\theta_1 + p_2\theta_2)) + t\theta_3)} \times \left(-\frac{\sqrt{-a\theta_1^2 - a\theta_2^2 - \theta_3}}{\sqrt{2}\sqrt{b_1\theta_1 + b_2\theta_2}} \right. \\ &\quad \left. - \frac{\sqrt{-a\theta_1^2 - a\theta_2^2 - \theta_3} \left(a\theta_1^2 + a\theta_2^2 + \frac{b_1\theta_1(-a\theta_1^2 - a\theta_2^2 - \theta_3)}{b_1\theta_1 + b_2\theta_2} + \frac{b_2\theta_2(-a\theta_1^2 - a\theta_2^2 - \theta_3)}{b_1\theta_1 + b_2\theta_2} + \theta_3 \right)}{\sqrt{2}a(c_1^2 + c_2^2) \left(1 + \frac{1}{z_2} [e^{z_2[\mu + d_0]} - z_1] \right) (z_2 - 2z_3) z_3 \sqrt{b_1\theta_1 + b_2\theta_2}} \right). \end{aligned} \tag{4.34}$$

Results and discussions

The results and solutions of considered two formats of generalized kudryashov method on SCNS equation is discussed below:

Figures 1 and 2 represents the solutions of applying generalized kudryashov method-1 with different values of parameters:

- Figure 1 shows kink soliton of absolute behavior for Eq. (4.2) with different parameters $b_0 = -1, b_1 = -1, z_1 = -0.5, z_2 = -2.5, p_1 = -5, p_2 = -0.5, \theta_1 = 1, \theta_2 = -1.25, \theta_3 = -0.5, \lambda = -1, y = -5$ and $\gamma = -0.5$.
- Figure 2 illustrates anti-kink soliton of absolute behavior for Eq. (4.3) with parameters $a_2 = 3, a_1 = 1.5, z_1 = -0.5, z_2 = 2, p_1 = 1.5, p_2 = -1.5, \theta_1 = 1.5, \theta_2 = -2, \theta_3 = -3, \lambda = -1, y = 0$ and $\gamma = -0.5$.

Figures 3, 4 and 5 represent the solutions of set A after applying generalized kudryashov method-2 with different values of paramters:

- Figure 3 shows periodic soliton for real behavior for Eq. (4.5) with different parameters $b_1 = 1.2, b_2 = 3, z_1 = -0.5, z_2 = 0.5, z_3 = -0.5, c_1 = 0.9, c_2 = 0.1, d_0 = -0.9, \mu = 0.5, p_1 = -2.3, c_1 = 0.9, c_2 = 0.1, d_0 = -0.9, \mu = 0.5, p_1 = -2.3, p_2 = -0.3, p_3 = 1.9, \theta_1 = -3, \theta_2 = 0.9, \theta_3 = 0.1, \lambda = 16, y = -0.9$ and $\gamma = -0.5$.
- Figure 4 shows V-shaped soliton for real behaviour for Eq. (4.6) with parameters $b_1 = 3.5, b_2 = 1, z_1 = -0.5, z_2 = -2, z_3 = -0.5, c_1 = -2, c_2 = 1, d_0 = 0, \mu = 0.5, p_1 = 0.9, p_2 = -0.5, p_3 = 0.08, \theta_1 = 2.5, \theta_2 = -3.5, \theta_3 = 0.5, \lambda = 4, y = 5$ and $\gamma = 1.4$.

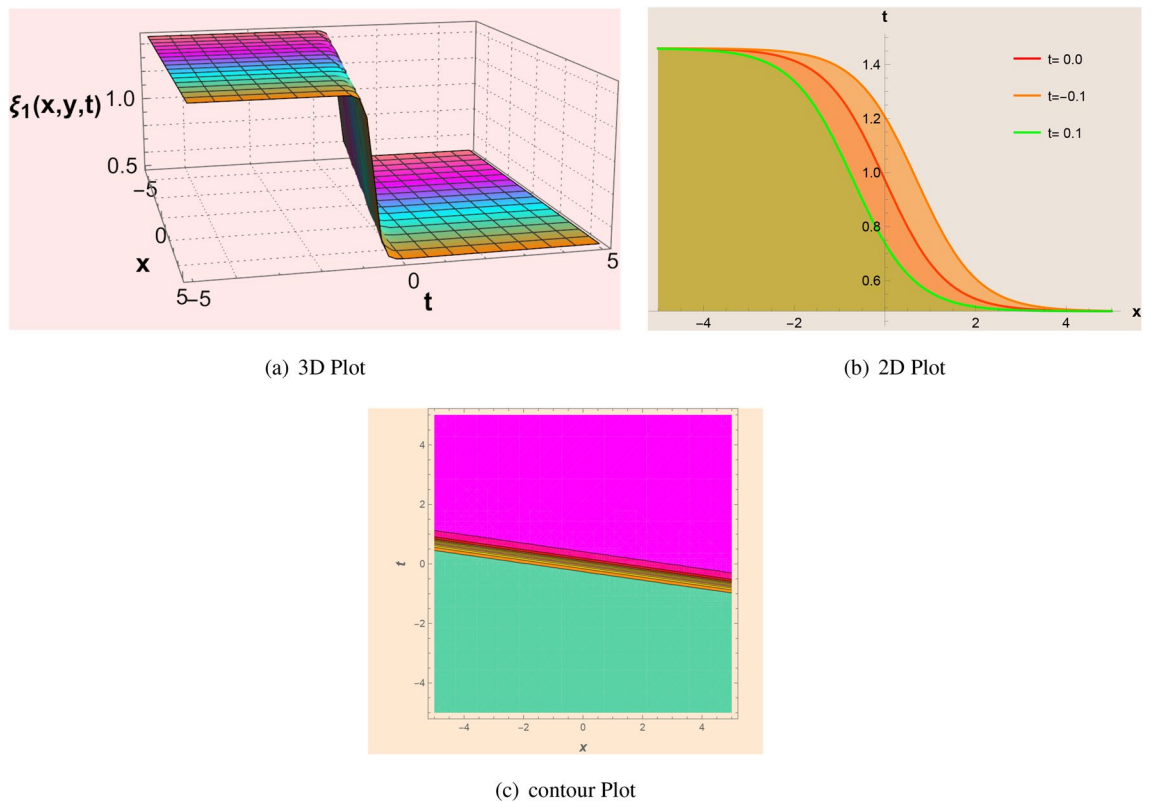


Fig. 1. 3D, 2D and contour representation of kink soliton of absolute behavior for Eq. (4.2) with parameters $b_0 = -1, b_1 = -1, z_1 = -0.5, z_2 = -2.5, p_1 = -5, p_2 = -0.5, \theta_1 = 1, \theta_2 = -1.25, \theta_3 = -0.5, \lambda = -1, y = -5$ and $\gamma = -0.5$.

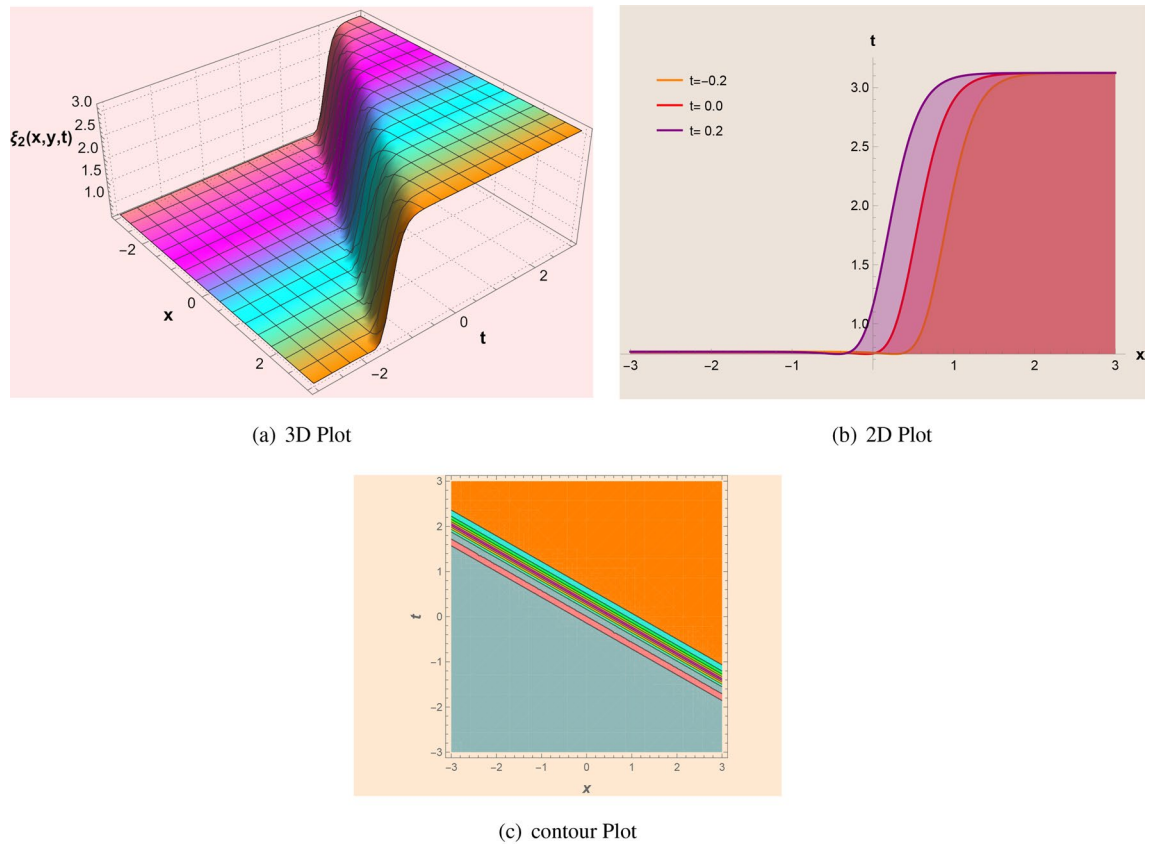


Fig. 2. 3D, 2D and contour representation of anti-kink soliton of absolute behavior for Eq. (4.3) with parameters $a_2 = 3, a_1 = 1.5, z_1 = -0.5, z_2 = 2, p_1 = 1.5, p_2 = -1.5, \theta_1 = 1.5, \theta_2 = -2, \theta_3 = -3, \lambda = -1, y = 0$ and $\gamma = -0.5$.

- Figure 5 shows Kink soliton for real behaviour for Eq. (4.9) with parameters $b_1 = 3.5, b_2 = 1, z_1 = -0.5, z_2 = -2, z_3 = -0.5, c_1 = 0.5, c_2 = 0, d_0 = 0, \mu = 0.5, p_1 = 0.9, p_2 = -0.5, p_3 = 0.1, \theta_1 = 2.5, \theta_2 = -3.5, \theta_3 = 0.5, \lambda = 4, y = -0.28$ and $\gamma = 3.5$.

Figures 6 and 7 illustrate the solutions of set B after applying generalized kudryashov method-2 with different values of parameters:

- Figure 6 shows M-shaped soliton for real behaviour for Eq. (4.20) with parameters $b_1 = 1.2, b_2 = 7, z_1 = -0.5, z_2 = 0.5, z_3 = -0.5, c_1 = 0.9, c_2 = 0.1, a = 1, d_0 = -0.9, \mu = 2.5, p_1 = 1, p_2 = -1, p_3 = -0.01, \theta_1 = -6, \theta_2 = 0.9, \theta_3 = 0.09, \lambda = 16, y = 2$ and $\gamma = -4$.
- Figure 7 shows W-shaped soliton for real behaviour for Eq. (4.21) with parameters $b_1 = 1.2, b_2 = 1, z_1 = -0.5, z_2 = 0.5, z_3 = -0.5, c_1 = 0.9, c_2 = 0.5, a = 0.5, d_0 = -0.9, \mu = 1.5, p_1 = 1, p_2 = -1, p_3 = 1, \theta_1 = -6, \theta_2 = 0.9, \theta_3 = 0.09, \lambda = 16, y = 0.5$ and $\gamma = 0$.

Qualitative dynamical analysis

In this section, thorough examination of the dynamical characteristics of the suggested system is occurred, paying attention on bifurcation analysis and chaotic behaviour and determining the sensitivity of the system to initial conditions.

Bifurcation analysis

A mathematical method called bifurcation analysis^{31,32} is used to examine how a system’s qualitative behavior changes as a parameter changes. In dynamical systems, where even minor adjustments to parameters can cause significant changes in the behavior of the system, such as the move from stability to chaos, it is very helpful. It is a technique of graphical representation showing how periodic trajectories vary with respect to parameters. Also it helps to visualize stable and unstable solutions. Eq (2.2) gives,

$$\Omega'' = M_1\Omega^3 - M_2\Omega, \tag{5.1}$$

where $M_1 = \frac{2(b_1\theta_1 + b_2\theta_2)}{a(c_1^2 + c_2^2)}$ and $M_2 = \frac{\theta_3 + a(\theta_1^2 + \theta_2^2)}{a(c_1^2 + c_2^2)}$.

Applying Galilean transformation on Eq. (5.1) gives

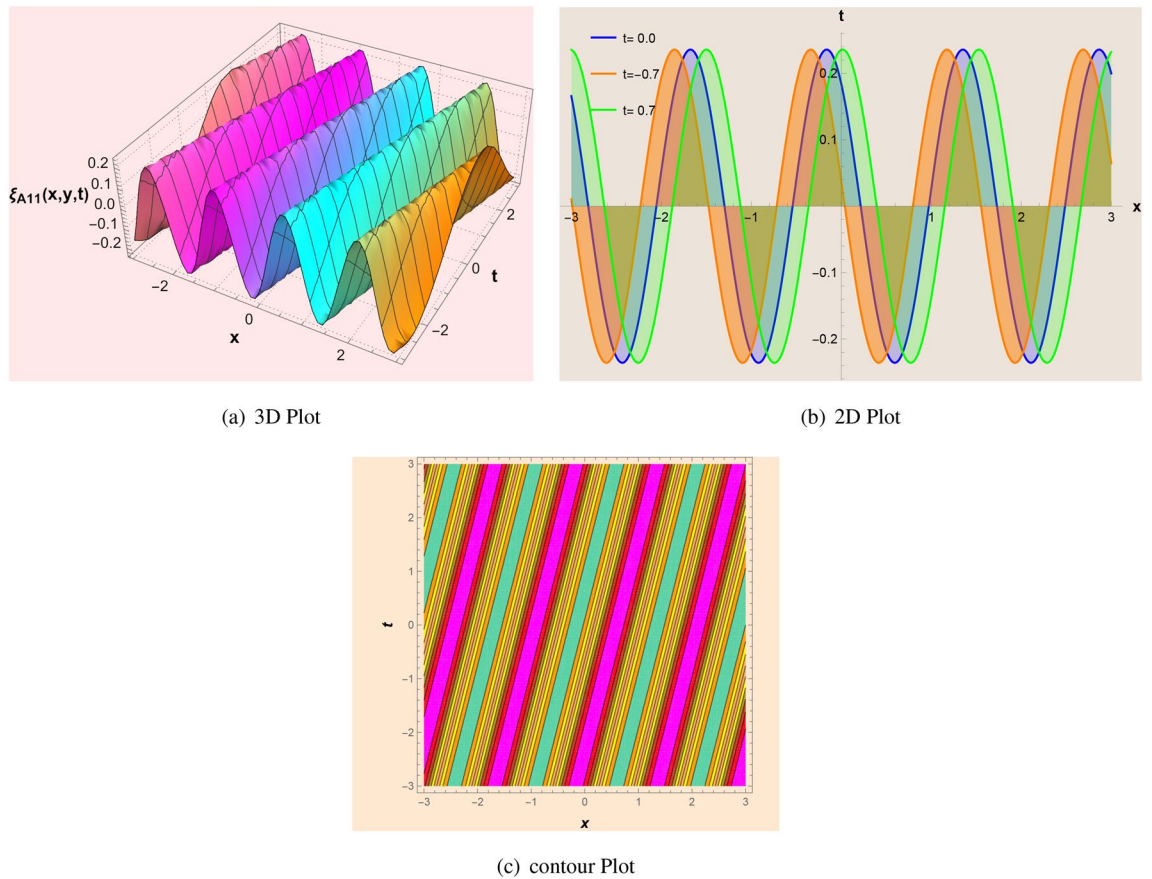


Fig. 3. 3D, 2D and contour representation of periodic soliton for real behavior for Eq. (4.5) with parameters $b_1 = 1.2, b_2 = 3, z_1 = -0.5, z_2 = 0.5, z_3 = -0.5, c_1 = 0.9, c_2 = 0.1, d_0 = -0.9, \mu = 0.5, p_1 = -2.3, p_2 = -0.3, p_3 = 1.9, \theta_1 = -3, \theta_2 = 0.9, \theta_3 = 0.1, \lambda = 16, y = -0.9$ and $\gamma = -0.5$.

$$\begin{cases} Z = \Omega', \\ Z' = M_1\Omega^3 - M_2\Omega. \end{cases} \tag{5.2}$$

And its Hamiltonian system is,

$$H(\Omega, Z) = \frac{Z^2}{2} + M_1 \frac{\Omega^4}{4} - M_2 \frac{\Omega^2}{2},$$

Where $M_1 = \frac{2(b_1\theta_1 + b_2\theta_2)}{a(c_1^2 + c_2^2)}$ and $M_2 = \frac{\theta_3 + a(\theta_1^2 + \theta_2^2)}{a(c_1^2 + c_2^2)}$. To determine Eq. (5.2) equilibrium points, the equation

$$\begin{cases} Z = 0, \\ M_1\Omega^3 - M_2\Omega = 0. \end{cases} \tag{5.3}$$

has three equilibrium points

$$\begin{aligned} x_1 &= (0, 0), \\ x_2 &= \left(\sqrt{\frac{M_2}{M_1}}, 0\right), \\ x_3 &= \left(-\sqrt{\frac{M_2}{M_1}}, 0\right). \end{aligned}$$

Jacobian matrix J for this system is

$$J(\Omega, Z) = \begin{bmatrix} 0 & 1 \\ 3M_1\Omega^2 - M_2 & 0 \end{bmatrix} = M_2 - 3M_1\Omega^2.$$

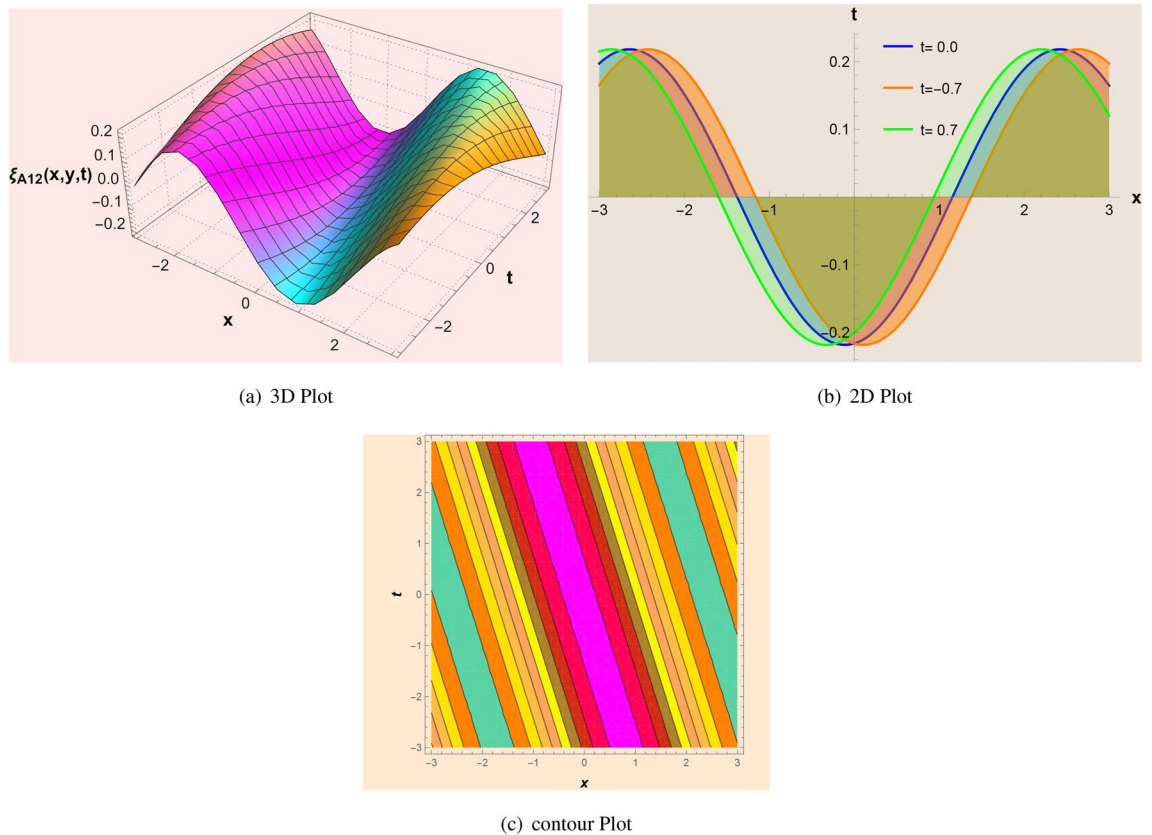


Fig. 4. 3D, 2D and contour representation of V-shaped soliton for real behaviour for Eq. (4.6) with parameters $b_1 = 3.5, b_2 = 1, z_1 = -0.5, z_2 = -2, z_3 = -0.5, c_1 = -2, c_2 = 1, d_0 = 0, \mu = 0.5, p_1 = 0.9, p_2 = -0.5, p_3 = 0.08, \theta_1 = 2.5, \theta_2 = -3.5, \theta_3 = 0.5, \lambda = 4, y = 5$ and $\gamma = 1.4$.

Let's examine different parameter configurations for system (5.3) in order to analyze the dynamical system's phase plane. Different parameter values and equilibrium points are represented by phase portraits for system (5.3) in Figs. 8 and 9.

Case-1:

$M_1 > 0$ and $M_2 > 0$

In this case, assume that the parameter values are $a=2, b_1 = 1, b_2 = 1, \theta_1 = 2, \theta_2 = 2, \theta_3 = 16, c_1 = 1$ and $c_2 = 1$ with three equilibrium points $x_1 = (0, 0), x_2 = (2, 0)$ and $x_3 = (-2, 0)$.

Case-2:

$M_1 < 0$ and $M_2 < 0$

In this case, assume that the parameter values are $a = 2, b_1 = 1, b_2 = 1, \theta_1 = -2, \theta_2 = -2, \theta_3 = -88, c_1 = 1$ and $c_2 = 1$ with three equilibrium points $x_1 = (0, 0), x_2 = (3, 0)$ and $x_3 = (-3, 0)$.

Case-3:

$M_1 < 0$ and $M_2 > 0$

In this case, assume that the parameter values are $a = 2, b_1 = 1, b_2 = 1, \theta_1 = -2, \theta_2 = -2, \theta_3 = 16, c_1 = 1$ and $c_2 = 1$ with three equilibrium points $x_1 = (0, 0), x_2 = (2i, 0)$ and $x_3 = (-2i, 0)$.

Case-4:

$M_1 > 0$ and $M_2 < 0$

In this case, assume that the parameter values are $a = 2, b_1 = 1, b_2 = 1, \theta_1 = 2, \theta_2 = 2, \theta_3 = -24, c_1 = 1$ and $c_2 = 1$ with three equilibrium points $x_1 = (0, 0), x_2 = (i, 0)$ and $x_3 = (-i, 0)$.

As the parameters are changed, these several instances highlight bifurcations and stability shifts among equilibrium points, showcasing the system's diverse dynamical nature.

Discussion of chaotic patterns

The SCNLS equation's chaotic behavior resembles the bifurcation analysis's predicted transitions. The system exhibits predictable pathways to chaos as it transitions from periodic to chaotic phases when parameters change. By confirming the system's sensitivity and complexity, these patterns strengthen the planning ability of bifurcation structures. Additionally, this coherence demonstrates that multistability exists within the same parameter range.

To examine the formation of quasi-periodic and chaotic behaviors, we add an external periodic force to Eq. (5.2). This external disturbance is incorporated into the improved system. Using the same initial conditions,

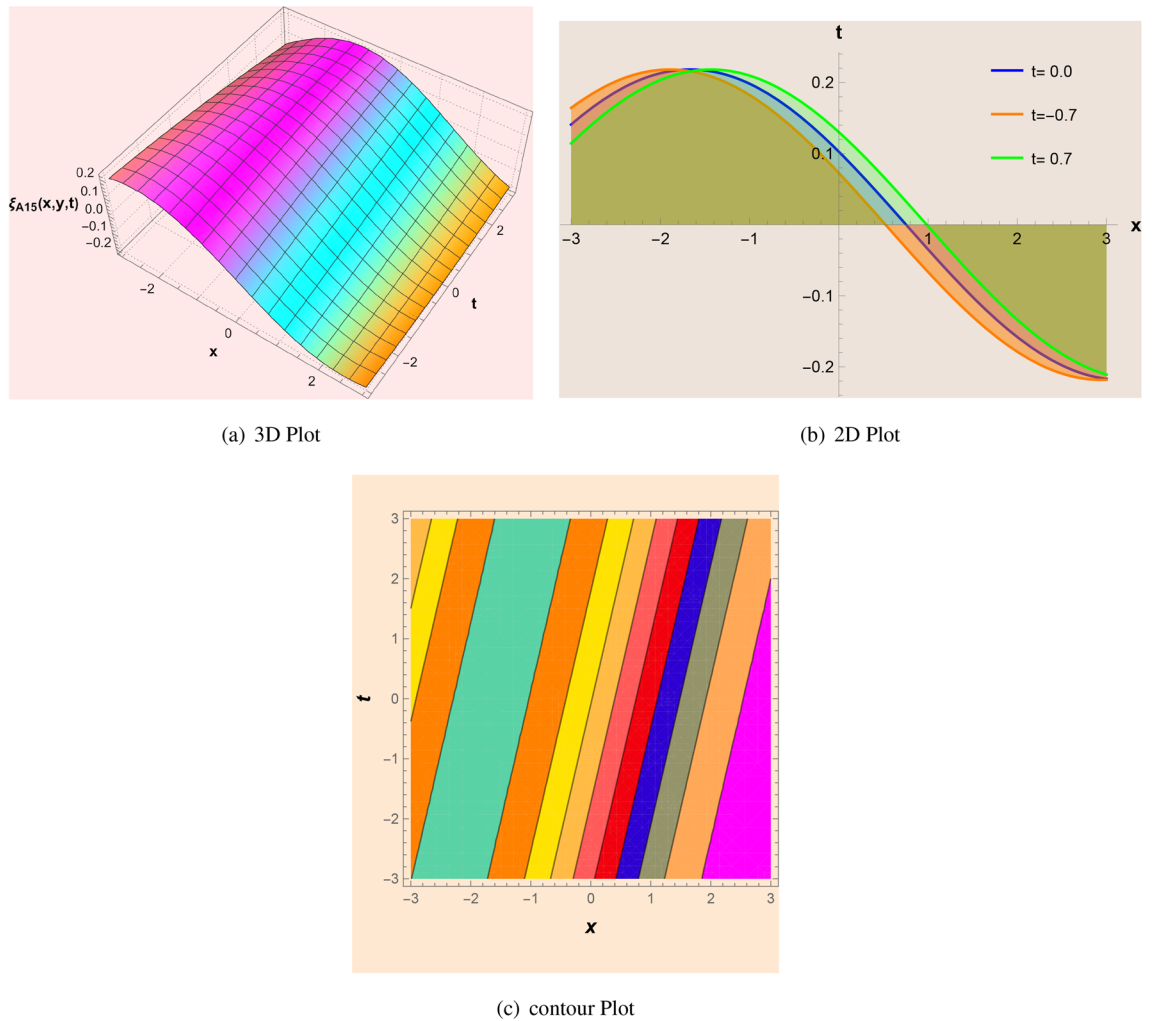


Fig. 5. 3D, 2D and contour representation of Kink soliton for real behaviour for Eq. (4.9) with parameters $b_1 = 3.5, b_2 = 1, z_1 = -0.5, z_2 = -2, z_3 = -0.5, c_1 = 0.5, c_2 = 0, d_0 = 0, \mu = 0.5, p_1 = 0.9, p_2 = -0.5, p_3 = 0.1, \theta_1 = 2.5, \theta_2 = -3.5, \theta_3 = 0.5, \lambda = 4, y = -0.28$ and $\gamma = 3.5$.

we examined various behaviors by varying the values of the parameters. This section will examine the chaotic behaviours of the modified system described below:

$$\begin{cases} Z = \Omega', \\ Z' = M_1\Omega^3 - M_2\Omega + k_0 \cos(f\rho). \end{cases}$$

In this study, f shows the frequency of vibration, while k_0 indicates the degree of the external disturbances. Figure 10 shows the chaotic behavior of the system under given parameters:

$$a = 2, b_1 = 1, b_2 = 1, \theta_1 = 2, \theta_2 = 2, \theta_3 = -24, c_1 = 1, c_2 = 1, k_0 = 0.1 \text{ and } f = 2$$

For these particular conditions, as shown, the system maintains a stable, demonstrating the ordered structure of the dynamics at low frequencies.

Figure 11 shows the system's chaotic behavior with little change in $k_0 = 3$. This modification illustrates how even a little change in parameter can cause significant change in the behavior of the system. Moreover, Fig. 12 shows the system's chaotic behavior with little increase in frequency $f = 10$. Figure's chaotic patterns are typical of non linear arrangements, where even little change in parameters can produce complicated and unpredictable behaviors. This modification highlights the system's reactivity to outside disturbances and demonstrates the complex relationship between system parameters and the initial stages of chaotic.

Multistability

Multistability is the coexistence of several stable states or attractors under the same system parameters. For the $(2 + 1)$ -dimensional SCNS equation, multistability occurs as a result of interaction among nonlinearity, chirality, and random perturbations. Multistability means that the system can develop qualitatively different stable states

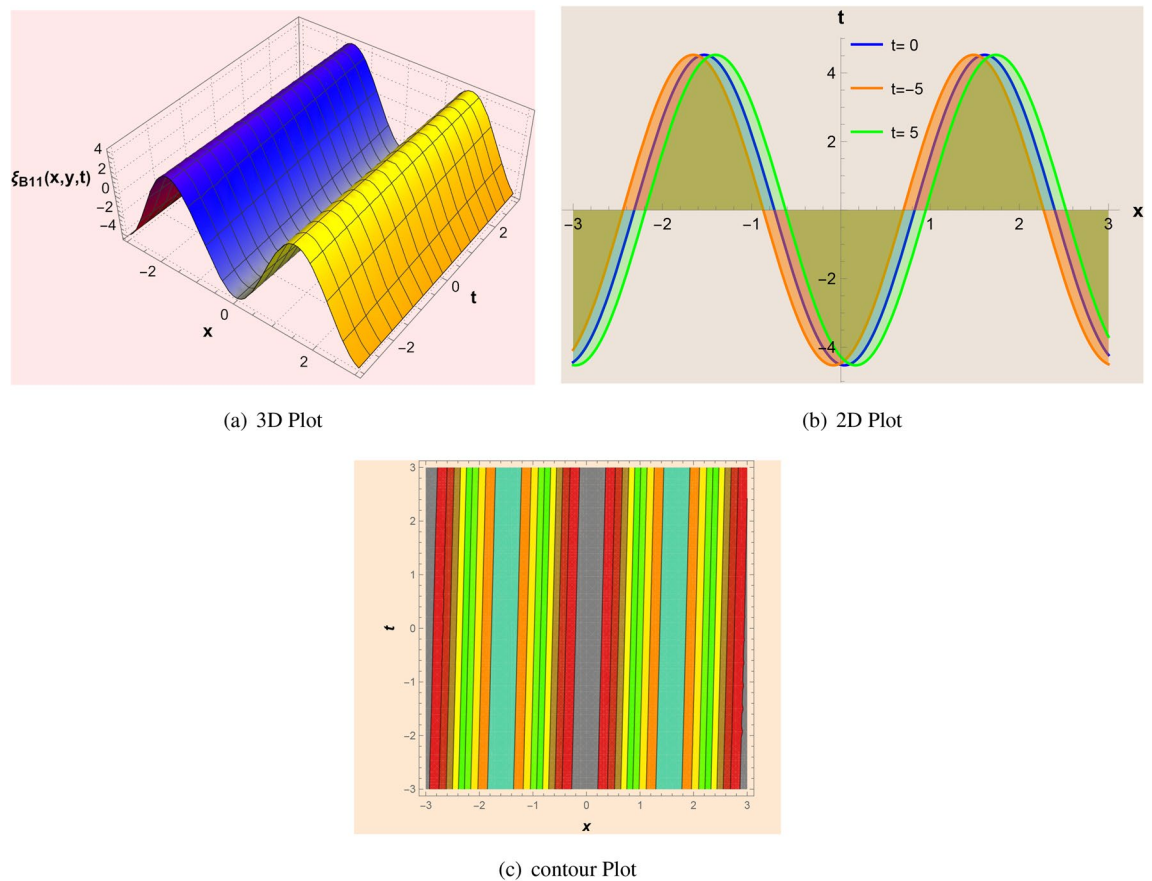


Fig. 6. 3D, 2D and contour representation of M-shaped soliton for real behaviour for Eq. (4.20) with parameters $b_1 = 1.2$, $b_2 = 7$, $z_1 = -0.5$, $z_2 = 0.5$, $z_3 = -0.5$, $c_1 = 0.9$, $c_2 = 0.1$, $a = 1$, $d_0 = -0.9$, $\mu = 2.5$, $p_1 = 1$, $p_2 = -1$, $p_3 = -0.01$, $\theta_1 = -6$, $\theta_2 = 0.9$, $\theta_3 = 0.09$, $\lambda = 16$, $y = 2$ and $\gamma = -4$.

based on initial conditions or small parameter changes, even if the governing parameters are kept constant. With bifurcation analysis and phase portraits, we recognize different attractors for periodic, quasi-periodic, and chaotic patterns, which confirm the existence of multistable behavior. The sensitivity analysis also confirms this, with small changes in initial conditions results in switching between these stable states. Such multistable dynamics play a key role in explaining the unpredictability and complexity of nonlinear wave interactions in chiral and noisy media and emphasizes initial state control importance in real-world applications such as signal propagation, optical switching, and information processing in nonlinear media. This model's multistability is important from both a practical and a theoretical perspective. In theory, it represents the rich structure of the SCNS equation's solution space under chiral and stochastic influences.

Sensitivity analysis

Sensitivity analysis characterizes the system's response to alterations in its original state. Low sensitivity is demonstrated by the system if only minor shifts to the original conditions have minimal effects. In contrast, a system is regarded as extremely sensitive if little adjustments result in major alterations to its behavior. To determine the system's physical properties and examine the ways in which frequency and the force of perturbation affect the model, the influence of the frequency term will be investigated. By making minor changes to the initial conditions, we conducted sensitivity analysis to further confirm multistability. Here, the sensitivity analysis of system (5.3) is performed using the parametric values $a = 2$, $b_1 = 1$, $b_2 = 1$, $\theta_1 = 2$, $\theta_2 = 2$, $\theta_3 = -24$, $c_1 = 1$, $c_2 = 1$, with the initial conditions represented by the red curve(0.7, 0.2) and the blue curve(0.7, 1.5) as shown in Fig. 13. In Fig. 14, the parameters are adjusted such that the initial conditions are employed as: red(0.7, 0.1) and green(0.9, 0.3) and the initial conditions represented by the green curve(0.9, 0.9) and the blue curve(0.6, 0.3) are in Fig. 15. And, In Fig. 16, the parameters are adjusted in the way such that three initial conditions are employed: blue (0.7, 0.8), red (0.5, 0.3) and green (0.3, 0.1). As can be inferred from the observations of figures, it is evident that slight variations in the initial values result in extreme fluctuations in the behavior of the system. This indicates that the system has a lesser level of sensitivity to variations in the initial conditions.

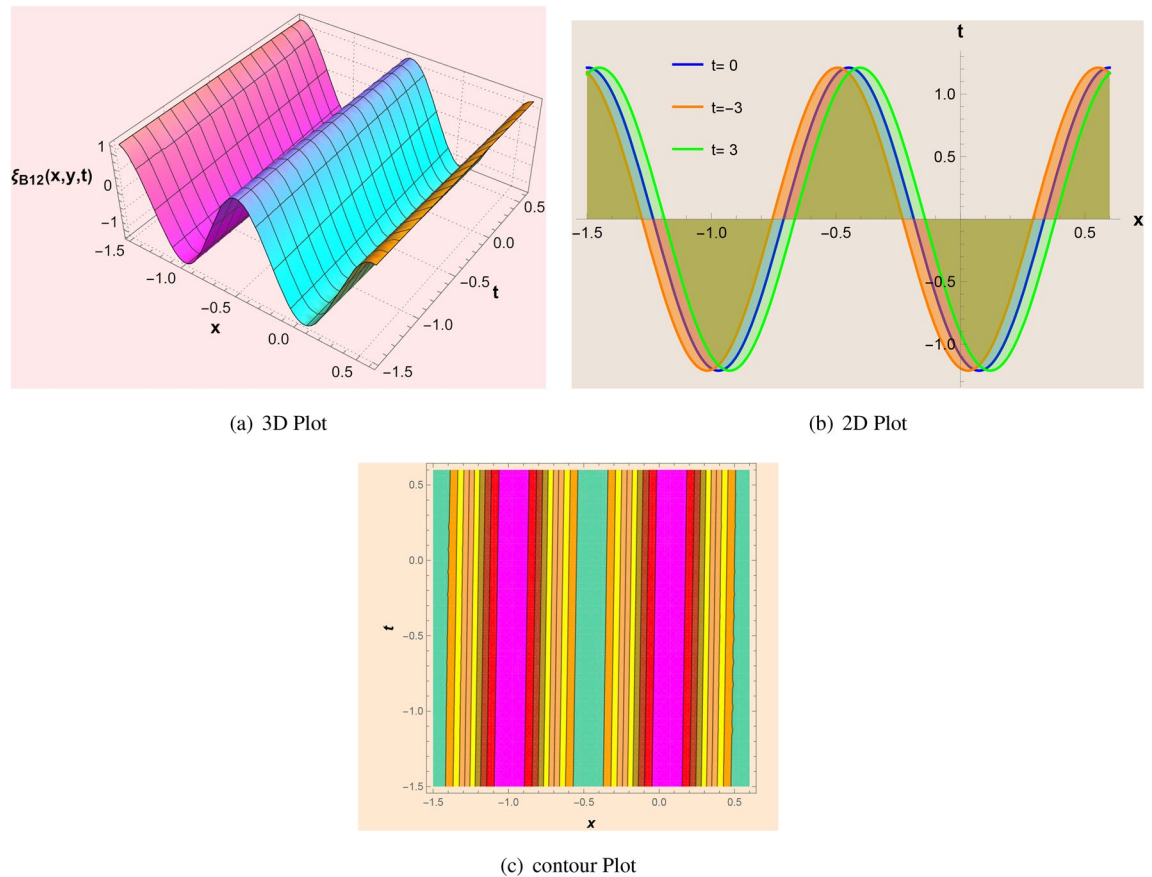


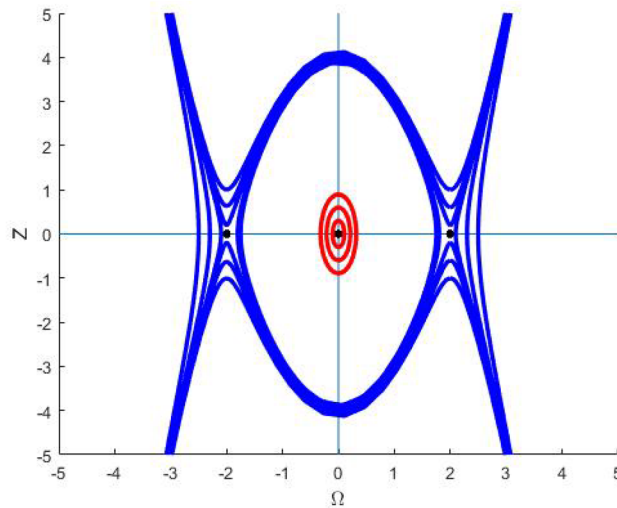
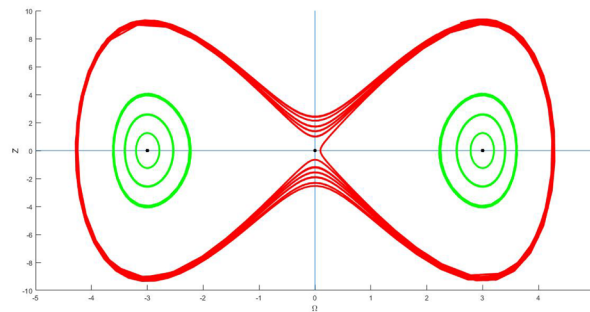
Fig. 7. 3D, 2D and contour representation of W-shaped soliton for real behaviour for Eq. (4.21) with parameters $b_1 = 1.2$, $b_2 = 1$, $z_1 = -0.5$, $z_2 = 0.5$, $z_3 = -0.5$, $c_1 = 0.9$, $c_2 = 0.5$, $a = 0.5$, $d_0 = -0.9$, $\mu = 1.5$, $p_1 = 1$, $p_2 = -1$, $p_3 = 1$, $\theta_1 = -6$, $\theta_2 = 0.9$, $\theta_3 = 0.09$, $\lambda = 16$, $y = 0.5$ and $\gamma = 0$.

Physical significance of proposed study

Figures 1 and 5 show kink soliton of absolute behavior for Eq. (4.2) and for real behaviour for Eq. (4.9) respectively, a kink solution in the SCNS equation characterizes a temporal or spatial change between two different system states. This can be equivalent to a shift in polarization, phase, or density across a material, for instance, in optical or condensed matter systems. A kink solution in nonlinear optics could be equivalent to a domain wall dividing areas with varying polarization states or refractive indices. It could signify a change between distinct quantum phases in Bose-Einstein condensates or quantum Hall systems. Then, Fig. 2 illustrates anti-kink soliton of absolute behavior for Eq. (4.3), anti-kink graphs are a type of solitary wave solutions to the SCNS equation that exhibit an abrupt, localized transition in the field variable, but in the opposite direction of a kink. Depending on the convention, an anti-kink links a higher asymptotic state to a lower one, whereas a kink usually links a lower asymptotic state to a higher one. When modeling optical solitons in fibers, transitions in Bose-Einstein condensates, and domain barriers in condensed matter systems—particularly when noise or disorder is present—anti-kink solutions are important.

Figure 3 shows periodic soliton for real behavior for Eq. (4.5). In SCNS equation, periodic graphs are solutions in which the field variable, despite stochastic (random) influences, shows regular, repeated oscillations in space and/or time. Regular oscillations in fluid mechanics and plasma physics, such as capillary waves or periodic changes in plasma density, are modeled using periodic solutions. While Fig. 4 shows V-shaped soliton for real behaviour for Eq. (4.6), V-shaped graphs represent abrupt, localized defects or phase slips, including intensity drops, density dips, or phase discontinuities, that are physically present even when noise is present. These characteristics are crucial in quantum fluids, nonlinear optics, and other systems that are represented by the nonlinear Schrödinger equation. They show how nonlinearity, dispersion, chirality, and stochastic effects interact to shape resilient, localized phenomena.

Figure 6 shows M-shaped soliton for real behaviour for Eq. (4.20). In the SCNS equation, M-shaped graphs physically depict multi-peak solitonic structures or limit states that result from nonlinear interactions and may be influenced by chirality and noise. In a variety of physical systems, such as quantum fluids and optics, these solutions are pertinent and represent intricate dynamical or quantum phenomena including interference, pattern generation, and multi-mode coexistence. Whereas Fig. 7 shows W-shaped soliton for real behaviour for Eq. (4.21), W-shaped graphs, which are influenced by noise and chirality and arise from nonlinear interactions,

(a) $M_1 > 0$ and $M_2 > 0$ (b) $M_1 < 0$ and $M_2 < 0$ **Fig. 8.** Phase portrait for case 1 and case2 of bifurcation.

physically depict multi-hump solitonic structures or bound states in the SCNS equation. These solutions describe intricate dynamical or quantum phenomena including pattern generation, interference, and multi-mode coexistence and are pertinent to a variety of physical systems, such as optics and quantum fluids.

The SCNS equation is used to model the dynamics of wave evolution in nonlinear and dispersive media subjected to chirality and randomness. The equation finds applications in understanding intricate physical systems like optical fibers, Bose-Einstein condensates and plasma physics, where wave propagation is governed by nonlinearity, random perturbations and chirality-induced effects. Through the generalized Kudryashov method, the research seeks to derive exact or approximate soliton solutions that facilitate predictions of the response of nonlinear waves to stochastic forces. The solutions provide insight into waveforms' stability and structure and are useful for providing vital information on energy localization, pulse shaping, and transmission in physical environments. In addition, bifurcation analysis enables us to investigate the qualitative wave dynamics changes caused by changes in system parameters. This analysis is essential for the comprehension of phase transitions, chaotic behavior, sensitivity analysis and the appearance of solitonic structures in various physical systems.

In general, this work contributes to the general understanding of nonlinear wave phenomena in stochastic environments, providing potential applications in nonlinear optics, fluid dynamics and quantum mechanics.

Limitation of methods

The two different formats of generalized kudryashov method are used above in this paper, the generalized kudryashov method-1 is the standard rational format and the generalized kudryashov method-2 is the polynomial format. In standard rational format, the ansatz solution is a ratio of two polynomials in $\Omega(\rho)$. It is capable of modeling single, or rational solutions and good for equations that may have singularities and significant nonlinearities. Whereas polynomial format has solution which is just a polynomial in $\Omega(\rho)$. It simulates bounded, regular, and smooth solutions and is ideal for PDEs with minimal nonlinearity and smooth solutions.

Among the drawbacks of the standard rational format includes some rational ansatzs can collapse into trivial solutions, finding coefficients a_j and b_i results in large nonlinear algebraic systems and sometimes the rational structure is "too much" for simple PDEs, creating artificial complexity. While polynomial format only works with

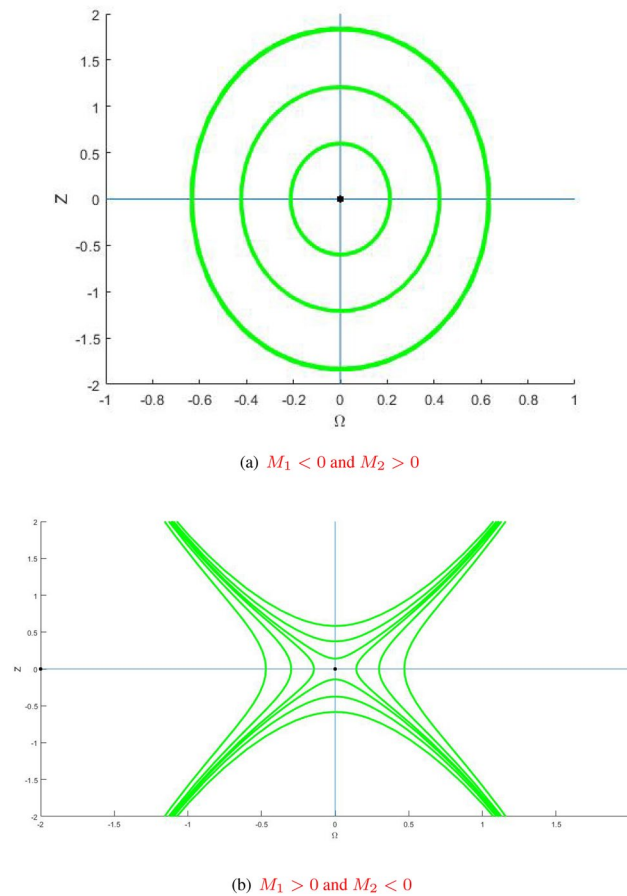


Fig. 9. Phase portrait for case 3 and case 4 of bifurcation.

smooth solutions, it may overlook some physically important solutions, such as peakons or rational solitons, and is unable to capture sharp kink, poles, or explosive behaviors.

The standard rational format of gKM is generally preferred for solving the $(2 + 1)$ -dimensional SNCS equation because it can better capture localized wave packets and random dispersions, allows for more degrees of freedom (because of the numerator and denominator polynomials), models stochastic instabilities, and has sharp interfaces when necessary.

Conclusion

This study has been investigated the $(2 + 1)$ -dimensional stochastic chiral nonlinear Schrödinger equation using two different variations of the generalized Kudryashov approach. Numerous precise soliton solutions have been derived, representing various structural dynamics like kink, anti-kink, periodic, M-shaped, W-shaped, and V-shaped waveforms. The 2D, 3D, and contour plots gave the direct visual representations of the solution behavior.

Bifurcation analysis has been conducted to study critical transitions and complex dynamical structures of the system. Chaotic behavior and sensitivity analysis to initial conditions have also been investigated, with a focus on the significant impact of stochastic effects on long-term behavior and stability.

The results further provided understanding of stochastic and chiral effects in nonlinear wave equations and illustrated the validity of the generalized Kudryashov method in obtaining valuable soliton solutions of complex systems. The results could potentially find use in nonlinear optics, fluid mechanics, and complex physical phenomena modeling with effects of randomness and chirality. Other higher-dimensional and multi-component stochastic nonlinear systems may be targeted in future research with this technique to further investigate their complex dynamical characteristics.

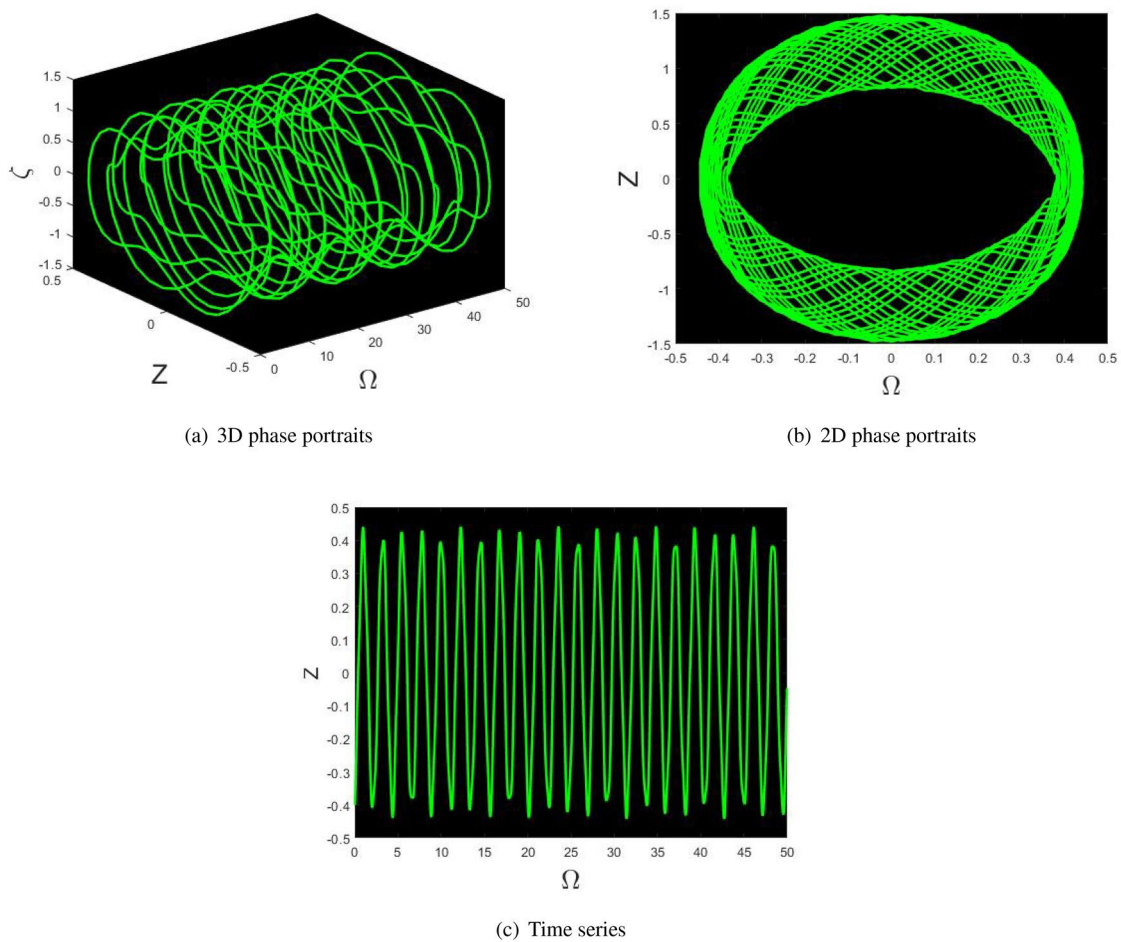


Fig. 10. Characterization and Detection of Chaotic Dynamics of Eq. (5.3) when $k_0 = 0.1$ and $f = 2$.

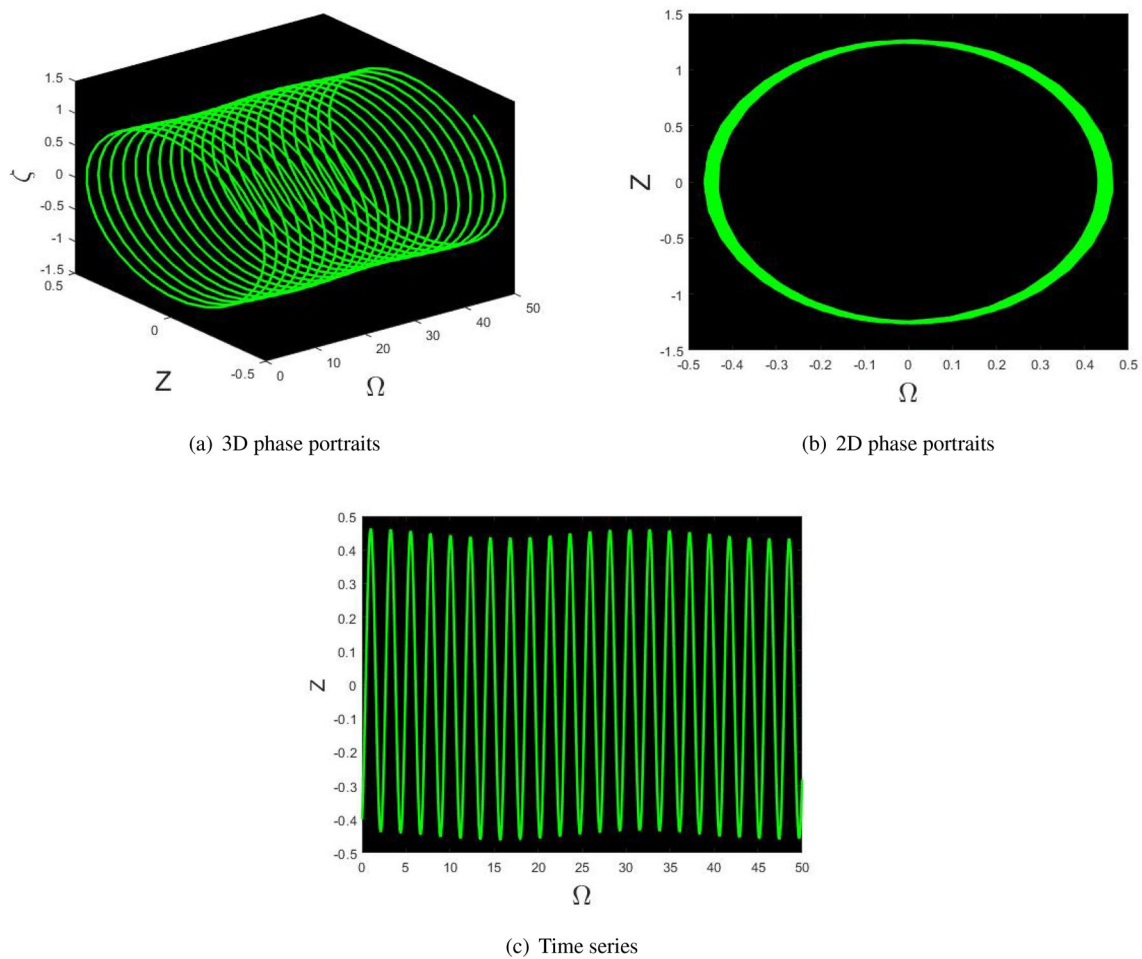


Fig. 11. Characterization and Detection of Chaotic Dynamics of Eq. (5.3) when $k_0 = 3$ and $f = 2\pi$.

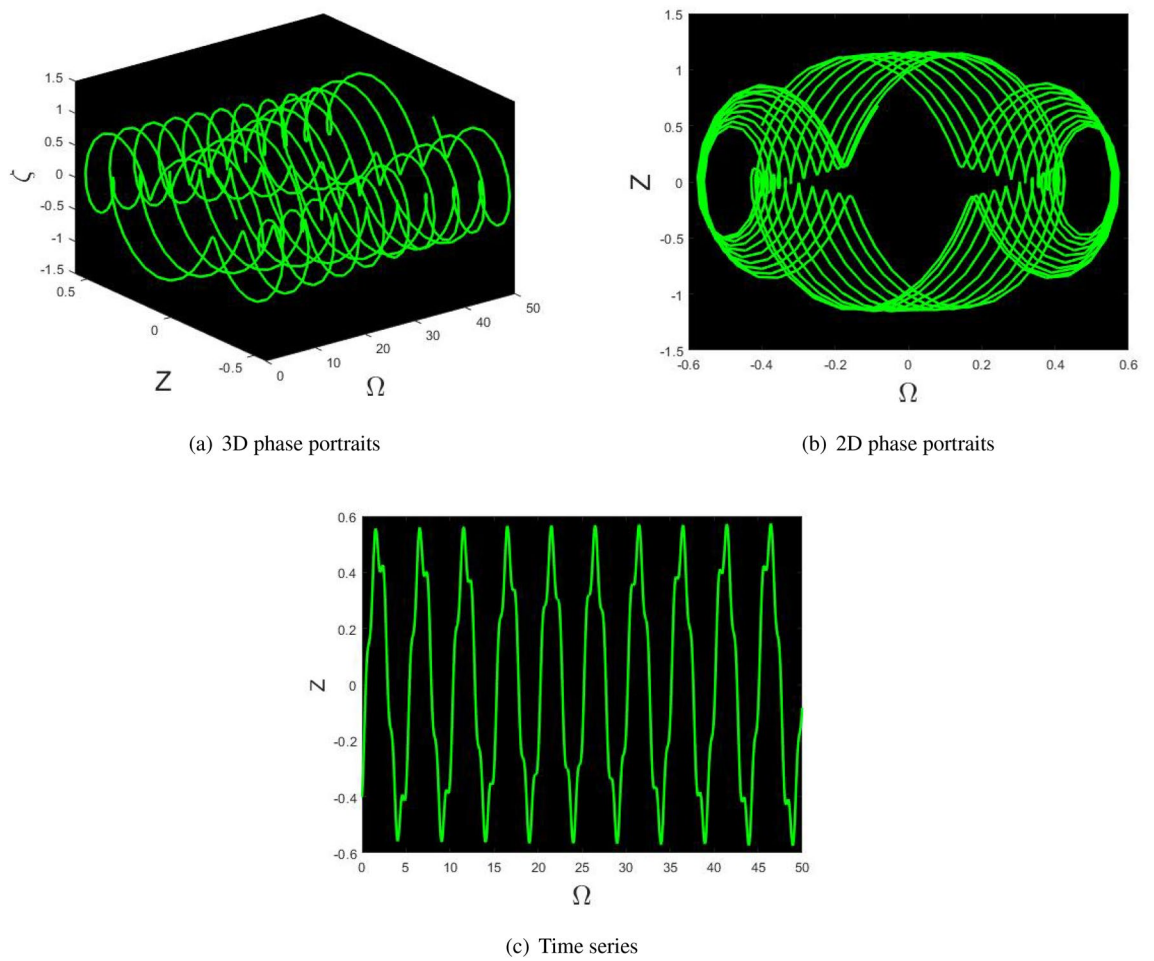


Fig. 12. Characterization and Detection of Chaotic Dynamics of Eq. (5.3) when $k_0 = 3$ and $f = 10$.

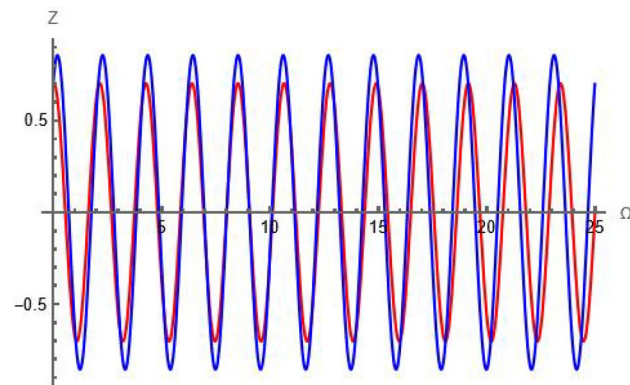


Fig. 13. The system characterized by Eq. (5.3) was subjected to sensitivity analysis with considering two distinct initial conditions: blue (0.7, 1.5) and red (0.7, 0.2).

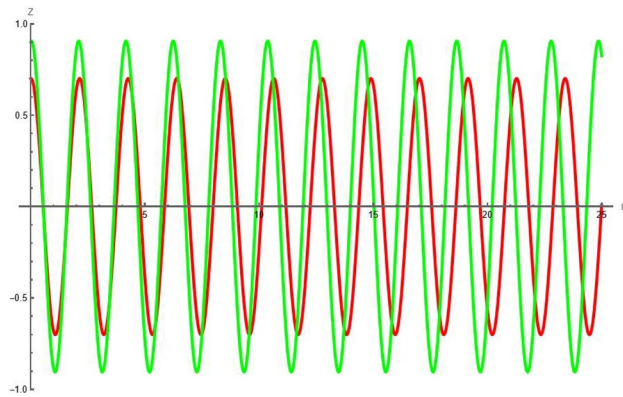


Fig. 14. The system characterized by Eq. (5.3) was subjected to sensitivity analysis with considering two distinct initial conditions:red colour(0.7, 0.1) and green colour (0.9, 0.3).

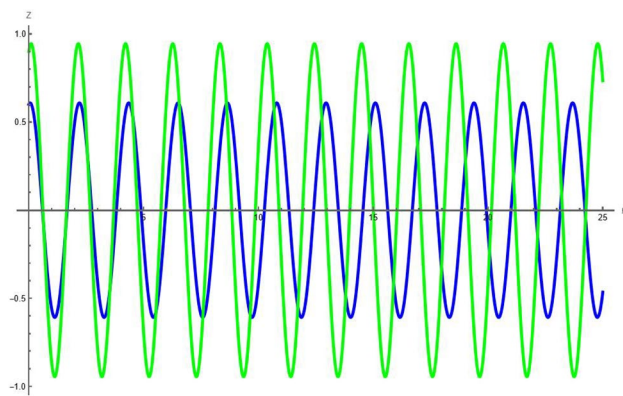


Fig. 15. The system characterized by Eq. (5.3) was subjected to sensitivity analysis with considering two distinct initial conditions:blue colour(0.6, 0.3) and green colour (0.9, 0.9).

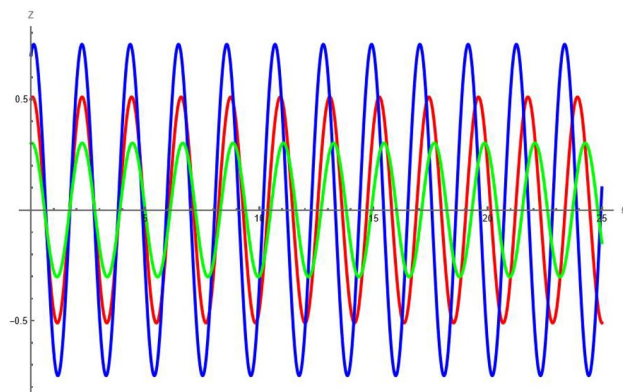


Fig. 16. The system characterized by Eq. (5.3) was subjected to sensitivity analysis with considering two distinct initial conditions: red colour (0.5, 0.3), green colour (0.3, 0.1) and blue colour(0.7, 0.8).

Data availability

The data and materials used to support the findings of this study are included in this article.

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Author contributions

Manal Alqhtani: Formal analysis, Visualization, Writing—review & editing. Afifa Shahbaz: Writing—original draft, Methodology. Muhammad Abbas: Supervision, Methodology, Writing—original draft. Khaled M. Saad: Formal analysis, Visualization, Writing—review & editing. Asnake Birhanu: Visualization, Software, Writing—original draft, Writing—original draft. Muhammad Zain Yousaf: Visualization, Methodology, Writing—original draft. All authors have read and agreed to publish the manuscript.

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Declarations

Conflicts of interest

There is no conflict of interest.

Ethical approval

We hereby declare that this manuscript is the result of our independent creation. This manuscript does not contain any research achievements that have been published or written by other individuals or groups.

Additional information

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