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Network design for bypass roads using interval valued fuzzy outerplanar graphs

Deivanai Jaisankar[1], Sujatha Ramalingam[1✉], Nagarajan Deivanayagampillai[2] & Tadesse Walelign[3✉]

This paper presents a novel approach to bypass road network design using interval valued fuzzy outerplanar graphs (IVFOGs), addressing the increasing demands of vehicular growth and evolving lifestyles. The uncertainty and variability inherent in urban traffic regulation are effectively managed through the application of interval valued fuzzy sets, which capture linguistic and imprecise traffic parameters. The study investigates the structural properties of IVFOGs and their corresponding duals, providing a solid theoretical foundation for modeling bypass networks. The concepts of vertex and edge deletion are explored to construct and analyze optimal bypass routes that avoid congested in urban centers. Furthermore, examples are provided to study the maximal and maximum interval valued fuzzy outerplanar graphs determined by both vertex and edge deletion. This framework enhances mobility, optimizes commuter travel time, and contributes to efficient road transport planning by offering a flexible, uncertainty-tolerant model tailored for real-world traffic scenarios.

Fuzzy set theory, introduced by Lotfi A. Zadeh[1] in 1965, extends classical set theory by allowing for degrees of membership. This framework is particularly useful in dealing with situations where the boundaries of sets are not well-defined. A significant early contribution came from Azriel Rosenfeld[2] pioneered the use of fuzzy set theory in image analysis and pattern recognition, laying the groundwork for further advancements. Fuzzy graph theory emerged as an extension of classical graph theory to handle ambiguous and uncertain relationships among elements. Algorithms have been developed for fuzzy shortest paths, clustering, isomorphism detection, and more, enabling applications in decision-making, social network analysis, and systems modeling. Abdul-Jabbar[3] introduced the concept of fuzzy planar graphs and fuzzy dual graphs, showing that the dual of a dual fuzzy graph is the original fuzzy graph and establishing that the dual of a fuzzy bipartite graph is Eulerian.

Contributions by Harary[4] emphasized key features of planar, non-planar, and dual graphs. Berthold et al.[5] built on Kuratowski's property, which states that a graph is non-planar if it contains subdivisions of K_5 or $K_{3,3}$. Additionally, Chartrand et al.[6] identified outerplanar graphs as those that do not contain a subgraph homeomorphic to $K_4 - x$, where x is an edge of K_4 and discussed the concept of permutation graphs. These graphs are created by taking two identical copies of a labeled graph G and connecting them based on a permutation α of the vertices of G. The study focuses on determining the conditions under which such permutation graphs are planar and provides a criterion for this. The Petersen graph is cited as an example of a non-planar permutation graph. Further exploration of outerplanar graphs was presented[7].

In parallel, Kulli[8] has worked on minimally nonouterplanar graphs, focusing on their properties and characteristics. The study throws some pertinent lights on their structure as well as consequences of these attributes in graph theory. Their work enhanced understanding of structural and computational properties within outerplanar graphs. Mitchell[9] had suggested linear algorithms that identify maximal outerplanar and outerplanar graphs. These algorithms improved upon previous methods by leveraging the characteristics of biconnected graphs and 2-vertices. Fleischner et al.[10] concentrated exclusively on developing a criterion to assess the planarity of permutation graphs, without exploring the characteristics of outerplanar graphs or their weak duals. Syslo[11] investigated outerplanar graphs, providing characterizations and developing methods for

[1]Mathematics, School of Science and Humanities, Shiv Nadar University Chennai, Rajiv Gandhi Salai (OMR), Chengalpattu District, Kalavakkam 603110, Tamil Nadu, India. [2]Department of Mathematics, Chettinad Institute of Technology, Chettinad Academy of Research and Education, Chennai, India. [3]Department of Mathematics, Debre Tabor University, Debre Tabor, Ethiopia. ✉email: sujathar@chennai.edu.in; tadelenyosy@gmail.com

testing, coding, and counting these graphs. Furthermore, investigated the subgraph isomorphism problem for outerplanar graphs, proving that it remains NP-complete even under strong connectivity constraints, except for the induced subgraph isomorphism problem on 2-connected outerplanar graphs, for which a polynomial-time algorithm is provided. This algorithm leverages a unique correspondence between 2-connected outerplanar graphs and plane trees to efficiently verify induced subgraph relations are described by[12].

Samanta et al.[13,14] presented extended these ideas to the fuzzy domain by introducing fuzzy multigraphs, fuzzy planar graphs, and fuzzy dual graphs, incorporating planarity values and defining strong fuzzy planar graphs to model real-world networks like subways and tunnels. Innovative concepts pertaining to fuzzy planar graphs, focusing on the characteristics of the fuzzy planarity value are presented and interesting theorems related to fuzzy planar graphs are investigated[15]. Shriram et al.[16] defined the concept of fuzzy combinatorial dual graphs, closely related to fuzzy dual graphs, and establishes several key results regarding their properties. This work contributes to the broader field of fuzzy graph theory, emphasizing its relevance and applications in various domains. Model for forming fuzzy planar subgraphs aimed at partitioning large-scale integration networks. Their approach emphasizes vertex-deletion and edge-deletion operations to create fuzzy planar subgraphs. They also proposed a new Planar Partition subgraph method to enhance the efficiency of identifying these subgraphs and discussed the concept of thickness value within fuzzy graphs. The results are demonstrated through examples and applied to network partitioning in VLSI design are introduced[17].

Parvathi and Karunambigai[18] introduced a new definition for intuitionistic fuzzy graphs (IFGs) and explores their properties, including concepts like semi-μ strong paths and bridges and discussed the potential extension of these concepts to other types of intuitionistic fuzzy sets and their components. Three new product operations: direct product, lexicographic product, and strong product of interval-valued intuitionistic $(s, t)-$ fuzzy graphs, enhancing the framework for analyzing these structures are proposed. And also defined the concepts such as regular and totally regular interval-valued intuitionistic fuzzy graphs, along with busy and free vertices, emphasizing their significance in applications like reliable communication networks and fuzzy social networks are introduced[19]. Rehman et al.[20] introduced the concept of domination in intuitionistic fuzzy influence graphs (IFIGs), extending fuzzy graph theory by incorporating intuitionistic fuzzy sets to better handle uncertainty through membership and non-membership degrees and it's defined several key notions such as strong influence pairs, influence cut nodes, and the strong pair domination number within IFIGs. Furthermore, various related concepts such as strong and weak q-fractional fuzzy influence pairs, q-fractional fuzzy influence cut-nodes, and domination numbers, and apply these to locate and control smog areas, demonstrating the model's practical effectiveness and comparing it favourably with existing methods and decision-making techniques are introduced[21]. Zuo[22] introduced the concept of picture fuzzy graphs (PFGs), extending fuzzy and intuitionistic fuzzy graphs by incorporating positive, neutral, and negative membership degrees to better model uncertainty in real-life scenarios and defined the various types of PFGs, operations on them, and demonstrates their application, particularly in social networks.

The application of bipolar fuzzy sets to graph structures, defining concepts such as bipolar fuzzy graph structures, bipolar fuzzy cycles, trees, cut vertices, and bridges, while exploring their properties through various examples. Additionally, the authors are investigated complementation types within bipolar fuzzy graph structures, enhancing the understanding of their theoretical framework are introduced[23]. Regular and totally regular bipolar fuzzy graphs and establishes necessary and sufficient conditions for their equivalence are presented. Additionally, discussed the bipolar fuzzy line graphs and their properties, including conditions for isomorphism between bipolar fuzzy graphs and their corresponding line graphs are investigated[24]. Akram[25] introduced bipolar fuzzy planar graphs: such graphs may well form a category that is different from traditional fuzzy planar graphs. Furthermore, the bipolar fuzzy planarity value has been defined to numerically value the degree of planarity in these graphs. Pramanik et al.[26] defined the properties of bipolar fuzzy planar graphs and bipolar fuzzy dual graphs are examined, and an application of these concepts is discussed for image shrinking layout. Three new operations on bipolar fuzzy graphs: direct product, semi-strong product, and strong product, along with sufficient conditions for their completeness, then demonstrated that any product of strong bipolar fuzzy graphs remains a strong bipolar fuzzy graph, contributing to the theoretical framework of this growing research area are introduced[27].

Ghorai et al.[28,29] introduced m-polar fuzzy multi-graphs to overcome edge crossings that exist in a planar graph. Edge crossings in various applications are problematic; they defined the key concepts such as m-polar fuzzy multi-sets, planar graphs, and strong edges and went on in the exploration of the properties. Mondal et al.[30] generalized m-polar fuzzy graphs are explored along with their properties, and this includes the study of generalized m-polar fuzzy planar graphs and generalized m-polar fuzzy dual graphs, an illustrative example is provided to demonstrate the application of generalized m-polar fuzzy graphs to a social group network problem.

Recent developments include interval-valued fuzzy graphs, where vertex and edge memberships are represented by intervals. Mahapatra et al.[31] conducted a study on interval-valued m-polar fuzzy graphs, and significant contributions involve the establishment of various concepts associated with IVmPF graphs, such as complete IVmPF graphs, strong IVmPF graphs, and the faces and dual graphs of IVmPF planar graphs. Additionally, the authors presented a practical application of IVmPF planar graphs. The idea of Generalized Neutrosophic Planar Graphs (GNPG) was proposed Mahapatra et al.[32]. Additionally, they illustrated the practical uses of GNPG in urban traffic management are achieved.

The interval-valued fuzzy line graphs, expanding traditional fuzzy graph theory and also established the conditions for isomorphism between interval-valued fuzzy graphs and their line graphs, highlighting applications in database theory and decision-making systems are presented[33]. Introduced the various types of interval-valued fuzzy graphs, including balanced and highly irregular variants, and explores their properties and relationships with intuitionistic fuzzy graphs, necessary and sufficient conditions for the equivalence of certain irregular interval-valued fuzzy graphs are also established[34]. The simplified interval-valued Pythagorean fuzzy

sets (SIVPFS) to streamline the calculations of interval-valued Pythagorean fuzzy sets (IVPFS) and developed a novel concept of simplified interval-valued Pythagorean fuzzy graphs for representing uncertain information in decision-making processes. Furthermore, the authors presented the aggregation operators and apply the SIVPFG in multi-agent decision-making scenarios with a numerical example to demonstrate its practicality are introduced[24]. Arif et al.[35] presented interval-valued picture (S, T)-fuzzy graphs (IVP- (S, T)-fuzzy graphs) as an extension of picture fuzzy graphs, aimed at effectively addressing problems involving uncertainty. Various operations and structural properties are explored, culminating in applications for multiple attribute decision-making (MADM). Pal et al.[36] introduced the theoretical concepts related to interval-valued fuzzy graphs, focusing on the degree of an edge and its total degree, derived from operations like Cartesian product and composition, emphasizes their applications in computer science areas such as data mining and networking. The concept of interval-valued fuzzy graphs, extending traditional fuzzy sets to represent uncertainty more effectively through interval membership degrees and explored the various operations and properties of these graphs, alongside their applications in fields like fuzzy control and approximate reasoning are discussed[37].

Additionally, researchers like Rashmanlou et al.[27] introduced three new operations on interval-valued fuzzy graphs: direct product, semi-strong product, and strong product, providing sufficient conditions for their completeness, established that if any of these products is complete, then at least one of the factors must be a complete interval-valued fuzzy graph. the concepts of antipodal interval-valued fuzzy graphs and self-median interval-valued fuzzy graphs, exploring their isomorphism properties. And emphasized the application of these graphs in enhancing precision and flexibility in various fields, including computer science and soft computing are introduced[38]. The ring sum and tensor product of product interval-valued fuzzy graphs, establishing key theorems on their relationships, then defined and analysed balanced and strictly balanced interval-valued fuzzy graphs are investigated[39]. Isometric relationships, demonstrating that isometry is an equivalence relation that preserves the interval-valued memberships of vertices are defined[40]. Three new operations: strong product, tensor product, and lexicographic product on interval-valued fuzzy graphs, highlighting their flexibility and compatibility for various applications in computer science, investigated the degrees of vertices resulting from these operations, contributing to a deeper understanding of the structural properties of interval-valued fuzzy graphs and their utility in fields like data mining and networking are defined[41.] Talebi et al.[42] introduced the concepts of regular and totally regular interval-valued fuzzy graphs, emphasizing their flexibility over traditional fuzzy graphs by allowing membership degrees to be represented as intervals. And also defined busy and free vertices, explores their properties under isomorphism, and discussed self μ-complementary interval-valued fuzzy graphs, providing necessary conditions for their existence.

In this context, outerplanar graphs originally defined in classical settings are extended into the fuzzy domain with interval-valued membership functions. This paper explores interval-valued fuzzy outerplanar graphs and their various properties. The main objective of this paper is to develop and analyze the structural properties of interval-valued fuzzy outerplanar graphs, and to demonstrate their applicability in real-world transportation network modeling. The specific aims are as follows:

- To define and characterize interval-valued fuzzy outerplanar graphs and explore their properties in relation to classical outerplanar graphs.
- To study the process of forming subgraphs of IVFOGs via vertex and edge deletion, and to differentiate between maximal and maximum IVFOG subgraphs.
- To introduce and investigate the concept of interval-valued fuzzy dual graphs, particularly within the context of outerplanar structures.
- To present a case study on bypass road networks, illustrating how IVFOGs can model uncertainty in urban planning and infrastructure design.

By addressing these objectives, this study not only fills a notable gap in fuzzy graph theory but also highlights the practical advantages of IVFOGs in contemporary applications, particularly where uncertainty and adaptability are critical.

Motivation

Fuzzy set theory was first introduced by Lotfi A. Zadeh[1] in 1965, extending classical set theory by allowing elements to have degrees of membership. This foundational concept is particularly effective for modeling situations with ambiguous or imprecise boundaries. Building on this idea, Azriel Rosenfeld[2] was among the first to apply fuzzy set theory to graphs, marking the beginning of fuzzy graph theory.

Further advancements came with Abdul-Jabbar[3], who introduced the concept of fuzzy planar graphs, incorporating fuzziness into the structure of planar graphs and extending classical notions of planarity. As research progressed, more refined models were proposed to capture uncertainty more effectively. One such development was the introduction of interval-valued fuzzy graphs by Akram et al.[43], who explored various operations such as Cartesian product, composition, union, and join, along with structural properties like completeness and self-complementarity. These graphs allow each vertex and edge to have an interval of membership values, adding flexibility in representing uncertain systems.

To better assess the spatial structure of fuzzy graphs, Pramanik et al.[44] introduced the concept of interval-valued fuzzy planar graphs and proposed the "degree of planarity" as a metric. They also studied related properties such as strong edges, fuzzy faces, and interval-valued fuzzy dual graphs, providing deeper insights into the topological features of fuzzy graphs.

In a more recent development, Deivanai et al.[45] introduced fuzzy outerplanar graphs, extending the classical outerplanar graph concept into the fuzzy domain. Their work examined how fuzziness can model ambiguous relationships in real-world networks, especially in the context of transportation systems. They investigated the

properties of maximal and maximum fuzzy outerplanar subgraphs, highlighting their usefulness in network optimization.

Building upon these contributions, this paper aims to investigate the notion of outerplanarity within interval-valued fuzzy graphs, combining the flexibility of interval-valued memberships with the structural simplicity of outerplanar graphs. The study explores their theoretical properties and practical applications, particularly in modeling and optimizing transport networks under uncertain and dynamic conditions. Basic notations and meaning given in the Table 1.

Preliminaries

This section explains the basic fundamental concepts associated with interval-valued fuzzy graphs and interval valued fuzzy planar graphs.

Definition 1 Ref[43]. Let $\mathbb{V}$ be a non-empty set. For $i = 1, 2, \ldots, p$, consider mappings $\tau_i^- : \mathbb{V} \to [0,1]$ and $\tau_i^+ : \mathbb{V} \to [0,1]$ such that $\tau_i^-(x) \leq \tau_i^+(x)$ for all $x \in \mathbb{V}$. An interval-valued fuzzy multiset on $\mathbb{V}$ is represented as $(\mathbb{V}, [\tau_i^-, \tau_i^+] = \{(x, [\tau_i^-, \tau_i^+]) \mid x \in \mathbb{V}, i = 1, 2, \ldots, p\}$.

Definition 2 Ref[43]. Let $\mathbb{V}$ be a non-empty set. Define the mappings $\tau_i^- : \mathbb{V} \to [0,1]$ and $\tau_i^+ : \mathbb{V} \to [0,1]$ such that $\tau_i^-(x) \leq \tau_i^+(x)$ for all $x \in \mathbb{V}$ and $i = 1, 2, \ldots, p$, which constitute an IVFMS on $\mathbb{V}$. Additionally, let $\delta_j^- : \mathbb{V} \times \mathbb{V} \to [0,1]$ be mappings such that $\mathbb{B} = \{(\mathbb{V} \times \mathbb{V}, [\delta_j^-, \delta_j^+]), j = 1, 2, \ldots, q\}$, forms an IVFMS on $\mathbb{V} \times \mathbb{V}$. $(\mathbb{A}, \mathbb{B})$ is termed an IVFMG if the conditions $\delta_j^-(x, y) \leq \min\{\tau_i^-(x), \tau_i^-(y)\}$ and $\delta_j^+(x, y) \leq \min\{\tau_i^+(x), \tau_i^+(y)\}$ hold for all $x, y \in \mathbb{V}$, for every $i = 1, 2, \ldots, p, j = 1, 2, \ldots, q$.

Definition 3 Ref[44]. The strength of an edge $(\mathfrak{a}, \mathfrak{b})$ is represented by an interval $I_{(\mathfrak{a}, \mathfrak{b})} = [I^-_{(\mathfrak{a}, \mathfrak{b})}, I^+_{(\mathfrak{a}, \mathfrak{b})}]$, where $I^-_{(\mathfrak{a}, \mathfrak{b})} = \frac{\delta^-(\mathfrak{a}, \mathfrak{b})}{\min\{\tau^+(\mathfrak{a}), \tau^+(\mathfrak{b})\}}$ and $I^+_{(\mathfrak{a}, \mathfrak{b})} = \frac{\delta^+(\mathfrak{a}, \mathfrak{b})}{\min\{\tau^+(\mathfrak{a}), \tau^+(\mathfrak{b})\}}$. An edge $(\mathfrak{a}, \mathfrak{b})$ is considered non-trivial if $I^-_{(\mathfrak{a}, \mathfrak{b})} > 0$.

Definition 4 Ref[44]. Let $\xi = (\mathbb{A}, \mathbb{B})$ be an IVFMG. An edge $(\mathfrak{a}, \mathfrak{b})$ in ξ is called interval-valued fuzzy strong if $I_{(\mathfrak{a}, \mathfrak{b})} \geq [0.5, 0.5]$; otherwise, it is considered interval-valued fuzzy weak.

Definition 5 Ref[44]. In an interval-valued fuzzy multigraph (IVFMG) denoted as $\xi = (\mathbb{A}, \mathbb{B})$, the set $\mathbb{B}$ contains two edges: $((\mathfrak{a}, \mathfrak{b}), [\delta^-(\mathfrak{a}, \mathfrak{b}), \delta^+(\mathfrak{a}, \mathfrak{b})])$ and $((\mathfrak{c}, \mathfrak{d}), [\delta^-(\mathfrak{c}, \mathfrak{d}), \delta^+(\mathfrak{c}, \mathfrak{d})])$, which intersect at a point P. To

Notations	Meanings
G	Crisp graph
ξ	an interval valued fuzzy graph
$\mathbb{V}$	set of vertices of ξ
$\mathbb{E}$	set of edges of ξ
$\mathbb{X}$	universal set
$\mathfrak{m}$	A membership function that maps elements from a crisp set to the interval [0,1]
$\mathbb{A}$	a fuzzy set
τ	The function that assigns membership values to the vertices
δ	The function that assigns membership values to the edges
$[\tau^-, \tau^+]$	the membership value of a vertex, τ^- and τ^+ represents the left end and right end points respectively
$[\delta^-, \delta^+]$	the membership value of an edge, δ^- and δ^+ represents the left end and right end points respectively
$[\tau_i^-, \tau_i^+]$	the i th membership value of a vertex, τ_i^- and τ_i^+ represents the left end and right end points respectively
$[\delta_i^-, \delta_i^+]$	The value of membership for the edge connecting j th two vertices is represented by δ_i^- and δ_i^+, denoted the left and right end points respectively
$I_{(\mathfrak{a}, \mathfrak{b})}$	The strength of the edge $(\mathfrak{a}, \mathfrak{b})$ is an interval number with the left end point $I_{(\mathfrak{a}, \mathfrak{b})} = \frac{\delta^-(\mathfrak{a}, \mathfrak{b})}{\{\tau^+(\mathfrak{a}) \wedge \tau^+(\mathfrak{b})\}}$. The right end point is $I_{(\mathfrak{a}, \mathfrak{b})} = \frac{\delta^+(\mathfrak{a}, \mathfrak{b})}{\min\{\tau^+(\mathfrak{a}) \wedge \tau^+(\mathfrak{b})\}}$
$I_{\mathscr{P}}$	Intersecting number at the point of crossing of two edges $(\mathfrak{a}, \mathfrak{b})$ and $(\mathfrak{c}, \mathfrak{d})$, which is an interval number whose left end point is $I_{\mathcal{P}}^- = \frac{I^-(\mathfrak{a}, \mathfrak{b}) + I^-(\mathfrak{c}, \mathfrak{d})}{2}$ and the right end point is $I_{\mathcal{P}}^+ = \frac{I^+(\mathfrak{a}, \mathfrak{b}) + I^+(\mathfrak{c}, \mathfrak{d})}{2}$
	The degree of planarity in an interval-valued fuzzy graph refers to the characteristics of an interval number, specifically its left endpoint is $\mathfrak{f}^- = \frac{1}{1 + \{I^+_{P_1} + I^+_{P_2} + \cdots + I^+_{P_k}\}}$ and the right end point is $\mathfrak{f}^+ = \frac{1}{1 + \{I^-_{P_1} + I^-_{P_2} + \cdots + I^-_{P_k}\}}$, k represent the number of intersection points

Table 1. Some basic notions.

analyse these edges, then calculate their respective interval numbers $I_{(a,b)}$ and $I_{(c,\partial)}$. The intersecting number at point P is defined as:

$$I_P = \left[I_P^-, I_P^+ \right] = \left[\frac{I^-_{(a,b)} + I^-_{(c,\partial)}}{2}, \frac{I^+_{(a,b)} + I^+_{(c,\partial)}}{2} \right]$$

Here, I_P represents an interval number within the range of [0,1]. Next, let consider the concept of planarity. In a traditional sense, a planar graph is defined as one that does not have any edge crossings or intersections. This means that the planarity of such a graph is considered 'full'. Consequently, as the number of intersection points in an IVFMG increases, the degree of planarity decreases. Thus, for IVFMG, it can be concluded that I_P is inversely proportional to the degree of planarity.

Definition 6 Ref[44]. In an interval-valued fuzzy multigraph (IVFMG) denoted as ξ, the points of intersection between edges in a geometrical representation are labeled as $P_1, P_2, \ldots, P_k$, where k is an integer. This configuration allows us to classify ξ as an interval-valued fuzzy planar graph (IVFGP), characterized by its degree of planarity expressed as $\mathfrak{f} = \left(\mathfrak{f}^-, \mathfrak{f}^+ \right)$. The calculations for the endpoints are defined as follows:

The left endpoint $\mathfrak{f}^-$ is given by:

$$\mathfrak{f}^- = \frac{1}{1 + \{ I_{P_1}^+ + I_{P_2}^+ + \cdots + I_{P_k}^+ \}}$$

The right endpoint $\mathfrak{f}^+$ is calculated using:

$$\mathfrak{f}^+ = \frac{1}{1 + \left\{ I_{P_1}^- + I_{P_2}^- + \cdots + I_{P_k}^- \right\}}$$

It is clear that   is bounded and belongs to the set $\mathfrak{f} \in D[0,1]$, which consists of all subintervals of the interval [0,1]. In cases where there are no intersection points in the geometrical representation of an IVFGP, its degree of planarity is represented as [1,1]. This indicates that if the underlying crisp graph corresponding to this IVFGP is a planar graph, it maintains full planarity. Therefore, the presence of intersection points reduces the degree of planarity in the IVFGP.

Definition 7 Ref[44]. Let $\xi = (\mathbb{A}, \mathbb{B})$ be a strong IVFPG characterized by a degree of planarity of [1,1] on a vertex set $\mathbb{V}$. A face of ξ is defined as a region bounded by a subset of edges $\mathbb{E}\prime \subset \mathbb{V} \times \mathbb{V}$ in its geometric representation. The strength of a face is determined by the interval:

$$\text{Strength} = \{[\min \left\{ I^-_{(x, y)} \right\}, \min \left\{ I^+_{(x, y)} \right\}] | (x, y) \in E\prime \}$$

A face is classified as an interval-valued strong fuzzy face if its strength is greater than of [0.5,0.5]; otherwise, it is termed an interval-valued weak fuzzy face. Every interval valued strong IVFPG has an infinite region which is called outer face. Other faces are called inner faces.

Definition 8 Ref[44]. let $\xi = (\mathbb{A}, \mathbb{B})$ be an IVFPG characterized by a degree of planarity represented as [1,1]. This indicates that the graph has no edge crossings. The graph has interval-valued strong faces denoted as $\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_k$, which correspond to a new IVFPG denoted as $\xi_1 = (\mathbb{A}_1, \mathbb{B}_1)$ where $\mathbb{A}_1 = \{(\mathbb{V}_1, [\tau^-, \tau^+]), \mathbb{B}_1 = \{(\mathbb{V}_1 \times \mathbb{V}_1, [\mathfrak{v}_l^-, \mathfrak{v}_l^+])$. In this new graph, the set of vertices is defined as $\mathbb{V}_1 = \{x_i, i = 1, 2, \ldots, k\}$, where each vertex x_i corresponds to the face $\mathfrak{f}_i$ of the original graph ξ.

The membership values of these vertices are determined by the mappings: $\tau^- : \mathbb{V}_1 \to [0,1]$ such that $\tau^-(x_i) = \max \{\delta^-(\mathfrak{u}, \mathfrak{v}) | (\mathfrak{u}, \mathfrak{v})$ is an edge of the boundary of the interval valued fuzzy face $\mathfrak{f}_i\}$. Also, $\tau^+ : \mathbb{V}_1 \to [0,1]$ such that $\tau^+(x_i) = \max \{\delta^+(\mathfrak{u}, \mathfrak{v}) | (\mathfrak{u}, \mathfrak{v})$ is an edge of the boundary of the interval valued fuzzy face $\mathfrak{f}_i\}$.

Between two interval-valued fuzzy faces $\mathfrak{f}_i$ and $\mathfrak{f}_j$ of ξ , there may exist more than one common edges. Consequently, in the dual graph ξ_1, there may also exist multiple edges between the corresponding vertices x_i and x_j. These edges are characterized by their membership degrees denoted as: For the   th edges between vertices x_i and x_j:

The lower membership degree is given by

$$\mathfrak{v}_l^- (x_i, x_j) = \delta^- (\mathfrak{u}, \mathfrak{v})$$

The upper membership degree is given by

$$\mathfrak{v}_l^+ (x_i, x_j) = \delta^+ (\mathfrak{u}, \mathfrak{v})$$

where $(\mathfrak{u}, \mathfrak{v})$ represents a common edge between the two interval-valued fuzzy faces $\mathfrak{f}_i$ and $\mathfrak{f}_j$ and $l = 1, 2, \ldots, s$. with s being the total number of common edges between these faces or the number of edges connecting vertices $\varkappa_i$ and $\varkappa_j$.

Definition 9 Ref[46]. Let $\xi = (\mathbb{A}, \mathbb{B})$ represent an interval-valued fuzzy graph, where $\mathbb{A} = [\delta_{\mathbb{A}-}, \delta_{\mathbb{A}+}]$ and $\mathbb{B} = [\delta_{\mathbb{B}-}, \delta_{\mathbb{B}+}]$ are two interval-valued fuzzy sets defined over a non-empty finite set $\mathbb{V}$ and a subset $\mathbb{E} \subseteq \mathbb{V} \times \mathbb{V}$ representing the edges. The order of the graph ξ is denoted as $\mathbb{O}(\xi)$ and is defined by the interval: $\mathbb{O}(\xi) = [\mathbb{O}^-(\xi), \mathbb{O}^+(\xi)]$ Where the lower bound of the order $\mathbb{O}^-(\xi)$ is calculated as:

$$\mathbb{O}^-(\xi) = \sum_{u \in \mathbb{V}} \delta_{\mathbb{A}-}(\mathfrak{u})$$

The upper bound of the order $\mathbb{O}^+(\xi)$ is given by:

$$\mathbb{O}^+(\xi) = \sum_{u \in \mathbb{V}} \delta_{\mathbb{A}+}(\mathfrak{u})$$

Definition 10 Ref[46]. Let $\xi = (\mathbb{A}, \mathbb{B})$ represent an interval-valued fuzzy graph, where $\mathbb{A} = [\delta_{\mathbb{A}-}, \delta_{\mathbb{A}+}]$ and $\mathbb{B} = [\delta_{\mathbb{B}-}, \delta_{\mathbb{B}+}]$ are two interval-valued fuzzy sets defined over a non-empty finite set $\mathbb{V}$ and a subset $\mathbb{E} \subseteq \mathbb{V} \times \mathbb{V}$ representing the edges. The size of the graph ξ is denoted as $\mathbb{S}(\xi)$ and is defined by the interval: $\mathbb{S}(\xi) = [\mathbb{S}^-(\xi), \mathbb{S}^+(\xi)]$

Where the lower bound of the size $\mathbb{S}^-(\xi)$ is calculated as:

$$\mathbb{S}^-(\xi) = \sum_{u \in \mathbb{V}} \delta_{\mathbb{B}-}(\mathfrak{u}\mathfrak{v})$$

The upper bound of the size $\mathbb{S}^+(\xi)$ is given by:

$$\mathbb{S}^+(\xi) = \sum_{u \in \mathbb{V}} \delta_{\mathbb{B}+}(\mathfrak{u}\mathfrak{v})$$

Interval valued fuzzy outerplanar graph

In this section, introduced the concept of interval valued fuzzy outerplanar graphs, which extend the classical notion of outerplanarity to graphs with interval valued fuzzy values. The focus is on defining the graph structure and understanding its properties in the context of interval valued fuzziness.

Definition 11 An interval valued fuzzy graph ξ qualifies as an interval valued fuzzy outerplanar graph when ξ is embedded in the plane such that every vertex lies on the exterior boundary of the region.

Let $i(\xi)$ represent the number of vertices in a planar embedding that do not depend on the boundary of the outer region. Therefore, $i(\xi) = 0$ for an interval valued fuzzy outerplanar graph.

The graphs given in Fig. 1 and Fig. 2 are examples of interval valued fuzzy outerplanar graphs.
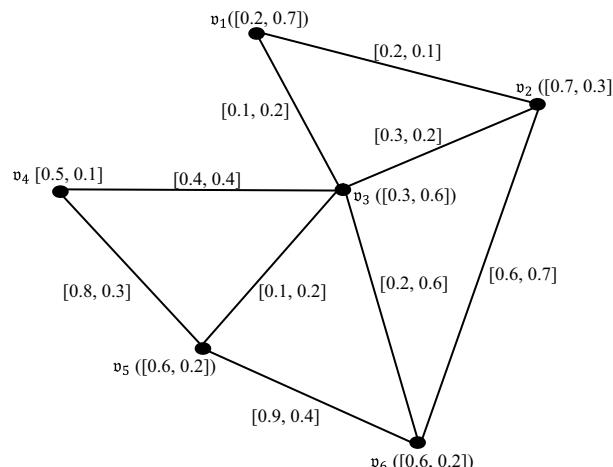

Fig. 1. Interval valued fuzzy outerplanar graph with cycle.
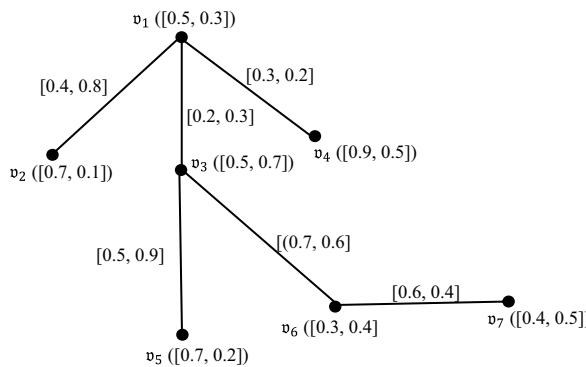
Fig. 2. Interval valued fuzzy outerplanar graph without cycle.
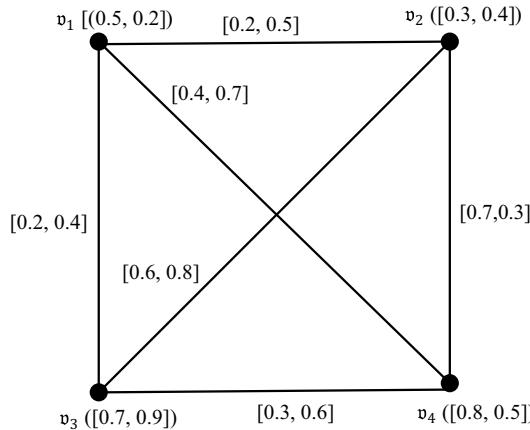
Fig. 3. Interval valued fuzzy graph K_4.

Definition 12 An Interval valued fuzzy graph that cannot be embedded in the plane so that each vertex lies on the exterior region's boundary is known as an Interval valued fuzzy non-outerplanar graph.

In other words, it's an Interval valued fuzzy graph that does not satisfy the property of being Interval valued fuzzy outerplanar, where some vertices are in located the interior of the drawing rather than exclusively on the outer boundary.

If $i(\xi) \neq 0$, then an Interval valued fuzzy planar graph is not an interval valued fuzzy outerplanar graph.

The graphs given in Fig. 3 and Fig. 4 are examples of Interval valued fuzzy non-outerplanar graphs.

### Characterization of interval valued fuzzy outerplanar graph

This section investigates the defining characteristics and structural criteria that characterize interval valued fuzzy outerplanar graphs.

Theorem 1 An Interval valued fuzzy graph ξ is interval valued fuzzy outer planar if and only if it contains no interval valued fuzzy subgraph homomorphic from K_4 or $K_{2,3}$.

Proof It is known that K_4 or $K_{2,3}$ are not Interval valued fuzzy outerplanar. Thus, no interval valued fuzzy graph having an interval valued fuzzy subgraph homomorphic from K_4 or $K_{2,3}$ is interval valued fuzzy outerplanar graphs shown in the Figs. 5 and 6.

Conversely,

Suppose ξ is an interval valued fuzzy graph containing no interval valued fuzzy subgraph homomorphic from K_4 or $K_{2,3}$. By kuratowski's theorem, ξ is planar. Assume ξ is not interval valued fuzzy outer planar graph. Thus, without loss of generality, we assume that ξ is a cyclic block which is not interval valued fuzzy outerplanar
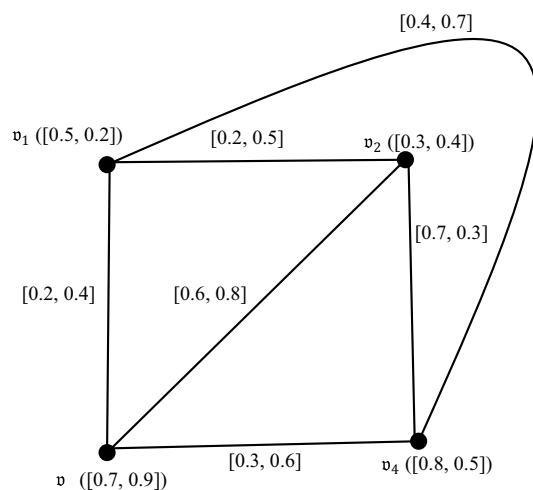
Fig. 4. Interval valued fuzzy planar embedding of K_4.
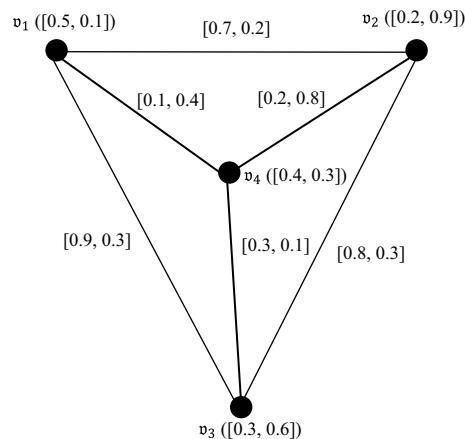
Fig. 5. Interval valued fuzzy graph K_4.
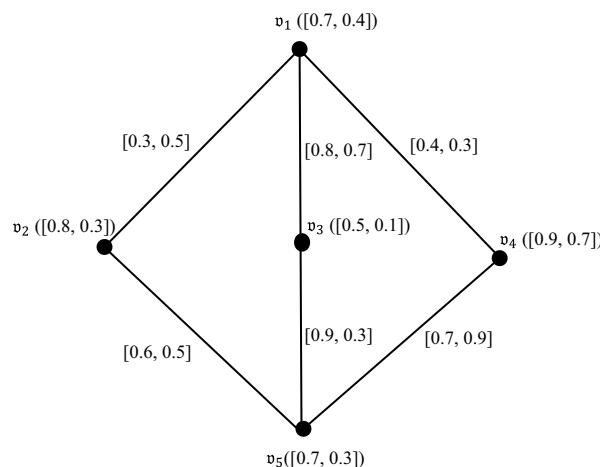
Fig. 6. Interval valued fuzzy graph $K_{2,3}$.

graph. Further we assume that ξ is so embedded in the fuzzy plane that the exterior region contains maximum number of vertices.

The exterior region is bounded by a cycle $\mathcal{C}$. Since not all vertices of ξ, lie on $\mathcal{C}$, there are one or more vertices lying interior to $\mathcal{C}$.

If there exist a vertex $\mathfrak{v}_3$ interior on $\mathcal{C}$ and three mutually disjoint paths between $\mathfrak{v}_3$ and three distinct vertices of $\mathcal{C}$, then ξ contains fuzzy subgraph homomorphic from K_4.

Otherwise, since ξ is a cyclic block, there must exists a vertex $\mathfrak{v}_3$ and two disjoint paths between $\mathfrak{v}_3$ and two disjoint vertices $\mathfrak{v}_1$ and $\mathfrak{v}_5$ on $\mathcal{C}$.

Moreover, from the above choice of $\mathcal{C}$, the edge $\mathfrak{v}_1\mathfrak{v}_5$ does not belong to $\mathcal{C}$. This implies ξ contains a bipolar fuzzy subgraph homomorphic from $K_{2,3}$. This is a contradiction. Therefore, Fig. 7 ξ is an interval valued fuzzy outerplanar graph. This completes the proof.

Theorem 2 If ξ is a connected interval valued fuzzy outerplanar graph, then it has a fuzzy dual ξ^* there exist a vertex $\mathfrak{v}$ such that $\xi^* - \mathfrak{v}$ contains no fuzzy cycle.

Figure 8 (a) and (b) Example for interval valued fuzzy outerplanar graph ξ has interval valued fuzzy dual graph ξ^* then there exist v such that $\xi^* - \mathfrak{v}$ contains no interval valued fuzzy cycle.

Proof Let $\xi = (\mathbb{V}, \tau, \delta)$ be a connected fuzzy outerplanar graph in Fig. 8 and there are no intersections of edges $E\prime \subset E$ within the graph, we can divide the fuzzy graph into a finite number of regions. Each of these regions corresponds to a $\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3, \ldots \mathfrak{f}_k$ be the fuzzy faces and the membership value for each face is determined[16].

$$\mathfrak{f} = [\mathfrak{f}^-, \mathfrak{f}^+] = \left[min\left\{ \frac{\delta^-\left(x, y\right)}{\tau^-(x) \wedge \tau^-(y)}, \frac{\delta^+\left(x, y\right)}{\tau^+(x) \wedge \tau^+(y)} : \delta^-\left(x, y\right) \in \mathbb{E}_i\prime \& \delta^+\left(x, y\right) \in \mathbb{E}_j\prime \right\} \right] \quad \text{Where} \quad \mathbb{E}_i\prime \text{ and}$$

$\mathbb{E}_j\prime$ is the region bounded by the interval valued fuzzy edges in interval valued fuzzy planar graph.

When the interval valued fuzzy graph divided into regions with interval valued fuzzy face and values, it becomes very easy to interval valued fuzzy dual graph ξ^* such that each face of ξ corresponds to a vertex and each edge of ξ corresponds to edge in $\xi^* = (\mathbb{V}\prime, \tau\prime, \delta\prime)$. Consequently, the interval valued fuzzy dual graph ξ^* exists for the interval valued fuzzy outerplanar graph.

Since ξ is interval valued fuzzy outerplanar graph. It is outer face is an interval valued fuzzy cycle. Therefore, the graph $\xi^* - \mathfrak{v}$ that results from removing the vertex $\mathfrak{v}$ from ξ^* in Fig. 8(a). And Fig. 8(b) the resulting graph $\xi^* - \mathfrak{v}$ contains no interval valued fuzzy cycle.

Example 1 Consider the interval valued fuzzy outerplanar graph $\xi = (\mathbb{V}, \tau, \delta)$ illustrated in the Fig. 8. If the dotted lines in Fig. 8(a) are superimposed interval valued fuzzy dual graph of ξ^* and separated diagram.

Here $\mathfrak{f}_1$, $\mathfrak{f}_2$, $\mathfrak{f}_3$ and $\mathfrak{f}_4$ are four fuzzy faces. $\mathfrak{f}_1$ is bounded by the edges $((\mathfrak{v}_1, \mathfrak{v}_2), [0.2, 0.3])$, $((\mathfrak{v}_2, \mathfrak{v}_3), [0.4, 0.5])$ and $((\mathfrak{v}_1, \mathfrak{v}_3), [0.4, 0.2])$; $\mathfrak{f}_2$ is bounded by the edges $((\mathfrak{v}_2, \mathfrak{v}_3), [0.4, 0.1])$, $((\mathfrak{v}_2, \mathfrak{v}_5), [0.4, 0.1])$, $((\mathfrak{v}_3, \mathfrak{v}_5), [0.3, 0.5])$ and $((\mathfrak{v}_2, \mathfrak{v}_5), [0.6, 0.7])$; $\mathfrak{f}_3$ is bounded by the edges $((\mathfrak{v}_2, \mathfrak{v}_4), [0.3, 0.3])$, $((\mathfrak{v}_2, \mathfrak{v}_5), [0.4, 0.1])$ and $((\mathfrak{v}_4, \mathfrak{v}_5), [0.3, 0.2])$; $\mathfrak{f}_4$ is the outer face $((\mathfrak{v}_3, \mathfrak{v}_5), [0.3, 0.5])$, $((\mathfrak{v}_1, \mathfrak{v}_3), [0.4, 0.2])$, $((\mathfrak{v}_1, \mathfrak{v}_2), [0.2, 0.3])$, $((\mathfrak{v}_2, \mathfrak{v}_4), [0.3, 0.3])$ and $((\mathfrak{v}_4, \mathfrak{v}_5), [0.3, 0.2])$. The membership value of $[\mathfrak{f}_1^-, \mathfrak{f}_1^+] = [0.4, 0.333]$, $[\mathfrak{f}_2^-, \mathfrak{f}_2^+] = [0.375, 0.333]$, $[\mathfrak{f}_3^-, \mathfrak{f}_3^+] = [0.6, 0.333]$, $[\mathfrak{f}_4^-, \mathfrak{f}_4^+] = (0.375, 0.333)$,
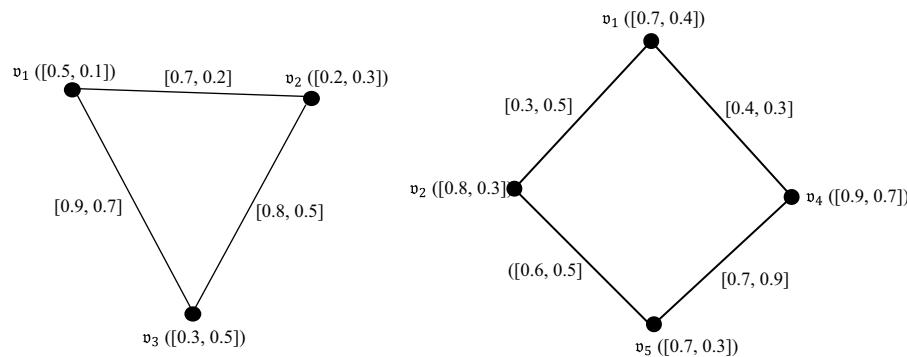
Fig. 7. Interval valued fuzzy Outerplanar graphs.
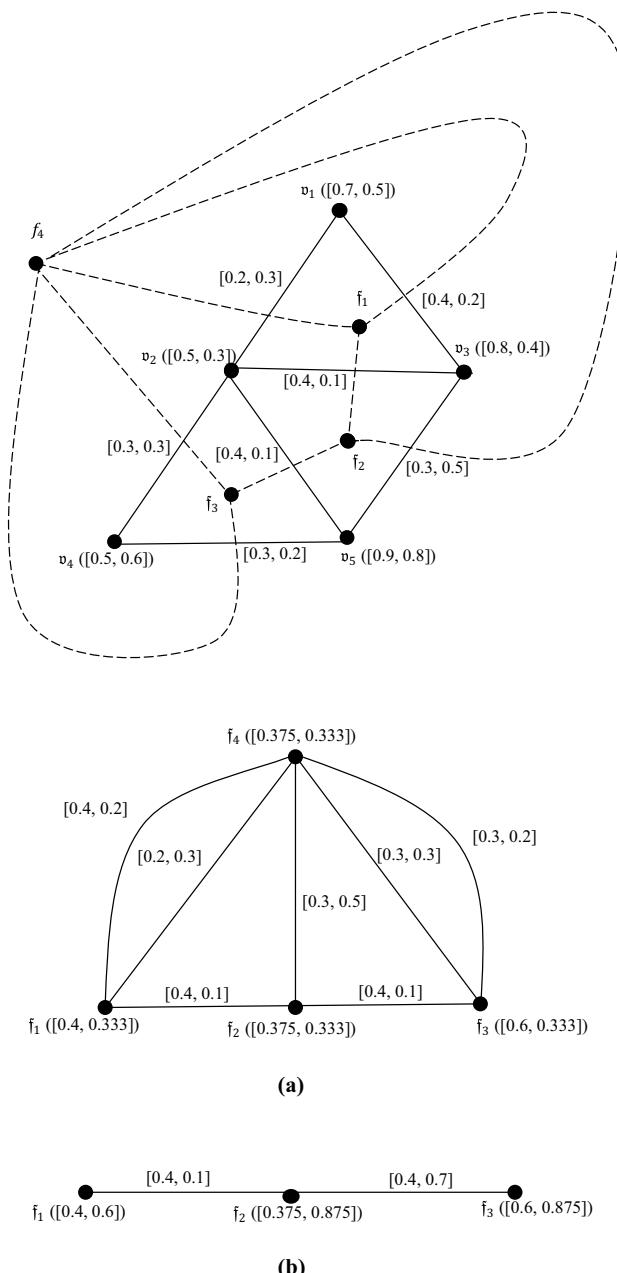
Fig. 8. Connected Interval valued fuzzy outerplanar graphs ξ. (**a**) Interval valued fuzzy dual graph ξ^*, (**b**) Not interval valued fuzzy cycle $\xi^* - \mathfrak{v}$.

S. No	Interval valued fuzzy faces	$\mathfrak{f}^-$	$\mathfrak{f}^+$
1	$\mathfrak{f}_1$	0.4	0.333
2	$\mathfrak{f}_2$	0.375	0.333
3	$\mathfrak{f}_3$	0.6	0.333
4	$\mathfrak{f}_4$	0.375	0.333

Table 2. Calculation of interval valued fuzzy faces.
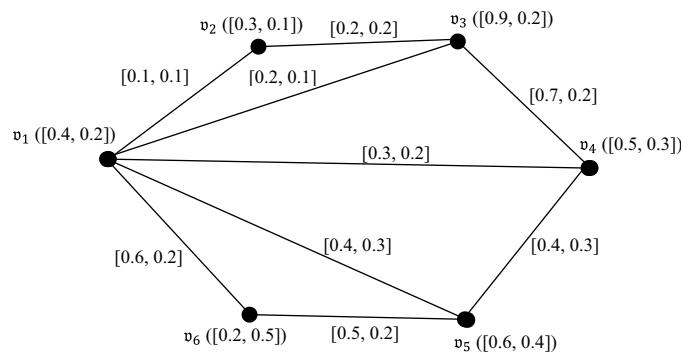
Fig. 9. Interval valued maximal fuzzy outerplanar graph.
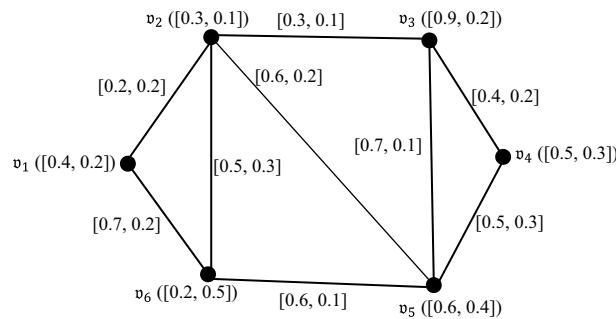
Fig. 10. Interval valued maximal fuzzy outerplanar graph.
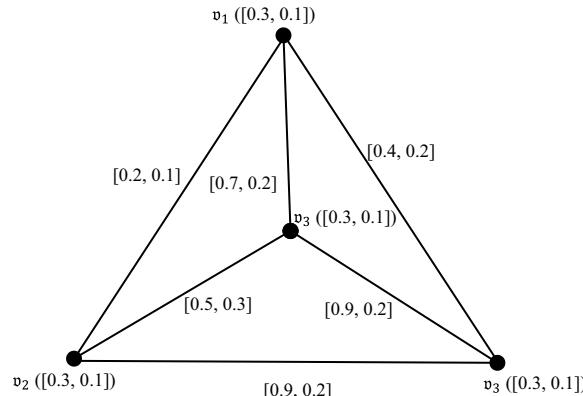
Fig. 11. Minimally interval valued fuzzy non-outerplanar graph K_4.

are presented in Table 2. Each interval valued face of ξ corresponds to a vertex and each edge of ψ corresponds to edge in $\xi^* = (\mathbb{V}\prime, \tau\prime, \delta\prime)$ Then $\mathfrak{v}$ exists in such a way that $\xi^* - \mathfrak{v}$ contains no interval valued fuzzy cycle.

Definition 13 If an edge cannot be added to an interval valued fuzzy outerplanar graph ξ without sacrificing its outer planarity property, then the graph is interval valued maximal fuzzy outerplanar graph.

The graphs given in Fig. 9 and Fig. 10 are examples of Interval valued maximal fuzzy outerplanar graphs.

Definition 14 An interval valued fuzzy planar graph ξ is considered minimally interval valued fuzzy non-outerplanar if $i\,(\xi) \neq 0$ with atmost one vertex $\mathfrak{v} \in \xi$ such that $\tau(\mathscr{x}) > 0$ in the interior region.

The graphs given in Fig. 11 and Fig. 12 are examples of minimally interval valued fuzzy non- outerplanar graphs.
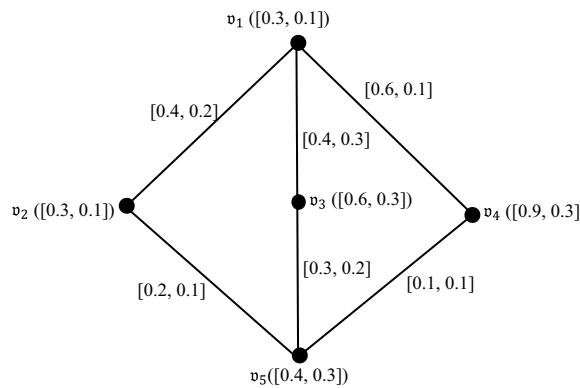
Fig. 12. Minimally interval valued fuzzy non-outerplanar graph $K_{2,3}$.

Vertex deletion interval valued fuzzy outerplanar subgraph

This section discusses the subgraph resulting from eliminating specified vertices from the interval valued fuzzy graph, as well as establishing interval valued fuzzy outerplanar graphs and providing examples that are relevant.

Definition 15 Assume ξ is an Interval valued fuzzy planar graph. If $\xi\prime$ is the Vertex Deleted interval valued fuzzy subgraph possessing outerplanarity property, then it is referred to as the Vertex Deleted interval valued fuzzy outerplanar subgraph of ξ''.

Example 2 Let's look at the Interval valued fuzzy graph $\xi = (\mathbb{V}, \mathbb{A}, \mathbb{B})$ shown in Fig. 13. The set of vertices in ξ is $\mathbb{V} = \{\mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4, \mathfrak{v}_5, \mathfrak{v}_6\}$ and their membership values of these are as follows: $\tau^-(\mathfrak{v}_1) = 0.6, \tau^+(\mathfrak{v}_1)=0.5, \tau^-(\mathfrak{v}_2) = 0.85, \tau^+(\mathfrak{v}_2)=0.9, \tau^-(\mathfrak{v}_3) = 0.4, \tau^+(\mathfrak{v}_3) =0.5, \tau^-(\mathfrak{v}_4) = 0.4, \tau^+(\mathfrak{v}_4)=0.1, \tau^-(\mathfrak{v}_5) = 0.7, \tau^+(\mathfrak{v}_5)=0.8, \tau^-(\mathfrak{v}_6) = 0.6, \tau^+(\mathfrak{v}_6)=0.7$ and the edges $\mathbb{B}$ is $\delta^-(\mathfrak{v}_1, \mathfrak{v}_2) = 0.65, \delta^+(\mathfrak{v}_1, \mathfrak{v}_2) = 0.1, \delta^-(\mathfrak{v}_1, \mathfrak{v}_3) = 0.35, \delta^+(\mathfrak{v}_1, \mathfrak{v}_3) = 0.4, \delta^-(\mathfrak{v}_2, \mathfrak{v}_3) = 0.25, \delta^+(\mathfrak{v}_2, \mathfrak{v}_3) = 0.3\delta^-(\mathfrak{v}_2, \mathfrak{v}_5) = 0.6\delta^+(\mathfrak{v}_2, \mathfrak{v}_5) = 0.7\delta^-(\mathfrak{v}_3, \mathfrak{v}_6) = 0.35\delta^+(\mathfrak{v}_3, \mathfrak{v}_6) = 0.4\delta^-(\mathfrak{v}_5, \mathfrak{v}_6) = 0.55, \delta^+(\mathfrak{v}_5, \mathfrak{v}_6) = 0.6, \delta^-(\mathfrak{v}_6, \mathfrak{v}_5) = 0.4, \delta^+(\mathfrak{v}_6, \mathfrak{v}_5) = 0.2, \delta^-(\mathfrak{v}_2, \mathfrak{v}_4) = 0.4, \delta^+(\mathfrak{v}_2, \mathfrak{v}_4) = 0.4, \delta^-(\mathfrak{v}_3, \mathfrak{v}_4) = 0.4, \delta^+(\mathfrak{v}_3, \mathfrak{v}_4) = 0.2, \delta^-(\mathfrak{v}_4, \mathfrak{v}_5) = 0.3, \delta^+(\mathfrak{v}_4, \mathfrak{v}_5) = 0.5, \delta^-(\mathfrak{v}_4, \mathfrak{v}_6) = 0.4, \delta^+(\mathfrak{v}_4, \mathfrak{v}_6) = 0.2$.

The edges intersect between the sets $\tau^-(\mathfrak{v}_2, \mathfrak{v}_6), \tau^+(\mathfrak{v}_2, \mathfrak{v}_6)$, and $\tau^-(\mathfrak{v}_3, \mathfrak{v}_5), \tau^+(\mathfrak{v}_3, \mathfrak{v}_5)$ within interval valued fuzzy graph ξ. Let us define subsets $\mathbb{X}$ and $\mathbb{Y}$ in $\mathbb{V}$, where $\mathbb{X} = \{(\mathfrak{v}_1([0.6,0.5])\}$ and $\mathbb{Y} = \{\mathfrak{v}_4([0.4,0.1])\}$ in the interval valued fuzzy graph ξ.

The Interval valued fuzzy subgraphs $\xi - \mathbb{X}$, represented as ψ_1 can be seen in Fig. 13(a). Similarly, the interval valued fuzzy subgraph $\xi - \mathbb{Y}$ represented as ξ_2 is shown in Fig. 13(b). It can be noted that, interval valued subgraph ξ_1 is categorized as a Vertex Deletion interval valued fuzzy subgraph, while ξ_2 is classified as a Vertex Deleted interval valued fuzzy outerplanar subgraph of ξ.

Note 1 It is not necessary for the Vertex Deletion interval valued fuzzy subgraph of ξ to be the Vertex Deletion interval valued fuzzy outerplanar subgraph of ξ. The interval valued graphs in Figs.13(a) and (b) allow for the observation of this.

Theorem 3 An interval valued fuzzy outerplanar graph ξ always has a vertex deleted interval valued fuzzy outerplanar subgraph that is also a vertex deleted interval valued fuzzy subgraph of ξ.

Proof Let the interval valued fuzzy outerplanar graph be. and $\mathbb{H}$ be the vertex deleted interval valued fuzzy subgraph of ξ. The vertices of the interval valued fuzzy graph ξ will all be in the outer region since it is outerplanar. As a result, the interval valued subgraph that was created by deleting vertices still has outerplanarity. Any vertex deleted interval valued subgraph in ξ is therefore a vertex deletion interval valued fuzzy outerplanar subgraph of ξ.

Theorem 4 Let ξ be an interval valued fuzzy outerplanar graph which is connected and let $\mathbb{W}$ be a subset of its vertices, such that $\mathbb{W} \subseteq \mathbb{V}$. The interval valued fuzzy outerplanar subgraph of ξ with a vertex deleted is $\xi\prime = \xi\backslash\mathbb{W}$. Its interval valued fuzzy dual graph is ξ''.

Proof Let us consider a connected interval valued fuzzy outerplanar subgraph ξ in Fig. 14 and $\mathbb{W} \subseteq \mathbb{V}$. let $\xi - \mathbb{W}$ form a Vertex deleted intervsal valued fuzzy outerplanar subgraph denoted $\xi\prime$ Since there is no intersection of interval valued fuzzy edges in the subgraph $\xi\prime$, we can divide the interval valued fuzzy graph into a finite number of regions. Each of these regions corresponds to an interval valued fuzzy face, and the membership values for each face is determined[16].
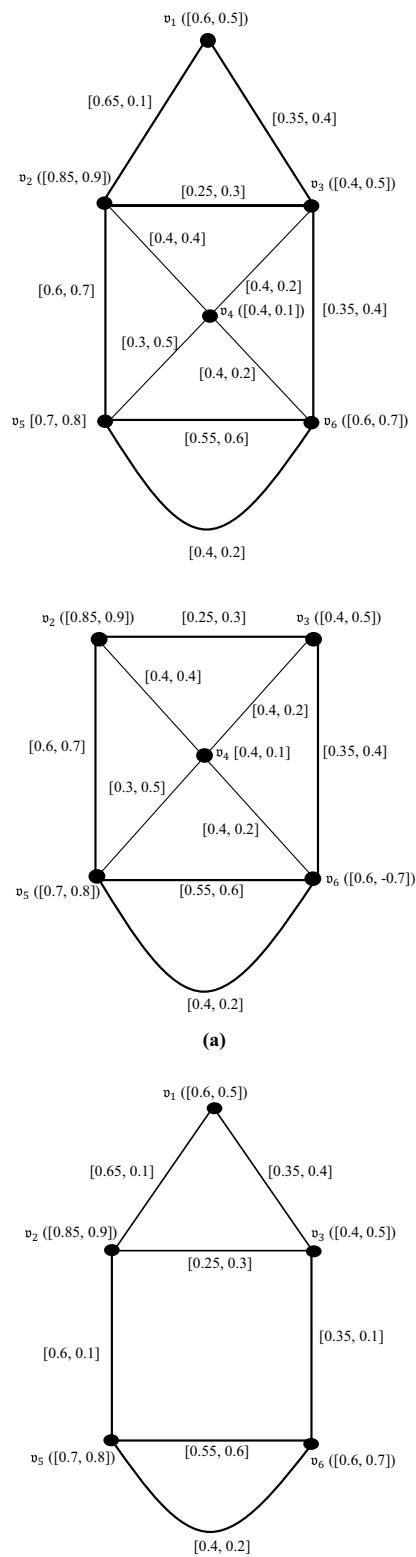
Fig. 13. Interval valued fuzzy graph. (**a**) Vertex Deleted Interval valued fuzzy subgraph, (**b**). Vertex deletion Interval valued fuzzy outerplanar subgraph.
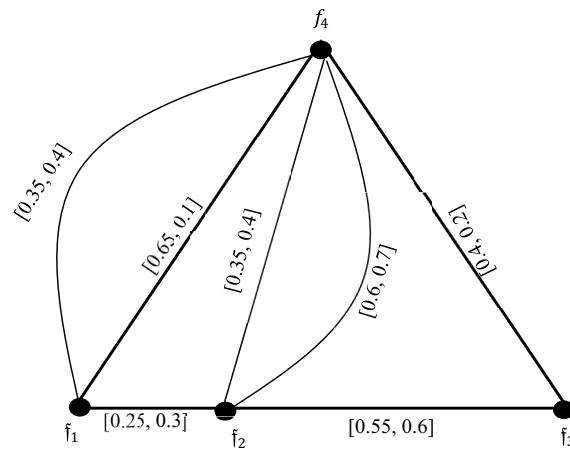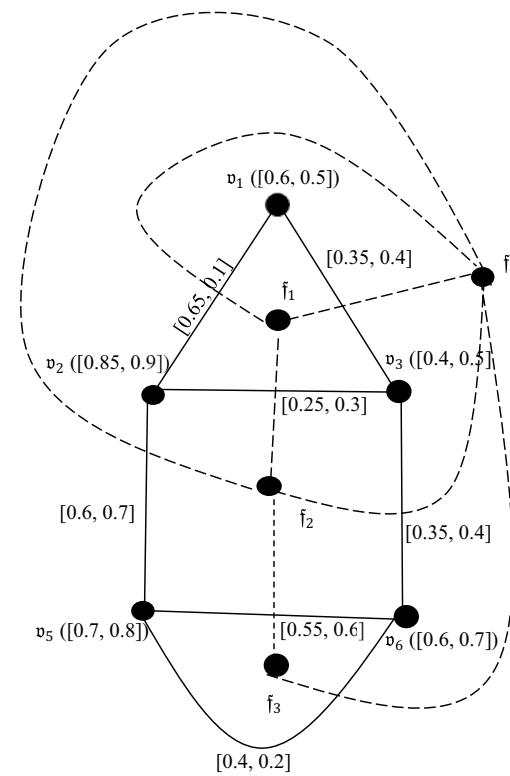
(a)

Fig. 14. Vertex Deletion interval valued fuzzy outerplanar subgraph. (**a**) Interval valued fuzzy dual graph.

$$\mathfrak{f} = [\mathfrak{f}^-, \mathfrak{f}^+] = \left[min \left\{ \frac{\delta^- \left(x, y \right)}{\tau^- (x) \wedge \tau^- \left(y \right)}, \frac{\delta^+ \left(x, y \right)}{\tau^+ (x) \wedge \tau^+ \left(y \right)} : \delta^- \left(x, y \right) \in \mathbb{E}_i \prime \& \delta^+ \left(x, y \right) \in \mathbb{E}_j \prime \right\} \right]$$ Where $\mathbb{E}_i \prime$ and

$\mathbb{E}_j \prime$ is the region bounded by the interval valued fuzzy edges in interval valued fuzzy planar graph.

Using the fact that each interval valued fuzzy graph with planarity value as 1 has interval valued fuzzy dual graph, $\xi - \mathbb{W}$ has an interval valued fuzzy dual graph ξ'' in Fig. 14(a). Consequently, the bipolar fuzzy dual graph exists for the Vertex Deleted interval valued fuzzy outerplanar subgraph of ξ.

Example 3 From Example 2. it is evident that the interval valued subgraph ξ'' is the Vertex Deletion interval valued fuzzy outerplanar subgraph of ξ. The dotted lines represent the superimposed interval valued fuzzy dual graph of ξ'' in Fig. Fig. 14(a).

S. No	Interval valued fuzzy faces	$\mathfrak{f}^-$	$\mathfrak{f}^+$
1	$\mathfrak{f}_1$	0.625	0.8
2	$\mathfrak{f}_2$	0.625	0.777
3	$\mathfrak{f}_3$	0.857	0.285
4	$\mathfrak{f}_4$	0.666	0.2

Table 3. Calculation of interval valued fuzzy faces.

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Here $\mathfrak{f}_1$, $\mathfrak{f}_2$, $\mathfrak{f}_3$ and $\mathfrak{f}_4$ are four fuzzy faces. $\mathfrak{f}_1$ is bounded by the edges $((\mathfrak{v}_1, \mathfrak{v}_2), [0.65, 0.1])$, $((\mathfrak{v}_2, \mathfrak{v}_3), [0.25, 0.3])$ and $((\mathfrak{v}_1, \mathfrak{v}_3), [0.35, 0.4])$; $\mathfrak{f}_2$ is bounded by the edges $((\mathfrak{v}_2, \mathfrak{v}_3), [0.25, 0.3])$, $((\mathfrak{v}_3, \mathfrak{v}_6), [0.35, 0.4])$, $((\mathfrak{v}_5, \mathfrak{v}_6), [0.55, 0.6])$ and $((\mathfrak{v}_2, \mathfrak{v}_5), [0.6, 0.7])$; $\mathfrak{f}_3$ is bounded by the edges $((\mathfrak{v}_5, \mathfrak{v}_6), [0.55, 0.6])$, $((\mathfrak{v}_6, \mathfrak{v}_5), [0.4, 0.2])$; $\mathfrak{f}_4$ is the outer face $((\mathfrak{v}_1, \mathfrak{v}_3), [0.35, 0.4])$, $((\mathfrak{v}_1, \mathfrak{v}_2), [0.65, 0.1])$, $((\mathfrak{v}_2, \mathfrak{v}_5), [0.6, 0.7])$, $((\mathfrak{v}_5, \mathfrak{v}_6), [0.35, 0.4])$ and $((\mathfrak{v}_3, \mathfrak{v}_6), [0.35, 0.4])$. The interval valued fuzzy dual graph is associated with interval valued fuzzy face values, with all interval valued fuzzy faces having membership values of $[\mathfrak{f}_1{}^-, \mathfrak{f}_1{}^+] = [0.625, 0.2]$, $[\mathfrak{f}_2{}^-, \mathfrak{f}_2{}^+] = [0.625, 0.6]$, $[\mathfrak{f}_3{}^-, \mathfrak{f}_3{}^+] = [0.857, 0.285]$ and $[\mathfrak{f}_4{}^-, \mathfrak{f}_4{}^+] = [0.666, 0.2]$, are presented in Table 3. Therefore, the Vertex Deletion interval valued fuzzy outerplanar subgraph ξ'' indeed possesses a interval valued fuzzy dual graph.

Maximum and maximal vertex deletion interval valued fuzzy outerplanar subgraphs

This section introduced the concepts of maximum and maximal vertex deletion interval valued fuzzy outerplanar subgraphs and investigates the relationship between these two ideas using appropriate examples.

Definition 16 If $\xi\prime = (\mathbb{V}\prime, \tau\prime, \delta\prime)$ is the Vertex Deletion interval valued fuzzy outerplanar subgraph of ξ such that there is no other Vertex deleted interval valued fuzzy outerplanar subgraph of ξ with maximum order and size compared to the interval valued subgraph $\xi\prime$, then $\xi\prime$ is said to be the maximum Vertex deleted interval valued fuzzy outerplanar subgraph of ξ.

Note 2 Let order and size of the Vertex deleted interval valued fuzzy outerplanar subgraphs $\xi\prime$ and ξ'' of ξ be $[\mathbb{O}^-\prime, \mathbb{O}^+\prime] > [\mathbb{O}^{-''}, \mathbb{O}^{+''}]$ and $[\mathbb{S}^-\prime, \mathbb{S}^+\prime] > [\mathbb{S}^{-''}, \mathbb{S}^{+''}]$ respectively. The interval valued subgraph $\xi\prime$ is said to be maximum Vertex deleted interval valued fuzzy outerplanar subgraph of ξ if $[\mathbb{O}^-\prime, \mathbb{S}^-\prime] > [\mathbb{O}^{-''}, \mathbb{S}^{-''}]$ and $[\mathbb{O}^+\prime, \mathbb{S}^+\prime] > [\mathbb{O}^{+''}, \mathbb{S}^{+''}]$. Suppose two Vertex deleted interval valued fuzzy outerplanar subgraphs ξ_1 and ξ_2 are obtained in the interval valued fuzzy graph ξ. If $\xi_1 \neq \xi_2$ but the order and size of two interval valued subgraphs are equal (i.e.), $[\mathbb{O}^-\prime, \mathbb{S}^-\prime] > [\mathbb{O}^{-''}, \mathbb{S}^{-''}]$ and $[\mathbb{O}^+\prime, \mathbb{S}^+\prime] > [\mathbb{O}^{+''}, \mathbb{S}^{+''}]$., then the two interval valued subgraphs are maximum Vertex deleted interval valued fuzzy outerplanar subgraph of ξ.

Example 4 Let's look at the Interval valued fuzzy graph $\xi = (\mathbb{V}, \mathbb{A}, \mathbb{B})$ shown in Fig. 15. The set of vertices in ξ is $\mathbb{V} = \{\mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4, \mathfrak{v}_5, \mathfrak{v}_6\}$ and their membership values of these are as follows: $\tau^-(\mathfrak{v}_1) = 0.25, \tau^+(\mathfrak{v}_1) = 0.3, \tau^-(\mathfrak{v}_2) = 0.35, \tau^+(\mathfrak{v}_2) = 0.7, \tau^-(\mathfrak{v}_3) = 0.4, \tau^+(\mathfrak{v}_3) = 0.5, \tau^-(\mathfrak{v}_4) = 0.7, \tau^+(\mathfrak{v}_4) = 0.8, \tau^-(\mathfrak{v}_5) = 0.15, \tau^+(\mathfrak{v}_5) = 0.1, \tau^-(\mathfrak{v}_6) = 0.45, \tau^+(\mathfrak{v}_6) = 0.5.$ and the edges $\delta^-(\mathfrak{v}_1, \mathfrak{v}_6) = 0.1, \delta^+(\mathfrak{v}_1, \mathfrak{v}_6) = 0.2, \delta^-(\mathfrak{v}_2, \mathfrak{v}_6) = 0.2, \delta^+(\mathfrak{v}_2, \mathfrak{v}_6) = 0.3, \delta^-(\mathfrak{v}_4, \mathfrak{v}_6) = 0.3, \delta^+(\mathfrak{v}_4, \mathfrak{v}_6) = \delta 0.6, \delta(\mathfrak{v}_5, \mathfrak{v}_6) = \delta 0.7, \delta(\mathfrak{v}_5, \mathfrak{v}_6) = \delta 0.8, \delta(\mathfrak{v}_1, \mathfrak{v}_2) = \delta 0.5, \delta(\mathfrak{v}_1, \mathfrak{v}_2) = \delta 0.6, \delta(\mathfrak{v}_2, \mathfrak{v}_3) = \delta 0.5, \delta(\mathfrak{v}_2, \mathfrak{v}_3) = \delta 0.6, \delta(\mathfrak{v}_3, \mathfrak{v}_4) = 0.4, \delta^+(\mathfrak{v}_3, \mathfrak{v}_4) = 0.5, \delta^-(\mathfrak{v}_4, \mathfrak{v}_5) = 0.7, \delta^+(\mathfrak{v}_4, \mathfrak{v}_5) = 0.8, \delta^-(\mathfrak{v}_5, \mathfrak{v}_1) = 0.25, \delta^+(\mathfrak{v}_5, \mathfrak{v}_1) = 0.35.$

Figure 15 (a)-(f) Maximum Vertex Deletion interval valued fuzzy outerplanar subgraphs.

The vertex deleted of interval valued fuzzy outerplanar subgraphs of ξ are shown in Fig. 15(a)-(f). From the Table 4. Note that Fig. 15(c) is the maximum vertex deleted interval valued fuzzy outerplanar subgraph of ξ.

Theorem 5 The interval valued fuzzy outerplanar subgraph of Maximum Vertex Deletion has an interval valued fuzzy dual graph, but the converse need not be true.

Proof We can easily conclude that maximum Vertex Deletion interval valued fuzzy outerplanar subgraph has an interval valued fuzzy dual and that the converse need not be true based on the proof of Theorem 4.

Definition 17 A Vertex Deletion interval valued fuzzy outerplanar subgraph $\xi\prime = (\mathbb{V}\prime, \tau\prime, \delta\prime)$ induced on ξ and all other interval valued subgraphs of ξ induced by the vertex set $\mathbb{V}'' = \mathbb{V}\prime \cup \{\mathfrak{v}\}$ (where $\mathfrak{v} \in \mathbb{V}\setminus\mathbb{V}\prime$) that does not satisfy the outerplanarity property is the maximal Vertex Deletion interval valued fuzzy outerplanar subgraph of $\xi\prime$ is.

Theorem 6 The interval valued fuzzy outerplanar subgraph of each maximum vertex deletion is the maximal vertex deletion interval valued fuzzy outerplanar subgraph of ξ.

Proof Let $\xi\prime$ denote the maximum interval valued fuzzy outerplanar subgraph of ξ resulting from Vertex Deletion. A subset $\mathbb{W} \neq \varnothing \subseteq \mathbb{V}$ exists, in which some of the ξ vertices have been deleted.
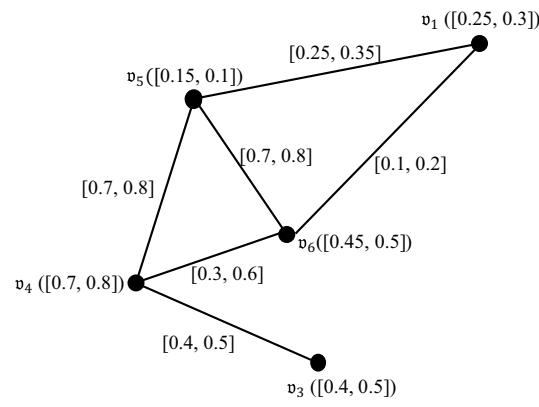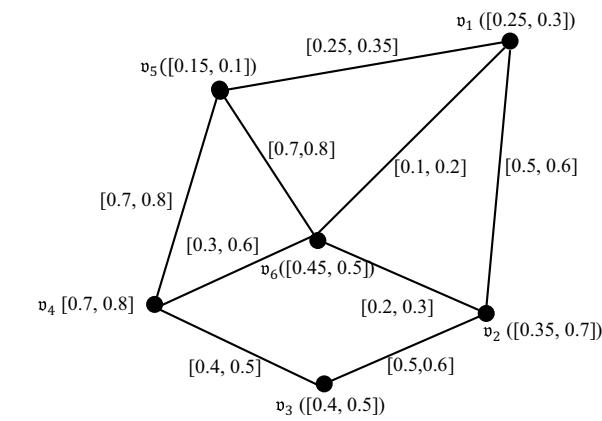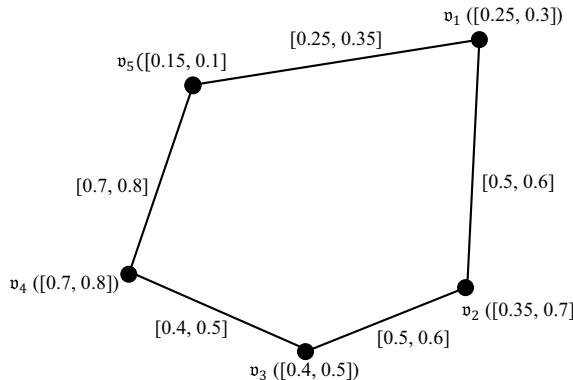
Fig. 15. Interval valued fuzzy graph. (**a**) $\xi - \mathfrak{v}_2$, (**b**) $\xi - \mathfrak{v}_6$, (**c**) $\xi - \mathfrak{v}_5$, (**d**). $\xi - \mathfrak{v}_3$, (**e**). $\xi - \mathfrak{v}_1$, (**f**). $\xi - \mathfrak{v}_4$.

Choose a vertex $\mathfrak{u}$ in $\mathbb{W}$, and consider $\xi'' = \xi' \cup \{\mathfrak{u}\}$ where $\xi'' = \{x \in \mathbb{V}' \cup \frac{\{\mathfrak{u}\}}{\delta(x,y)} \neq \varnothing, x, y \in \mathbb{V} \backslash \mathbb{W} \cup \{\mathfrak{u}\}\}$.

Since ξ' is a maximum induced interval valued outerplanar subgraph, adding any of the vertex's results $\mathfrak{u}$. Hence, ξ' is the maximal induced vertex elimination interval valued fuzzy outerplanar subgraph of ξ. Therefore, every maximum induced vertex deleted interval valued fuzzy outer graph is a maximal induced vertex deleted interval valued fuzzy outerplanar subgraph of ξ.
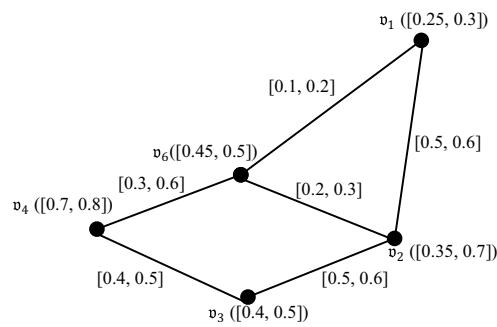
(c)
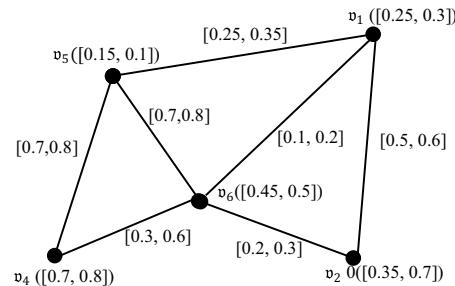


(d)
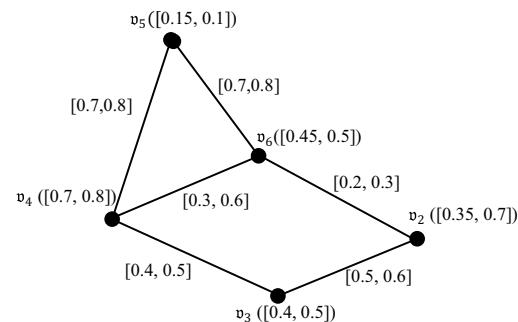
(e)
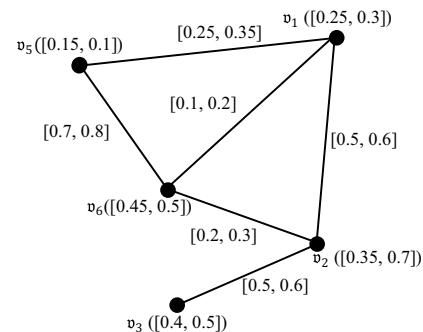
(f)

Fig. 15. (continued)

S.No	Subset	Vertex deleted Interval Valued fuzzy Outerplanar Subgraph	$\mathbb{O}^-(\xi)$	$\mathbb{O}^+(\xi)$	$\mathbb{S}^-(\xi)$	$\mathbb{S}^+(\xi)$
1	$\mathbb{W}_1 = \mathfrak{v}_1([0.25, 0.3])$	$\xi - \mathbb{W}_1$	2.05	2.6	0.85	1.15
2	$\mathbb{W}_2 = \mathfrak{v}_2([0.35, 0.7])$	$\xi - \mathbb{W}_2$	1.95	2.2	1.2	1.5
3	$\mathbb{W}_3 = \mathfrak{v}_3(0.4, 0.5)$	$\xi - \mathbb{W}_3$	1.9	2.4	0.9	1.1
4	$\mathbb{W}_4 = \mathfrak{v}_4([0.7, 0.8])$	$\xi - \mathbb{W}_4$	1.6	2.1	1.4	1.9
5	$\mathbb{W}_5 = \mathfrak{v}_5([0.15, 0.1])$	$\xi - \mathbb{W}_5$	2.15	2.8	1.65	1.95
6	$\mathbb{W}_6 = \mathfrak{v}_6([0.45, 0.5])$	$\xi - \mathbb{W}_6$	1.85	2.4	1.3	1.9

Table 4. Calculation of order and size of these vertex deletion interval valued fuzzy outerplanar subgraphs.

Theorem 7 The answer to Theorem 6. is not necessarily true.

Proof Let ξ be the interval valued fuzzy graph and, $\xi\prime$ and ξ'' are the Vertex Deletion interval valued fuzzy outerplanar subgraphs of ξ. Let $\mathbb{W}_1, \mathbb{W}_2$ be the subsets of vertex set $\mathbb{V}$ in ξ. $\mathbb{W}_1$ and $\mathbb{W}_2$ contains the deleted vertices from ξ respectively. Then $\xi\prime$ and ξ'' are said to be maximal interval valued fuzzy outerplanar subgraph if $\xi\prime \cup \{\mathfrak{u}\}$ where $\mathfrak{u} \in \mathbb{W}_1$ and $\xi'' \cup \{\mathfrak{v}\}$ where $\mathfrak{v} \in \mathbb{W}_2$ are interval valued fuzzy non outerplanar.

Therefore, these interval valued subgraphs are interval valued fuzzy outerplanar subgraphs with maximal vertex deletion of ξ. However, fuzzy interval valued subgraphs have different order and size, so a vertex deletion interval valued fuzzy outerplanar subgraph is a maximal vertex deletion interval valued fuzzy outer subgraph with highest order and size. So $\xi\prime$ or ξ'' is a maximal vertex deletion interval valued fuzzy outerplanar subgraph, but two interval valued fuzzy graphs are maximal. Therefore, the interval valued fuzzy outerplanar subgraph of every maximal vertex deletion need not be a maximum vertex deletion interval valued fuzzy outerplanar subgraph.

Note 3 Only when both interval valued subgraphs are maximum vertex Deletion interval valued fuzzy outerplanar subgraphs as defined by Note 2. are two Vertex Deletion interval valued fuzzy outerplanar subgraphs maximal.

Example 5 Let's look at the Interval valued fuzzy graph $\xi = (\mathbb{V}, \tau, \delta)$ shown in Fig. 16. The set of vertices in ξ is $\mathbb{V} = \{\mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4, \mathfrak{v}_5\}$ and their membership values of these are as follows: $\tau^-(\mathfrak{v}_1) = 0.35, \tau^+(\mathfrak{v}_1) = 0.7$, $\tau^-(\mathfrak{v}_2) = 0.45, \tau^+(\mathfrak{v}_2) = 0.9$, $\tau^-(\mathfrak{v}_3) = 0.5, \tau^+(\mathfrak{v}_3) = 0.6$, $\tau^-(\mathfrak{v}_4) = 0.7, \tau^+(\mathfrak{v}_4) = 0.8, \tau^-(\mathfrak{v}_5) = 0.15, \tau^+(\mathfrak{v}_5) = 0.2$ and the edges $\mathbb{B}$ is $\delta^-(\mathfrak{v}_1, \mathfrak{v}_5) = 0.15$, $\delta^+(\mathfrak{v}_1, \mathfrak{v}_5) = 0.25$, $\delta^-(\mathfrak{v}_2, \mathfrak{v}_5) = 0.7, \delta^+(\mathfrak{v}_2, \mathfrak{v}_5) = 0.8, \delta^-(\mathfrak{v}_3, \mathfrak{v}_5) = 0.17, \delta^+(\mathfrak{v}_3, \mathfrak{v}_5) = 0.18, \delta^-(\mathfrak{v}_1, \mathfrak{v}_2) = \delta^-(\mathfrak{v}_2, \mathfrak{v}_1) = 0.1$, $\delta^+(\mathfrak{v}_1, \mathfrak{v}_2) = \delta^+(\mathfrak{v}_2, \mathfrak{v}_1) = 0.2, \delta^-(\mathfrak{v}_2, \mathfrak{v}_3) = \delta^-(\mathfrak{v}_3, \mathfrak{v}_2) = 0.2, \delta^+(\mathfrak{v}_2, \mathfrak{v}_3) = \delta^+(\mathfrak{v}_3, \mathfrak{v}_2) = 0.3, \delta^-(\mathfrak{v}_3, \mathfrak{v}_4) = \delta^-(\mathfrak{v}_4, \mathfrak{v}_3) = 0.3$ $\delta^+(\mathfrak{v}_3, \mathfrak{v}_4) = \delta^+(\mathfrak{v}_4, \mathfrak{v}_3) = 0.2, \delta^-(\mathfrak{v}_4, \mathfrak{v}_1) = \delta^-(\mathfrak{v}_1, \mathfrak{v}_4) = 0.2, \delta^+(\mathfrak{v}_4, \mathfrak{v}_1) = \delta^+(\mathfrak{v}_1, \mathfrak{v}_4) = 0.3$.

All the graphs in Fig. 16(a)-(e) are maximal outerplanar of interval valued fuzzy subgraph of ξ. But Fig. 16(a) is the maximum interval valued fuzzy outerplanar subgraph of ξ in the Table 5. Then it can be observed that maximal interval valued fuzzy outerplanar subgraph need not be maximum interval valued fuzzy outerplanar subgraph of ξ.

Edge deletion interval valued fuzzy outerplanar subgraphs
In this section, we covered the subgraph formed by removing specified edges from an interval valued fuzzy graph, as well as constructing interval valued fuzzy outerplanar graphs and presenting illustrations.

Definition 18 If ξ is an interval valued fuzzy planar graph and $\xi\prime$ is its Edge Deletion interval valued fuzzy subgraph, then $\xi\prime$ is the Edge Deletion interval valued fuzzy outerplanar subgraph of ξ if and only if it preserves an interval valued fuzzy outerplanar.

Note 4 There is no need for an edge deletion interval valued fuzzy outerplanar subgraph of ξ to be an edge deletion interval valued fuzzy subgraph of ξ.

Theorem 8 In an interval valued fuzzy outerplanar graph ξ, every Edge Deletion interval valued fuzzy subgraph is always an Edge Deletion interval valued fuzzy outerplanar subgraph of ξ.

Proof Let ξ be an interval valued fuzzy outerplanar graph and let $\mathbb{H}$ be an edge deletion interval valued fuzzy subgraph of ξ. Since an interval valued fuzzy graph ξ is outerplanar, all its vertices will be in the outer region. Thus, the subgraph obtained by edge deletion still possess outerplanarity property. Therefore, any edge-deleted interval valued fuzzy subgraphs in ξ are edge-deleted interval valued fuzzy outerplanar subgraphs of ξ.

Theorem 9 Let $\mathbb{W}$ be the subset of edges of a connected interval valued fuzzy outerplanar graph such that $\mathbb{W} \subseteq \mathbb{E}$. If $\xi\prime$ is an edge deletion interval valued fuzzy outerplanar subgraph of ξ if the resulting graph connects $\xi\prime$ to the interval valued fuzzy dual graph.
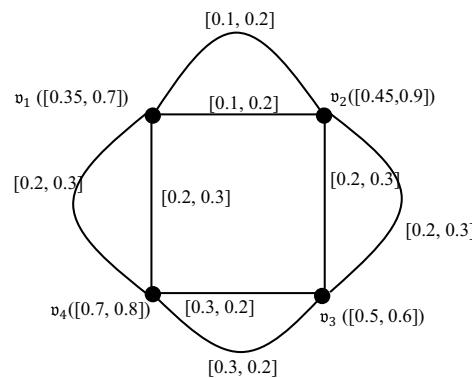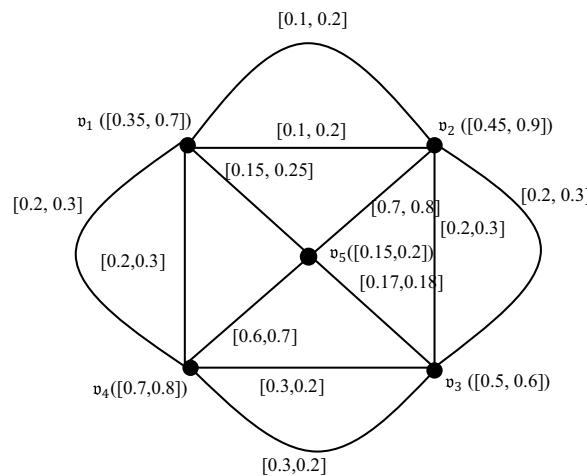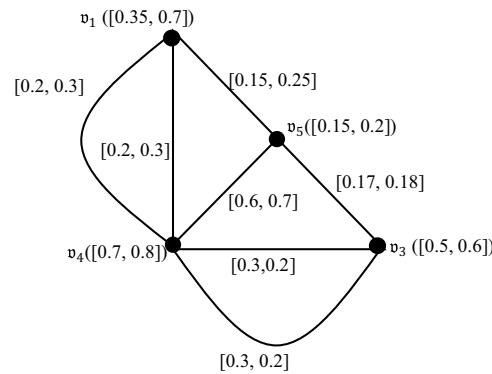
Fig. 16. Example for maximal Vertex Deletion Interval valued fuzzy graph ξ. (**a**) $\xi - \mathfrak{v}_5$, (**b**). $\xi - \mathfrak{v}_2$, (**c**). $\xi - \mathfrak{v}_3$, (**d**). $\xi - \mathfrak{v}_1$, (**e**). $\xi - \mathfrak{v}_4$.

Proof By using the same proof as Theorem 4. and which is obviously true for the case of edges.

Maximum and maximal edge deletion interval valued fuzzy outerplanar subgraph

This section introduced the ideas of maximum and maximal edge deletion interval valued fuzzy outerplanar subgraphs and analyzes the relationship between these two notions, using appropriate examples.
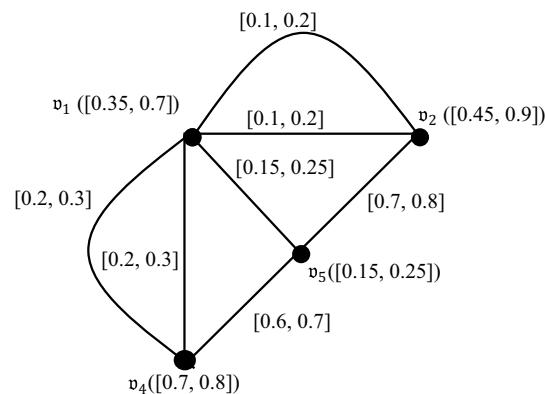
(c)
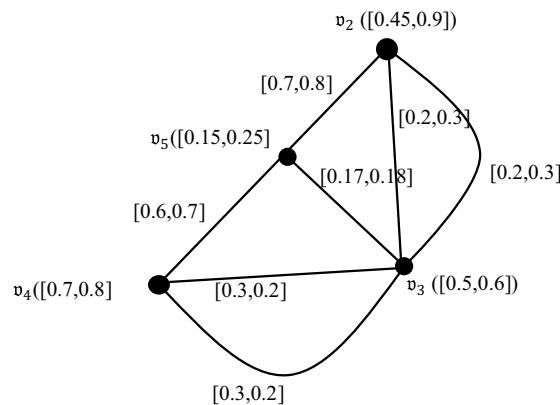
(d)
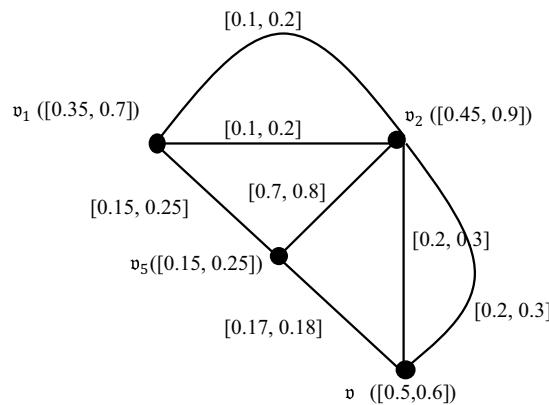
(e)

Fig. 16. (continued)

Definition 19 Maximum ED- interval valued fuzzy outerplanar subgraph.

If $\xi\prime = (\mathbb{V}\prime, \tau\prime, \delta\prime)$ with size $[\mathbb{S}^-, \mathbb{S}^+]$ is the Edge Deletion interval valued fuzzy outerplanar subgraph of non-outerplanar interval valued fuzzy graph $\xi = (\mathbb{V}, \tau, \delta)$ such that there is no other Edge Deletion interval valued fuzzy outerplanar subgraph ξ'' of size $[\mathbb{S}^{-''}, \mathbb{S}^{+''}]$ of ξ such that $\left[\mathbb{S}^{-''}, \mathbb{S}^{+''}\right] > [\mathbb{S}^{-\prime}, \mathbb{S}^{+\prime}]$, then $\xi\prime$ is the maximum interval valued fuzzy outerplanar subgraph of ξ.

S.No		VD Interval Valued Fuzzy outerplanar subgraphs	$\mathbb{O}^-(\xi)$	$\mathbb{O}^+(\xi)$	$\mathbb{S}^-(\xi)$	$\mathbb{S}^+(\xi)$
1	Figure 16(a)	$\xi-\{\mathfrak{v}_5\ ([0.15, 0.2])\}$	2	2.3	1.62	1.93
2	Figure 16(b)	$\xi-\{\mathfrak{v}_2\ ([0.45, 0.9])\}$	1.7	1.6	1.3	1.8
3	Figure 16(c)	$\xi-\{\mathfrak{v}_3\ ([0.5, 0.6])\}$	1.65	1.9	1.17	1.18
4	Figure 16(d)	$\xi-\{\mathfrak{v}_1\ ([0.35, 0.7])\}$	1.8	1.8	0.75	1.25
5	Figure 16(e)	$\xi-\{\mathfrak{v}_4\ ([0.7, 0.8])\}$	1.45	1.7	1.6	1.7

Table 5. Calculation of order and size of these vertex deletion interval valued fuzzy outerplanar subgraphs.
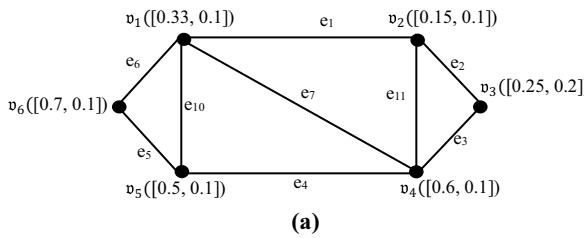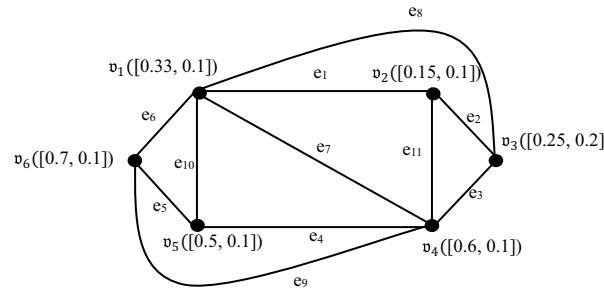
Fig. 17. Interval valued fuzzy graph. (**a**) Edge Deletion from an interval valued fuzzy subgraph.

Note 5 Suppose two Edge Deletion interval valued fuzzy outerplanar subgraphs ξ_1 and ξ_2 is obtained from the interval valued fuzzy graph ψ.If the sizes of the two interval valued subgraphs are equal but $\xi_1 \neq \xi_2$. (i.e.) $\left[\mathbb{S}^{-''}, \mathbb{S}^{+''}\right] > [\mathbb{S}^{-'}, \mathbb{S}^{+'}]$, then the two interval valued subgraphs are maximum Edge Deletion interval

valued fuzzy outerplanar subgraphs of ξ. This means that there is no unique maximum edge deletion interval valued fuzzy outerplanar subgraph of ξ.

Example 6 Let's look at the Interval valued fuzzy graph $\xi = (\mathbb{V}, \tau, \delta)$ shown in Fig. 17. The set of vertices in ξ is $\mathbb{V} = \{\mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4, \mathfrak{v}_5\}$ and their membership values of these are as follows: $\tau^-(\mathfrak{v}_1) = 0.33, \tau^+(\mathfrak{v}_1)=0.1, \tau^-(\mathfrak{v}_2) = 0.15, \tau^+(\mathfrak{v}_2)=0.1, \tau^-(\mathfrak{v}_3) = 0.25, \tau^+(\mathfrak{v}_3) =0.2, \tau^-(\mathfrak{v}_4) = 0.6, \tau^+(\mathfrak{v}_4)=0.1, \tau^-(\mathfrak{v}_5) = 0.5, \tau^+(\mathfrak{v}_5)=0.1 \quad , \tau^-(\mathfrak{v}_6) = 0.7, \tau^+(\mathfrak{v}_6)=0.1.$ For edges $([\dot{a}, \dot{b}]) = e_i\ \forall \delta([\dot{a}, \dot{b}]) \in \mathbb{E};\ i = 1, 2, 3, \ldots, 11$ and edge set with the values as $\mathbb{B}$ is

$\delta^-(\mathfrak{e}_1) = \delta^-(\mathfrak{e}_8) = \delta^-(\mathfrak{e}_9) = 0.5,\ \ \delta^+(\mathfrak{e}_1) = \delta^+(\mathfrak{e}_8) = \delta^+(\mathfrak{e}_9) = 0.1,\ \ \delta^-(\mathfrak{e}_2) = 0.35,\ \ \delta^-(\mathfrak{e}_2) = 0.2,$ $\delta^-(\mathfrak{e}_3) = 0.55,\ \delta^+(\mathfrak{e}_3) = 0.3, \delta^-(\mathfrak{e}_4) = 0.25,\ \delta^+(\mathfrak{e}_4) = 0.1, \delta^-(\mathfrak{e}_5) = 0.55,\ \delta^+(\mathfrak{e}_5) = 0.2, \delta^-(\mathfrak{e}_6) = 0.3,$ $\delta^+(\mathfrak{e}_6) = 0.1, \delta^-(\mathfrak{e}_7) = 0.8,\ \delta^+(\mathfrak{e}_7) = 0.3, \delta^-(\mathfrak{e}_{10}) = 0.65,\ \delta^+(\mathfrak{e}_{10}) = 0.2, \delta^-(\mathfrak{e}_{11}) = 0.2,\ \delta^+(\mathfrak{e}_{11}) = 0.1.$

Figure 17 (a) An example of interval valued fuzzy subgraphs with maximum and maximal Edge Deletion from an interval valued fuzzy graph ξ.

In Fig. 17 (a), the maximal Edge deleted interval valued fuzzy outerplanar subgraph are shown. Thus, a result, $\xi - \mathbb{U}$ is the greatest Edge Deleted interval valued fuzzy outerplanar subgraph of ξ, and interval valued fuzzy subgraph is maximal.

In the following section, an application of interval valued fuzzy outerplanar graphs in a bypass road network is shown.

Application of bypass road network in interval valued fuzzy outerplanar graphs

Bypass roads are designed to redirect heavy traffic away from city centers, effectively reducing traffic congestion. By diverting heavy-duty vehicles onto alternate routes, these roads alleviate pressure on the main city streets, leading to quicker travel times on both routes. This can result in reduced travel times, lower fuel consumption, and decreased emissions. Consequently, it improves accessibility to desired locations and positively impacts the local economy. A fuzzy graph can be used to model a traffic network where there is some certainty about the relative positions of the objects, but the information remains imprecise or uncertain. In this model, the nodes represent different city neighborhoods, while the edges illustrate the connections between them.

The fuzziness in the graph captures the uncertainty associated with factors like population distribution, traffic volumes, and road capacities. Managing this uncertainty is crucial for designing bypass roads that are both efficient and adaptable to changing conditions. By identifying an interval valued fuzzy outerplanar subgraph, planners can create routes that consider these uncertainties, ensuring that the bypass roads not only alleviate traffic in the city center but also adjust to shifts in traffic patterns and population growth. This strategy enables the development of a more resilient and flexible transportation network, capable of meeting future demands while maintaining optimal travel times and minimizing congestion in the urban core.

Moreover, the use of interval-valued fuzzy models allows for a more nuanced understanding of transportation dynamics. Unlike classical graphs, which operate with binary or crisp data, interval-valued fuzzy graphs account for a range of possibilities, making them suitable for complex urban systems where data is often incomplete, evolving, or imprecise. For instance, in real-world scenarios, traffic flow data may fluctuate throughout the day due to weather, special events, or accidents. Modeling these variations using interval values provides a buffer for unpredictability, allowing traffic engineers to design bypass networks that perform well under a range of conditions.

Additionally, the outerplanar nature of the graph ensures that the network remains relatively simple and computationally efficient. Outerplanar graphs are those that can be drawn in the plane without edge crossings such that all vertices lie on the outer face. This property is particularly beneficial in bypass road planning, where the goal is often to connect outer regions without interfering with the densely connected inner-city network. This simplifies not only the visualization of potential routes but also the optimization of infrastructure investments.

In practice, incorporating interval-valued fuzzy outerplanar subgraphs into geographic information systems (GIS) and traffic simulation tools can assist urban planners in visualizing multiple planning scenarios. For example, a bypass route might be modeled with intervals reflecting best-case and worst-case travel times or expected traffic volumes. These insights can guide the selection of routes that offer the best compromise between efficiency, cost, and long-term adaptability.

Ultimately, the integration of interval-valued fuzzy outerplanar graphs in traffic modeling offers a powerful toolset for dealing with uncertainty and complexity. As cities continue to expand and transportation demands grow more dynamic, such advanced graph-theoretic techniques will become increasingly vital for sustainable urban planning and smart infrastructure development.

Let $V = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, ... \mathcal{B}_n\}$ denote the set of towns where buses stop and let c_i represent the number of buses stopping at town $\mathcal{B}_i$. Suppose the number of buses arriving in town $\mathbb{B}_i$ during a time period T is a_i. The town $\mathcal{B}_i$ is given by the interval a_i/c_i, Expected Crowd in each in town $[C_i, C_j]$ for $i = 1,2,...,m, j = 1,2,...,n$.

The vertex membership is calculated based on the number of buses passing through the town, shopping at the town, Expected crowd at each town.

The membership value of a route $[\mathcal{B}_i, \mathcal{B}_j]$ is defined as follows in interval form:

$$\delta[\mathcal{B}_i, \mathcal{B}_j] = \begin{cases} \left[\frac{a_{ij}}{\mathcal{P}} \left(\delta\left(\mathcal{B}_i\right) \wedge \delta\left(\mathcal{B}_j\right) \right), \frac{a_{ij}}{\mathcal{P}} \left(\tau\left(\mathcal{B}_i\right) \wedge \tau\left(\mathcal{B}_j\right) \right) \right], & if\, 0 \leq a \leq \mathcal{P} \\ \left[\delta(\mathcal{B}_i) \wedge \delta\left(\mathcal{B}_j\right), \tau(\mathcal{B}_i) \wedge \tau\left(\mathcal{B}_j\right) \right], & if\, a > \mathcal{P} \end{cases}$$

where a_{ij} represents the number of passengers traveling on route $(\mathcal{B}_i, \mathcal{B}_j)$ during time T, and $\mathcal{P}$ is the threshold for a 'satisfied bus-stopping town', set to 30.

$\delta\left(\mathcal{B}_i\right)$ and $\delta\left(\mathcal{B}_j\right)$ are the lower bounds of the membership intervals for towns $\mathcal{B}_i$ and $\mathcal{B}_j$ respectively. $\tau(\mathcal{B}_i)$ and $\tau(\mathcal{B}_j)$ are the upper bounds of the membership intervals for towns $\mathcal{B}_i$ and $\mathcal{B}_j$ respectively. Thus, the interval membership value for the edge $(\mathcal{B}_i, \mathcal{B}_j)$ for both the lower and upper bounds of the number of passengers and their respective memberships.

The modelling to design an optimal bypass road network are presented as an algorithm:

Algorithm

Constructing an interval-valued fuzzy outerplanar graph for bypass road planning.

1. Consider the fuzzy transportation network of a city, where towns are represented as vertices and routes between them as edges.
2. Identify the set of towns $V = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \ldots \mathcal{B}_n\}$, and collect data:

   - a_i: Number of buses arriving at town $\mathcal{B}_i$ during time T
   - c_i: Number of buses stopping at town $\mathcal{B}_i$
   - s_{ij}: Number of passengers on route $(\mathcal{B}_i, \mathcal{B}_j)$
   - Threshold $\mathcal{P}$ for a satisfied bus-stopping town.

3. Compute vertex membership intervals for each town using:
   $$\delta(\mathcal{B}_i) = {}^{a_i}\!/_{c_i}$$
   $\tau(\mathcal{B}_i)$= Expected Crowd Estimate $(\mathcal{B}_i)$
4. For each edge $(\mathcal{B}_i, \mathcal{B}_j)$, compute the interval-valued membership using:

   If $0 \leq s_{ij} \leq \mathcal{P}$:

   $$\delta[\mathcal{B}_i, \mathcal{B}_j] = \left[\frac{s_{ij}}{\mathcal{P}}\left(\delta(\mathcal{B}_i) \wedge \delta(\mathcal{B}_j)\right), \frac{s_{ij}}{\mathcal{P}}\left(\tau(\mathcal{B}_i) \wedge \tau(\mathcal{B}_j)\right)\right]$$

   If $s_{ij} > \mathcal{P}$:

   $$\delta[\mathcal{B}_i, \mathcal{B}_j] = \left[\delta(\mathcal{B}_i) \wedge \delta(\mathcal{B}_j), \tau(\mathcal{B}_i) \wedge \tau(\mathcal{B}_j)\right]$$

5. Construct the interval-valued fuzzy graph $\xi = (\mathbb{V}, \tau, \delta)$.
6. Test ξ for outerplanarity under interval-valued fuzzy conditions: ξ is not outerplanar, iteratively delete edges or vertices with the lowest membership values: Continue until a subgraph $\xi' \subseteq \xi$ is outerplanar.
7. The resulting subgraph $\xi' \subseteq \xi$, after vertex deletion, is an interval-valued fuzzy outerplanar graph ξ'.
8. Use this ξ' to identify bypass routes that minimize urban congestion and accommodate uncertain traffic flows.

For illustration, Table 6 provides data on the number of buses stopping at each town and the number of passengers traveling through the towns. The routes (edges) for the problem include: $(\mathcal{B}_1, \mathcal{B}_2), (\mathcal{B}_1, \mathcal{B}_8), (\mathcal{B}_1, \mathcal{B}_9),$ $(\mathcal{B}_2, \mathcal{B}_3), (\mathcal{B}_2, \mathcal{B}_{10}), (\mathcal{B}_3, \mathcal{B}_4), (\mathcal{B}_3, \mathcal{B}_{12}), (\mathcal{B}_4, \mathcal{B}_5), (\mathcal{B}_4, \mathcal{B}_{12}), (\mathcal{B}_5, \mathcal{B}_6), (\mathcal{B}_5, \mathcal{B}_{12}), (\mathcal{B}_6, \mathcal{B}_7), (\mathcal{B}_6, \mathcal{B}_{11}), (\mathcal{B}_7, \mathcal{B}_8),$ $(\mathcal{B}_7, \mathcal{B}_{11}), (\mathcal{B}_8, \mathcal{B}_9), (\mathcal{B}_9, \mathcal{B}_{10}), (\mathcal{B}_{10}, \mathcal{B}_{11}), (\mathcal{B}_{10}, \mathcal{B}_{12}), (\mathcal{B}_{11}, \mathcal{B}_{12})$. The number of passengers for all routes and their membership values are listed in Table 7. The bus routes within the town are illustrated in Fig. 17.

For instance, consider the bypass road network and the corresponding interval-valued fuzzy graph shown in Figs. 18 and 19. Figure 20 does not initially represent an interval-valued fuzzy outerplanar graph due to the presence of overlapping connections and central congestion nodes that violate the conditions of outerplanarity. However, by strategically deleting certain vertices particularly those representing highly congested or low-traffic towns an interval-valued fuzzy outerplanar subgraph can be derived. This refined subgraph enables the

Vertex	Number of buses stopping at town a_i	Number of buses passing through town c_i	a_i/c_i	Expected Crowd in each town $\left(\frac{\tau_i}{c_i}\right) \times [C_i, C_j]$	Crowdness factor as Percentage [Vertex Membership value α]
$\mathcal{B}_1$	38	42	0.90	[35,45]	[0.32,0.41]
$\mathcal{B}_2$	39	45	0.87	[37,47]	[0.32,0.41]
$\mathcal{B}_3$	41	45	0.91	[43,45]	[0.39,0.41]
$\mathcal{B}_4$	41	45	0.91	[43,45]	[0.39,0.41]
$\mathcal{B}_5$	43	48	0.91	[43,45	[0.39,0.45]
$\mathcal{B}_6$	44	49	0.89	[41,48]	[0.36,0.43]
$\mathcal{B}_7$	44	49	0.89	[41,48]	[0.36,0.43]
$\mathcal{B}_8$	44	49	0.89	[41,48]	[0.36,0.43]
$\mathcal{B}_9$	19	35	0.54	[45,49]	[0.24,0.26]
$\mathcal{B}_{10}$	17	35	0.48	[47,48]	[0.22,0.23]
$\mathcal{B}_{11}$	19	35	0.54	[41,45]	[0.22,0.24]
$\mathcal{B}_{12}$	20	38	0.53	[46,48]	[0.24,0.25]

Table 6. Calculation of vertex membership value.

Edge	Number of passengers traveling ($\jmath$)	Edge membership value β
$(\mathcal{B}_1, \mathcal{B}_2)$	28	[0.30,0.38]
$(\mathcal{B}_1, \mathcal{B}_8)$	16	[0.17,0.22]
$(\mathcal{B}_1, \mathcal{B}_9)$	17	[0.14,0.15]
$(\mathcal{B}_2, \mathcal{B}_3)$	26	[0.28,0.36]
$(\mathcal{B}_2, \mathcal{B}_{10})$	15	[0.11,0.12]
$(\mathcal{B}_3, \mathcal{B}_4)$	27	[0.35,0.37]
$(\mathcal{B}_3, \mathcal{B}_{12})$	18	[0.14,0.15]
$(\mathcal{B}_4, \mathcal{B}_5)$	24	[0.31,0.33]
$(\mathcal{B}_4, \mathcal{B}_{12})$	17	[0.14,0.14]
$(\mathcal{B}_5, \mathcal{B}_6)$	21	[0.25,0.30]
$(\mathcal{B}_5, \mathcal{B}_{12})$	17	[0.14,0.14]
$(\mathcal{B}_6, \mathcal{B}_7)$	19	[0.23,0.27]
$(\mathcal{B}_6, \mathcal{B}_{11})$	18	[0.13,0.14]
$(\mathcal{B}_7, \mathcal{B}_8)$	26	[0.31,0.37]
$(\mathcal{B}_7, \mathcal{B}_{11})$	22	[0.16,0.17]
$(\mathcal{B}_8, \mathcal{B}_9)$	23	[0.18,0.19]
$(\mathcal{B}_9, \mathcal{B}_{10})$	20	[0.15,0.15]
$(\mathcal{B}_{10}, \mathcal{B}_{11})$	26	[0.19,0.20]
$(\mathcal{B}_{10}, \mathcal{B}_{12})$	21	[0.15,0.16]
$(\mathcal{B}_{11}, \mathcal{B}_{12})$	23	[0.17,0.18]

Table 7. Calculation of edge membership value.
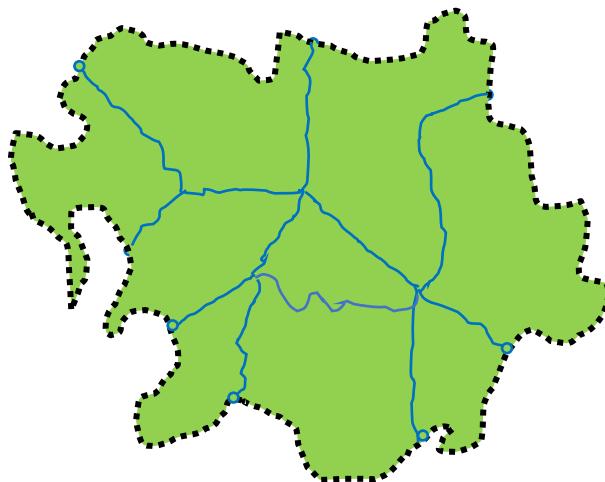
Fig. 18. Bypass road network mapping.

construction of an efficient bypass road system between key urban centers, as effectively illustrated in Fig. 18. Through this approach, the original graph, which may contain complex or non-planar structures, is transformed into a more manageable and application-ready model.

The city can be accessed using the bypass routes via the edges highlighted in Fig. 20, which represent feasible travel paths after simplification. As demonstrated in Fig. 19, the deletion of specific vertices not only reduces the complexity of the network but also results in a topology conducive to bypass development. These adjustments preserve the essential connectivity of the transportation network while removing redundancies and minimizing intersections.

Additionally, Fig. 20 provides an illustrative example of a fuzzy outerplanar graph structure, showcasing how vertices (representing towns) and edges (representing routes) can be effectively rearranged and optimized to support a more streamlined bypass network. The outerplanar configuration ensures that all vertices lie along
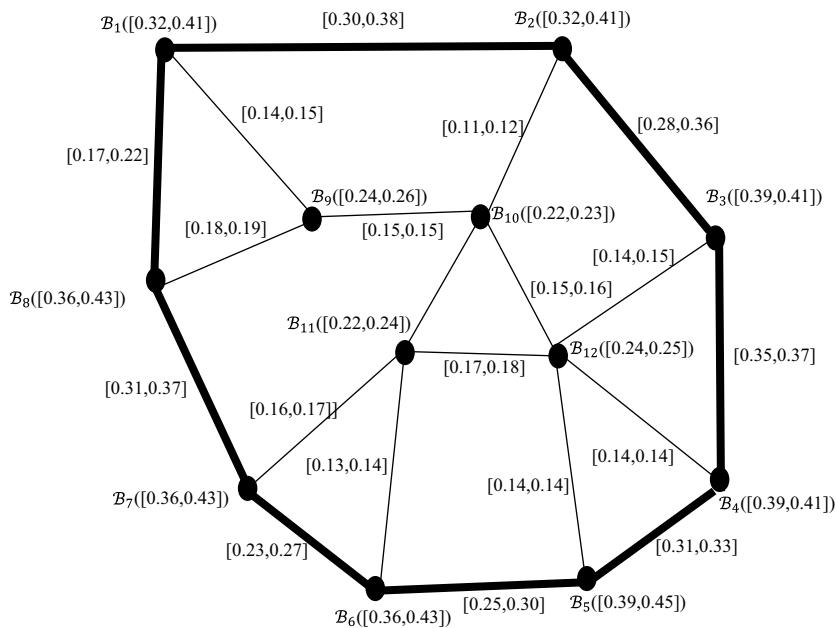
Fig. 19. Interval valued fuzzy non outerplanar graph.
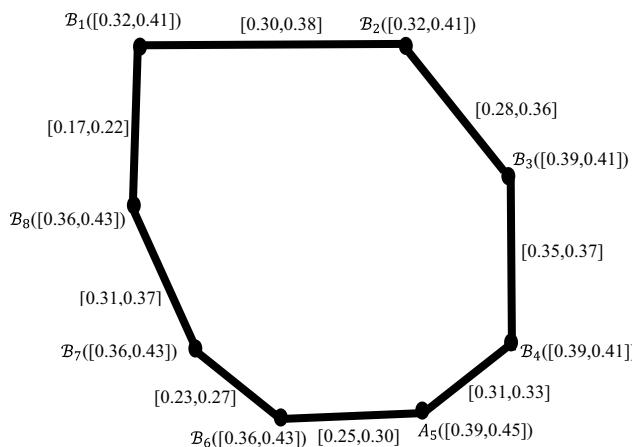
Fig. 20. Interval valued fuzzy outerplanar graph.

the external boundary, eliminating inner cycles that typically contribute to traffic congestion. This geometrical clarity allows for the development of direct and less conflicted routes that circumnavigate the city center.

By utilizing the inherent characteristics of fuzzy outerplanar graphs such as the ability to model ambiguity in road usage, traffic density, and infrastructural capacity urban planners can design bypass road systems that are not only capable of alleviating current traffic challenges but are also inherently adaptive to future fluctuations in population growth, vehicular load, and spatial expansion. The interval-valued fuzzy model, in particular, offers the flexibility to account for uncertainty and variation in transportation parameters, making it a powerful tool in smart urban planning. This methodology thus underscores the practical advantages and long-term sustainability of employing fuzzy graph theory in contemporary infrastructure development, enabling the creation of road networks that are resilient, efficient, and well-aligned with the dynamic needs of modern cities.

Conclusion
In this study, we have explored the properties and characteristics of interval-valued fuzzy outerplanar graphs. Through the analysis of subgraphs derived from these graphs via vertex and edge deletions, we have demonstrated methods for identifying maximal and maximum interval-valued fuzzy outerplanar subgraphs. The relationships among these concepts were rigorously examined, leading to several theorems that further our understanding of interval-valued fuzzy graph structures. Furthermore, we introduced interval-valued fuzzy dual graphs, highlighting their important characteristics and their close relationship with interval-valued fuzzy outerplanar graphs. This investigation has revealed the versatility and utility of these graphs in various applications due to

their ability to represent uncertainty and variability. Finally, we explore the application of interval-valued fuzzy outerplanar graphs to bypass road networks. By modeling the bypass road network with this approach, we can effectively capture and represent the uncertainty and variability in factors such as road conditions, traffic flow, and connectivity.

Data availability

All data generated or analysed during this study are included in this article. If someone wants to request the data from this study, it is possible to contact the corresponding author: Tadesse Walelign (tadelenyosy@gmail.com).

References

1. Zadeh, L. A. Fuzzy sets. *Infor. Cont.* **8**(3), 338–353 (1965).
2. Rosenfeld. A. Fuzzy graphs, fuzzy sets and their applications to cognitive and decision processes. *Proceed US-Japan. Sem. Fuzzy Sets App. Acad. Pre.* 77–95, (1975).
3. Abdul-Jabbar, N., Naoom, J. H. & Ouda, E. H. Fuzzy dual graphs. *J Al Nahrain Univ.* **12**(4), 168–171 (2009).
4. Harary, F. Graph theoretic methods in the management sciences. *Manag. Sci.* **5**(4), 387–403 (1959).
5. Berthold, M. R. & Huber, K. P. Constructing fuzzy graphs from examples. *Intell. Data Analy.* **3**(1), 37–53 (1999).
6. Chartrand. G, Harary. F. Planar permutation graphs. In Annales. *de l'institut. Henri. Poincaré. Section B Calcul prob et statis.* **3**(4), 433–438, (1967).
7. Sysło, M. M. Characterizations of outerplanar graphs. *Discrete Math.* **26**(1), 47–53 (1978).
8. Kulli, V. R. On minimally non outerplanar graphs. *Proceed. Ind. Nat. Sci. Acad.* **41**, 275–280 (1975).
9. Mitchell, S. L. Linear algorithms to recognize outerplanar and maximal outerplanar graphs. *Infor. Process. Lett.* **9**(5), 229–232 (1979).
10. Fleischner, H. J., Geller, D. P. & Harary, F. Outerplanar graphs and weak duals. *J. Indian Math. Soc.* **38**, 215–219 (1974).
11. Sysło, M. M. Outerplanar graphs: Characterizations, testing, coding, and counting. *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astr. Phys.* **26**, 675–684 (1978).
12. Sysło, M. M. The subgraph isomorphism problem for outerplanar graphs. *Theor. Comp. Sci.* **17**(1), 91–97 (1982).
13. Samanta, S., Pal, A. & Pal, M. Concept of fuzzy planar graphs. *Pro. Sci. Infor. Confer.* **7–9**, 557–563 (2013).
14. Samanta, S. & Pal, M. Fuzzy planar graphs. *IEEE Trans. Fuzzy Syst.* **23**(6), 1936–1942 (2016).
15. Samanta, S., Pal, M. & Pal, A. New concepts of fuzzy planar graph. *Int. J. Advan. Res. Arti. Intell.* **3**(1), 52–59 (2014).
16. Shriram, K., Sujatha, R. & Narasimman, S. Fuzzy combinatorial dual graphs. *Int. J. Pure App. Math.* **109**(7), 35–41 (2016).
17. Shriram, K., Sujatha, R. & Nagarajan, D. A fuzzy planar subgraph formation model for partitioning very large-scale integration networks. *Dec. Analy. Jour.* **9**, 100339 (2023).
18. Parvathi, R., & Karunambigai, M. G. Intuitionistic fuzzy graphs. In *Comput. Intel. theory. app: inter. Con 9th fuzzy days in Dortmund, Germany, Sept. 18–20, 2006 Proceed*, 139–150, Springer Berlin Heidelberg, (2006).
19. Rashmanlou, H., Borzooei, R. A., Samanta, S. & Pal, M. Properties of interval valued intuitionistic (s, t)–fuzzy graphs. *Pac. Sci. Review. Natur. Sci. Eng.* **18**(1), 30–37 (2016).
20. Rehman, F. U., Hussain, M. T. & Rashid, T. Strong pair domination number in intuitionistic fuzzy influence graphs with application for the selection of hospital having the optimal medical facilities. *Ex. Syst. App* **238**, 122169 (2024).
21. Rehman, F. U., Rashid, T. & Hussain, M. T. Q-fractional fuzzy influence pair domination number to locate and control smog area. *Ex. Syst. App* **274**, 126886 (2025).
22. Zuo, C., Pal, A. & Dey, A. New concepts of picture fuzzy graphs with application. *Mathematics* **7**(5), 470 (2019).
23. Akram, M. & Akmal, R. Application of bipolar fuzzy sets in graph structures. *App. Comput. Intell. Soft Comput.* **1**, 5859080 (2016).
24. Akram, M. & Dudek, W. A. Regular bipolar fuzzy graphs. *Neu. Comput. App.* **21**, 197–205 (2012).
25. Akram. M, Sarwar. M, Dudek. W. A. Bipolar Fuzzy Planar Graphs. Graphs. *Analy. Bipo. Fuzzy. Infor.* 223–251, (2021).
26. Pramanik, T., Pal, M., Mondal, S. & Samanta, S. A study on bipolar fuzzy planar graph and its application in image shrinking. *J. Intell. Fuzzy. Syst.* **34**, 1863–1874 (2018).
27. Rashmanlou, H. & Jun, Y. B. Complete interval-valued fuzzy graphs. *An. Fuzzy. Math. Infor.* **6**(3), 677–687 (2013).
28. Ghorai, G. & Pal, M. A study on m-polar fuzzy planar graphs. *Int. J. Comput. Sci. Math.* **7**(3), 283–292 (2016).
29. Ghorai, G. & Pal, M. Faces and dual of m-polar fuzzy planar graphs. *J. Intell. Fuzzy. Syst.* **31**(3), 2043–2049 (2016).
30. Mondal, U., Mahapatra, T., Xin, Q. & Pal, M. Generalized m-Polar Fuzzy Planar Graph and Its Application. *IEEE Access.* **11**, 138399–138413 (2023).
31. Mahapatra, T., Sahoo, S., Ghorai, G. & Pal, M. Interval valued m-polar fuzzy planar graph and its application. *Art. Intell. Review.* **54**, 1649–1675 (2021).
32. Mahapatra, T., Samanta, S. & Pal, M. Generalized neutrosophic planar graphs and its application. *J. App. Math. Comput.* **65**, 693–712 (2021).
33. Akram, M. Interval-valued fuzzy line graphs. *Neu. Comput. App.* **21**, 145–150 (2012).
34. Akram, M., Alshehri, N. O. & Dudek, W. A. Certain Types of Interval-Valued Fuzzy Graphs. *J. App. Math.* **1**, 857070 (2013).
35. Akram, M., Naz, S. & Davvaz, B. Simplified interval-valued Pythagorean fuzzy graphs with application. *Com. Intel. Syst.* **5**, 229–253 (2019).
36. Pal, M., Samanta, S. & Rashmanlou, H. Some results on interval-valued fuzzy graphs. *Int. J. Comp. Sci. Elec. Eng.* **3**(3), 205–211 (2015).
37. Mathew. S, Mordeson. J.N, Malik. D.S. Interval-valued fuzzy graphs. *Fuzzy. Gra. Theo.* 231–269, (2018).
38. Rashmanlou, H. & Pal, M. Antipodal interval-valued fuzzy graphs. *Inter. J. App. Fuzzy Sets. Arti. Intell.* **3**, 107–130 (2014).
39. Rashmanlou, H. & Pal, M. Balanced interval-valued fuzzy graphs. *J. Phy. Sci.* **17**, 43–57 (2013).
40. Rashmanlou, H. & Pal, M. Isometry on interval-valued fuzzy graphs. *Int. J. Fuzzy Math. Archi.* **3**, 28–35 (2014).
41. Rashmanlou, H., Pal, M., Borzooei, R. A., Mofidnakhaei, F. & Sarkar, B. Product of interval-valued fuzzy graphs and degree. *J. Intell. Fuzzy Syst.* **35**(6), 6443–6645 (2018).
42. Talebi, A. A., Rashmanlou, H. & Davvaz, B. New concepts of regular interval-valued fuzzy graphs. *J. App. Math. Infor.* **35**(12), 95–111 (2017).
43. Akram, M. & Dudek, W. A. Interval-valued fuzzy graphs. *Com & Math. App.* **61**(2), 289–299 (2011).
44. Pramanik, T., Samanta, S. & Pal, M. Interval-valued fuzzy planar graphs. *Int. J. Mach. Learn. Cyber.* **7**(4), 653–664 (2016).
45. Deivanai, J., Sujatha, R., Nagarajan, D. & Tadesse, W. Fuzzy outerplanar graphs and its applications. *In. J. Comput. Intell. Syst.* **17**(1), 231 (2024).
46. Pal. M, Rashmanlou. H Irregular interval valued fuzzy graphs. *arXiv preprint arXiv*,1407.61900, (2014).

Author contributions

All authors contributed to the study conception and design, Formal Analysis, Investigation, Data Curation, Writing-Original Draft: DJ and SR, Resources and Supervision: SR, Methodology: DJ, SR, TW and ND, Writing-Review & Editing, Visualization: DJ, SR, TW and ND. All authors have read and agreed to the published version of the manuscript.

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Correspondence and requests for materials should be addressed to S.R. or T.W.

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