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Pythagorean fuzzy N-bipolar soft sets-based multi-criteria decision-making framework for sustainability evaluation and risk assessment in manufacturing industries

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Sustainability evaluation in manufacturing industries is increasingly vital for promoting responsible growth and long-term competitiveness amid environmental, social, and economic challenges. Effective decision-making (DM) under uncertainty is crucial for managing multiple, often conflicting sustainability objectives. In this paper, we propose a novel hybrid model, termed Pythagorean fuzzy N-bipolar soft sets (PFNBSSs), which integrates Pythagorean fuzzy sets (PFSs), N-soft sets (NSSs), and bipolar soft sets (BSSs) within a unified multi-criteria decision-making (MCDM) framework. For theoretical purposes, we define basic operations and algebraic properties of PFNBSSs, supported by illustrative examples. To demonstrate practical applicability, the PFNBSS model is applied to assess sustainability practices in manufacturing industries through two numerical examples: one focusing on positive and negative sustainability indicators, and another emphasizing comparative sustainability risk assessment across diverse manufacturing sectors. Detailed interpretations of computational results and their relevance in practical DM are provided. This is followed by a comparative analysis confirming the superior discrimination power and expressive capability of the PFNBSS model over existing alternatives. The paper concludes with a critical evaluation of the model and suggestions for future research.

Keywords Pythagorean fuzzy N-bipolar soft sets, N-soft sets, Pythagorean fuzzy sets, MCDM, Sustainable Manufacturing Evaluation

In real-world DM, scenarios often involve complex evaluations where multiple factors, conflicting criteria, and uncertain information coexist. For instance, consider a manufacturing company aiming to select a new supplier based on sustainability practices. Decision-makers must assess not only positive attributes such as eco-friendly production and ethical sourcing but also negative concerns, including cost implications, potential supply chain disruptions, and sustainability risks. Furthermore, evaluations are rarely binary; suppliers may partially fulfill sustainability goals or present varying degrees of risk. However, many existing models lack the ability to simultaneously capture multilevel, bipolar, and uncertain information–particularly when balancing both positive drivers and risk factors–limiting their applicability in complex decision contexts. This gap motivates the development of enhanced soft computing frameworks capable of addressing these multifaceted challenges.

To tackle these issues effectively, various mathematical frameworks have been proposed to manage the uncertainty and vagueness inherent in real-world problems. One foundational approach is fuzzy set (FS) theory, introduced by Zadeh[1], which enables the representation of partial membership degrees (MDs) of elements

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in a set, offering a more nuanced treatment of uncertainty than classical set theory. Intuitionistic FSs (IFSs), introduced by Atanassov[2], extend this concept by incorporating non-membership degrees (NMDs), thus capturing both MDs and NMDs of an element. Further generalizations, such as Pythagorean fuzzy sets (PFSs) introduced by Yager[3], offer even greater flexibility by considering the square sum of MDs and NMDs. Due to their enhanced expressive power, PFSs have gained traction in both theoretical and applied domains, particularly in MCDM contexts where uncertainty is significant. For example, PFSs have been incorporated into TOPSIS-based models[4,5] and extended using Dombi operators[6].

Soft set (SS) theory, introduced by Molodtsov[7], has emerged as a robust mathematical tool for addressing uncertainty without requiring auxiliary conditions like parameter membership functions. Subsequent work has expanded SS theory through new operations[8] and algebraic enhancements[9]. Integrating FS theory led to fuzzy SSs (FSSs)[10], while intuitionistic FSSs (IFSSs)[11] were proposed to better capture hesitation in expert opinions. Pythagorean FSSs (PFSSs)[12] further extend this capability. A recent systematic review comprehensively analyzed the evolution and applications of SS theory, underscoring its growing relevance in modern DM frameworks[13]. The SS paradigm has undergone significant evolution, extending beyond classical structures to encompass hypersoft sets[14], which provide hierarchical representation of parameterization. Notably, the development of N-hypersoft sets by Musa et al.[15] introduces a novel extension that enhances the expressiveness of hypersoft frameworks for real-world applications. Further advancements include the incorporation of q-rung picture fuzzy environments into hypersoft models, enabling more robust and uncertain data handling, as demonstrated in recent works on intelligent transportation systems[16], sustainable smart technologies[17], and pattern recognition using similarity measures[18]. These contributions, among others, reflect a growing trend toward sophisticated and multidimensional DM models in the SS literature. Moreover, other advanced forms of SSs continue to emerge, offering additional avenues for exploration and application[19].

BSSs[20] have drawn increasing interest for their ability to model uncertainty and vagueness involving both positive and negative aspects of information. Various extensions have been proposed by integrating fuzzy, rough, and other uncertainty-based theories to handle bipolar information more effectively. Fuzzy BSSs (FBSSs)[21] have been examined for their algebraic structures and practical utility across domains. Hybrid models such as rough Pythagorean FBSSs (PFBSSs)[22] have also shown promise in complex decision scenarios. Moreover, the development of bipolar hypersoft sets[23] and fuzzy bipolar hypersoft sets[24] further enhances the expressiveness of bipolar models by introducing parameter hierarchies and increased fuzziness in soft DM frameworks.

NSSs[25] refine the classical SS framework by enabling multi-valued (multinary) parameterized representations, in which each parameter is assigned a specific value from a predefined domain. Unlike classical SSs that rely on binary associations, NSSs allow for a more granular and expressive classification of objects across multiple evaluation dimensions. Extensions include fuzzy NSSs (FNSSs)[26] that incorporate fuzziness, and intuitionistic FNSSs (IFNSSs)[27] that account for hesitancy. In group DM, multi-agent NSS frameworks[28] support the aggregation of diverse expert judgments. Separable NSSs[29] allow decomposition of parameter sets for finer-grained evaluation. Pythagorean FNSSs (PFNSSs)[30] generalize IFNSSs by relaxing the square-sum constraint, providing added flexibility. M-parameterized NSSs[31] further extend the paradigm by associating multiple parameter values with each object. Bipolar M-parameterized NSSs[32] unify bipolarity, multilevel evaluation, and parameterization into a powerful hybrid framework.

N-bipolar soft sets (NBSSs)[33] represent a compelling hybridization of BSSs and NSSs, integrating affirmative/negative judgments with multi-valued parameterization. This combination mirrors the nuanced nature of human DM, where evaluations span a spectrum of attitudes across multiple criteria. This foundational model has given rise to several notable extensions: N-bipolar soft expert sets[34] incorporate collective expert opinions; fuzzy NBSSs (FNBSSs)[35] model vagueness through MDs; and intuitionistic FNBSSs (IFNBSSs)[36] introduce a hesitation component to handle indecision. N-bipolar hypersoft sets[37] advance this structure by introducing parameter hierarchies for multi-level abstraction. Additionally, N-bipolar hypersoft topologies[38] offer a topological foundation for modeling continuity and separation within bipolar, multi-valued settings. Together, these models significantly enrich soft computing by enabling flexible, layered, and context-sensitive reasoning in complex decision environments. For additional related studies not discussed in this paper, interested readers may consult Paul et al.[39], El-Morsy[40], Chohan et al.[41], Gul and Tufail[42], Badi et al.[43], Garg[44], Hussain et al.[45], and Mahmood et al.[46,47].

Motivation and model development

Many existing DM models, including classical FSs and SSs, face critical limitations when applied to complex, real-world scenarios. Specifically, they struggle to represent multi-valued evaluations, integrate both positive and negative information (bipolarity), and effectively handle uncertainty. These limitations become especially problematic in domains such as sustainable manufacturing, where decisions must simultaneously weigh benefits and trade-offs under vague and imprecise information.

PFSs offer a stronger framework for capturing uncertainty compared to traditional FSs and IFSs, yet they lack parameterization and bipolar representation. Conversely, NSSs enable multi-parameter modeling but are not equipped to handle bipolar or high-order uncertainty. BSSs allow for positive and negative evaluations but often restrict analysis to binary scales and lack higher uncertainty handling.

To address these gaps, we propose the PFNBSS framework–a hybrid model that synergistically integrates the strengths of PFSs, NSSs, and BSSs. This integration enables rich, multigraded, and bipolar information representation while maintaining parameterized structure and superior uncertainty modeling.

The motivation behind this integration is not only conceptual but also supported by a comparative analysis of related models. A qualitative comparison is provided in Table 32, which illustrates the distinct capabilities and expressive advantages of the proposed PFNBSS model.

Research objectives and contributions

This study aims to establish a comprehensive DM model that effectively integrates multi-valued evaluations, bipolar information, and advanced uncertainty modeling based on PFSs. The main contributions of this work are summarized as follows:

- The introduction of the PFNBSS model that combines the advantages of PFSs with NBSSs to address complex decision problems.
- Development of formal definitions, algebraic operations, and illustrative examples to underpin the theoretical foundation of PFNBSS.
- Proposal of a robust DM procedure tailored to the PFNBSS framework.
- Demonstration of the model's applicability through a case study evaluating sustainability practices in manufacturing industries.
- Illustration of the model's versatility in handling risk-focused sustainability evaluations across diverse manufacturing sectors.
- Comparative analysis illustrating the superiority of the PFNBSS model over existing methods regarding flexibility, interpretability, and decision quality.

Structure of the paper

The paper is structured as follows: Section 2 presents a review of relevant foundational concepts. Section 3 introduces the PFNBSS model along with its formal structure, operations, and algebraic properties. Section 4 outlines a comprehensive DM methodology and demonstrates its application through two practical examples: (i) a sustainability evaluation of manufacturing companies and (ii) a comparative sustainability risk assessment across sectors. Section 5 presents the results and discussion through these two examples. The first offers a detailed quantitative evaluation and ranking of manufacturing companies based on sustainability criteria, while the second illustrates the model's adaptability in assessing sustainability-related risks. Together, they demonstrate the PFNBSS model's robustness, transparency, and effectiveness in MCDM contexts. Section 6 provides a comparative evaluation of the proposed model with existing approaches, highlighting its advantages and discussing its limitations. Finally, Section 7 summarizes the main contributions of the paper and suggests potential directions for future research.

Preliminaries and related concepts

This section revisits the essential definitions of several models that underpin the concepts used in our proposed framework. Throughout the paper, $\mathcal{L}$ denotes the universal set of alternatives (or objects), ρ represents the set of attributes (or parameters), and $R = \{0, 1, \ldots, N-1\}$ is the set of ordered grades, where $N \in \{2, 3, \ldots\}$. For clarity and ease of reference, the key symbols and abbreviations used throughout the paper are summarized in Table 1.

Definition 2.1 Let $\nu^+ : \mathcal{L} \to [0,1]$ and $\nu^- : \mathcal{L} \to [0,1]$ represent, respectively, the degrees of membership and non-membership of $\ell \in \mathcal{L}$. Then, $\Re = \left\{ \left\langle \ell, \nu^+(\ell), \nu^-(\ell) \right\rangle : \ell \in \mathcal{L} \right\}$ is called:

i. an FS[1], if for all $\ell \in \mathcal{L}$, $\nu^-(\ell) = 0$.
ii. an IFS[2], if for all $\ell \in \mathcal{L}$, $0 \le \nu^+(\ell) + \nu^-(\ell) \le 1$.
iii. a PFS[3], if for all $\ell \in \mathcal{L}$, $0 \le (\nu^+(\ell))^2 + (\nu^-(\ell))^2 \le 1$.

Definition 2.2 [5] Let $\Psi = (\alpha^+, \alpha^-)$ be a Pythagorean fuzzy number (PFN). Then,

i. the score value of Ψ is given by $\mathbb{S}(\Psi) = (\alpha^+)^2 - (\alpha^-)^2$, where $\mathbb{S}(\Psi) \in [-1, 1]$.
ii. the accuracy value of Ψ is given by $\mathbb{A}(\Psi) = (\alpha^+)^2 + (\alpha^-)^2$, where $\mathbb{A}(\Psi) \in [0, 1]$.

Definition 2.3 [5] Let $\Psi_1 = (\alpha_1^+, \alpha_1^-)$ and $\Psi_2 = (\alpha_2^+, \alpha_2^-)$ be any two PFNs. Let $\mathbb{S}(\Psi_1)$ and $\mathbb{S}(\Psi_2)$ denote the score values of Ψ_1 and Ψ_2, respectively, and let $\mathbb{A}(\Psi_1)$ and $\mathbb{A}(\Psi_2)$ denote their corresponding accuracy values. Then,

i. if $\mathbb{S}(\Psi_1) \succ \mathbb{S}(\Psi_2)$, then $\Psi_1 \succ \Psi_2$.
ii. if $\mathbb{S}(\Psi_1) = \mathbb{S}(\Psi_2)$, and:

- if $\mathbb{A}(\Psi_1) \succ \mathbb{A}(\Psi_2)$, then $\Psi_1 \succ \Psi_2$.
- if $\mathbb{A}(\Psi_1) = \mathbb{A}(\Psi_2)$, then $\Psi_1 = \Psi_2$.

Definition 2.4 A pair (μ, ρ) is called:

i. an SS[7], if $\mu : \rho \to 2^{\mathcal{L}}$, where $2^{\mathcal{L}}$ denotes the set of all crisp subsets of $\mathcal{L}$.
ii. an FSS[10], if $\mu : \rho \to \mathbb{F}^{\mathcal{L}}$, where $\mathbb{F}^{\mathcal{L}}$ denotes the set of all FSs over $\mathcal{L}$.
iii. an IFSS[11], if $\mu : \rho \to \mathbb{I}^{\mathcal{L}}$, where $\mathbb{I}^{\mathcal{L}}$ denotes the set of all IFSs over $\mathcal{L}$.
iv. a PFSS[12], if $\mu : \rho \to \mathbb{P}^{\mathcal{L}}$, where $\mathbb{P}^{\mathcal{L}}$ denotes the set of all PFSs over $\mathcal{L}$.

Definition 2.5 [8] Let $\rho = \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$ be a set of attributes. The NOT set of ρ, denoted by $\neg\rho$, is given by $\neg\rho = \{\neg\varepsilon_1, \neg\varepsilon_2, \ldots, \neg\varepsilon_n\}$, where each $\neg\varepsilon_i$ denotes the negation (i.e., the opposite) of the attribute ε_i, for $i = 1, 2, \ldots, n$.

Symbol	Meaning
$\mathcal{L}$	Universal set of alternatives (objects)
ρ	Set of attributes or decision parameters
R	Ordered set of evaluation grades
FS	Fuzzy set
IFS	Intuitionistic fuzzy set
PFS	Pythagorean fuzzy set
PFN	Pythagorean fuzzy number
SS	Soft set
FSS	Fuzzy soft set
IFSS	Intuitionistic fuzzy soft set
PFSS	Pythagorean fuzzy soft set
BSS	Bipolar soft set
FBSS	Fuzzy bipolar soft set
IFBSS	Intuitionistic fuzzy bipolar soft set
PFBSS	Pythagorean fuzzy bipolar soft set
NSS	N-soft set
FNSS	Fuzzy N-soft set
IFNSS	Intuitionistic fuzzy N-soft set
PFNSS	Pythagorean fuzzy N-soft set
NBSS	N-bipolar soft set
FNBSS	Fuzzy N-bipolar soft set
IFNBSS	Intuitionistic fuzzy N-bipolar soft set
PFNBSS	Pythagorean fuzzy N-bipolar soft set
DM	Decision-making
MCDM	Multi-criteria decision-making
MD	Membership degree
NMD	Non-membership degree
ν^+	Membership function for FS, IFS, and PFS
ν^-	Non-membership function for IFS and PFS
μ	Mapping for SS, FSS, IFSS, and PFSS
τ	Positive mapping for BSS, FBSS, IFBSS, and PFBSS
η	Negative mapping for BSS, FBSS, IFBSS, and PFBSS
β	Mapping for NSS, FNSS, IFNSS, and PFNSS
π	Positive mapping for NBSS, FNBSS, and IFNBSS
κ	Negative mapping for NBSS, FNBSS, and IFNBSS
ζ	Positive mapping for PFNBSS
ξ	Negative mapping for PFNBSS
$2^{\mathcal{L}}$	Set of all crisp subsets of $\mathcal{L}$
$\mathbb{F}^{\mathcal{L}}$	Set of all FSs over $\mathcal{L}$
$\mathbb{I}^{\mathcal{L}}$	Set of all IFSs over $\mathcal{L}$
$\mathbb{P}^{\mathcal{L}}$	Set of all PFSs over $\mathcal{L}$
$2^{\mathcal{L}\times R}$	Set of all crisp subsets of $\mathcal{L} \times R$
$\mathbb{F}^{\mathcal{L}\times R}$	Set of all FSs over $\mathcal{L} \times R$
$\mathbb{I}^{\mathcal{L}\times R}$	Set of all IFSs over $\mathcal{L} \times R$
$\mathbb{P}^{\mathcal{L}\times R}$	Set of all PFSs over $\mathcal{L} \times R$

Table 1. List of symbols and notations used in the paper.

Definition 2.6 A triple (τ, η, ρ) is called:

i. a BSS[20], if $\tau : \rho \to 2^{\mathcal{L}}$ and $\eta : \neg\rho \to 2^{\mathcal{L}}$ such that, for all $\varepsilon \in \rho$, $\tau(\varepsilon) \cap \eta(\neg\varepsilon) = \emptyset$, where $\tau(\varepsilon), \eta(\neg\varepsilon) \subseteq \mathcal{L}$.

ii. an FBSS[21], if $\tau : \rho \to \mathbb{F}^{\mathcal{L}}$ and $\eta : \neg\rho \to \mathbb{F}^{\mathcal{L}}$ such that, for all $\varepsilon \in \rho$ and $\ell \in \mathcal{L}$, the condition $0 \leq \tau(\varepsilon)(\ell) + \eta(\neg\varepsilon)(\ell) \leq 1$ holds, where $\tau(\varepsilon)(\ell), \eta(\neg\varepsilon)(\ell) \in [0, 1]$.

iii. an IFBSS[22], if $\tau : \rho \to \mathbb{I}^{\mathcal{L}}$ and $\eta : \neg\rho \to \mathbb{I}^{\mathcal{L}}$ such that, for all $\varepsilon \in \rho$ and $\ell \in \mathcal{L}$, the following conditions hold: $0 \le \tau^{+}(\varepsilon)(\ell) + \eta^{+}(\neg\varepsilon)(\ell) \le 1$ and $0 \le \tau^{-}(\varepsilon)(\ell) + \eta^{-}(\neg\varepsilon)(\ell) \le 1$, where $\tau^{+}(\varepsilon)(\ell), \eta^{+}(\neg\varepsilon)(\ell) \in [0,1]$ and $\tau^{-}(\varepsilon)(\ell), \eta^{-}(\neg\varepsilon)(\ell) \in [0,1]$ are the MDs and NMDs, respectively.
iv. a PFBSS[22], if $\tau : \rho \to \mathbb{P}^{\mathcal{L}}$ and $\eta : \neg\rho \to \mathbb{P}^{\mathcal{L}}$ such that, for all $\varepsilon \in \rho$ and $\ell \in \mathcal{L}$, the following conditions hold: $0 \le (\tau^{+}(\varepsilon)(\ell))^2 + (\eta^{+}(\neg\varepsilon)(\ell))^2 \le 1$ and $0 \le (\tau^{-}(\varepsilon)(\ell))^2 + (\eta^{-}(\neg\varepsilon)(\ell))^2 \le 1$, where $\tau^{+}(\varepsilon)(\ell), \tau^{-}(\varepsilon)(\ell), \eta^{+}(\neg\varepsilon)(\ell), \eta^{-}(\neg\varepsilon)(\ell) \in [0,1]$.

Definition 2.7 A triple (β, ρ, N) is called:

i. an NSS[25], if $\beta : \rho \to 2^{\mathcal{L}\times R}$, with the property that for each $\varepsilon \in \rho$, there exists a unique pair $(\ell, r_\varepsilon) \in \mathcal{L}\times R$ such that $(\ell, r_\varepsilon) \in \beta(\varepsilon)$, where $\ell \in \mathcal{L}$ and $r_\varepsilon \in R$. The set $2^{\mathcal{L}\times R}$ denotes all crisp sets of $\mathcal{L}\times R$.
ii. an FNSS[26], if $\beta : \rho \to \mathbb{F}^{\mathcal{L}\times R}$, with the property that for each $\varepsilon \in \rho$, there exists a unique pair $(\ell, r_\varepsilon) \in \mathcal{L}\times R$ such that $\langle(\ell, r_\varepsilon), \beta(\ell, r_\varepsilon)\rangle \in \beta(\varepsilon)$, where $\ell \in \mathcal{L}$ and $r_\varepsilon \in R$. The set $\mathbb{F}^{\mathcal{L}\times R}$ represents all FSs of $\mathcal{L}\times R$.
iii. an IFNSS[27], if $\beta : \rho \to \mathbb{I}^{\mathcal{L}\times R}$, with the property that for each $\varepsilon \in \rho$, there exists a unique pair $(\ell, r_\varepsilon) \in \mathcal{L}\times R$ such that $\langle(\ell, r_\varepsilon), \beta^{+}(\ell, r_\varepsilon), \beta^{-}(\ell, r_\varepsilon)\rangle \in \beta(\varepsilon)$, subject to the condition $0 \le \beta^{+}(\ell, r_\varepsilon) + \beta^{-}(\ell, r_\varepsilon) \le 1$, where $\ell \in \mathcal{L}, r_\varepsilon \in R$, and $\beta^{+}(\ell, r_\varepsilon), \beta^{-}(\ell, r_\varepsilon) \in [0,1]$. The set $\mathbb{I}^{\mathcal{L}\times R}$ denotes all IFSs of $\mathcal{L}\times R$.
iv. a PFNSS[30], if $\beta : \rho \to \mathbb{P}^{\mathcal{L}\times R}$, with the property that for each $\varepsilon \in \rho$, there exists a unique pair $(\ell, r_\varepsilon) \in \mathcal{L}\times R$ such that $\langle(\ell, r_\varepsilon), \beta^{+}(\ell, r_\varepsilon), \beta^{-}(\ell, r_\varepsilon)\rangle \in \beta(\varepsilon)$, subject to the condition $0 \le (\beta^{+}(\ell, r_\varepsilon))^2 + (\beta^{-}(\ell, r_\varepsilon))^2 \le 1$, where $\ell \in \mathcal{L}$, $r_\varepsilon \in R$, and $\beta^{+}(\ell, r_\varepsilon), \beta^{-}(\ell, r_\varepsilon) \in [0,1]$. The set $\mathbb{P}^{\mathcal{L}\times R}$ denotes all PFSs of $\mathcal{L}\times R$.

Definition 2.8 A quadruple (π, κ, ρ, N) is called:

i. an NBSS[33], if $\pi : \rho \to 2^{\mathcal{L}\times R}$ and $\kappa : \neg\rho \to 2^{\mathcal{L}\times R}$, with the property that for each $\varepsilon \in \rho$, there exists a unique pair $(\ell, r_\varepsilon) \in \mathcal{L}\times R$ such that $(\ell, r_\varepsilon) \in \pi(\varepsilon)$. Similarly, for each $\neg\varepsilon \in \neg\rho$, there exists a unique pair $(\ell, r_{\neg\varepsilon}) \in \mathcal{L}\times R$ such that $(\ell, r_{\neg\varepsilon}) \in \kappa(\neg\varepsilon)$, subject to the condition $r_\varepsilon + r_{\neg\varepsilon} \le N-1$, where $\ell \in \mathcal{L}$ and $r_\varepsilon, r_{\neg\varepsilon} \in R$.
ii. an FNBSS[35], if $\pi : \rho \to \mathbb{F}^{\mathcal{L}\times R}$ and $\kappa : \neg\rho \to \mathbb{F}^{\mathcal{L}\times R}$, with the property that for each $\varepsilon \in \rho$, there exists a unique pair $(\ell, r_\varepsilon) \in \mathcal{L}\times R$ such that $\langle(\ell, r_\varepsilon), \pi(\ell, r_\varepsilon)\rangle \in \pi(\varepsilon)$, and for each $\neg\varepsilon \in \neg\rho$, there exists a unique pair $(\ell, r_{\neg\varepsilon}) \in \mathcal{L}\times R$ such that $\langle(\ell, r_{\neg\varepsilon}), \kappa(\ell, r_{\neg\varepsilon})\rangle \in \kappa(\neg\varepsilon)$, subject to the condition $0 \le \pi(\ell, r_\varepsilon) + \kappa(\ell, r_{\neg\varepsilon}) \le 1$, where $\ell \in \mathcal{L}, r_\varepsilon, r_{\neg\varepsilon} \in R$, and $\pi(\ell, r_\varepsilon), \kappa(\ell, r_{\neg\varepsilon}) \in [0,1]$.
iii. an IFNBSS[36], if $\pi : \rho \to \mathbb{I}^{\mathcal{L}\times R}$ bipolar soft sh the property that for each $\varepsilon \in \rho$, there exists a unique pair $(\ell, r_\varepsilon) \in \mathcal{L}\times R$ such that $\langle(\ell, r_\varepsilon), \pi^{+}(\ell, r_\varepsilon), \pi^{-}(\ell, r_\varepsilon)\rangle \in \pi(\varepsilon)$, and for each $\neg\varepsilon \in \neg\rho$, there exists a unique pair $(\ell, r_{\neg\varepsilon}) \in \mathcal{L}\times R$ such that $\langle(\ell, r_{\neg\varepsilon}), \kappa^{+}(\ell, r_{\neg\varepsilon}), \kappa^{-}(\ell, r_{\neg\varepsilon})\rangle \in \kappa(\neg\varepsilon)$, subject to the conditions $0 \le \pi^{+}(\ell, r_\varepsilon) + \kappa^{+}(\ell, r_{\neg\varepsilon}) \le 1$ and $0 \le \pi^{-}(\ell, r_\varepsilon) + \kappa^{-}(\ell, r_{\neg\varepsilon}) \le 1$, where $\ell \in \mathcal{L}, r_\varepsilon, r_{\neg\varepsilon} \in R$, and $\pi^{+}(\ell, r_\varepsilon), \pi^{-}(\ell, r_\varepsilon), \kappa^{+}(\ell, r_{\neg\varepsilon}), \kappa^{-}(\ell, r_{\neg\varepsilon}) \in [0,1]$. Clearly, $\pi^{+}(\ell, r_\varepsilon)$ and $\kappa^{+}(\ell, r_{\neg\varepsilon})$ are MDs, while $\pi^{-}(\ell, r_\varepsilon)$ and $\kappa^{-}(\ell, r_{\neg\varepsilon})$ are NMDs.

Pythagorean Fuzzy N-Bipolar Soft Sets

In this section, we present the PFNBSS model and develop its core operations–namely the null and whole sets, complement, subset, equality, union, and intersection–each accompanied by their algebraic properties and illustrative examples.

Definition 3.1 A quadruple (ζ, ξ, ρ, N) is called a PFNBSS, where $\zeta : \rho \to \mathbb{P}^{\mathcal{L}\times R}$ and $\xi : \neg\rho \to \mathbb{P}^{\mathcal{L}\times R}$, with the property that for each $\varepsilon \in \rho$, there exists a unique pair $(\ell, r_\varepsilon) \in \mathcal{L}\times R$ such that $\langle(\ell, r_\varepsilon), \zeta^{+}(\ell, r_\varepsilon), \zeta^{-}(\ell, r_\varepsilon)\rangle \in \zeta(\varepsilon)$, and for each $\neg\varepsilon \in \neg\rho$, there exists a unique pair $(\ell, r_{\neg\varepsilon}) \in \mathcal{L}\times R$ such that $\langle(\ell, r_{\neg\varepsilon}), \xi^{+}(\ell, r_{\neg\varepsilon}), \xi^{-}(\ell, r_{\neg\varepsilon})\rangle \in \xi(\neg\varepsilon)$, subject to the following conditions:

$$0 \le (\zeta^{+}(\ell, r_\varepsilon))^2 + (\xi^{+}(\ell, r_{\neg\varepsilon}))^2 \le 1,$$
$$0 \le (\zeta^{-}(\ell, r_\varepsilon))^2 + (\xi^{-}(\ell, r_{\neg\varepsilon}))^2 \le 1,$$

where $\ell \in \mathcal{L}$, $r_\varepsilon, r_{\neg\varepsilon} \in R$, and $\zeta^{+}(\ell, r_\varepsilon), \zeta^{-}(\ell, r_\varepsilon), \xi^{+}(\ell, r_{\neg\varepsilon}), \xi^{-}(\ell, r_{\neg\varepsilon}) \in [0,1]$. Clearly, $\zeta^{+}(\ell, r_\varepsilon)$ and $\xi^{+}(\ell, r_{\neg\varepsilon})$ are MDs, while $\zeta^{-}(\ell, r_\varepsilon)$ and $\xi^{-}(\ell, r_{\neg\varepsilon})$ are NMDs.
Unless specified otherwise, both $\mathcal{L}$ and ρ are assumed to be finite. In such cases, the PFNBSS can be represented in a unified tabular form, where each cell contains a pair of tuples–one for $\langle r_{ij_{\varepsilon_j}}, \zeta^{+}_{ij}, \zeta^{-}_{ij}\rangle$, which corresponds to $\langle(\ell_i, r_{ij_{\varepsilon_j}}), \zeta^{+}(\ell_i, r_{ij_{\varepsilon_j}}), \zeta^{-}(\ell_i, r_{ij_{\varepsilon_j}})\rangle \in \zeta(\varepsilon_j)$; and one for $\langle r_{ij_{\neg\varepsilon_j}}, \xi^{+}_{ij}, \xi^{-}_{ij}\rangle$, which corresponds to $\langle(\ell_i, r_{ij_{\neg\varepsilon_j}}), \xi^{+}(\ell_i, r_{ij_{\neg\varepsilon_j}}), \xi^{-}(\ell_i, r_{ij_{\neg\varepsilon_j}})\rangle \in \xi(\neg\varepsilon_j)$, as shown in Table 2.

Now, we represent the PFNBSS (ζ, ξ, ρ, N), originally displayed in Table 2, using two separate tables: one for (ζ, ρ, N) with respect to the set of parameters ρ, and another for $(\xi, \neg\rho, N)$ with respect to the set of parameters $\neg\rho$, as provided in Tables 3 and 4, respectively.

To clarify the core features of our new model, let us examine the following example.

Example 3.1 Consider a technology company that is in the process of recruiting for a senior software engineering position. The selection committee aims to assess a group of candidates $\mathcal{L} = \{\ell_1, \ell_2, \ell_3\}$ based on a comprehensive set of attributes that reflect both technical skills and soft competencies. The attributes under consideration are defined as $\rho = \{\varepsilon_1 = \text{programming proficiency}, \ \varepsilon_2 = \text{system design skills}, \ \varepsilon_3 = \text{team collaboration}, \ \varepsilon_4 = \text{problem-solving aptitude}\}$

(ζ, ξ, ρ, N)	ε_1	ε_2	$\ldots$	ε_n
ℓ_1	$\langle r_{11\varepsilon_1}, \zeta_{11}^+, \zeta_{11}^- \rangle$ $\langle r_{11\neg\varepsilon_1}, \xi_{11}^+, \xi_{11}^- \rangle$	$\langle r_{12\varepsilon_2}, \zeta_{12}^+, \zeta_{12}^- \rangle$ $\langle r_{12\neg\varepsilon_2}, \xi_{12}^+, \xi_{12}^- \rangle$	$\ldots$	$\langle r_{1n\varepsilon_n}, \zeta_{1n}^+, \zeta_{1n}^- \rangle$ $\langle r_{1n\neg\varepsilon_n}, \xi_{1n}^+, \xi_{1n}^- \rangle$
ℓ_2	$\langle r_{21\varepsilon_1}, \zeta_{21}^+, \zeta_{21}^- \rangle$ $\langle r_{21\neg\varepsilon_1}, \xi_{21}^+, \xi_{21}^- \rangle$	$\langle r_{22\varepsilon_2}, \zeta_{22}^+, \zeta_{22}^- \rangle$ $\langle r_{22\neg\varepsilon_2}, \xi_{22}^+, \xi_{22}^- \rangle$	$\ldots$	$\langle r_{2n\varepsilon_n}, \zeta_{2n}^+, \zeta_{2n}^- \rangle$ $\langle r_{2n\neg\varepsilon_n}, \xi_{2n}^+, \xi_{2n}^- \rangle$
			$\ddots$	
ℓ_m	$\langle r_{m1\varepsilon_1}, \zeta_{m1}^+, \zeta_{m1}^- \rangle$ $\langle r_{m1\neg\varepsilon_1}, \xi_{m1}^+, \xi_{m1}^- \rangle$	$\langle r_{m2\varepsilon_2}, \zeta_{m2}^+, \zeta_{m2}^- \rangle$ $\langle r_{m2\neg\varepsilon_2}, \xi_{m2}^+, \xi_{m2}^- \rangle$	$\ldots$	$\langle r_{mn\varepsilon_n}, \zeta_{mn}^+, \zeta_{mn}^- \rangle$ $\langle r_{mn\neg\varepsilon_n}, \xi_{mn}^+, \xi_{mn}^- \rangle$

Table 2. Tabular form of the PFNBSS (ζ, ξ, ρ, N)

(ζ, ρ, N)	ε_1	ε_2	$\ldots$	ε_n
ℓ_1	$\langle r_{11\varepsilon_1}, \zeta_{11}^+, \zeta_{11}^- \rangle$	$\langle r_{12\varepsilon_2}, \zeta_{12}^+, \zeta_{12}^- \rangle$	$\ldots$	$\langle r_{1n\varepsilon_n}, \zeta_{1n}^+, \zeta_{1n}^- \rangle$
ℓ_2	$\langle r_{21\varepsilon_1}, \zeta_{21}^+, \zeta_{21}^- \rangle$	$\langle r_{22\varepsilon_2}, \zeta_{22}^+, \zeta_{22}^- \rangle$	$\ldots$	$\langle r_{2n\varepsilon_n}, \zeta_{2n}^+, \zeta_{2n}^- \rangle$
			$\ddots$	
ℓ_m	$\langle r_{m1\varepsilon_1}, \zeta_{m1}^+, \zeta_{m1}^- \rangle$	$\langle r_{m2\varepsilon_2}, \zeta_{m2}^+, \zeta_{m2}^- \rangle$	$\ldots$	$\langle r_{mn\varepsilon_n}, \zeta_{mn}^+, \zeta_{mn}^- \rangle$

Table 3. Tabular form of (ζ, ρ, N)

$(\xi, \neg\rho, N)$	$\neg\varepsilon_1$	$\neg\varepsilon_2$	$\ldots$	$\neg\varepsilon_n$
ℓ_1	$\langle r_{11\neg\varepsilon_1}, \xi_{11}^+, \xi_{11}^- \rangle$	$\langle r_{12\neg\varepsilon_2}, \xi_{12}^+, \xi_{12}^- \rangle$	$\ldots$	$\langle r_{1n\neg\varepsilon_n}, \xi_{1n}^+, \xi_{1n}^- \rangle$
ℓ_2	$\langle r_{21\neg\varepsilon_1}, \xi_{21}^+, \xi_{21}^- \rangle$	$\langle r_{22\neg\varepsilon_2}, \xi_{22}^+, \xi_{22}^- \rangle$	$\ldots$	$\langle r_{2n\neg\varepsilon_n}, \xi_{2n}^+, \xi_{2n}^- \rangle$
			$\ddots$	
ℓ_m	$\langle r_{m1\neg\varepsilon_1}, \xi_{m1}^+, \xi_{m1}^- \rangle$	$\langle r_{m2\neg\varepsilon_2}, \xi_{m2}^+, \xi_{m2}^- \rangle$	$\ldots$	$\langle r_{mn\neg\varepsilon_n}, \xi_{mn}^+, \xi_{mn}^- \rangle$

Table 4. Tabular form of $(\xi, \neg\rho, N)$

$\mathcal{L} \setminus \rho$	ε_1	ε_2	ε_3	ε_4
ℓ_1	$\star\star\star$ $\star$	$\star$ $\star\star$	$\star\star$ $\star\star$	$\star\star$ $\star$
ℓ_2	$\star\star\star$ $\star$	$\circ$ $\star\star\star\star$	$\star$ $\star\star\star$	$\star\star$ $\star$
ℓ_3	$\star$ $\star\star\star$	$\star$ $\star\star$	$\star\star$ $\star\star$	$\star\star\star$ $\star$

Table 5. Initial Evaluations

To ensure a balanced evaluation, the committee also considers the corresponding negative attributes (i.e., lack or weakness of the same capabilities), denoted by $\neg\rho = \{\neg\varepsilon_1 = \text{poor programming skills}, \ \neg\varepsilon_2 = \text{weak system design skills}, \ \neg\varepsilon_3 = \text{poor team collaboration}, \ \neg\varepsilon_4 = \text{low problem-solving ability}\}$ This bipolar perspective allows the decision-makers to capture both the positive and negative tendencies in each candidate's profile, enhancing the precision and fairness of the evaluation.

In this context, the performance of each candidate with respect to the given attributes (and their negations) is initially expressed using qualitative markers that indicate the strength or weakness of their qualifications, see Table 5. These markers are then used to construct the 5BSS, which serves as a foundational step for further DM procedures.

According to Definition 2.8 (i.), the 5BSS $(\pi, \kappa, \rho, 5)$ can be derived from the evaluations presented in Table 6, where:

$(\pi, \kappa, \rho, 5)$	ε_1	ε_2	ε_3	ε_4
ℓ_1	3 1	1 2	2 2	2 1
ℓ_2	3 1	0 4	1 3	2 1
ℓ_3	1 3	1 2	2 2	3 1

Table 6. Tabular form of the 5BSS $(\pi, \kappa, \rho, 5)$ in Example 3.1

Grade	Criterion
$r_\varepsilon = 0$	$0.0 \le (\zeta^+(\ell, r_\varepsilon))^2 + (\zeta^-(\ell, r_\varepsilon))^2 < 0.2$ $0.0 \le (\xi^+(\ell, r_{\neg\varepsilon}))^2 + (\xi^-(\ell, r_{\neg\varepsilon}))^2 < 0.2$
$r_\varepsilon = 1$	$0.2 \le (\zeta^+(\ell, r_\varepsilon))^2 + (\zeta^-(\ell, r_\varepsilon))^2 < 0.4$ $0.2 \le (\xi^+(\ell, r_{\neg\varepsilon}))^2 + (\xi^-(\ell, r_{\neg\varepsilon}))^2 < 0.4$
$r_\varepsilon = 2$	$0.4 \le (\zeta^+(\ell, r_\varepsilon))^2 + (\zeta^-(\ell, r_\varepsilon))^2 < 0.6$ $0.4 \le (\xi^+(\ell, r_{\neg\varepsilon}))^2 + (\xi^-(\ell, r_{\neg\varepsilon}))^2 < 0.6$
$r_\varepsilon = 3$	$0.6 \le (\zeta^+(\ell, r_\varepsilon))^2 + (\zeta^-(\ell, r_\varepsilon))^2 < 0.8$ $0.6 \le (\xi^+(\ell, r_{\neg\varepsilon}))^2 + (\xi^-(\ell, r_{\neg\varepsilon}))^2 < 0.8$
$r_\varepsilon = 4$	$0.8 \le (\zeta^+(\ell, r_\varepsilon))^2 + (\zeta^-(\ell, r_\varepsilon))^2 \le 1.0$ $0.8 \le (\xi^+(\ell, r_{\neg\varepsilon}))^2 + (\xi^-(\ell, r_{\neg\varepsilon}))^2 \le 1.0$

Table 7. Evaluation grades and corresponding criteria.

$(\zeta, \xi, \rho, 5)$	ε_1	ε_2	ε_3	ε_4
ℓ_1	$\langle 3, 0.6, 0.5\rangle$ $\langle 1, 0.3, 0.5\rangle$	$\langle 1, 0.3, 0.5\rangle$ $\langle 2, 0.5, 0.5\rangle$	$\langle 2, 0.6, 0.4\rangle$ $\langle 2, 0.5, 0.5\rangle$	$\langle 2, 0.5, 0.5\rangle$ $\langle 1, 0.3, 0.5\rangle$
ℓ_2	$\langle 3, 0.7, 0.4\rangle$ $\langle 1, 0.4, 0.4\rangle$	$\langle 0, 0.2, 0.2\rangle$ $\langle 4, 0.8, 0.6\rangle$	$\langle 1, 0.4, 0.4\rangle$ $\langle 3, 0.7, 0.5\rangle$	$\langle 2, 0.6, 0.4\rangle$ $\langle 1, 0.3, 0.5\rangle$
ℓ_3	$\langle 1, 0.4, 0.4\rangle$ $\langle 3, 0.6, 0.6\rangle$	$\langle 1, 0.3, 0.5\rangle$ $\langle 2, 0.5, 0.5\rangle$	$\langle 2, 0.6, 0.4\rangle$ $\langle 2, 0.5, 0.5\rangle$	$\langle 3, 0.6, 0.5\rangle$ $\langle 1, 0.3, 0.5\rangle$

Table 8. Tabular form of the PF5BSS $(\zeta, \xi, \rho, 5)$ in Example 3.1

- "$\circ$" represents inadequate performance.
- "$\star$" represents basic competency.
- "$\star\star$" represent moderate competency.
- "$\star\star\star$" represent high competency.
- "$\star\star\star\star$" represent exceptional competency.

This symbolic grading can be easily mapped to numerical values in $R = \{0, 1, 2, 3, 4\}$, where:

- 0 corresponds to $\circ$.
- 1 corresponds to $\star$.
- 2 corresponds to $\star\star$.
- 3 corresponds to $\star\star\star$.
- 4 corresponds to $\star\star\star\star$.

The tabular representation of the 5BSS $(\pi, \kappa, \rho, 5)$ is shown in Table 6.

This level of detail suffices for exact data. Yet, in situations involving ambiguity or uncertainty, the PFNBSS framework is essential to interpret the grading of candidates. Using the established grade scale, the selection committee then allocates MDs and NMDs according to Pythagorean fuzzy principles, as exemplified in Table 7.

Therefore, Table 8 shows the final MDs and NMDs within a Pythagorean fuzzy environment for each applicant under each attribute and its negation.

We next define, for theoretical purposes, a collection of basic operations on PFNBSSs–together with their algebraic properties–and illustrate each with examples. These operations comprise the null and whole sets, complement, subset relation, equality, union, and intersection.

$(\zeta,\xi,\rho,5)^{\check{c}}$	ε_1	ε_2	ε_3	ε_4
ℓ_1	$\langle 1,0.3,0.5\rangle$ $\langle 3,0.6,0.5\rangle$	$\langle 2,0.5,0.5\rangle$ $\langle 1,0.3,0.5\rangle$	$\langle 2,0.5,0.5\rangle$ $\langle 2,0.6,0.4\rangle$	$\langle 1,0.3,0.5\rangle$ $\langle 2,0.5,0.5\rangle$
ℓ_2	$\langle 1,0.4,0.4\rangle$ $\langle 3,0.7,0.4\rangle$	$\langle 4,0.8,0.6\rangle$ $\langle 0,0.2,0.2\rangle$	$\langle 3,0.7,0.5\rangle$ $\langle 1,0.4,0.4\rangle$	$\langle 1,0.3,0.5\rangle$ $\langle 2,0.6,0.4\rangle$
ℓ_3	$\langle 3,0.6,0.6\rangle$ $\langle 1,0.4,0.4\rangle$	$\langle 2,0.5,0.5\rangle$ $\langle 1,0.3,0.5\rangle$	$\langle 2,0.5,0.5\rangle$ $\langle 2,0.6,0.4\rangle$	$\langle 1,0.3,0.5\rangle$ $\langle 3,0.6,0.5\rangle$

Table 9. The complement of the PF5BSS $(\zeta,\xi,\rho,5)$ in Example 3.1

$(\zeta_1,\xi_1,\rho_1,5)$	ε_1	ε_2	ε_3
ℓ_1	$\langle 0,0.1,0.4\rangle$ $\langle 3,0.8,0.2\rangle$	$\langle 1,0.0,0.6\rangle$ $\langle 3,0.8,0.2\rangle$	$\langle 0,0.1,0.1\rangle$ $\langle 4,0.9,0.4\rangle$
ℓ_2	$\langle 3,0.2,0.8\rangle$ $\langle 1,0.4,0.4\rangle$	$\langle 1,0.1,0.6\rangle$ $\langle 2,0.7,0.0\rangle$	$\langle 0,0.0,0.4\rangle$ $\langle 3,0.8,0.2\rangle$
ℓ_3	$\langle 1,0.2,0.4\rangle$ $\langle 3,0.8,0.2\rangle$	$\langle 1,0.3,0.5\rangle$ $\langle 2,0.7,0.1\rangle$	$\langle 2,0.2,0.6\rangle$ $\langle 2,0.7,0.3\rangle$

Table 10. Tabular form of PF5BSS $(\zeta_1,\xi_1,\rho_1,5)$ in Example 3.3

Definition 3.2 A PFNBSS $(\zeta^{\mathbb{N}},\xi^{\mathbb{N}},\rho,N)$ is defined as a relative null if, for each $\varepsilon\in\rho$ and $\ell\in\mathcal{L}$, we have $\zeta^{\mathbb{N}}(\varepsilon)(\ell)=\langle 0,0.0,1.0\rangle$, and for each $\neg\varepsilon\in\neg\rho$ and $\ell\in\mathcal{L}$, we have $\xi^{\mathbb{N}}(\neg\varepsilon)(\ell)=\langle N-1,1.0,0.0\rangle$.

Definition 3.3 A PFNBSS $(\zeta^{\mathbb{U}},\xi^{\mathbb{U}},\rho,N)$ is referred to as a relative whole if, for each $\varepsilon\in\rho$ and $\ell\in\mathcal{L}$, we have $\zeta^{\mathbb{U}}(\varepsilon)(\ell)=\langle N-1,1.0,0.0\rangle$, and for each $\neg\varepsilon\in\neg\rho$ and $\ell\in\mathcal{L}$, we have $\xi^{\mathbb{U}}(\neg\varepsilon)(\ell)=\langle 0,0.0,1.0\rangle$.

Definition 3.4 The complement of (ζ,ξ,ρ,N), denoted as $(\zeta,\xi,\rho,N)^{\check{c}}$, is given by $(\zeta,\xi,\rho,N)^{\check{c}}=(\zeta^{\check{c}},\xi^{\check{c}},\rho,N)$, where for every $\varepsilon\in\rho$ and $\ell\in\mathcal{L}$, it follows that $\zeta^{\check{c}}(\varepsilon)=\xi(\neg\varepsilon)$, i.e., $r_\varepsilon^{\check{c}}=r_{\neg\varepsilon}$, $\zeta^{+\check{c}}(\ell,r_\varepsilon)=\xi^{+}(\ell,r_{\neg\varepsilon})$ and $\zeta^{-\check{c}}(\ell,r_\varepsilon)=\xi^{-}(\ell,r_{\neg\varepsilon})$. Similarly, for every $\neg\varepsilon\in\neg\rho$ and $\ell\in\mathcal{L}$, we have $\xi^{\check{c}}(\neg\varepsilon)=\zeta(\varepsilon)$, i.e., $r_{\neg\varepsilon}^{\check{c}}=r_\varepsilon$, $\xi^{+\check{c}}(\ell,r_{\neg\varepsilon})=\zeta^{+}(\ell,r_\varepsilon)$ and $\xi^{-\check{c}}(\ell,r_{\neg\varepsilon})=\zeta^{-}(\ell,r_\varepsilon)$.

Example 3.2 Let us consider the PF5BSS $(\zeta,\xi,\rho,5)$ presented in Table 8 of Example 3.1. The corresponding complement is shown in Table 9.

Proposition 3.1 Let (ζ,ξ,ρ,N) be a PFNBSS, and let $(\zeta^{\mathbb{N}},\xi^{\mathbb{N}},\rho,N)$ and $(\zeta^{\mathbb{U}},\xi^{\mathbb{U}},\rho,N)$ denote the relative null set and the relative whole set, respectively. Then,

1. $((\zeta,\xi,\rho,N)^{\check{c}})^{\check{c}}=(\zeta,\xi,\rho,N)$.
2. $(\zeta^{\mathbb{N}},\xi^{\mathbb{N}},\rho,N)^{\check{c}}=(\zeta^{\mathbb{U}},\xi^{\mathbb{U}},\rho,N)$.
3. $(\zeta^{\mathbb{U}},\xi^{\mathbb{U}},\rho,N)^{\check{c}}=(\zeta^{\mathbb{N}},\xi^{\mathbb{N}},\rho,N)$.

Proof
1. Follows directly from Definition 3.4.
2. Follows from Definitions 3.2, 3.3, and 3.4.
3. Follows from the same definitions as part 2.□

Definition 3.5 A PFNBSS (ζ_1,ξ_1,ρ_1,N) is said to be a subset of (ζ_2,ξ_2,ρ_2,N), denoted as $(\zeta_1,\xi_1,\rho_1,N)\,\check{\subseteq}\,(\zeta_2,\xi_2,\rho_2,N)$, if the following conditions are satisfied:

1. $\rho_1\subseteq\rho_2$.
2. For each $\varepsilon\in\rho_1$ and $\ell\in\mathcal{L}$, it holds that $r_{1\varepsilon}\le r_{2\varepsilon}$, $\zeta_1^{+}(\ell,r_{1\varepsilon})\le\zeta_2^{+}(\ell,r_{2\varepsilon})$, and $\zeta_2^{-}(\ell,r_{2\varepsilon})\le\zeta_1^{-}(\ell,r_{1\varepsilon})$, where $\langle(\ell,r_{1\varepsilon}),\zeta_1^{+}(\ell,r_{1\varepsilon}),\zeta_1^{-}(\ell,r_{1\varepsilon})\rangle\in\zeta_1(\varepsilon)$ and $\langle(\ell,r_{2\varepsilon}),\zeta_2^{+}(\ell,r_{2\varepsilon}),\zeta_2^{-}(\ell,r_{2\varepsilon})\rangle\in\zeta_2(\varepsilon)$.
3. For each $\neg\varepsilon\in\neg\rho_1$ and $\ell\in\mathcal{L}$, we have $r_{2\neg\varepsilon}\le r_{1\neg\varepsilon}$, $\xi_2^{+}(\ell,r_{2\neg\varepsilon})\le\xi_1^{+}(\ell,r_{1\neg\varepsilon})$, and $\xi_1^{-}(\ell,r_{1\neg\varepsilon})\le\xi_2^{-}(\ell,r_{2\neg\varepsilon})$, where $\langle(\ell,r_{1\neg\varepsilon}),\xi_1^{+}(\ell,r_{1\neg\varepsilon}),\xi_1^{-}(\ell,r_{1\neg\varepsilon})\rangle\in\xi_1(\neg\varepsilon)$ and $\langle(\ell,r_{2\neg\varepsilon}),\xi_2^{+}(\ell,r_{2\neg\varepsilon}),\xi_2^{-}(\ell,r_{2\neg\varepsilon})\rangle\in\xi_2(\neg\varepsilon)$.

Example 3.3 Referring to Example 3.1, consider two PF5BSSs $(\zeta_1,\xi_1,\rho_1,5)$ and $(\zeta_2,\xi_2,\rho_2,5)$, presented in Tables 10 and 11, respectively. It is clear that $(\zeta_1,\xi_1,\rho_1,5)\,\check{\subseteq}\,(\zeta_2,\xi_2,\rho_2,5)$.

Definition 3.6 Two PFNBSSs (ζ_1,ξ_1,ρ_1,N) and (ζ_2,ξ_2,ρ_2,N) are said to be equal if both $(\zeta_1,\xi_1,\rho_1,N)\,\check{\subseteq}\,(\zeta_2,\xi_2,\rho_2,N)$ and $(\zeta_2,\xi_2,\rho_2,N)\,\check{\subseteq}\,(\zeta_1,\xi_1,\rho_1,N)$ are satisfied.

$(\zeta_2, \xi_2, \rho_2, 5)$	ε_1	ε_2	ε_3	ε_4
ℓ_1	$\langle 1, 0.2, 0.4\rangle$ $\langle 2, 0.6, 0.4\rangle$	$\langle 1, 0.1, 0.6\rangle$ $\langle 3, 0.8, 0.3\rangle$	$\langle 0, 0.2, 0.3\rangle$ $\langle 4, 0.9, 0.0\rangle$	$\langle 1, 0.5, 0.3\rangle$ $\langle 1, 0.2, 0.5\rangle$
ℓ_2	$\langle 3, 0.3, 0.8\rangle$ $\langle 1, 0.4, 0.4\rangle$	$\langle 2, 0.3, 0.6\rangle$ $\langle 2, 0.7, 0.2\rangle$	$\langle 0, 0.1, 0.4\rangle$ $\langle 4, 0.9, 0.1\rangle$	$\langle 3, 0.5, 0.6\rangle$ $\langle 1, 0.2, 0.5\rangle$
ℓ_3	$\langle 1, 0.3, 0.4\rangle$ $\langle 3, 0.6, 0.6\rangle$	$\langle 2, 0.6, 0.3\rangle$ $\langle 2, 0.7, 0.2\rangle$	$\langle 2, 0.6, 0.2\rangle$ $\langle 2, 0.7, 0.2\rangle$	$\langle 2, 0.5, 0.5\rangle$ $\langle 2, 0.2, 0.6\rangle$

Table 11. Tabular form of PF5BSS $(\zeta_2, \xi_2, \rho_2, 5)$ in Example 3.3

$(\zeta_1, \xi_1, \rho_1, 4)$	ε_1	ε_2	ε_3
ℓ_1	$\langle 0, 0.2, 0.3\rangle$ $\langle 3, 0.5, 0.6\rangle$	$\langle 2, 0.6, 0.4\rangle$ $\langle 1, 0.3, 0.5\rangle$	$\langle 2, 0.7, 0.0\rangle$ $\langle 0, 0.3, 0.3\rangle$
ℓ_2	$\langle 1, 0.3, 0.5\rangle$ $\langle 2, 0.4, 0.6\rangle$	$\langle 1, 0.4, 0.2\rangle$ $\langle 1, 0.2, 0.4\rangle$	$\langle 1, 0.5, 0.2\rangle$ $\langle 2, 0.2, 0.7\rangle$
ℓ_3	$\langle 1, 0.3, 0.4\rangle$ $\langle 0, 0.3, 0.3\rangle$	$\langle 2, 0.7, 0.0\rangle$ $\langle 0, 0.0, 0.1\rangle$	$\langle 3, 0.6, 0.5\rangle$ $\langle 0, 0.4, 0.1\rangle$

Table 12. Tabular form of PF4BSS $(\zeta_1, \xi_1, \rho_1, 4)$ in Example 3.4

$(\zeta_2, \xi_2, \rho_2, 5)$	ε_1	ε_2	ε_4
ℓ_1	$\langle 1, 0.2, 0.4\rangle$ $\langle 3, 0.8, 0.3\rangle$	$\langle 2, 0.6, 0.3\rangle$ $\langle 2, 0.3, 0.6\rangle$	$\langle 0, 0.0, 0.0\rangle$ $\langle 4, 1.0, 0.0\rangle$
ℓ_2	$\langle 1, 0.2, 0.5\rangle$ $\langle 1, 0.4, 0.2\rangle$	$\langle 4, 0.2, 0.9\rangle$ $\langle 0, 0.1, 0.2\rangle$	$\langle 1, 0.0, 0.5\rangle$ $\langle 3, 0.7, 0.4\rangle$
ℓ_3	$\langle 1, 0.2, 0.4\rangle$ $\langle 0, 0.1, 0.4\rangle$	$\langle 1, 0.1, 0.6\rangle$ $\langle 2, 0.7, 0.1\rangle$	$\langle 2, 0.5, 0.5\rangle$ $\langle 2, 0.7, 0.3\rangle$

Table 13. Tabular form of PF5BSS $(\zeta_2, \xi_2, \rho_2, 5)$ in Example 3.4

Definition 3.7 The extended union of $(\zeta_1, \xi_1, \rho_1, N_1)$ and $(\zeta_2, \xi_2, \rho_2, N_2)$ is denoted and defined as $(\zeta_1, \xi_1, \rho_1, N_1)\, \check{\cup}_e\, (\zeta_2, \xi_2, \rho_2, N_2) = (\zeta, \xi, \rho_1 \cup \rho_2, \max(N_1, N_2))$, where for all $\varepsilon \in \rho_1 \cup \rho_2$:

$$\zeta(\varepsilon) = \begin{cases} \zeta_1(\varepsilon), & \text{if } \varepsilon \in \rho_1 \setminus \rho_2, \\ \zeta_2(\varepsilon), & \text{if } \varepsilon \in \rho_2 \setminus \rho_1, \\ \left\langle \begin{matrix} (\ell, \max\{r_{1\varepsilon}, r_{2\varepsilon}\}), \\ \max\{\zeta_1^+(\ell, r_{1\varepsilon}), \zeta_2^+(\ell, r_{2\varepsilon})\}, \\ \min\{\zeta_1^-(\ell, r_{1\varepsilon}), \zeta_2^-(\ell, r_{2\varepsilon})\} \end{matrix} \right\rangle, & \text{if } \varepsilon \in \rho_1 \cap \rho_2, \end{cases}$$

where $\langle (\ell, r_{1\varepsilon}), \zeta_1^+(\ell, r_{1\varepsilon}), \zeta_1^-(\ell, r_{1\varepsilon})\rangle \in \zeta_1(\varepsilon)$ and $\langle (\ell, r_{2\varepsilon}), \zeta_2^+(\ell, r_{2\varepsilon}), \zeta_2^-(\ell, r_{2\varepsilon})\rangle \in \zeta_2(\varepsilon)$.

Similarly, for all $\neg\varepsilon \in \neg\rho_1 \cup \neg\rho_2$:

$$\xi(\neg\varepsilon) = \begin{cases} \xi_1(\neg\varepsilon), & \text{if } \neg\varepsilon \in \neg\rho_1 \setminus \neg\rho_2, \\ \xi_2(\neg\varepsilon), & \text{if } \neg\varepsilon \in \neg\rho_2 \setminus \neg\rho_1, \\ \left\langle \begin{matrix} (\ell, \min\{r_{1\neg\varepsilon}, r_{2\neg\varepsilon}\}), \\ \min\{\xi_1^+(\ell, r_{1\neg\varepsilon}), \xi_2^+(\ell, r_{2\neg\varepsilon})\}, \\ \max\{\xi_1^-(\ell, r_{1\neg\varepsilon}), \xi_2^-(\ell, r_{2\neg\varepsilon})\} \end{matrix} \right\rangle, & \text{if } \neg\varepsilon \in \neg\rho_1 \cap \neg\rho_2, \end{cases}$$

where $\langle (\ell, r_{1\neg\varepsilon}), \xi_1^+(\ell, r_{1\neg\varepsilon}), \xi_1^-(\ell, r_{1\neg\varepsilon})\rangle \in \xi_1(\neg\varepsilon)$ and $\langle (\ell, r_{2\neg\varepsilon}), \xi_2^+(\ell, r_{2\neg\varepsilon}), \xi_2^-(\ell, r_{2\neg\varepsilon})\rangle \in \xi_2(\neg\varepsilon)$.

Example 3.4 Refer again to Example 3.1. Consider $(\zeta_1, \xi_1, \rho_1, 4)$ and $(\zeta_2, \xi_2, \rho_2, 5)$ as the PF4BSS and PF5BSS, respectively, presented in Tables 12 and 13. The resulting extended union is shown in Table 14.

Proposition 3.2 Let $(\zeta_1, \xi_1, \rho_1, N_1)$, $(\zeta_2, \xi_2, \rho_2, N_2)$, and $(\zeta_3, \xi_3, \rho_3, N_3)$ be PFN_1BSS, PFN_2BSS, and PFN_3 BSS, respectively. Then,

1. $(\zeta_1, \xi_1, \rho_1, N_1)\, \check{\cup}_e\, (\zeta_2, \xi_2, \rho_2, N_2) = (\zeta_2, \xi_2, \rho_2, N_2)\, \check{\cup}_e\, (\zeta_1, \xi_1, \rho_1, N_1)$.
2. $(\zeta_1, \xi_1, \rho_1, N_1)\, \check{\cup}_e\, \big((\zeta_2, \xi_2, \rho_2, N_2)\, \check{\cup}_e\, (\zeta_3, \xi_3, \rho_3, N_3)\big) = \big((\zeta_1, \xi_1, \rho_1, N_1)\, \check{\cup}_e\, (\zeta_2, \xi_2, \rho_2, N_2)\big)\, \check{\cup}_e\, (\zeta_3, \xi_3, \rho_3, N_3)$.

$(\zeta_3,\xi_3,\rho_1\cup\rho_2,5)$	ε_1	ε_2	ε_3	ε_4
ℓ_1	$\langle 1,0.2,0.3\rangle$ $\langle 3,0.5,0.6\rangle$	$\langle 2,0.6,0.3\rangle$ $\langle 1,0.3,0.6\rangle$	$\langle 2,0.7,0.0\rangle$ $\langle 0,0.3,0.3\rangle$	$\langle 0,0.0,0.0\rangle$ $\langle 4,1.0,0.0\rangle$
ℓ_2	$\langle 1,0.3,0.5\rangle$ $\langle 1,0.4,0.6\rangle$	$\langle 4,0.4,0.2\rangle$ $\langle 0,0.1,0.4\rangle$	$\langle 1,0.5,0.2\rangle$ $\langle 2,0.2,0.7\rangle$	$\langle 1,0.0,0.5\rangle$ $\langle 3,0.7,0.4\rangle$
ℓ_3	$\langle 1,0.3,0.4\rangle$ $\langle 0,0.1,0.4\rangle$	$\langle 2,0.7,0.0\rangle$ $\langle 0,0.0,0.1\rangle$	$\langle 3,0.6,0.5\rangle$ $\langle 0,0.4,0.1\rangle$	$\langle 2,0.5,0.5\rangle$ $\langle 2,0.7,0.3\rangle$

Table 14. The extended union $(\zeta_1,\xi_1,\rho_1,4)\check{\cup}_e(\zeta_2,\xi_2,\rho_2,5)$=$(\zeta_3,\xi_3,\rho_1\cup\rho_2,5)$ in Example 3.4

Proof 1. Let $(\zeta_1,\xi_1,\rho_1,N_1)\check{\cup}_e(\zeta_2,\xi_2,\rho_2,N_2)=(\zeta_3,\xi_3,\rho_1\cup\rho_2,\max(N_1,N_2))$. Then, for all $\varepsilon\in\rho_1\cup\rho_2$:

$$\zeta_3(\varepsilon)=\begin{cases}\zeta_1(\varepsilon), & \text{if } \varepsilon\in\rho_1\setminus\rho_2,\\ \zeta_2(\varepsilon), & \text{if } \varepsilon\in\rho_2\setminus\rho_1,\\ \left\langle\begin{array}{l}(\ell,\max\{r_{1\varepsilon},r_{2\varepsilon}\}),\\ \max\{\zeta_1^+(\ell,r_{1\varepsilon}),\zeta_2^+(\ell,r_{2\varepsilon})\},\\ \min\{\zeta_1^-(\ell,r_{1\varepsilon}),\zeta_2^-(\ell,r_{2\varepsilon})\}\end{array}\right\rangle, & \text{if } \varepsilon\in\rho_1\cap\rho_2,\end{cases}$$

where $\langle(\ell,r_{1\varepsilon}),\zeta_1^+(\ell,r_{1\varepsilon}),\zeta_1^-(\ell,r_{1\varepsilon})\rangle\in\zeta_1(\varepsilon)$ and $\langle(\ell,r_{2\varepsilon}),\zeta_2^+(\ell,r_{2\varepsilon}),\zeta_2^-(\ell,r_{2\varepsilon})\rangle\in\zeta_2(\varepsilon)$. Similarly for all $\neg\varepsilon\in\neg\rho_1\cup\neg\rho_2$:

$$\xi_3(\neg\varepsilon)=\begin{cases}\xi_1(\neg\varepsilon), & \text{if } \neg\varepsilon\in\neg\rho_1\setminus\neg\rho_2,\\ \xi_2(\neg\varepsilon), & \text{if } \neg\varepsilon\in\neg\rho_2\setminus\neg\rho_1,\\ \left\langle\begin{array}{l}(\ell,\min\{r_{1\neg\varepsilon},r_{2\neg\varepsilon}\}),\\ \min\{\xi_1^+(\ell,r_{1\neg\varepsilon}),\xi_2^+(\ell,r_{2\neg\varepsilon})\},\\ \max\{\xi_1^-(\ell,r_{1\neg\varepsilon}),\xi_2^-(\ell,r_{2\neg\varepsilon})\}\end{array}\right\rangle, & \text{if } \neg\varepsilon\in\neg\rho_1\cap\neg\rho_2,\end{cases}$$

where $\langle(\ell,r_{1\neg\varepsilon}),\xi_1^+(\ell,r_{1\neg\varepsilon}),\xi_1^-(\ell,r_{1\neg\varepsilon})\rangle\in\xi_1(\neg\varepsilon)$ and $\langle(\ell,r_{2\neg\varepsilon}),\xi_2^+(\ell,r_{2\neg\varepsilon}),\xi_2^-(\ell,r_{2\neg\varepsilon})\rangle\in\xi_2(\neg\varepsilon)$. On the other hand, let $(\zeta_2,\xi_2,\rho_2,N_2)\check{\cup}_e(\zeta_1,\xi_1,\rho_1,N_1)=(\zeta_4,\xi_4,\rho_2\cup\rho_1,\max(N_2,N_1))$. Then, for all $\varepsilon\in\rho_2\cup\rho_1$:

$$\zeta_4(\varepsilon)=\begin{cases}\zeta_2(\varepsilon), & \text{if } \varepsilon\in\rho_2\setminus\rho_1,\\ \zeta_1(\varepsilon), & \text{if } \varepsilon\in\rho_1\setminus\rho_2,\\ \left\langle\begin{array}{l}(\ell,\max\{r_{2\varepsilon},r_{1\varepsilon}\}),\\ \max\{\zeta_2^+(\ell,r_{2\varepsilon}),\zeta_1^+(\ell,r_{1\varepsilon})\},\\ \min\{\zeta_2^-(\ell,r_{2\varepsilon}),\zeta_1^-(\ell,r_{1\varepsilon})\}\end{array}\right\rangle, & \text{if } \varepsilon\in\rho_2\cap\rho_1,\end{cases}$$

Similarly for all $\neg\varepsilon\in\neg\rho_2\cup\neg\rho_1$:

$$\xi_4(\neg\varepsilon)=\begin{cases}\xi_2(\neg\varepsilon), & \text{if } \neg\varepsilon\in\neg\rho_2\setminus\neg\rho_1,\\ \xi_1(\neg\varepsilon), & \text{if } \neg\varepsilon\in\neg\rho_1\setminus\neg\rho_2,\\ \left\langle\begin{array}{l}(\ell,\min\{r_{2\neg\varepsilon},r_{1\neg\varepsilon}\}),\\ \min\{\xi_2^+(\ell,r_{2\neg\varepsilon}),\xi_1^+(\ell,r_{1\neg\varepsilon})\},\\ \max\{\xi_2^-(\ell,r_{2\neg\varepsilon}),\xi_1^-(\ell,r_{1\neg\varepsilon})\}\end{array}\right\rangle, & \text{if } \neg\varepsilon\in\neg\rho_2\cap\neg\rho_1,\end{cases}$$

Since $(\zeta_3,\xi_3,\rho_1\cup\rho_2,max(N_1,N_2))$ and $(\zeta_4,\xi_4,\rho_2\cup\rho_1,max(N_2,N_1))$ are equivalent for all $\varepsilon\in\rho_1\cup\rho_2$ and $\neg\varepsilon\in\neg\rho_1\cup\neg\rho_2$, the proof follows.

2. Let $(\zeta_2,\xi_2,\rho_2,N_2)\check{\cup}_e(\zeta_3,\xi_3,\rho_3,N_3)=(\zeta_4,\xi_4,\rho_2\cup\rho_3,\max(N_2,N_3))$. Then, for all $\varepsilon\in\rho_2\cup\rho_3$:

$$\zeta_4(\varepsilon)=\begin{cases}\zeta_2(\varepsilon), & \text{if } \varepsilon\in\rho_2\setminus\rho_3,\\ \zeta_3(\varepsilon), & \text{if } \varepsilon\in\rho_3\setminus\rho_2,\\ \left\langle\begin{array}{l}(\ell,\max\{r_{2\varepsilon},r_{3\varepsilon}\}),\\ \max\{\zeta_2^+(\ell,r_{2\varepsilon}),\zeta_3^+(\ell,r_{3\varepsilon})\},\\ \min\{\zeta_2^-(\ell,r_{2\varepsilon}),\zeta_3^-(\ell,r_{3\varepsilon})\}\end{array}\right\rangle, & \text{if } \varepsilon\in\rho_2\cap\rho_3,\end{cases}$$

where $\langle(\ell,r_{2\varepsilon}),\zeta_2^+(\ell,r_{2\varepsilon}),\zeta_2^-(\ell,r_{2\varepsilon})\rangle\in\zeta_2(\varepsilon)$ and $\langle(\ell,r_{3\varepsilon}),\zeta_3^+(\ell,r_{3\varepsilon}),\zeta_3^-(\ell,r_{3\varepsilon})\rangle\in\zeta_3(\varepsilon)$. Similarly for all $\neg\varepsilon\in\neg\rho_2\cup\neg\rho_3$:

$$\xi_4(\neg\varepsilon)=\begin{cases}\xi_2(\neg\varepsilon), & \text{if } \neg\varepsilon\in\neg\rho_2\setminus\neg\rho_3,\\ \xi_3(\neg\varepsilon), & \text{if } \neg\varepsilon\in\neg\rho_3\setminus\neg\rho_2,\\ \left\langle\begin{array}{l}(\ell,\min\{r_{2\neg\varepsilon},r_{3\neg\varepsilon}\}),\\ \min\{\xi_2^+(\ell,r_{2\neg\varepsilon}),\xi_3^+(\ell,r_{3\neg\varepsilon})\},\\ \max\{\xi_2^-(\ell,r_{2\neg\varepsilon}),\xi_3^-(\ell,r_{3\neg\varepsilon})\}\end{array}\right\rangle, & \text{if } \neg\varepsilon\in\neg\rho_2\cap\neg\rho_3,\end{cases}$$

where $\langle(\ell,r_{2\neg\varepsilon}),\xi_2^+(\ell,r_{2\neg\varepsilon}),\xi_2^-(\ell,r_{2\neg\varepsilon})\rangle\in\xi_2(\neg\varepsilon)$ and $\langle(\ell,r_{3\neg\varepsilon}),\xi_3^+(\ell,r_{3\neg\varepsilon}),\xi_3^-(\ell,r_{3\neg\varepsilon})\rangle\in\xi_3(\neg\varepsilon)$. Now, let $(\zeta_1,\xi_1,\rho_1,N_1)\,\breve{\cup}_e\,(\zeta_4,\xi_4,\rho_2\cup\rho_3,max(N_2,N_3))=(\zeta_5,\xi_5,\rho_1\cup(\rho_2\cup\rho_3),max(N_1,max(N_2,N_3))$ $=(\zeta_5,\xi_5,\rho_1\cup(\rho_2\cup\rho_3),max(N_1,N_2,N_3))$. Then, for all $\varepsilon\in\rho_1\cup(\rho_2\cup\rho_3)$:

$$\zeta_5(\varepsilon)=\begin{cases}\zeta_1(\varepsilon), & \text{if } \varepsilon\in\rho_1\setminus(\rho_2\cup\rho_3),\\ \zeta_4(\varepsilon), & \text{if } \varepsilon\in(\rho_2\cup\rho_3)\setminus\rho_1,\\ \left\langle\begin{array}{l}(\ell,\max\{r_{1\varepsilon},\max\{r_{2\varepsilon},r_{3\varepsilon}\}\}),\\ \max\left\{\zeta_1^+(\ell,r_{1\varepsilon}),\max\left\{\zeta_2^+(\ell,r_{2\varepsilon}),\zeta_3^+(\ell,r_{3\varepsilon})\right\}\right\},\\ \min\left\{\zeta_1^-(\ell,r_{1\varepsilon}),\min\left\{\zeta_2^-(\ell,r_{2\varepsilon}),\zeta_3^-(\ell,r_{3\varepsilon})\right\}\right\}\end{array}\right\rangle, & \text{if } \varepsilon\in\rho_1\cap(\rho_2\cup\rho_3),\end{cases}$$

where $\langle(\ell,r_{1\varepsilon}),\zeta_1^+(\ell,r_{1\varepsilon}),\zeta_1^-(\ell,r_{1\varepsilon})\rangle\in\zeta_1(\varepsilon)$ Similarly, for all $\neg\varepsilon\in\neg\rho_1\cup(\neg\rho_2\cup\neg\rho_3)$:

$$\xi_5(\neg\varepsilon)=\begin{cases}\xi_1(\neg\varepsilon), & \text{if } \neg\varepsilon\in\neg\rho_1\setminus(\neg\rho_2\cup\neg\rho_3),\\ \xi_3(\neg\varepsilon), & \text{if } \neg\varepsilon\in(\neg\rho_2\cup\neg\rho_3)\setminus\neg\rho_1,\\ \left\langle\begin{array}{l}(\ell,\min\{r_{1\neg\varepsilon},\min\{r_{2\neg\varepsilon},r_{3\neg\varepsilon}\}\}),\\ \min\left\{\xi_1^+(\ell,r_{1\neg\varepsilon}),\min\left\{\xi_2^+(\ell,r_{2\neg\varepsilon}),\xi_3^+(\ell,r_{3\neg\varepsilon})\right\}\right\},\\ \max\left\{\xi_1^-(\ell,r_{1\neg\varepsilon}),\max\left\{\xi_2^-(\ell,r_{2\neg\varepsilon}),\xi_3^-(\ell,r_{3\neg\varepsilon})\right\}\right\}\end{array}\right\rangle, & \text{if } \neg\varepsilon\in\neg\rho_1\cap(\neg\rho_2\cup\neg\rho_3),\end{cases}$$

where $\langle(\ell,r_{1\neg\varepsilon}),\xi_1^+(\ell,r_{1\neg\varepsilon}),\xi_1^-(\ell,r_{1\neg\varepsilon})\rangle\in\xi_1(\neg\varepsilon)$. On the other hand, let $(\zeta_1,\xi_1,\rho_1,N_1)\breve{\cup}_e(\zeta_2,\xi_2,\rho_2,N_2)=(\zeta_6,\xi_6,\rho_1\cup\rho_2,\max(N_1,N_2))$. Then, for all $\varepsilon\in\rho_1\cup\rho_2$:

$$\zeta_6(\varepsilon)=\begin{cases}\zeta_1(\varepsilon), & \text{if } \varepsilon\in\rho_1\setminus\rho_2,\\ \zeta_2(\varepsilon), & \text{if } \varepsilon\in\rho_2\setminus\rho_1,\\ \left\langle\begin{array}{l}(\ell,\max\{r_{1\varepsilon},r_{2\varepsilon}\}),\\ \max\{\zeta_1^+(\ell,r_{1\varepsilon}),\zeta_2^+(\ell,r_{2\varepsilon})\},\\ \min\{\zeta_1^-(\ell,r_{1\varepsilon}),\zeta_2^-(\ell,r_{2\varepsilon})\}\end{array}\right\rangle, & \text{if } \varepsilon\in\rho_1\cap\rho_2,\end{cases}$$

where $\langle(\ell,r_{1\varepsilon}),\zeta_1^+(\ell,r_{1\varepsilon}),\zeta_1^-(\ell,r_{1\varepsilon})\rangle\in\zeta_1(\varepsilon)$ and $\langle(\ell,r_{2\varepsilon}),\zeta_2^+(\ell,r_{2\varepsilon}),\zeta_2^-(\ell,r_{2\varepsilon})\rangle\in\zeta_2(\varepsilon)$. Similarly for all $\neg\varepsilon\in\neg\rho_1\cup\neg\rho_2$:

$$\xi_6(\neg\varepsilon)=\begin{cases}\xi_1(\neg\varepsilon), & \text{if } \neg\varepsilon\in\neg\rho_1\setminus\neg\rho_2,\\ \xi_2(\neg\varepsilon), & \text{if } \neg\varepsilon\in\neg\rho_2\setminus\neg\rho_1,\\ \left\langle\begin{array}{l}(\ell,\min\{r_{1\neg\varepsilon},r_{2\neg\varepsilon}\}),\\ \min\{\xi_1^+(\ell,r_{1\neg\varepsilon}),\xi_2^+(\ell,r_{2\neg\varepsilon})\},\\ \max\{\xi_1^-(\ell,r_{1\neg\varepsilon}),\xi_2^-(\ell,r_{2\neg\varepsilon})\}\end{array}\right\rangle, & \text{if } \neg\varepsilon\in\neg\rho_1\cap\neg\rho_2,\end{cases}$$

where $\langle(\ell,r_{1\neg\varepsilon}),\xi_1^+(\ell,r_{1\neg\varepsilon}),\xi_1^-(\ell,r_{1\neg\varepsilon})\rangle\in\xi_1(\neg\varepsilon)$ and $\langle(\ell,r_{2\neg\varepsilon}),\xi_2^+(\ell,r_{2\neg\varepsilon}),\xi_2^-(\ell,r_{2\neg\varepsilon})\rangle\in\xi_2(\neg\varepsilon)$. Now, let $(\zeta_6,\xi_6,\rho_1\cup\rho_2,max(N_1,N_2))\,\breve{\cup}_e\,(\zeta_3,\xi_3,\rho_3,N_3)=(\zeta_7,\xi_7,(\rho_1\cup\rho_2)\cup\rho_3,max(max(N_1,N_2),N_3)$ $=(\zeta_7,\xi_7,(\rho_1\cup\rho_2)\cup\rho_3,max(N_1,N_2,N_3))$. Then, for all $\varepsilon\in(\rho_1\cup\rho_2)\cup\rho_3$:

$$\zeta_7(\varepsilon)=\begin{cases}\zeta_3(\varepsilon), & \text{if } \varepsilon\in\rho_3\setminus(\rho_1\cup\rho_2),\\ \zeta_6(\varepsilon), & \text{if } \varepsilon\in(\rho_1\cup\rho_2)\setminus\rho_3,\\ \left\langle\begin{array}{l}(\ell,\max\{r_{3\varepsilon},\max\{r_{1\varepsilon},r_{2\varepsilon}\}\}),\\ \max\left\{\zeta_3^+(\ell,r_{3\varepsilon}),\max\left\{\zeta_1^+(\ell,r_{1\varepsilon}),\zeta_2^+(\ell,r_{2\varepsilon})\right\}\right\},\\ \min\left\{\zeta_3^-(\ell,r_{3\varepsilon}),\min\left\{\zeta_1^-(\ell,r_{1\varepsilon}),\zeta_2^-(\ell,r_{2\varepsilon})\right\}\right\}\end{array}\right\rangle, & \text{if } \varepsilon\in\rho_3\cap(\rho_1\cup\rho_2),\end{cases}$$

where $\langle(\ell,r_{3\varepsilon}),\zeta_3^+(\ell,r_{3\varepsilon}),\zeta_3^-(\ell,r_{3\varepsilon})\rangle\in\zeta_3(\varepsilon)$. Similarly, for all $\neg\varepsilon\in\neg\rho_3\cup(\neg\rho_1\cup\neg\rho_2)$:

$$\xi_7(\neg\varepsilon)=\begin{cases}\xi_3(\neg\varepsilon), & \text{if } \neg\varepsilon\in\neg\rho_3\setminus(\neg\rho_1\cup\neg\rho_2),\\ \xi_6(\neg\varepsilon), & \text{if } \neg\varepsilon\in(\neg\rho_1\cup\neg\rho_2)\setminus\neg\rho_3,\\ \left\langle\begin{array}{l}(\ell,\min\{r_{3\neg\varepsilon},\min\{r_{1\neg\varepsilon},r_{2\neg\varepsilon}\}\}),\\ \min\left\{\xi_3^+(\ell,r_{3\neg\varepsilon}),\min\left\{\xi_1^+(\ell,r_{1\neg\varepsilon}),\xi_2^+(\ell,r_{2\neg\varepsilon})\right\}\right\},\\ \max\left\{\xi_3^-(\ell,r_{3\neg\varepsilon}),\max\left\{\xi_1^-(\ell,r_{1\neg\varepsilon}),\xi_2^-(\ell,r_{2\neg\varepsilon})\right\}\right\}\end{array}\right\rangle, & \text{if } \neg\varepsilon\in\neg\rho_3\cap(\neg\rho_1\cup\neg\rho_2),\end{cases}$$

where $\langle(\ell,r_{3\neg\varepsilon}),\xi_3^+(\ell,r_{3\neg\varepsilon}),\xi_3^-(\ell,r_{3\neg\varepsilon})\rangle\in\xi_3(\neg\varepsilon)$. Since $(\zeta_5,\xi_5,\rho_1\cup(\rho_2\cup\rho_3),max(N_1,N_2,N_3))$ and $(\zeta_7,\xi_7,(\rho_1\cup\rho_2)\cup\rho_3,max(N_1,N_2,N_3))$ are equivalent for all $\varepsilon\in\rho_1\cup(\rho_2\cup\rho_3)$ and $\neg\varepsilon\in\neg\rho_1\cup(\neg\rho_2\cup\neg\rho_3)$, the proof follows.□

Definition 3.8 The extended intersection of $(\zeta_1, \xi_1, \rho_1, N_1)$ and $(\zeta_2, \xi_2, \rho_2, N_2)$ is denoted and defined as $(\zeta_1, \xi_1, \rho_1, N_1) \check{\cap}_e (\zeta_2, \xi_2, \rho_2, N_2) = (\zeta, \xi, \rho_1 \cup \rho_2, \max(N_1, N_2))$, where for all $\varepsilon \in \rho_1 \cup \rho_2$:

$$\zeta(\varepsilon) = \begin{cases} \zeta_1(\varepsilon), & \text{if } \varepsilon \in \rho_1 \setminus \rho_2, \\ \zeta_2(\varepsilon), & \text{if } \varepsilon \in \rho_2 \setminus \rho_1, \\ \left\langle \begin{array}{l} (\ell, \min\{r_{1\varepsilon}, r_{2\varepsilon}\}), \\ \min\{\zeta_1^+(\ell, r_{1\varepsilon}), \zeta_2^+(\ell, r_{2\varepsilon})\}, \\ \max\{\zeta_1^-(\ell, r_{1\varepsilon}), \zeta_2^-(\ell, r_{2\varepsilon})\} \end{array} \right\rangle, & \text{if } \varepsilon \in \rho_1 \cap \rho_2, \end{cases}$$

where $\langle (\ell, r_{1\varepsilon}), \zeta_1^+(\ell, r_{1\varepsilon}), \zeta_1^-(\ell, r_{1\varepsilon}) \rangle \in \zeta_1(\varepsilon)$ and $\langle (\ell, r_{2\varepsilon}), \zeta_2^+(\ell, r_{2\varepsilon}), \zeta_2^-(\ell, r_{2\varepsilon}) \rangle \in \zeta_2(\varepsilon)$.

Similarly, for all $\neg\varepsilon \in \neg\rho_1 \cup \neg\rho_2$:

$$\xi(\neg\varepsilon) = \begin{cases} \xi_1(\neg\varepsilon), & \text{if } \neg\varepsilon \in \neg\rho_1 \setminus \neg\rho_2, \\ \xi_2(\neg\varepsilon), & \text{if } \neg\varepsilon \in \neg\rho_2 \setminus \neg\rho_1, \\ \left\langle \begin{array}{l} (\ell, \max\{r_{1\neg\varepsilon}, r_{2\neg\varepsilon}\}), \\ \max\{\xi_1^+(\ell, r_{1\neg\varepsilon}), \xi_2^+(\ell, r_{2\neg\varepsilon})\}, \\ \min\{\xi_1^-(\ell, r_{1\neg\varepsilon}), \xi_2^-(\ell, r_{2\neg\varepsilon})\} \end{array} \right\rangle, & \text{if } \neg\varepsilon \in \neg\rho_1 \cap \neg\rho_2, \end{cases}$$

where $\langle (\ell, r_{1\neg\varepsilon}), \xi_1^+(\ell, r_{1\neg\varepsilon}), \xi_1^-(\ell, r_{1\neg\varepsilon}) \rangle \in \xi_1(\neg\varepsilon)$ and $\langle (\ell, r_{2\neg\varepsilon}), \xi_2^+(\ell, r_{2\neg\varepsilon}), \xi_2^-(\ell, r_{2\neg\varepsilon}) \rangle \in \xi_2(\neg\varepsilon)$.

Example 3.5 Based on the PF4BSS $(\zeta_1, \xi_1, \rho_1, 4)$ and PF5BSS $(\zeta_2, \xi_2, \rho_2, 5)$ given in Tables 12 and 13, their extended intersection is detailed in Table 15.

Proposition 3.3 Let $(\zeta_1, \xi_1, \rho_1, N_1)$, $(\zeta_2, \xi_2, \rho_2, N_2)$, and $(\zeta_3, \xi_3, \rho_3, N_3)$ be PFN_1BSS, PFN_2BSS, and PFN_3 BSS, respectively. Then,

1. $(\zeta_1, \xi_1, \rho_1, N_1) \check{\cap}_e (\zeta_2, \xi_2, \rho_2, N_2) = (\zeta_2, \xi_2, \rho_2, N_2) \check{\cap}_e (\zeta_1, \xi_1, \rho_1, N_1)$.
2. $(\zeta_1, \xi_1, \rho_1, N_1) \check{\cap}_e \big((\zeta_2, \xi_2, \rho_2, N_2) \check{\cap}_e (\zeta_3, \xi_3, \rho_3, N_3)\big) = \big((\zeta_1, \xi_1, \rho_1, N_1) \check{\cap}_e (\zeta_2, \xi_2, \rho_2, N_2)\big) \check{\cap}_e (\zeta_3, \xi_3, \rho_3, N_3)$.

Proof
1. Similar to the proof of Proposition 3.2 (1).
2. Similar to the proof of Proposition 3.2 (2).□

Definition 3.9 The restricted union of $(\zeta_1, \xi_1, \rho_1, N_1)$ and $(\zeta_2, \xi_2, \rho_2, N_2)$ is denoted and defined as $(\zeta_1, \xi_1, \rho_1, N_1) \check{\cup}_r (\zeta_2, \xi_2, \rho_2, N_2) = (\zeta, \xi, \rho_1 \cap \rho_2, \max(N_1, N_2))$, where for all $\varepsilon \in \rho_1 \cap \rho_2 \neq \emptyset$:

$$\zeta(\varepsilon) = \langle (\ell, \max\{r_{1\varepsilon}, r_{2\varepsilon}\}), \max\{\zeta_1^+(\ell, r_{1\varepsilon}), \zeta_2^+(\ell, r_{2\varepsilon})\}, \min\{\zeta_1^-(\ell, r_{1\varepsilon}), \zeta_2^-(\ell, r_{2\varepsilon})\} \rangle,$$

where $\langle (\ell, r_{1\varepsilon}), \zeta_1^+(\ell, r_{1\varepsilon}), \zeta_1^-(\ell, r_{1\varepsilon}) \rangle \in \zeta_1(\varepsilon)$ and $\langle (\ell, r_{2\varepsilon}), \zeta_2^+(\ell, r_{2\varepsilon}), \zeta_2^-(\ell, r_{2\varepsilon}) \rangle \in \zeta_2(\varepsilon)$.

Similarly, for all $\neg\varepsilon \in \neg\rho_1 \cap \neg\rho_2 \neq \emptyset$:

$$\xi(\neg\varepsilon) = \langle (\ell, \min\{r_{1\neg\varepsilon}, r_{2\neg\varepsilon}\}), \min\{\xi_1^+(\ell, r_{1\neg\varepsilon}), \xi_2^+(\ell, r_{2\neg\varepsilon})\}, \max\{\xi_1^-(\ell, r_{1\neg\varepsilon}), \xi_2^-(\ell, r_{2\neg\varepsilon})\} \rangle,$$

where $\langle (\ell, r_{1\neg\varepsilon}), \xi_1^+(\ell, r_{1\neg\varepsilon}), \xi_1^-(\ell, r_{1\neg\varepsilon}) \rangle \in \xi_1(\neg\varepsilon)$ and $\langle (\ell, r_{2\neg\varepsilon}), \xi_2^+(\ell, r_{2\neg\varepsilon}), \xi_2^-(\ell, r_{2\neg\varepsilon}) \rangle \in \xi_2(\neg\varepsilon)$.

Example 3.6 Consider again the PF4BSS $(\zeta_1, \xi_1, \rho_1, 4)$ and PF5BSS $(\zeta_2, \xi_2, \rho_2, 5)$ presented in Tables 12 and 13, respectively. Their restricted union is shown in Table 16.

Proposition 3.4 Let $(\zeta_1, \xi_1, \rho_1, N_1)$, $(\zeta_2, \xi_2, \rho_2, N_2)$, and $(\zeta_3, \xi_3, \rho_3, N_3)$ be PFN_1BSS, PFN_2BSS, and PFN_3 BSS, respectively. Then,

1. $(\zeta_1, \xi_1, \rho_1, N_1) \check{\cup}_r (\zeta_2, \xi_2, \rho_2, N_2) = (\zeta_2, \xi_2, \rho_2, N_2) \check{\cup}_r (\zeta_1, \xi_1, \rho_1, N_1)$.
2. $(\zeta_1, \xi_1, \rho_1, N_1) \check{\cup}_r \big((\zeta_2, \xi_2, \rho_2, N_2) \check{\cup}_r (\zeta_3, \xi_3, \rho_3, N_3)\big) = \big((\zeta_1, \xi_1, \rho_1, N_1) \check{\cup}_r (\zeta_2, \xi_2, \rho_2, N_2)\big) \check{\cup}_r (\zeta_3, \xi_3, \rho_3, N_3)$.

$(\zeta_4, \xi_4, \rho_1 \cup \rho_2, 5)$	ε_1	ε_2	ε_3	ε_4
ℓ_1	$\langle 0, 0.2, 0.4 \rangle$ $\langle 3, 0.8, 0.3 \rangle$	$\langle 2, 0.6, 0.4 \rangle$ $\langle 2, 0.3, 0.5 \rangle$	$\langle 2, 0.7, 0.0 \rangle$ $\langle 0, 0.3, 0.3 \rangle$	$\langle 0, 0.0, 0.0 \rangle$ $\langle 4, 1.0, 0.0 \rangle$
ℓ_2	$\langle 1, 0.2, 0.5 \rangle$ $\langle 2, 0.4, 0.2 \rangle$	$\langle 1, 0.2, 0.9 \rangle$ $\langle 1, 0.2, 0.2 \rangle$	$\langle 1, 0.5, 0.2 \rangle$ $\langle 2, 0.2, 0.7 \rangle$	$\langle 1, 0.0, 0.5 \rangle$ $\langle 3, 0.7, 0.4 \rangle$
ℓ_3	$\langle 1, 0.2, 0.4 \rangle$ $\langle 0, 0.3, 0.3 \rangle$	$\langle 1, 0.1, 0.6 \rangle$ $\langle 2, 0.7, 0.1 \rangle$	$\langle 3, 0.6, 0.5 \rangle$ $\langle 0, 0.4, 0.1 \rangle$	$\langle 2, 0.5, 0.5 \rangle$ $\langle 2, 0.7, 0.3 \rangle$

Table 15. The extended intersection $(\zeta_1, \xi_1, \rho_1, 4) \check{\cap}_e (\zeta_2, \xi_2, \rho_2, 5) = (\zeta_4, \xi_4, \rho_1 \cup \rho_2, 5)$ in Example 3.5

$(\zeta_5, \xi_5, \rho_1 \cap \rho_2, 5)$	ε_1	ε_2
ℓ_1	$\langle 1, 0.2, 0.3\rangle$ $\langle 3, 0.5, 0.6\rangle$	$\langle 2, 0.6, 0.3\rangle$ $\langle 1, 0.3, 0.6\rangle$
ℓ_2	$\langle 1, 0.3, 0.5\rangle$ $\langle 1, 0.4, 0.6\rangle$	$\langle 4, 0.4, 0.2\rangle$ $\langle 0, 0.1, 0.4\rangle$
ℓ_3	$\langle 1, 0.3, 0.4\rangle$ $\langle 0, 0.1, 0.4\rangle$	$\langle 2, 0.7, 0.0\rangle$ $\langle 0, 0.0, 0.1\rangle$

Table 16. The restricted union $(\zeta_1, \xi_1, \rho_1, 4)\check{\cup}_r(\zeta_2, \xi_2, \rho_2, 5)$=$(\zeta_5, \xi_5, \rho_1 \cap \rho_2, 5)$ in Example 3.6

Proof 1. Let $(\zeta_1, \xi_1, \rho_1, N_1)\check{\cup}_r(\zeta_2, \xi_2, \rho_2, N_2) = (\zeta_3, \xi_3, \rho_1 \cap \rho_2, \max(N_1, N_2))$. For all $\varepsilon \in \rho_1 \cap \rho_2 \neq \emptyset$:

$$\zeta_3(\varepsilon) = \langle(\ell, \max\{r_{1\varepsilon}, r_{2\varepsilon}\}), \max\{\zeta_1^+(\ell, r_{1\varepsilon}), \zeta_2^+(\ell, r_{2\varepsilon})\}, \min\{\zeta_1^-(\ell, r_{1\varepsilon}), \zeta_2^-(\ell, r_{2\varepsilon})\}\rangle,$$

where $\langle(\ell, r_{1\varepsilon}), \zeta_1^+(\ell, r_{1\varepsilon}), \zeta_1^-(\ell, r_{1\varepsilon})\rangle \in \zeta_1(\varepsilon)$ and $\langle(\ell, r_{2\varepsilon}), \zeta_2^+(\ell, r_{2\varepsilon}), \zeta_2^-(\ell, r_{2\varepsilon})\rangle \in \zeta_2(\varepsilon)$. Similarly, for all $\neg\varepsilon \in \neg\rho_1 \cap \neg\rho_2 \neq \emptyset$:

$$\xi_3(\neg\varepsilon) = \langle(\ell, \min\{r_{1\neg\varepsilon}, r_{2\neg\varepsilon}\}), \min\{\xi_1^+(\ell, r_{1\neg\varepsilon}), \xi_2^+(\ell, r_{2\neg\varepsilon})\}, \max\{\xi_1^-(\ell, r_{1\neg\varepsilon}), \xi_2^-(\ell, r_{2\neg\varepsilon})\}\rangle,$$

where $\langle(\ell, r_{1\neg\varepsilon}), \xi_1^+(\ell, r_{1\neg\varepsilon}), \xi_1^-(\ell, r_{1\neg\varepsilon})\rangle \in \xi_1(\neg\varepsilon)$ and $\langle(\ell, r_{2\neg\varepsilon}), \xi_2^+(\ell, r_{2\neg\varepsilon}), \xi_2^-(\ell, r_{2\neg\varepsilon})\rangle \in \xi_2(\neg\varepsilon)$. On the other hand, let $(\zeta_2, \xi_2, \rho_2, N_2)\check{\cup}_r(\zeta_1, \xi_1, \rho_1, N_1) = (\zeta_4, \xi_4, \rho_2 \cap \rho_1, \max(N_2, N_1))$. Then, for all $\varepsilon \in \rho_2 \cap \rho_1 \neq \emptyset$:

$$\zeta_4(\varepsilon) = \langle(\ell, \max\{r_{2\varepsilon}, r_{1\varepsilon}\}), \max\{\zeta_2^+(\ell, r_{2\varepsilon}), \zeta_1^+(\ell, r_{1\varepsilon})\}, \min\{\zeta_2^-(\ell, r_{2\varepsilon}), \zeta_1^-(\ell, r_{1\varepsilon})\}\rangle.$$

Similarly, for all $\neg\varepsilon \in \neg\rho_2 \cap \neg\rho_1 \neq \emptyset$:

$$\xi_4(\neg\varepsilon) = \langle(\ell, \min\{r_{2\neg\varepsilon}, r_{1\neg\varepsilon}\}), \min\{\xi_2^+(\ell, r_{2\neg\varepsilon}), \xi_1^+(\ell, r_{1\neg\varepsilon})\}, \max\{\xi_2^-(\ell, r_{2\neg\varepsilon}), \xi_1^-(\ell, r_{1\neg\varepsilon})\}\rangle.$$

Since $(\zeta_3, \xi_3, \rho_1 \cap \rho_2, \max(N_1, N_2))$ and $(\zeta_4, \xi_4, \rho_2 \cap \rho_1, \max(N_2, N_1))$ are equivalent for all $\varepsilon \in \rho_1 \cap \rho_2$ and $\neg\varepsilon \in \neg\rho_1 \cap \neg\rho_2$, the proof follows.

2. Let $(\zeta_2, \xi_2, \rho_2, N_2)\check{\cup}_r(\zeta_3, \xi_3, \rho_3, N_3) = (\zeta_4, \xi_4, \rho_2 \cap \rho_3, \max(N_2, N_3))$. Then, for all $\varepsilon \in \rho_2 \cap \rho_3 \neq \emptyset$,

$$\zeta_4(\varepsilon) = \langle(\ell, \max\{r_{2\varepsilon}, r_{3\varepsilon}\}), \max\{\zeta_2^+(\ell, r_{2\varepsilon}), \zeta_3^+(\ell, r_{3\varepsilon})\}, \min\{\zeta_2^-(\ell, r_{2\varepsilon}), \zeta_3^-(\ell, r_{3\varepsilon})\}\rangle,$$

where $\langle(\ell, r_{2\varepsilon}), \zeta_2^+(\ell, r_{2\varepsilon}), \zeta_2^-(\ell, r_{2\varepsilon})\rangle \in \zeta_2(\varepsilon)$ and $\langle(\ell, r_{3\varepsilon}), \zeta_3^+(\ell, r_{3\varepsilon}), \zeta_3^-(\ell, r_{3\varepsilon})\rangle \in \zeta_3(\varepsilon)$. Similarly, for all $\neg\varepsilon \in \neg\rho_2 \cap \neg\rho_3 \neq \emptyset$,

$$\xi_4(\neg\varepsilon) = \langle(\ell, \min\{r_{2\neg\varepsilon}, r_{3\neg\varepsilon}\}), \min\{\xi_2^+(\ell, r_{2\neg\varepsilon}), \xi_3^+(\ell, r_{3\neg\varepsilon})\}, \max\{\xi_2^-(\ell, r_{2\neg\varepsilon}), \xi_3^-(\ell, r_{3\neg\varepsilon})\}\rangle,$$

where $\langle(\ell, r_{2\neg\varepsilon}), \xi_2^+(\ell, r_{2\neg\varepsilon}), \xi_2^-(\ell, r_{2\neg\varepsilon})\rangle \in \xi_2(\neg\varepsilon)$ and $\langle(\ell, r_{3\neg\varepsilon}), \xi_3^+(\ell, r_{3\neg\varepsilon}), \xi_3^-(\ell, r_{3\neg\varepsilon})\rangle \in \xi_3(\neg\varepsilon)$. Now, let $(\zeta_1, \xi_1, \rho_1, N_1)\check{\cup}_r(\zeta_4, \xi_4, \rho_2 \cap \rho_3, \max(N_2, N_3)) = (\zeta_5, \xi_5, \rho_1 \cap (\rho_2 \cap \rho_3), \max(N_1, \max(N_2, N_3)))$. Then, for all $\varepsilon \in \rho_1 \cap (\rho_2 \cap \rho_3) \neq \emptyset$,

$$\zeta_5(\varepsilon) = \left\langle \begin{array}{l} (\ell, \max\{r_{1\varepsilon}, \max\{r_{2\varepsilon}, r_{3\varepsilon}\}\}), \\ \max\{\zeta_1^+(\ell, r_{1\varepsilon}), \max\{\zeta_2^+(\ell, r_{2\varepsilon}), \zeta_3^+(\ell, r_{3\varepsilon})\}\}, \\ \min\{\zeta_1^-(\ell, r_{1\varepsilon}), \min\{\zeta_2^-(\ell, r_{2\varepsilon}), \zeta_3^-(\ell, r_{3\varepsilon})\}\} \end{array} \right\rangle,$$

where $\langle(\ell, r_{1\varepsilon}), \zeta_1^+(\ell, r_{1\varepsilon}), \zeta_1^-(\ell, r_{1\varepsilon})\rangle \in \zeta_1(\varepsilon)$. Similarly, for all $\neg\varepsilon \in \neg\rho_1 \cap (\neg\rho_2 \cap \neg\rho_3) \neq \emptyset$,

$$\xi_5(\neg\varepsilon) = \left\langle \begin{array}{l} (\ell, \min\{r_{1\neg\varepsilon}, \min\{r_{2\neg\varepsilon}, r_{3\neg\varepsilon}\}\}), \\ \min\{\xi_1^+(\ell, r_{1\neg\varepsilon}), \min\{\xi_2^+(\ell, r_{2\neg\varepsilon}), \xi_3^+(\ell, r_{3\neg\varepsilon})\}\}, \\ \max\{\xi_1^-(\ell, r_{1\neg\varepsilon}), \max\{\xi_2^-(\ell, r_{2\neg\varepsilon}), \xi_3^-(\ell, r_{3\neg\varepsilon})\}\} \end{array} \right\rangle,$$

where $\langle(\ell, r_{1\neg\varepsilon}), \xi_1^+(\ell, r_{1\neg\varepsilon}), \xi_1^-(\ell, r_{1\neg\varepsilon})\rangle \in \xi_1(\neg\varepsilon)$. On the other hand, let $(\zeta_1, \xi_1, \rho_1, N_1)\breve{\cup}_r(\zeta_2, \xi_2, \rho_2, N_2) = (\zeta_6, \xi_6, \rho_1 \cap \rho_2, \max(N_1, N_2))$. Then, for all $\varepsilon \in \rho_1 \cap \rho_2 \neq \emptyset$,

$$\zeta_6(\varepsilon) = \langle(\ell, \max\{r_{1\varepsilon}, r_{2\varepsilon}\}), \max\{\zeta_1^+(\ell, r_{1\varepsilon}), \zeta_2^+(\ell, r_{2\varepsilon})\}, \min\{\zeta_1^-(\ell, r_{1\varepsilon}), \zeta_2^-(\ell, r_{2\varepsilon})\}\rangle,$$

where $\langle(\ell, r_{1\varepsilon}), \zeta_1^+(\ell, r_{1\varepsilon}), \zeta_1^-(\ell, r_{1\varepsilon})\rangle \in \zeta_1(\varepsilon)$ and $\langle(\ell, r_{2\varepsilon}), \zeta_2^+(\ell, r_{2\varepsilon}), \zeta_2^-(\ell, r_{2\varepsilon})\rangle \in \zeta_2(\varepsilon)$. Similarly, for all $\neg\varepsilon \in \neg\rho_1 \cap \neg\rho_2 \neq \emptyset$,

$$\xi_6(\neg\varepsilon) = \langle(\ell, \min\{r_{1\neg\varepsilon}, r_{2\neg\varepsilon}\}), \min\{\xi_1^+(\ell, r_{1\neg\varepsilon}), \xi_2^+(\ell, r_{2\neg\varepsilon})\}, \max\{\xi_1^-(\ell, r_{1\neg\varepsilon}), \xi_2^-(\ell, r_{2\neg\varepsilon})\}\rangle,$$

where $\langle(\ell, r_{1\neg\varepsilon}), \xi_1^+(\ell, r_{1\neg\varepsilon}), \xi_1^-(\ell, r_{1\neg\varepsilon})\rangle \in \xi_1(\neg\varepsilon)$ and $\langle(\ell, r_{2\neg\varepsilon}), \xi_2^+(\ell, r_{2\neg\varepsilon}), \xi_2^-(\ell, r_{2\neg\varepsilon})\rangle \in \xi_2(\neg\varepsilon)$. Now, let $(\zeta_6, \xi_6, \rho_1 \cup \rho_2, \max(N_1, N_2))\breve{\cup}_e(\zeta_3, \xi_3, \rho_3, N_3) = (\zeta_7, \xi_7, (\rho_1 \cup \rho_2) \cup \rho_3, \max(\max(N_1, N_2), N_3))$. Then, for all $\varepsilon \in (\rho_1 \cap \rho_2) \cap \rho_3 \neq \emptyset$,

$$\zeta_7(\varepsilon) = \left\langle \begin{matrix} (\ell, \max\{r_{3\varepsilon}, \max\{r_{1\varepsilon}, r_{2\varepsilon}\}\}), \\ \max\{\zeta_3^+(\ell, r_{3\varepsilon}), \max\{\zeta_1^+(\ell, r_{1\varepsilon}), \zeta_2^+(\ell, r_{2\varepsilon})\}\}, \\ \min\{\zeta_3^-(\ell, r_{3\varepsilon}), \min\{\zeta_1^-(\ell, r_{1\varepsilon}), \zeta_2^-(\ell, r_{2\varepsilon})\}\} \end{matrix} \right\rangle,$$

where $\langle(\ell, r_{3\varepsilon}), \zeta_3^+(\ell, r_{3\varepsilon}), \zeta_3^-(\ell, r_{3\varepsilon})\rangle \in \zeta_3(\varepsilon)$. Similarly, for all $\neg\varepsilon \in \neg\rho_3 \cap (\neg\rho_1 \cap \neg\rho_2) \neq \emptyset$,

$$\xi_7(\neg\varepsilon) = \left\langle \begin{matrix} (\ell, \min\{r_{3\neg\varepsilon}, \min\{r_{1\neg\varepsilon}, r_{2\neg\varepsilon}\}\}), \\ \min\{\xi_3^+(\ell, r_{3\neg\varepsilon}), \min\{\xi_1^+(\ell, r_{1\neg\varepsilon}), \xi_2^+(\ell, r_{2\neg\varepsilon})\}\}, \\ \max\{\xi_3^-(\ell, r_{3\neg\varepsilon}), \max\{\xi_1^-(\ell, r_{1\neg\varepsilon}), \xi_2^-(\ell, r_{2\neg\varepsilon})\}\} \end{matrix} \right\rangle,$$

where $\langle(\ell, r_{3\neg\varepsilon}), \xi_3^+(\ell, r_{3\neg\varepsilon}), \xi_3^-(\ell, r_{3\neg\varepsilon})\rangle \in \xi_3(\neg\varepsilon)$. Since $(\zeta_5, \xi_5, \rho_1 \cap (\rho_2 \cap \rho_3), \max(N_1, N_2, N_3))$ and $(\zeta_7, \xi_7, (\rho_1 \cap \rho_2) \cap \rho_3, \max(N_1, N_2, N_3))$ are equivalent for all $\varepsilon \in \rho_1 \cap (\rho_2 \cap \rho_3)$ and $\neg\varepsilon \in \neg\rho_1 \cap (\neg\rho_2 \cap \neg\rho_3)$, the proof follows.□

Definition 3.10 The restricted intersection of $(\zeta_1, \xi_1, \rho_1, N_1)$ and $(\zeta_2, \xi_2, \rho_2, N_2)$ is denoted and defined as $(\zeta_1, \xi_1, \rho_1, N_1) \breve{\cap}_r (\zeta_2, \xi_2, \rho_2, N_2) = (\zeta, \xi, \rho_1 \cap \rho_2, \max(N_1, N_2))$, where for all $\varepsilon \in \rho_1 \cap \rho_2 \neq \emptyset$:

$$\zeta(\varepsilon) = \langle(\ell, \min\{r_{1\varepsilon}, r_{2\varepsilon}\}), \min\{\zeta_1^+(\ell, r_{1\varepsilon}), \zeta_2^+(\ell, r_{2\varepsilon})\}, \max\{\zeta_1^-(\ell, r_{1\varepsilon}), \zeta_2^-(\ell, r_{2\varepsilon})\}\rangle,$$

where $\langle(\ell, r_{1\varepsilon}), \zeta_1^+(\ell, r_{1\varepsilon}), \zeta_1^-(\ell, r_{1\varepsilon})\rangle \in \zeta_1(\varepsilon)$ and $\langle(\ell, r_{2\varepsilon}), \zeta_2^+(\ell, r_{2\varepsilon}), \zeta_2^-(\ell, r_{2\varepsilon})\rangle \in \zeta_2(\varepsilon)$.

Similarly, for all $\neg\varepsilon \in \neg\rho_1 \cap \neg\rho_2 \neq \emptyset$:

$$\xi(\neg\varepsilon) = \langle(\ell, \max\{r_{1\neg\varepsilon}, r_{2\neg\varepsilon}\}), \max\{\xi_1^+(\ell, r_{1\neg\varepsilon}), \xi_2^+(\ell, r_{2\neg\varepsilon})\}, \min\{\xi_1^-(\ell, r_{1\neg\varepsilon}), \xi_2^-(\ell, r_{2\neg\varepsilon})\}\rangle,$$

where $\langle(\ell, r_{1\neg\varepsilon}), \xi_1^+(\ell, r_{1\neg\varepsilon}), \xi_1^-(\ell, r_{1\neg\varepsilon})\rangle \in \xi_1(\neg\varepsilon)$ and $\langle(\ell, r_{2\neg\varepsilon}), \xi_2^+(\ell, r_{2\neg\varepsilon}), \xi_2^-(\ell, r_{2\neg\varepsilon})\rangle \in \xi_2(\neg\varepsilon)$.

Example 3.7 Consider again the PF4BSS $(\zeta_1, \xi_1, \rho_1, 4)$ and PF5BSS $(\zeta_2, \xi_2, \rho_2, 5)$, as shown in Tables 12 and 13, respectively. The restricted intersection of these sets is presented in Table 17.

Proposition 3.5 Let $(\zeta_1, \xi_1, \rho_1, N_1)$, $(\zeta_2, \xi_2, \rho_2, N_2)$, and $(\zeta_3, \xi_3, \rho_3, N_3)$ be PFN_1BSS, PFN_2BSS, and PFN_3 BSS, respectively. Then,

1. $(\zeta_1, \xi_1, \rho_1, N_1) \breve{\cap}_r (\zeta_2, \xi_2, \rho_2, N_2) = (\zeta_2, \xi_2, \rho_2, N_2) \breve{\cap}_r (\zeta_1, \xi_1, \rho_1, N_1)$.
2. $(\zeta_1, \xi_1, \rho_1, N_1) \breve{\cap}_r \big((\zeta_2, \xi_2, \rho_2, N_2) \breve{\cap}_r (\zeta_3, \xi_3, \rho_3, N_3)\big) = \big((\zeta_1, \xi_1, \rho_1, N_1) \breve{\cap}_r (\zeta_2, \xi_2, \rho_2, N_2)\big) \breve{\cap}_r (\zeta_3, \xi_3, \rho_3, N_3)$.

$(\zeta_6, \xi_6, \rho_1 \cap \rho_2, 5)$	ε_1	ε_2
ℓ_1	$\langle 0, 0.2, 0.4\rangle$ $\langle 3, 0.8, 0.3\rangle$	$\langle 2, 0.6, 0.4\rangle$ $\langle 2, 0.3, 0.5\rangle$
ℓ_2	$\langle 1, 0.2, 0.5\rangle$ $\langle 2, 0.4, 0.2\rangle$	$\langle 1, 0.2, 0.9\rangle$ $\langle 1, 0.2, 0.2\rangle$
ℓ_3	$\langle 1, 0.2, 0.4\rangle$ $\langle 0, 0.3, 0.3\rangle$	$\langle 1, 0.1, 0.6\rangle$ $\langle 2, 0.7, 0.1\rangle$

Table 17. The restricted intersection $(\zeta_1, \xi_1, \rho_1, 4)\breve{\cap}_r(\zeta_2, \xi_2, \rho_2, 5)$=$(\zeta_6, \xi_6, \rho_1 \cap \rho_2, 5)$ in Example 3.7

Proof 1. Similar to the proof of Proposition 3.4 (1).
2. Similar to the proof of Proposition 3.4 (2).□

We now present the relationships between the extended union, extended intersection, restricted union, and restricted intersection within the PFNBSS framework.

Proposition 3.6 Let $(\zeta_1, \xi_1, \rho, N)$ and $(\zeta_2, \xi_2, \rho, N)$ be two PFNBSSs. Then,

1. $(\zeta_1, \xi_1, \rho, N) \check{\cup}_e (\zeta_2, \xi_2, \rho, N) = (\zeta_1, \xi_1, \rho, N) \check{\cup}_r (\zeta_2, \xi_2, \rho, N)$.
2. $(\zeta_1, \xi_1, \rho, N) \check{\cap}_e (\zeta_2, \xi_2, \rho, N) = (\zeta_1, \xi_1, \rho, N) \check{\cap}_r (\zeta_2, \xi_2, \rho, N)$.

Proof 1. Follows from the fact that the set of parameters is only ρ; hence, by Definitions 3.7 and 3.9, the extended and restricted unions between two PFNBSSs are identical.
2. Follows from the fact that the set of parameters is only ρ; hence, by Definitions 3.8 and 3.10, the extended and restricted intersections between two PFNBSSs are identical.□

Proposition 3.7 Let $(\zeta_1, \xi_1, \rho_1, N)$ and $(\zeta_2, \xi_2, \rho_2, N)$ be two PFNBSSs. Then,

1. $\left((\zeta_1, \xi_1, \rho_1, N) \check{\cup}_e (\zeta_2, \xi_2, \rho_2, N)\right)^{\check{c}} = (\zeta_1, \xi_1, \rho_1, N)^{\check{c}} \check{\cap}_e (\zeta_2, \xi_2, \rho_2, N)^{\check{c}}$.
2. $\left((\zeta_1, \xi_1, \rho_1, N) \check{\cap}_e (\zeta_2, \xi_2, \rho_2, N)\right)^{\check{c}} = (\zeta_1, \xi_1, \rho_1, N)^{\check{c}} \check{\cup}_e (\zeta_2, \xi_2, \rho_2, N)^{\check{c}}$.
3. $\left((\zeta_1, \xi_1, \rho_1, N) \check{\cup}_r (\zeta_2, \xi_2, \rho_2, N)\right)^{\check{c}} = (\zeta_1, \xi_1, \rho_1, N)^{\check{c}} \check{\cap}_r (\zeta_2, \xi_2, \rho_2, N)^{\check{c}}$.
4. $\left((\zeta_1, \xi_1, \rho_1, N) \check{\cap}_r (\zeta_2, \xi_2, \rho_2, N)\right)^{\check{c}} = (\zeta_1, \xi_1, \rho_1, N)^{\check{c}} \check{\cup}_r (\zeta_2, \xi_2, \rho_2, N)^{\check{c}}$.

Proof 1. Let $(\zeta_1, \xi_1, \rho_1, N) \check{\cup}_e (\zeta_2, \xi_2, \rho_2, N) = (\zeta_3, \xi_3, \rho_1 \cup \rho_2, N)$. Then, $\left((\zeta_1, \xi_1, \rho_1, N) \check{\cup}_e (\zeta_2, \xi_2, \rho_2, N)\right)^{\check{c}} = (\zeta_3, \xi_3, \rho_1 \cup \rho_2, N)^{\check{c}} = (\zeta_3^{\check{c}}, \xi_3^{\check{c}}, \rho_1 \cup \rho_2, N)$. For all $\varepsilon \in \rho_1 \cup \rho_2$:

$$\zeta_3(\varepsilon) = \begin{cases} \zeta_1(\varepsilon), & \text{if } \varepsilon \in \rho_1 \setminus \rho_2, \\ \zeta_2(\varepsilon), & \text{if } \varepsilon \in \rho_2 \setminus \rho_1, \\ \left\langle \begin{array}{l} (\ell, \max\{r_{1\varepsilon}, r_{2\varepsilon}\}), \\ \max\{\zeta_1^+(\ell, r_{1\varepsilon}), \zeta_2^+(\ell, r_{2\varepsilon})\}, \\ \min\{\zeta_1^-(\ell, r_{1\varepsilon}), \zeta_2^-(\ell, r_{2\varepsilon})\} \end{array} \right\rangle, & \text{if } \varepsilon \in \rho_1 \cap \rho_2. \end{cases}$$

where $\langle(\ell, r_{1\varepsilon}), \zeta_1^+(\ell, r_{1\varepsilon}), \zeta_1^-(\ell, r_{1\varepsilon})\rangle \in \zeta_1(\varepsilon)$ and $\langle(\ell, r_{2\varepsilon}), \zeta_2^+(\ell, r_{2\varepsilon}), \zeta_2^-(\ell, r_{2\varepsilon})\rangle \in \zeta_2(\varepsilon)$. Similarly, for all $\neg\varepsilon \in \neg\rho_1 \cup \neg\rho_2$:

$$\xi_3(\neg\varepsilon) = \begin{cases} \xi_1(\neg\varepsilon), & \text{if } \neg\varepsilon \in \neg\rho_1 \setminus \neg\rho_2, \\ \xi_2(\neg\varepsilon), & \text{if } \neg\varepsilon \in \neg\rho_2 \setminus \neg\rho_1, \\ \left\langle \begin{array}{l} (\ell, \min\{r_{1\neg\varepsilon}, r_{2\neg\varepsilon}\}), \\ \min\{\xi_1^+(\ell, r_{1\neg\varepsilon}), \xi_2^+(\ell, r_{2\neg\varepsilon})\}, \\ \max\{\xi_1^-(\ell, r_{1\neg\varepsilon}), \xi_2^-(\ell, r_{2\neg\varepsilon})\} \end{array} \right\rangle, & \text{if } \neg\varepsilon \in \neg\rho_1 \cap \neg\rho_2. \end{cases}$$

where $\langle(\ell, r_{1\neg\varepsilon}), \xi_1^+(\ell, r_{1\neg\varepsilon}), \xi_1^-(\ell, r_{1\neg\varepsilon})\rangle \in \xi_1(\neg\varepsilon)$ and $\langle(\ell, r_{2\neg\varepsilon}), \xi_2^+(\ell, r_{2\neg\varepsilon}), \xi_2^-(\ell, r_{2\neg\varepsilon})\rangle \in \xi_2(\neg\varepsilon)$. Then, for all $\varepsilon \in \rho_1 \cup \rho_2$:

$$\zeta_3^{\check{c}}(\varepsilon) = \xi_3(\neg\varepsilon) = \begin{cases} \xi_1(\neg\varepsilon), & \text{if } \varepsilon \in \rho_1 \setminus \rho_2, \\ \xi_2(\neg\varepsilon), & \text{if } \varepsilon \in \rho_2 \setminus \rho_1, \\ \left\langle \begin{array}{l} (\ell, \min\{r_{1\neg\varepsilon}, r_{2\neg\varepsilon}\}), \\ \min\{\xi_1^+(\ell, r_{1\neg\varepsilon}), \xi_2^+(\ell, r_{2\neg\varepsilon})\}, \\ \max\{\xi_1^-(\ell, r_{1\neg\varepsilon}), \xi_2^-(\ell, r_{2\neg\varepsilon})\} \end{array} \right\rangle, & \text{if } \varepsilon \in \rho_1 \cap \rho_2. \end{cases}$$

Similarly, for all $\neg\varepsilon \in \neg\rho_1 \cup \neg\rho_2$:

$$\xi_3^{\check{c}}(\neg\varepsilon) = \zeta_3(\varepsilon) = \begin{cases} \zeta_1(\varepsilon), & \text{if } \neg\varepsilon \in \neg\rho_1 \setminus \neg\rho_2, \\ \zeta_2(\varepsilon), & \text{if } \neg\varepsilon \in \neg\rho_2 \setminus \neg\rho_1, \\ \left\langle \begin{array}{l} (\ell, \max\{r_{1\varepsilon}, r_{2\varepsilon}\}), \\ \max\{\zeta_1^+(\ell, r_{1\varepsilon}), \zeta_2^+(\ell, r_{2\varepsilon})\}, \\ \min\{\zeta_1^-(\ell, r_{1\varepsilon}), \zeta_2^-(\ell, r_{2\varepsilon})\} \end{array} \right\rangle, & \text{if } \neg\varepsilon \in \neg\rho_1 \cap \neg\rho_2. \end{cases}$$

On the other hand, let $(\zeta_1, \xi_1, \rho_1, N)^{\check{c}} \check{\cap}_e (\zeta_2, \xi_2, \rho_2, N)^{\check{c}} = (\zeta_4, \xi_4, \rho_1 \cup \rho_2, N)$. For all $\varepsilon \in \rho_1 \cup \rho_2$:

$$\zeta_4(\varepsilon)=\begin{cases}\zeta_1^{\check{c}}(\varepsilon), & \text{if } \varepsilon\in\rho_1\setminus\rho_2,\\ \zeta_2^{\check{c}}(\varepsilon), & \text{if } \varepsilon\in\rho_2\setminus\rho_1,\\ \left\langle\begin{array}{l}(\ell,\min\{r_{1\varepsilon}^{\check{c}},r_{2\varepsilon}^{\check{c}}\}),\\ \min\left\{\zeta_1^{+\check{c}}(\ell,r_{1\varepsilon}^{\check{c}}),\zeta_2^{+\check{c}}(\ell,r_{2\varepsilon}^{\check{c}})\right\},\\ \max\left\{\zeta_1^{-\check{c}}(\ell,r_{1\varepsilon}^{\check{c}}),\zeta_2^{-\check{c}}(\ell,r_{2\varepsilon}^{\check{c}})\right\}\end{array}\right\rangle, & \text{if } \varepsilon\in\rho_1\cap\rho_2.\end{cases}$$

$$=\begin{cases}\xi_1(\neg\varepsilon), & \text{if } \varepsilon\in\rho_1\setminus\rho_2,\\ \xi_2(\neg\varepsilon), & \text{if } \varepsilon\in\rho_2\setminus\rho_1,\\ \left\langle\begin{array}{l}(\ell,\min\{r_{1\neg\varepsilon},r_{2\neg\varepsilon}\}),\\ \min\left\{\xi_1^{+}(\ell,r_{1\neg\varepsilon}),\xi_2^{+}(\ell,r_{2\neg\varepsilon})\right\},\\ \max\left\{\xi_1^{-}(\ell,r_{1\neg\varepsilon}),\xi_2^{-}(\ell,r_{2\neg\varepsilon})\right\}\end{array}\right\rangle, & \text{if } \varepsilon\in\rho_1\cap\rho_2.\end{cases}$$

Similarly, for all $\neg\varepsilon\in\neg\rho_1\cup\neg\rho_2$:

$$\xi_4(\neg\varepsilon)=\begin{cases}\xi_1^{\check{c}}(\neg\varepsilon), & \text{if } \neg\varepsilon\in\neg\rho_1\setminus\neg\rho_2,\\ \xi_2^{\check{c}}(\neg\varepsilon), & \text{if } \neg\varepsilon\in\neg\rho_2\setminus\neg\rho_1,\\ \left\langle\begin{array}{l}(\ell,\max\{r_{1\neg\varepsilon}^{\check{c}},r_{2\neg\varepsilon}^{\check{c}}\}),\\ \max\left\{\xi_1^{+\check{c}}(\ell,r_{1\neg\varepsilon}^{\check{c}}),\xi_2^{+\check{c}}(\ell,r_{2\neg\varepsilon}^{\check{c}})\right\},\\ \min\left\{\xi_1^{-\check{c}}(\ell,r_{1\neg\varepsilon}^{\check{c}}),\xi_2^{-\check{c}}(\ell,r_{2\neg\varepsilon}^{\check{c}})\right\}\end{array}\right\rangle, & \text{if } \neg\varepsilon\in\neg\rho_1\cap\neg\rho_2.\end{cases}$$

$$=\begin{cases}\zeta_1(\varepsilon), & \text{if } \neg\varepsilon\in\neg\rho_1\setminus\neg\rho_2,\\ \zeta_2(\varepsilon), & \text{if } \neg\varepsilon\in\neg\rho_2\setminus\neg\rho_1,\\ \left\langle\begin{array}{l}(\ell,\max\{r_{1\varepsilon},r_{2\varepsilon}\}),\\ \max\left\{\zeta_1^{+}(\ell,r_{1\varepsilon}),\zeta_2^{+}(\ell,r_{2\varepsilon})\right\},\\ \min\left\{\zeta_1^{-}(\ell,r_{1\varepsilon}),\zeta_2^{-}(\ell,r_{2\varepsilon})\right\}\end{array}\right\rangle, & \text{if } \neg\varepsilon\in\neg\rho_1\cap\neg\rho_2.\end{cases}$$

Since $(\zeta_3,\xi_3,\rho_1\cup\rho_2,N)^{\check{c}}$ and $(\zeta_4,\xi_4,\rho_1\cup\rho_2,N)$ are equivalent for all $\varepsilon\in\rho_1\cup\rho_2$ and $\neg\varepsilon\in\neg\rho_1\cup\neg\rho_2$, the proof follows.

The other parts can be illustrated in the same way. □

Proposition 3.8 Let (ζ_1,ξ_1,ρ_1,N) and (ζ_2,ξ_2,ρ_2,N) be two PFNBSSs. Then

1. $(\zeta_1,\xi_1,\rho_1,N)\,\check{\cup}_e\left((\zeta_1,\xi_1,\rho_1,N)\,\check{\cap}_r\,(\zeta_2,\xi_2,\rho_2,N)\right)=(\zeta_1,\xi_1,\rho_1,N)$.
2. $(\zeta_1,\xi_1,\rho_1,N)\,\check{\cap}_e\left((\zeta_1,\xi_1,\rho_1,N)\,\check{\cup}_r\,(\zeta_2,\xi_2,\rho_2,N)\right)=(\zeta_1,\xi_1,\rho_1,N)$.
3. $(\zeta_1,\xi_1,\rho_1,N)\,\check{\cup}_r\left((\zeta_1,\xi_1,\rho_1,N)\,\check{\cap}_e\,(\zeta_2,\xi_2,\rho_2,N)\right)=(\zeta_1,\xi_1,\rho_1,N)$.
4. $(\zeta_1,\xi_1,\rho_1,N)\,\check{\cap}_r\left((\zeta_1,\xi_1,\rho_1,N)\,\check{\cup}_e\,(\zeta_2,\xi_2,\rho_2,N)\right)=(\zeta_1,\xi_1,\rho_1,N)$.

Proof 1. Suppose that $(\zeta_1,\xi_1,\rho_1,N)\,\check{\cap}_r\,(\zeta_2,\xi_2,\rho_2,N)=(\zeta_3,\xi_3,\rho_1\cap\rho_2,N)$. Then, for all $\varepsilon\in\rho_1\cap\rho_2$:

$$\zeta_3(\varepsilon)=\langle(\ell,\min\{r_{1\varepsilon},r_{2\varepsilon}\}),\min\{\zeta_1^{+}(\ell,r_{1\varepsilon}),\zeta_2^{+}(\ell,r_{2\varepsilon})\},\max\{\zeta_1^{-}(\ell,r_{1\varepsilon}),\zeta_2^{-}(\ell,r_{2\varepsilon})\}\rangle,$$

where $\langle(\ell,r_{1\varepsilon}),\zeta_1^{+}(\ell,r_{1\varepsilon}),\zeta_1^{-}(\ell,r_{1\varepsilon})\rangle\in\zeta_1(\varepsilon)$ and $\langle(\ell,r_{2\varepsilon}),\zeta_2^{+}(\ell,r_{2\varepsilon}),\zeta_2^{-}(\ell,r_{2\varepsilon})\rangle\in\zeta_2(\varepsilon)$. Similarly, for all $\neg\varepsilon\in\neg\rho_1\cup\neg\rho_2$:

$$\xi_3(\neg\varepsilon)=\langle(\ell,\max\{r_{1\neg\varepsilon},r_{2\neg\varepsilon}\}),\max\{\xi_1^{+}(\ell,r_{1\neg\varepsilon}),\xi_2^{+}(\ell,r_{2\neg\varepsilon})\},\min\{\xi_1^{-}(\ell,r_{1\neg\varepsilon}),\xi_2^{-}(\ell,r_{2\neg\varepsilon})\}\})\rangle,$$

where $\langle(\ell,r_{1\neg\varepsilon}),\xi_1^{+}(\ell,r_{1\neg\varepsilon}),\xi_1^{-}(\ell,r_{1\neg\varepsilon})\rangle\in\xi_1(\neg\varepsilon)$ and $\langle(\ell,r_{2\neg\varepsilon}),\xi_2^{+}(\ell,r_{2\neg\varepsilon}),\xi_2^{-}(\ell,r_{2\neg\varepsilon})\rangle\in\xi_2(\neg\varepsilon)$. Now, let $(\zeta_1,\xi_1,\rho_1,N)\,\check{\cup}_e\,(\zeta_3,\xi_3,\rho_1\cap\rho_2,N)=(\zeta_4,\xi_4,\rho_1\cup(\rho_1\cap\rho_2),N)=(\zeta_4,\xi_4,\rho_1,N)$. Then, for all $\varepsilon\in\rho_1\cup(\rho_1\cap\rho_2)$:

$$\zeta_4(\varepsilon)=\begin{cases}\zeta_1(\varepsilon), & \text{if } \varepsilon\in\rho_1\setminus(\rho_1\cap\rho_2),\\ \zeta_3(\varepsilon), & \text{if } \varepsilon\in(\rho_1\cap\rho_2)\setminus\rho_1=\emptyset,\\ \left\langle\begin{array}{l}(\ell,\max\{r_{1\varepsilon},r_{3\varepsilon}\}),\\ \max\left\{\zeta_1^{+}(\ell,r_{1\varepsilon}),\zeta_3^{+}(\ell,r_{3\varepsilon})\right\},\\ \min\left\{\zeta_1^{-}(\ell,r_{1\varepsilon}),\zeta_3^{-}(\ell,r_{3\varepsilon})\right\}\end{array}\right\rangle, & \text{if } \varepsilon\in\rho_1\cap(\rho_1\cap\rho_2).\end{cases}$$

$$=\begin{cases}\zeta_1(\varepsilon), & \text{if } \varepsilon\in\rho_1\setminus(\rho_1\cap\rho_2),\\ \left\langle\begin{array}{l}(\ell,\max\{r_{1\varepsilon},\min\{r_{1\varepsilon},r_{2\varepsilon}\}\}),\\ \max\left\{\zeta_1^{+}(\ell,r_{1\varepsilon}),\min\left\{\zeta_1^{+}(\ell,r_{1\varepsilon}),\zeta_2^{+}(\ell,r_{2\varepsilon})\right\}\right\},\\ \min\left\{\zeta_1^{-}(\ell,r_{1\varepsilon}),\max\left\{\zeta_1^{-}(\ell,r_{1\varepsilon}),\zeta_2^{-}(\ell,r_{2\varepsilon})\right\}\right\}\end{array}\right\rangle, & \text{if } \varepsilon\in\rho_1\cap(\rho_1\cap\rho_2).\end{cases}$$

where $\langle(\ell,r_{3\varepsilon}),\zeta_3^{+}(\ell,r_{3\varepsilon}),\zeta_3^{-}(\ell,r_{3\varepsilon})\rangle\in\zeta_3(\varepsilon)$. Similarly, for all $\neg\varepsilon\in\neg\rho_1\cup(\neg\rho_1\cap\neg\rho_2)$:

$$\xi_4(\neg\varepsilon)=\begin{cases}\xi_1(\neg\varepsilon), & \text{if } \neg\varepsilon\in\neg\rho_1\setminus(\neg\rho_1\cap\neg\rho_2),\\ \xi_3(\neg\varepsilon), & \text{if } \neg\varepsilon\in(\neg\rho_1\cap\neg\rho_2)\setminus\neg\rho_1=\emptyset,\\ \left\langle\begin{array}{l}(\ell,\min\{r_{1\neg\varepsilon},r_{3\neg\varepsilon}\}),\\ \min\left\{\xi_1^+(\ell,r_{1\neg\varepsilon}),\xi_3^+(\ell,r_{3\neg\varepsilon})\right\},\\ \max\left\{\xi_1^-(\ell,r_{1\neg\varepsilon}),\xi_3^-(\ell,r_{3\neg\varepsilon})\right\}\end{array}\right\rangle, & \text{if } \neg\varepsilon\in\neg\rho_1\cap(\neg\rho_1\cap\neg\rho_2).\end{cases}$$

$$=\begin{cases}\xi_1(\neg\varepsilon), & \text{if } \neg\varepsilon\in\neg\rho_1\setminus(\neg\rho_1\cap\neg\rho_2),\\ \left\langle\begin{array}{l}(\ell,\min\{r_{1\neg\varepsilon},\max\{r_{1\neg\varepsilon},r_{2\neg\varepsilon}\}\}),\\ \min\left\{\xi_1^+(\ell,r_{1\neg\varepsilon}),\max\left\{\xi_1^+(\ell,r_{1\neg\varepsilon}),\xi_2^+(\ell,r_{2\neg\varepsilon})\right\}\right\},\\ \max\left\{\xi_1^-(\ell,r_{1\neg\varepsilon}),\min\left\{\xi_1^-(\ell,r_{1\neg\varepsilon}),\xi_2^-(\ell,r_{2\neg\varepsilon})\right\}\right\}\end{array}\right\rangle, & \text{if } \neg\varepsilon\in\neg\rho_1\cap(\neg\rho_1\cap\neg\rho_2).\end{cases}$$

where $\langle(\ell,r_{3\neg\varepsilon}),\xi_3^+(\ell,r_{3\neg\varepsilon}),\xi_3^-(\ell,r_{3\neg\varepsilon})\rangle\in\xi_3(\neg\varepsilon)$. Hence,

$$\zeta_4(\varepsilon)=\begin{cases}\zeta_1(\varepsilon), & \text{if } \varepsilon\in\rho_1\setminus(\rho_1\cap\rho_2)\\ \zeta_1(\varepsilon), & \text{if } \varepsilon\in\rho_1\cap\rho_2\end{cases}$$

and

$$\xi_4(\neg\varepsilon)=\begin{cases}\xi_1(\neg\varepsilon), & \text{if } \neg\varepsilon\in\neg\rho_1\setminus(\neg\rho_1\cap\neg\rho_2)\\ \xi_1(\neg\varepsilon), & \text{if } \neg\varepsilon\in\neg\rho_1\cap\neg\rho_2.\end{cases}$$

Therefore, $(\zeta_1,\xi_1,\rho_1,N)\,\check{\cup}_e\,\big((\zeta_1,\xi_1,\rho_1,N)\,\check{\cap}_r\,(\zeta_2,\xi_2,\rho_2,N)\big)=(\zeta_1,\xi_1,\rho_1,N)$.

The other parts can be illustrated in the same way. □

Proposition 3.9 Let $(\zeta_1,\xi_1,\rho_1,N_1)$, $(\zeta_2,\xi_2,\rho_2,N_2)$, and $(\zeta_3,\xi_3,\rho_3,N_3)$ be PFN_1BSS, PFN_2BSS, and PFN_3 BSS, respectively. Then,

1. $(\zeta_1,\xi_1,\rho_1,N_1)\,\check{\cup}_e\,\big((\zeta_2,\xi_2,\rho_2,N_2)\,\check{\cap}_r\,(\zeta_3,\xi_3,\rho_3,N_3)\big)=\big((\zeta_1,\xi_1,\rho_1,N_1)\,\check{\cup}_e\,(\zeta_2,\xi_2,\rho_2,N_2)\big)\,\check{\cap}_r\,\big((\zeta_1,\xi_1,\rho_1,N_1)\,\check{\cup}_e\,(\zeta_3,\xi_3,\rho_3,N_3)\big)$.
2. $(\zeta_1,\xi_1,\rho_1,N_1)\,\check{\cap}_e\,\big((\zeta_2,\xi_2,\rho_2,N_2)\,\check{\cup}_r\,(\zeta_3,\xi_3,\rho_3,N_3)\big)=\big((\zeta_1,\xi_1,\rho_1,N_1)\,\check{\cap}_e\,(\zeta_2,\xi_2,\rho_2,N_2)\big)\,\check{\cup}_r\,\big((\zeta_1,\xi_1,\rho_1,N_1)\,\check{\cap}_e\,(\zeta_3,\xi_3,\rho_3,N_3)\big)$.
3. $(\zeta_1,\xi_1,\rho_1,N_1)\,\check{\cup}_r\,\big((\zeta_2,\xi_2,\rho_2,N_2)\,\check{\cap}_e\,(\zeta_3,\xi_3,\rho_3,N_3)\big)=\big((\zeta_1,\xi_1,\rho_1,N_1)\,\check{\cup}_r\,(\zeta_2,\xi_2,\rho_2,N_2)\big)\,\check{\cap}_e\,\big((\zeta_1,\xi_1,\rho_1,N_1)\,\check{\cup}_r\,(\zeta_3,\xi_3,\rho_3,N_3)\big)$.
4. $(\zeta_1,\xi_1,\rho_1,N_1)\,\check{\cap}_r\,\big((\zeta_2,\xi_2,\rho_2,N_2)\,\check{\cup}_e\,(\zeta_3,\xi_3,\rho_3,N_3)\big)=\big((\zeta_1,\xi_1,\rho_1,N_1)\,\check{\cap}_r\,(\zeta_2,\xi_2,\rho_2,N_2)\big)\,\check{\cup}_e\,\big((\zeta_1,\xi_1,\rho_1,N_1)\,\check{\cap}_r\,(\zeta_3,\xi_3,\rho_3,N_3)\big)$.
5. $(\zeta_1,\xi_1,\rho_1,N_1)\,\check{\cup}_r\,\big((\zeta_2,\xi_2,\rho_2,N_2)\,\check{\cap}_r\,(\zeta_3,\xi_3,\rho_3,N_3)\big)=\big((\zeta_1,\xi_1,\rho_1,N_1)\,\check{\cup}_r\,(\zeta_2,\xi_2,\rho_2,N_2)\big)\,\check{\cap}_r\,\big((\zeta_1,\xi_1,\rho_1,N_1)\,\check{\cup}_r\,(\zeta_3,\xi_3,\rho_3,N_3)\big)$.
6. $(\zeta_1,\xi_1,\rho_1,N_1)\,\check{\cap}_r\,\big((\zeta_2,\xi_2,\rho_2,N_2)\,\check{\cup}_r\,(\zeta_3,\xi_3,\rho_3,N_3)\big)=\big((\zeta_1,\xi_1,\rho_1,N_1)\,\check{\cap}_r\,(\zeta_2,\xi_2,\rho_2,N_2)\big)\,\check{\cup}_r\,\big((\zeta_1,\xi_1,\rho_1,N_1)\,\check{\cap}_r\,(\zeta_3,\xi_3,\rho_3,N_3)\big)$.

Proof 1. 3. Suppose that $\big((\zeta_2,\xi_2,\rho_2,N_2)\,\check{\cap}_e\,(\zeta_3,\xi_3,\rho_3,N_3)\big)=(\zeta_4,\xi_4,\rho_2\cup\rho_3,\max(N_2,N_3))$. Then, for all $\varepsilon\in\rho_2\cup\rho_3$:

$$\zeta_4(\varepsilon)=\begin{cases}\zeta_2(\varepsilon), & \text{if } \varepsilon\in\rho_2\setminus\rho_3,\\ \zeta_3(\varepsilon), & \text{if } \varepsilon\in\rho_3\setminus\rho_2,\\ \left\langle\begin{array}{l}(\ell,\min\{r_{2\varepsilon},r_{3\varepsilon}\}),\\ \min\left\{\zeta_2^+(\ell,r_{2\varepsilon}),\zeta_3^+(\ell,r_{3\varepsilon})\right\},\\ \max\left\{\zeta_2^-(\ell,r_{2\varepsilon}),\zeta_3^-(\ell,r_{3\varepsilon})\right\}\end{array}\right\rangle, & \text{if } \varepsilon\in\rho_2\cap\rho_3.\end{cases}$$

where $\langle(\ell,r_{2\varepsilon}),\zeta_2^+(\ell,r_{2\varepsilon}),\zeta_2^-(\ell,r_{2\varepsilon})\rangle\in\zeta_2(\varepsilon)$ and $\langle(\ell,r_{3\varepsilon}),\zeta_3^+(\ell,r_{3\varepsilon}),\zeta_3^-(\ell,r_{3\varepsilon})\rangle\in\zeta_3(\varepsilon)$. Similarly, for all $\neg\varepsilon\in\neg\rho_2\cup\neg\rho_3$:

$$\xi_4(\neg\varepsilon)=\begin{cases}\xi_2(\neg\varepsilon), & \text{if } \neg\varepsilon\in\neg\rho_2\setminus\neg\rho_3,\\ \xi_3(\neg\varepsilon), & \text{if } \neg\varepsilon\in\neg\rho_3\setminus\neg\rho_2,\\ \left\langle\begin{array}{l}(\ell,\max\{r_{2\neg\varepsilon},r_{3\neg\varepsilon}\}),\\ \max\left\{\xi_2^+(\ell,r_{2\neg\varepsilon}),\xi_3^+(\ell,r_{3\neg\varepsilon})\right\},\\ \min\left\{\xi_2^-(\ell,r_{2\neg\varepsilon}),\xi_3^-(\ell,r_{3\neg\varepsilon})\right\}\end{array}\right\rangle, & \text{if } \neg\varepsilon\in\neg\rho_2\cap\neg\rho_3.\end{cases}$$

where $\langle(\ell,r_{2\neg\varepsilon}),\xi_2^+(\ell,r_{2\neg\varepsilon}),\xi_2^-(\ell,r_{2\neg\varepsilon})\rangle\in\xi_2(\neg\varepsilon)$ and $\langle(\ell,r_{3\neg\varepsilon}),\xi_3^+(\ell,r_{3\neg\varepsilon}),\xi_3^-(\ell,r_{3\neg\varepsilon})\rangle\in\xi_3(\neg\varepsilon)$.

Let $(\zeta_1, \xi_1, \rho_1, N_1) \check{\cup}_r (\zeta_4, \xi_4, \rho_2 \cup \rho_3, \max(N_2, N_3)) = (\zeta_5, \xi_5, \rho_1 \cap (\rho_2 \cup \rho_3), \max(N_1, \max(N_2, N_3)) = (\zeta_5, \xi_5, \lambda_1 \cup \lambda_2, \max(N_1, N_2, N_3))$ where $\lambda_1 = \rho_1 \cap \rho_2$ and $\lambda_2 = \rho_1 \cap \rho_3$. Then, for all $\varepsilon \in \lambda_1 \cup \lambda_2$:

$$\zeta_5(\varepsilon) = \langle(\ell, \max\{r_{1\varepsilon}, r_{4\varepsilon}\}), \max\{\zeta_1^+(\ell, r_{1\varepsilon}), \zeta_4^+(\ell, r_{4\varepsilon})\}, \min\{\zeta_1^-(\ell, r_{1\varepsilon}), \zeta_4^-(\ell, r_{4\varepsilon})\}\rangle,$$

where $\langle(\ell, r_{1\varepsilon}), \zeta_1^+(\ell, r_{1\varepsilon}), \zeta_1^-(\ell, r_{1\varepsilon})\rangle \in \zeta_1(\varepsilon)$ and $\langle(\ell, r_{4\varepsilon}), \zeta_4^+(\ell, r_{4\varepsilon}), \zeta_4^-(\ell, r_{4\varepsilon})\rangle \in \zeta_4(\varepsilon)$.
Similarly, for all $\neg\varepsilon \in \neg\lambda_1 \cup \neg\lambda_2$:

$$\xi_5(\neg\varepsilon) = \langle(\ell, \min\{r_{1\neg\varepsilon}, r_{4\neg\varepsilon}\}), \min\{\xi_1^+(\ell, r_{1\neg\varepsilon}), \xi_4^+(\ell, r_{4\neg\varepsilon})\}, \max\{\xi_1^-(\ell, r_{1\neg\varepsilon}), \xi_4^-(\ell, r_{4\neg\varepsilon})\}\rangle,$$

where $\langle(\ell, r_{1\neg\varepsilon}), \xi_1^+(\ell, r_{1\neg\varepsilon}), \xi_1^-(\ell, r_{1\neg\varepsilon})\rangle \in \xi_1(\neg\varepsilon)$ and $\langle(\ell, r_{4\neg\varepsilon}), \xi_4^+(\ell, r_{4\neg\varepsilon}), \xi_4^-(\ell, r_{4\neg\varepsilon})\rangle \in \xi_4(\neg\varepsilon)$.
Hence, for all $\varepsilon \in \lambda_1 \cup \lambda_2$:

$$\zeta_5(\varepsilon) = \begin{cases} \left\langle \begin{array}{l} (\ell, \max\{r_{1\varepsilon}, r_{2\varepsilon}\}), \\ \max\left\{\zeta_1^+(\ell, r_{1\varepsilon}), \zeta_2^+(\ell, r_{2\varepsilon})\right\}, \\ \min\left\{\zeta_1^-(\ell, r_{1\varepsilon}), \zeta_2^-(\ell, r_{2\varepsilon})\right\} \end{array} \right\rangle, & \text{if } \varepsilon \in \lambda_1 \setminus \lambda_2, \\ \left\langle \begin{array}{l} (\ell, \max\{r_{1\varepsilon}, r_{3\varepsilon}\}), \\ \max\left\{\zeta_1^+(\ell, r_{1\varepsilon}), \zeta_3^+(\ell, r_{3\varepsilon})\right\}, \\ \min\left\{\zeta_1^-(\ell, r_{1\varepsilon}), \zeta_3^-(\ell, r_{3\varepsilon})\right\} \end{array} \right\rangle, & \text{if } \varepsilon \in \lambda_2 \setminus \lambda_1, \\ \left\langle \begin{array}{l} (\ell, \max\{r_{1\varepsilon}, \min\{r_{2\varepsilon}, r_{3\varepsilon}\}\}), \\ \max\left\{\zeta_1^+(\ell, r_{1\varepsilon}), \min\left\{\zeta_2^+(\ell, r_{2\varepsilon}), \zeta_3^+(\ell, r_{3\varepsilon})\right\}\right\}, \\ \min\left\{\zeta_1^-(\ell, r_{1\varepsilon}), \max\left\{\zeta_2^-(\ell, r_{2\varepsilon}), \zeta_3^-(\ell, r_{3\varepsilon})\right\}\right\} \end{array} \right\rangle, & \text{if } \varepsilon \in \lambda_1 \cap \lambda_2. \end{cases}$$

Similarly, for all $\neg\varepsilon \in \neg\lambda_1 \cup \neg\lambda_2$:

$$\xi_5(\neg\varepsilon) = \begin{cases} \left\langle \begin{array}{l} (\ell, \min\{r_{1\neg\varepsilon}, r_{2\neg\varepsilon}\}), \\ \min\left\{\xi_1^+(\ell, r_{1\neg\varepsilon}), \xi_2^+(\ell, r_{2\neg\varepsilon})\right\}, \\ \max\left\{\xi_1^-(\ell, r_{1\neg\varepsilon}), \xi_2^-(\ell, r_{2\neg\varepsilon})\right\} \end{array} \right\rangle, & \text{if } \neg\varepsilon \in \neg\lambda_1 \setminus \neg\lambda_2, \\ \left\langle \begin{array}{l} (\ell, \min\{r_{1\neg\varepsilon}, r_{3\neg\varepsilon}\}), \\ \min\left\{\xi_1^+(\ell, r_{1\neg\varepsilon}), \xi_3^+(\ell, r_{3\neg\varepsilon})\right\}, \\ \max\left\{\xi_1^-(\ell, r_{1\neg\varepsilon}), \xi_3^-(\ell, r_{3\neg\varepsilon})\right\} \end{array} \right\rangle, & \text{if } \neg\varepsilon \in \neg\lambda_2 \setminus \neg\lambda_1, \\ \left\langle \begin{array}{l} \langle(\ell, \min\{r_{1\neg\varepsilon}, \max\{r_{2\neg\varepsilon}, r_{3\neg\varepsilon}\}\}), \\ \min\left\{\xi_1^+(\ell, r_{1\neg\varepsilon}), \max\left\{\xi_2^+(\ell, r_{2\neg\varepsilon}), \xi_3^+(\ell, r_{3\neg\varepsilon})\right\}\right\}, \\ \max\left\{\xi_1^-(\ell, r_{1\neg\varepsilon}), \min\left\{\xi_2^-(\ell, r_{2\neg\varepsilon}), \xi_3^-(\ell, r_{3\neg\varepsilon})\right\}\right\}\rangle \end{array} \right\rangle, & \text{if } \neg\varepsilon \in \neg\lambda_1 \cap \neg\lambda_2. \end{cases}$$

On the other hand, let $(\zeta_1, \xi_1, \rho_1, N_1) \check{\cup}_r (\zeta_2, \xi_2, \rho_2, N_2) = (\zeta_6, \xi_6, \rho_1 \cap \rho_2, \max(N_1, N_2))$. Then, for all $\varepsilon \in \rho_1 \cap \rho_2$:

$$\zeta_6(\varepsilon) = \langle(\ell, \max\{r_{1\varepsilon}, r_{2\varepsilon}\}), \max\{\zeta_1^+(\ell, r_{1\varepsilon}), \zeta_2^+(\ell, r_{2\varepsilon})\}, \min\{\zeta_1^-(\ell, r_{1\varepsilon}), \zeta_2^-(\ell, r_{2\varepsilon})\}\rangle.$$

Similarly, for all $\neg\varepsilon \in \neg\lambda_1 \cup \neg\lambda_2$:

$$\xi_6(\neg\varepsilon) = \langle(\ell, \min\{r_{1\neg\varepsilon}, r_{2\neg\varepsilon}\}), \min\{\xi_1^+(\ell, r_{1\neg\varepsilon}), \xi_2^+(\ell, r_{2\neg\varepsilon})\}, \max\{\xi_1^-(\ell, r_{1\neg\varepsilon}), \xi_2^-(\ell, r_{2\neg\varepsilon})\}\rangle.$$

Next, let $(\zeta_1, \xi_1, \rho_1, N_1) \check{\cup}_r (\zeta_3, \xi_3, \rho_3, N_3) = (\zeta_7, \xi_7, \rho_1 \cap \rho_2, \max(N_1, N_2))$. Then, for all $\varepsilon \in \rho_1 \cap \rho_3$:

$$\zeta_7(\varepsilon) = \langle(\ell, \max\{r_{1\varepsilon}, r_{3\varepsilon}\}), \max\{\zeta_1^+(\ell, r_{1\varepsilon}), \zeta_3^+(\ell, r_{3\varepsilon})\}, \min\{\zeta_1^-(\ell, r_{1\varepsilon}), \zeta_3^-(\ell, r_{3\varepsilon})\}\rangle.$$

Similarly, for all $\neg\varepsilon \in \neg\lambda_1 \cup \neg\lambda_2$:

$$\xi_7(\neg\varepsilon)(\ell) = \langle(\ell, \min\{r_{1\neg\varepsilon}, r_{3\neg\varepsilon}\}), \min\{\xi_1^+(\ell, r_{1\neg\varepsilon}), \xi_3^+(\ell, r_{3\neg\varepsilon})\}, \max\{\xi_1^-(\ell, r_{1\neg\varepsilon}), \xi_3^-(\ell, r_{3\neg\varepsilon})\}\rangle.$$

Now, suppose that $(\zeta_6, \xi_6, \rho_1 \cap \rho_2, \max(N_1, N_2)) \check{\cap}_e (\zeta_7, \xi_7, \rho_1 \cap \rho_3, \max(N_1, N_3)) = (\zeta_8, \xi_8, \lambda_1 \cup \lambda_2, \max(N_1, N_2, N_3))$ where $\lambda_1 = \rho_1 \cap \rho_2$ and $\lambda_2 = \rho_1 \cap \rho_3$. Then, for all $\varepsilon \in \lambda_1 \cup \lambda_2$:

$$\zeta_8(\varepsilon)=\begin{cases}\zeta_6(\varepsilon), & \text{if } \varepsilon\in\lambda_1\setminus\lambda_2,\\ \zeta_7(\varepsilon), & \text{if } \varepsilon\in\lambda_2\setminus\lambda_1,\\ \left\langle\begin{array}{l}(\ell,\min\{r_{6\varepsilon},r_{7\varepsilon}\}),\\ \min\left\{\zeta_6^+(\ell,r_{6\varepsilon}),\zeta_7^+(\ell,r_{7\varepsilon})\right\},\\ \max\left\{\zeta_6^-(\ell,r_{6\varepsilon}),\zeta_7^-(\ell,r_{7\varepsilon})\right\}\end{array}\right\rangle, & \text{if } \varepsilon\in\lambda_1\cap\lambda_2.\end{cases}$$

$$=\begin{cases}\left\langle\begin{array}{l}(\ell,\max\{r_{1\varepsilon},r_{2\varepsilon}\}),\\ \max\left\{\zeta_1^+(\ell,r_{1\varepsilon}),\zeta_2^+(\ell,r_{2\varepsilon})\right\},\\ \min\left\{\zeta_1^-(\ell,r_{1\varepsilon}),\zeta_2^-(\ell,r_{2\varepsilon})\right\}\end{array}\right\rangle, & \text{if } \varepsilon\in\lambda_1\setminus\lambda_2,\\ \left\langle\begin{array}{l}(\ell,\max\{r_{1\varepsilon},r_{3\varepsilon}\}),\\ \max\left\{\zeta_1^+(\ell,r_{1\varepsilon}),\zeta_3^+(\ell,r_{3\varepsilon})\right\},\\ \min\left\{\zeta_1^-(\ell,r_{1\varepsilon}),\zeta_3^-(\ell,r_{3\varepsilon})\right\}\end{array}\right\rangle, & \text{if } \varepsilon\in\lambda_2\setminus\lambda_1,\\ \left\langle\begin{array}{l}(\ell,\min\{\max\{r_{1\varepsilon},r_{2\varepsilon}\},\max\{r_{1\varepsilon},r_{3\varepsilon}\}\}),\\ \min\left\{\max\left\{\zeta_1^+(\ell,r_{1\varepsilon}),\zeta_2^+(\ell,r_{2\varepsilon})\right\},\max\left\{\zeta_1^+(\ell,r_{1\varepsilon}),\zeta_3^+(\ell,r_{3\varepsilon})\right\}\right\},\\ \max\left\{\min\left\{\zeta_1^-(\ell,r_{1\varepsilon}),\zeta_2^-(\ell,r_{2\varepsilon})\right\},\min\left\{\zeta_1^-(\ell,r_{1\varepsilon}),\zeta_3^-(\ell,r_{3\varepsilon})\right\}\right\}\end{array}\right\rangle, & \text{if } \varepsilon\in\lambda_1\cap\lambda_2.\end{cases}$$

where $\langle(\ell,r_{6\varepsilon}),\zeta_6^+(\ell,r_{6\varepsilon}),\zeta_6^-(\ell,r_{6\varepsilon})\rangle\in\zeta_6(\varepsilon)$ and $\langle(\ell,r_{7\varepsilon}),\zeta_7^+(\ell,r_{7\varepsilon}),\zeta_7^-(\ell,r_{7\varepsilon})\rangle\in\zeta_7(\varepsilon)$. Similarly, for all $\neg\varepsilon\in\neg\lambda_1\cup\neg\lambda_2$:

$$\xi_8(\neg\varepsilon)=\begin{cases}\xi_6(\neg\varepsilon), & \text{if } \neg\varepsilon\in\neg\lambda_1\setminus\neg\lambda_2,\\ \xi_7(\neg\varepsilon), & \text{if } \neg\varepsilon\in\neg\lambda_2\setminus\neg\lambda_1,\\ \left\langle\begin{array}{l}(\ell,\max\{r_{6\neg\varepsilon},r_{7\neg\varepsilon}\}),\\ \max\left\{\xi_6^+(\ell,r_{6\neg\varepsilon}),\xi_7^+(\ell,r_{7\neg\varepsilon})\right\},\\ \min\left\{\xi_6^-(\ell,r_{6\neg\varepsilon}),\xi_7^-(\ell,r_{7\neg\varepsilon})\right\}\end{array}\right\rangle, & \text{if } \neg\varepsilon\in\neg\lambda_1\cap\neg\lambda_2.\end{cases}$$

$$=\begin{cases}\left\langle\begin{array}{l}(\ell,\min\{r_{1\neg\varepsilon},r_{2\neg\varepsilon}\}),\\ \min\left\{\xi_1^+(\ell,r_{1\neg\varepsilon}),\xi_2^+(\ell,r_{2\neg\varepsilon})\right\},\\ \max\left\{\xi_1^-(\ell,r_{1\neg\varepsilon}),\xi_2^-(\ell,r_{2\neg\varepsilon})\right\}\end{array}\right\rangle, & \text{if } \neg\varepsilon\in\neg\lambda_1\setminus\neg\lambda_2,\\ \left\langle\begin{array}{l}(\ell,\min\{r_{1\neg\varepsilon},r_{3\neg\varepsilon}\}),\\ \min\left\{\xi_1^+(\ell,r_{1\neg\varepsilon}),\xi_3^+(\ell,r_{3\neg\varepsilon})\right\},\\ \max\left\{\xi_1^-(\ell,r_{1\neg\varepsilon}),\xi_3^-(\ell,r_{3\neg\varepsilon})\right\}\end{array}\right\rangle, & \text{if } \neg\varepsilon\in\neg\lambda_2\setminus\neg\lambda_1,\\ \left\langle\begin{array}{l}(\ell,\max\{\min\{r_{1\neg\varepsilon},r_{2\neg\varepsilon}\},\min\{r_{1\neg\varepsilon},r_{3\neg\varepsilon}\}\}),\\ \max\left\{\min\left\{\xi_1^+(\ell,r_{1\neg\varepsilon}),\xi_2^+(\ell,r_{2\neg\varepsilon})\right\},\min\left\{\xi_1^+(\ell,r_{1\neg\varepsilon}),\xi_3^+(\ell,r_{3\neg\varepsilon})\right\}\right\},\\ \min\left\{\max\left\{\xi_1^-(\ell,r_{1\neg\varepsilon}),\xi_2^-(\ell,r_{2\neg\varepsilon})\right\},\max\left\{\xi_1^-(\ell,r_{1\neg\varepsilon}),\xi_3^-(\ell,r_{3\neg\varepsilon})\right\}\right\}\end{array}\right\rangle, & \text{if } \neg\varepsilon\in\neg\lambda_1\cap\neg\lambda_2.\end{cases}$$

where $\langle(\ell,r_{6\neg\varepsilon}),\xi_6^+(\ell,r_{6\neg\varepsilon}),\xi_6^-(\ell,r_{6\neg\varepsilon})\rangle\in\xi_6(\neg\varepsilon)$ and $\langle(\ell,r_{7\neg\varepsilon}),\xi_7^+(\ell,r_{7\neg\varepsilon}),\xi_7^-(\ell,r_{7\neg\varepsilon})\rangle\in\xi_7(\neg\varepsilon)$. Since $(\zeta_5,\xi_5,\lambda_1\cup\lambda_1,\max(N_1,N_2,N_3))$ and $(\zeta_8,\xi_8,\lambda_1\cup\lambda_2,\max(N_1,N_2,N_3))$ are equivalent for all $\varepsilon\in\lambda_1\cup\lambda_2$ and $\neg\varepsilon\in\neg\lambda_1\cup\neg\lambda_2$, the proof follows.

The other parts can be illustrated in the same way. □

Decision-making framework and application

This section presents a comprehensive overview of the DM framework based on the proposed PFNBSS model and demonstrates its application in sustainability evaluation within the manufacturing industry. The framework systematically evaluates alternatives by accounting for both positive sustainability indicators and potential risk factors, relying on the structural decomposition principles inherent in the PFNBSS model.

To illustrate the versatility and practical relevance of the model, two examples are presented. The first is a symbolic, illustrative case used to clarify the operational mechanics of the model in a simplified setting. The second example focuses on a comparative sustainability risk assessment scenario involving manufacturing enterprises and is structured to reflect typical real-world evaluation practices. Although both examples use simulated data, the second is constructed around realistic sustainability indicators, challenges, and expert-based assessment logic commonly encountered in industrial decision environments.

The section is organized as follows. First, we introduce the PFNBSS-based DM procedure, which outlines the algorithmic steps for calculating aggregate and net scores to identify the most suitable alternative. Then, we apply the framework in two DM contexts. These examples collectively demonstrate both the internal workings of the model and its applicability to sustainability-related evaluations in manufacturing domains.

Decision-making procedure

In this subsection, we introduce the DM procedure based on the proposed model, PFNBSS. The procedure utilizes the structural decomposition of the PFNBSS framework to evaluate alternatives. It involves calculating aggregate score values from substructures to assess each alternative. The algorithm identifies the best alternative by computing net scores derived from the individual contributions of each substructure. The entire process is visually represented in a flowchart, providing a clear illustration of the DM steps involved.

To facilitate understanding of the computational framework, we provide the following explanation prior to presenting the algorithm. This explanation bridges the representation in Table 2 with the computations in the algorithm by noting that each cell entry–comprising two tuples for the positive and negative evaluations–is first processed by computing corresponding score values. These individual scores, derived from the components $\langle r_{ij_{\varepsilon_j}}, \zeta_{ij}^+, \zeta_{ij}^- \rangle$ and $\langle r_{ij_{\neg\varepsilon_j}}, \xi_{ij}^+, \xi_{ij}^- \rangle$, are mapped to numerical values denoted by z_{ij}. Thus, in the algorithm, the symbols z_{ij} refer to the scalar score values that represent the overall evaluation of alternative ℓ_i with respect to attribute ε_j, after transforming the PFN entries into decision-relevant quantities. This transformation enables the calculation of aggregate and net scores and ensures consistency between the tabular representation and the algorithmic steps given in Figure 1.

Algorithm: Decision-Making Based on PFNBSS.

1: **Input:**
- $\mathcal{L}$: A set of alternatives.
- ρ: A set of decision attributes.
- (ζ, ξ, ρ, N): The PFNBSS structure.

2: **Steps:**

i. Decompose the PFNBSS structure (ζ, ξ, ρ, N) into two substructures: (ζ, ρ, N) and $(\xi, \neg\rho, N)$, as shown in Tables 3 and 4, respectively.

ii. For each substructure, identify the corresponding PFNs and compute their score values using Definition 2.2.

iii. For each alternative ℓ_i, calculate the aggregate score $a_i = \sum_j z_{ij}$ from the substructure (ζ, ρ, N).

iv. Likewise, compute the aggregate score $b_i = \sum_j z_{ij}$ from the substructure $(\xi, \neg\rho, N)$.

v. Determine the final net score $z_i = a_i - b_i$ for each alternative ℓ_i, where $i = 1, 2, \ldots, n$.

3: **Output:** Select the best alternative ℓ_q such that $z_q = \max\{z_i\}$.

Algorithm 1Decision-Making Based on PFNBSS.

Practical example 1: sustainability evaluation in manufacturing industries

In today's global economy, manufacturing industries are under increasing pressure to adopt sustainable practices that minimize environmental impacts, optimize energy consumption, and foster social responsibility. As industries work towards reducing their ecological footprint, integrating sustainability into their operations is seen not only as a regulatory requirement but also as a strategic advantage. The assessment of sustainability in these industries involves evaluating several critical aspects, including resource efficiency, environmental protection, and corporate social responsibility.

This first example serves as a conceptual illustration of the proposed PFNBSS model, using simplified data to demonstrate the DM process step by step. We evaluate the sustainability performance of five representative manufacturing companies $\mathcal{L} = \{\ell_1, \ell_2, \ell_3, \ell_4, \ell_5\}$, each reflecting a typical industrial profile (e.g., automotive parts, food processing, chemicals, textiles, electronics). The assessment is based on simulated expert ratings and captures key sustainability enablers and barriers in a stylized format.

Although the data used is synthetic, it is designed to resemble general sustainability evaluation scenarios and highlight the inner workings of the PFNBSS-based DM framework.

Key sustainability indicators

The following attributes are used to measure the sustainability practices of the manufacturing companies:

- ε_1: Energy efficiency - This attribute evaluates how effectively the company reduces energy consumption during the manufacturing process without compromising product quality.
- ε_2: Waste reduction - Assesses the company's ability to minimize, reuse, or recycle waste materials generated throughout production.
- ε_3: Use of renewable resources - Measures the extent to which the company integrates renewable materials into its operations, contributing to a more sustainable production process.
- ε_4: Social responsibility - This factor gauges the company's commitment to community engagement, fair labor practices, and contributing to the welfare of society at large.

Challenges to sustainability implementation

In addition to the positive sustainability outcomes, several challenges may hinder the successful implementation of sustainable practices. These challenges are represented by the set $\neg\rho$:

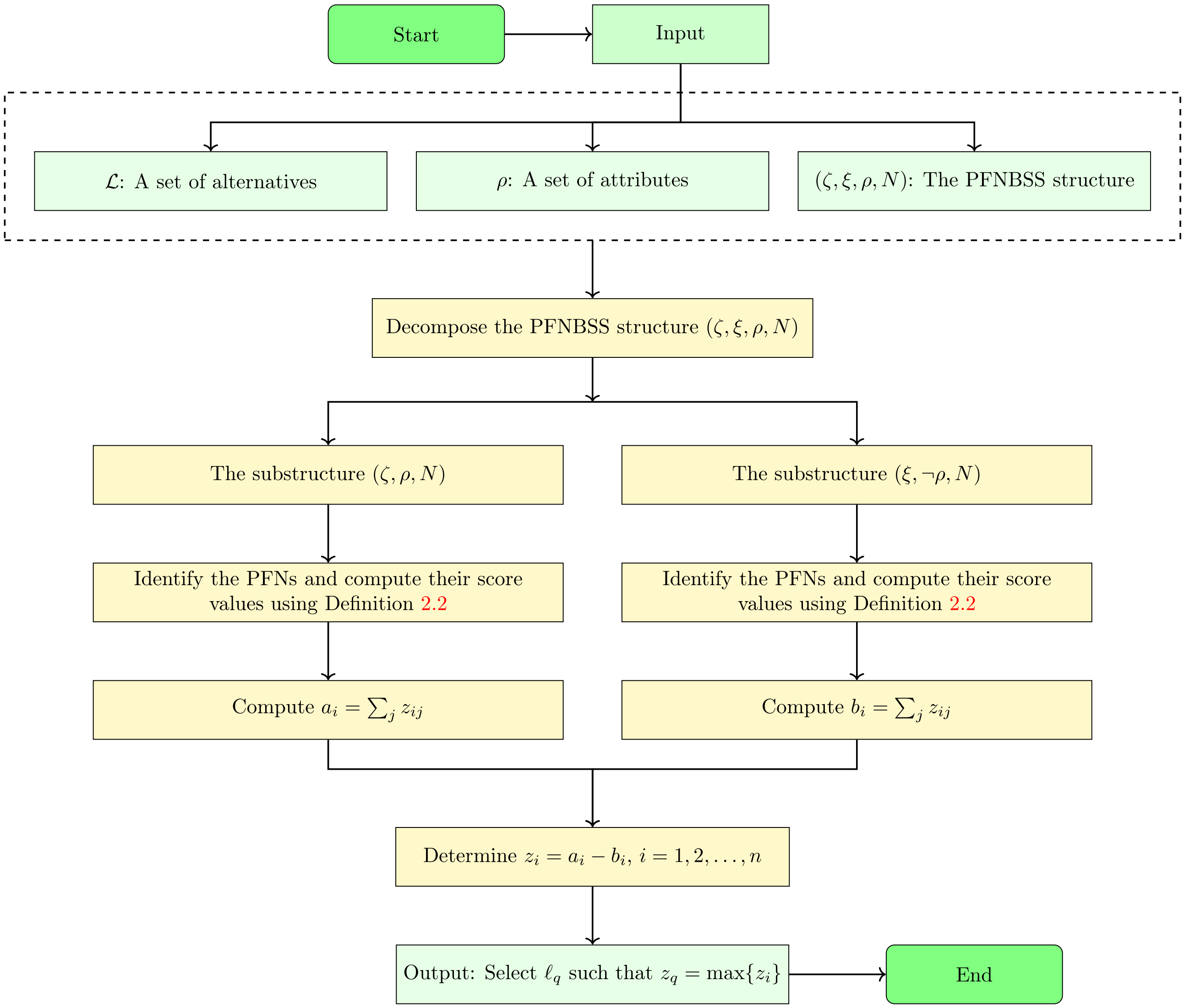

Fig. 1. Flowchart for the proposed algorithm.

- $\neg\varepsilon_1$: High energy consumption - This signifies inefficiencies in energy usage that lead to higher costs and increased environmental impact.
- $\neg\varepsilon_2$: Inefficient waste management - Reflects the company's failure to implement effective waste reduction and recycling strategies.
- $\neg\varepsilon_3$: Reliance on non-renewable resources - Highlights the dependence on non-renewable materials, such as fossil fuels, which undermine long-term sustainability goals.
- $\neg\varepsilon_4$: Poor labor practices - Indicates the company's failure to uphold ethical labor standards and its lack of community engagement, affecting both its reputation and operational sustainability.

Sustainability evaluation framework

The sustainability of each company is assessed based on the presence of the positive sustainability attributes and the challenges identified. Table 18 summarizes the evaluations, where:

- $\circ$ denotes areas with weak compliance to sustainability practices.
- Multiple $\star$ symbols indicate varying degrees of efficiencies in a given aspect of sustainability.

The check-marks from the previous evaluations are converted into numerical values ranging from 0 to 4 using the same technique as in Example 3.1, and the results are presented in Table 19.

We now identify each of $(\zeta, \rho, 5)$ and $(\xi, \neg\rho, 5)$, as presented in Tables 20 and 21, respectively.

By Definition 2.2, we compute the score values for each of the PFNs in Tables 20 and 21, respectively. The corresponding results are presented in Tables 22 and 23.

Based on the developed algorithm, we are now able to recommend the most suitable alternative. Table 24 is constructed directly from Tables 22 and 23.

From Table 24, it is evident that $\max z_i = z_4$; hence, ℓ_4 is identified as the most suitable option.

$\mathcal{L} \setminus \rho$	ε_1	ε_2	ε_3	ε_4
ℓ_1	⋆⋆ ⋆	⋆ ⋆ ⋆ ⋆	⋆⋆ ⋆	⋆⋆ ⋆⋆
ℓ_2	⋆ ⋆ ⋆ ⋆	○ ⋆ ⋆ ⋆⋆	⋆ ⋆⋆	⋆⋆ ⋆
ℓ_3	⋆ ⋆ ⋆ ⋆	⋆ ⋆ ⋆ ⋆	⋆⋆ ⋆⋆	⋆ ⋆ ⋆ ⋆
ℓ_4	⋆ ⋆ ⋆ ⋆	⋆ ⋆ ⋆⋆ ○	⋆ ⋆ ⋆ ⋆	⋆ ⋆
ℓ_5	⋆ ⋆	⋆ ⋆⋆	⋆⋆ ⋆	⋆ ⋆ ⋆ ⋆

Table 18. Initial Evaluations

$(\zeta, \xi, \rho, 5)$	ε_1	ε_2	ε_3	ε_4
ℓ_1	$\langle 2, 0.7, 0.2\rangle$ $\langle 1, 0.3, 0.4\rangle$	$\langle 1, 0.5, 0.3\rangle$ $\langle 3, 0.8, 0.0\rangle$	$\langle 2, 0.5, 0.5\rangle$ $\langle 1, 0.5, 0.3\rangle$	$\langle 2, 0.5, 0.4\rangle$ $\langle 2, 0.7, 0.1\rangle$
ℓ_2	$\langle 3, 0.7, 0.5\rangle$ $\langle 1, 0.0, 0.6\rangle$	$\langle 0, 0.4, 0.1\rangle$ $\langle 4, 0.7, 0.7\rangle$	$\langle 1, 0.4, 0.4\rangle$ $\langle 2, 0.3, 0.7\rangle$	$\langle 2, 0.7, 0.3\rangle$ $\langle 1, 0.3, 0.5\rangle$
ℓ_3	$\langle 3, 0.5, 0.7\rangle$ $\langle 1, 0.6, 0.1\rangle$	$\langle 1, 0.5, 0.1\rangle$ $\langle 3, 0.7, 0.4\rangle$	$\langle 2, 0.6, 0.2\rangle$ $\langle 2, 0.6, 0.4\rangle$	$\langle 3, 0.6, 0.6\rangle$ $\langle 1, 0.5, 0.0\rangle$
ℓ_4	$\langle 3, 0.8, 0.3\rangle$ $\langle 1, 0.5, 0.3\rangle$	$\langle 4, 1.0, 0.0\rangle$ $\langle 0, 0.0, 0.4\rangle$	$\langle 1, 0.4, 0, 4\rangle$ $\langle 3, 0.3, 0.8\rangle$	$\langle 1, 0.4, 0.2\rangle$ $\langle 1, 0.3, 0.4\rangle$
ℓ_5	$\langle 1, 0.5, 0.3\rangle$ $\langle 1, 0.5, 0.3\rangle$	$\langle 1, 0.1, 0.6\rangle$ $\langle 2, 0.6, 0.4\rangle$	$\langle 2, 0.6, 0.3\rangle$ $\langle 1, 0.6, 0.0\rangle$	$\langle 3, 0.6, 0.6\rangle$ $\langle 1, 0.3, 0.4\rangle$

Table 19. Tabular form of PF5BSS $(\zeta, \xi, \rho, 5)$

$(\zeta, \rho, 5)$	ε_1	ε_2	ε_3	ε_4
ℓ_1	$\langle 2, 0.7, 0.2\rangle$	$\langle 1, 0.5, 0.3\rangle$	$\langle 2, 0.5, 0.5\rangle$	$\langle 2, 0.5, 0.4\rangle$
ℓ_2	$\langle 3, 0.7, 0.5\rangle$	$\langle 0, 0.4, 0.1\rangle$	$\langle 1, 0.4, 0.4\rangle$	$\langle 2, 0.7, 0.3\rangle$
ℓ_3	$\langle 3, 0.5, 0.7\rangle$	$\langle 1, 0.5, 0.1\rangle$	$\langle 2, 0.6, 0.2\rangle$	$\langle 3, 0.6, 0.6\rangle$
ℓ_4	$\langle 3, 0.8, 0.3\rangle$	$\langle 4, 1.0, 0.0\rangle$	$\langle 1, 0.4, 0.4\rangle$	$\langle 1, 0.4, 0.2\rangle$
ℓ_5	$\langle 1, 0.5, 0.3\rangle$	$\langle 1, 0.1, 0.6\rangle$	$\langle 2, 0.6, 0.3\rangle$	$\langle 3, 0.6, 0.6\rangle$

Table 20. Tabular form of $(\zeta, \rho, 5)$

$(\xi, \neg\rho, 5)$	$\neg\varepsilon_1$	$\neg\varepsilon_2$	$\neg\varepsilon_3$	$\neg\varepsilon_4$
ℓ_1	$\langle 1, 0.3, 0.4\rangle$	$\langle 3, 0.8, 0.0\rangle$	$\langle 1, 0.5, 0.3\rangle$	$\langle 2, 0.7, 0.1\rangle$
ℓ_2	$\langle 1, 0.0, 0.6\rangle$	$\langle 4, 0.7, 0.7\rangle$	$\langle 2, 0.3, 0.7\rangle$	$\langle 1, 0.3, 0.5\rangle$
ℓ_3	$\langle 1, 0.6, 0.1\rangle$	$\langle 3, 0.7, 0.4\rangle$	$\langle 2, 0.6, 0.4\rangle$	$\langle 1, 0.5, 0.0\rangle$
ℓ_4	$\langle 1, 0.5, 0.3\rangle$	$\langle 0, 0.0, 0.4\rangle$	$\langle 3, 0.3, 0.8\rangle$	$\langle 1, 0.3, 0.4\rangle$
ℓ_5	$\langle 1, 0.5, 0.3\rangle$	$\langle 2, 0.6, 0.4\rangle$	$\langle 1, 0.6, 0.0\rangle$	$\langle 1, 0.3, 0.4\rangle$

Table 21. Tabular form of $(\xi, \neg\rho, 5)$

Practical example 2: comparative risk assessment in sustainable manufacturing

As industries intensify their efforts to integrate sustainable practices, there remains a pressing need to evaluate the risk factors that threaten long-term sustainability objectives. These risks, if not properly assessed and mitigated, can undermine environmental efforts, increase operational costs, and damage stakeholder trust. Sustainable manufacturing, therefore, demands not only proactive adoption of positive attributes but also robust mechanisms for identifying and responding to sustainability risks.

$(\zeta, \rho, 5)$	ε_1	ε_2	ε_3	ε_4	$a_i = \sum_j z_{ij}$
ℓ_1	0.45	0.16	0.00	0.09	0.70
ℓ_2	0.24	0.15	0.00	0.40	0.79
ℓ_3	-0.24	0.24	0.32	0.00	0.32
ℓ_4	0.55	1.00	0.00	0.12	1.67
ℓ_5	0.16	-0.35	0.27	0.00	0.08

Table 22. Score values of PFNs in $(\zeta, \rho, 5)$ and the corresponding calculations of $a_i = \sum_j z_{ij}$.

$(\xi, \neg\rho, 5)$	$\neg\varepsilon_1$	$\neg\varepsilon_2$	$\neg\varepsilon_3$	$\neg\varepsilon_4$	$b_i = \sum_j z_{ij}$
ℓ_1	-0.07	0.64	0.16	0.48	1.21
ℓ_2	-0.36	0.00	-0.40	-0.16	-0.92
ℓ_3	0.35	0.17	0.20	0.25	0.97
ℓ_4	0.16	-0.16	-0.55	-0.07	-0.62
ℓ_5	0.16	0.20	0.36	-0.07	0.65

Table 23. Score values of PFNs in $(\xi, \neg\rho, 5)$ and the corresponding calculations of $b_i = \sum_j z_{ij}$.

$a_i = \sum_j z_{ij}$	$b_i = \sum_j z_{ij}$	$z_i = a_i - b_i$
$a_1 = 0.70$	$b_1 = 1.21$	$z_1 = -0.51$
$a_2 = 0.79$	$b_2 = -0.92$	$z_2 = 1.71$
$a_3 = 0.32$	$b_3 = 0.97$	$z_3 = -0.65$
$a_4 = 1.67$	$b_4 = -0.62$	$z_4 = 2.29$
$a_5 = 0.08$	$b_5 = 0.65$	$z_5 = -0.57$

Table 24. Final score table

This second example illustrates a more industry-oriented application of the PFNBSS model. It involves a comparative sustainability risk assessment of five manufacturing enterprises $\mathcal{L} = \{\ell_1, \ell_2, \ell_3, \ell_4, \ell_5\}$ operating in sectors such as packaging, pharmaceuticals, steel, renewable energy, and electronics. The goal is to assess not only sustainable practices but also the extent of underlying risks that may hinder sustainability performance.

While the data is also synthetically generated, it is grounded in real-world-inspired sustainability indicators and expert evaluation structures. This example better reflects practical assessment logic and is intended to demonstrate the model's suitability for more realistic industrial decision environments.

Positive sustainability drivers
The following set of positive indicators reflects core sustainability enablers used in the assessment:

- ε_1: Emissions reduction strategies – Evaluates the company's efforts in limiting greenhouse gas emissions through clean technologies and carbon offsetting.
- ε_2: Sustainable supply chain – Assesses the degree to which the company integrates environmental and ethical considerations across its supply chain.
- ε_3: Eco-friendly product innovation – Measures the company's investments in designing products that reduce lifecycle environmental impact.
- ε_4: Employee engagement in sustainability – Captures the company's success in involving its workforce in sustainability initiatives and awareness programs.

Sustainability risk factors
The corresponding set $\neg\rho$ represents key risks or challenges that may hinder sustainable performance:

- $\neg\varepsilon_1$: High carbon footprint – Indicates that the company still relies on emission-intensive processes without adequate mitigation.
- $\neg\varepsilon_2$: Unsustainable supplier practices – Reflects the lack of transparency and environmental standards within the supply chain.
- $\neg\varepsilon_3$: Resistance to innovation – Points to organizational inertia or limited investment in sustainable Research and Development activities.

$\mathcal{L} \setminus \rho$	ε_1	ε_2	ε_3	ε_4
ℓ_1	$\star\star\star$ $\star$	$\star\star$ $\star\star$	$\star\star$ $\circ$	$\star$ $\star\star$
ℓ_2	$\circ$ $\star\star\star\star$	$\star\star\star$ $\star$	$\star$ $\star\star\star$	$\star\star$ $\star\star$
ℓ_3	$\star$ $\star\star\star$	$\star\star$ $\star\star$	$\star\star$ $\star\star$	$\star\star\star$ $\circ$
ℓ_4	$\star\star\star$ $\star$	$\star\star\star\star$ $\circ$	$\star$ $\star\star\star$	$\star$ $\star\star$
ℓ_5	$\star$ $\star$	$\star$ $\star\star\star$	$\star\star$ $\star$	$\star\star\star$ $\star$

Table 25. Initial Evaluations

$(\zeta, \xi, \rho, 5)$	ε_1	ε_2	ε_3	ε_4
ℓ_1	$\langle 3, 0.2, 0.8\rangle$ $\langle 1, 0.3, 0.5\rangle$	$\langle 2, 0.5, 0.5\rangle$ $\langle 2, 0.6, 0.2\rangle$	$\langle 2, 0.6, 0.4\rangle$ $\langle 0, 0.1, 0.3\rangle$	$\langle 1, 0.2, 0.4\rangle$ $\langle 2, 0.7, 0.1\rangle$
ℓ_2	$\langle 0, 0.4, 0.1\rangle$ $\langle 4, 0.8, 0.5\rangle$	$\langle 3, 0.4, 0.7\rangle$ $\langle 1, 0.3, 0.5\rangle$	$\langle 1, 0.4, 0.4\rangle$ $\langle 3, 0.3, 0.8\rangle$	$\langle 2, 0.7, 0.3\rangle$ $\langle 2, 0.2, 0.7\rangle$
ℓ_3	$\langle 1, 0.1, 0.6\rangle$ $\langle 3, 0.6, 0.5\rangle$	$\langle 2, 0.5, 0.4\rangle$ $\langle 2, 0.6, 0.4\rangle$	$\langle 2, 0.6, 0.2\rangle$ $\langle 2, 0.2, 0.7\rangle$	$\langle 3, 0.7, 0.5\rangle$ $\langle 0, 0.1, 0.4\rangle$
ℓ_4	$\langle 3, 0.8, 0.3\rangle$ $\langle 1, 0.4, 0.3\rangle$	$\langle 4, 1.0, 0.0\rangle$ $\langle 0, 0.0, 0.4\rangle$	$\langle 1, 0.1, 0, 6\rangle$ $\langle 3, 0.3, 0.8\rangle$	$\langle 1, 0.4, 0.2\rangle$ $\langle 2, 0.3, 0.7\rangle$
ℓ_5	$\langle 1, 0.5, 0.3\rangle$ $\langle 1, 0.5, 0.3\rangle$	$\langle 1, 0.1, 0.6\rangle$ $\langle 3, 0.7, 0.5\rangle$	$\langle 2, 0.6, 0.3\rangle$ $\langle 1, 0.1, 0.5\rangle$	$\langle 3, 0.8, 0.1\rangle$ $\langle 1, 0.3, 0.4\rangle$

Table 26. Tabular form of PF5BSS $(\zeta, \xi, \rho, 5)$

- $\neg\varepsilon_4$: Low internal sustainability awareness – Suggests insufficient training or communication efforts related to environmental responsibility.

Risk evaluation framework
Each company is evaluated using the PFNBSS model, which incorporates both the presence of sustainability drivers and the prevalence of associated risks. The evaluations are presented in Table 25, where:

- $\circ$ represents areas where sustainability engagement is lacking or inconsistent.
- $\star$, $\star\star$, $\star\star\star$ and $\star\star\star\star$ symbolize incremental levels of success in managing sustainability factors or mitigating risks.

The symbolic check-marks from the earlier assessments have been translated into numerical values on a scale from 0 to 4 following the method applied in Example 3.1; the corresponding results are displayed in Table 26.

We proceed by extracting the positive substructure $(\zeta, \rho, 5)$ and the negative substructure $(\xi, \neg\rho, 5)$, which are detailed separately in Tables 27 and 28, respectively.

Using Definition 2.2, the score values for the PFNs listed in Tables 27 and 28 are calculated. The resulting scores are summarized in Tables 29 and 30, respectively.

Following the proposed algorithm, we identify the most appropriate alternative. Table 31 compiles the final results derived from the score values presented in Tables 29 and 30.

Table 31 clearly shows that the highest net score is z_4, indicating that ℓ_4 is the optimal choice among the alternatives.

Results and discussion

The proposed DM procedure based on the PFNBSS model demonstrates a structured and effective approach for handling complex sustainability evaluations in manufacturing industries. The framework decomposes the overall assessment into two complementary substructures: positive sustainability attributes (ζ, ρ, N) and negative challenges $(\xi, \neg\rho, N)$. This bipolar decomposition facilitates a nuanced analysis that balances the benefits and barriers associated with each alternative.

The first numerical example involving five manufacturing companies illustrates this approach concretely. Key sustainability indicators such as energy efficiency, waste reduction, use of renewable resources, and social responsibility form the basis of the evaluation. Simultaneously, recognized challenges–including high energy consumption, inefficient waste management, reliance on non-renewable resources, and poor labor practices–are explicitly accounted for through the negative substructure.

The initial qualitative ratings, transformed into quantitative PFNs, allow the incorporation of uncertainty and vagueness inherent in expert evaluations. These fuzzy evaluations reflect degrees of membership and non-

$(\zeta, \rho, 5)$	ε_1	ε_2	ε_3	ε_4
ℓ_1	$\langle 3, 0.2, 0.8\rangle$	$\langle 2, 0.5, 0.5\rangle$	$\langle 2, 0.6, 0.4\rangle$	$\langle 1, 0.2, 0.4\rangle$
ℓ_2	$\langle 0, 0.4, 0.1\rangle$	$\langle 3, 0.4, 0.7\rangle$	$\langle 1, 0.4, 0.4\rangle$	$\langle 2, 0.7, 0.3\rangle$
ℓ_3	$\langle 1, 0.1, 0.6\rangle$	$\langle 2, 0.5, 0.4\rangle$	$\langle 2, 0.6, 0.2\rangle$	$\langle 3, 0.7, 0.5\rangle$
ℓ_4	$\langle 3, 0.8, 0.3\rangle$	$\langle 4, 1.0, 0.0\rangle$	$\langle 1, 0.1, 0, 6\rangle$	$\langle 1, 0.4, 0.2\rangle$
ℓ_5	$\langle 1, 0.5, 0.3\rangle$	$\langle 1, 0.1, 0.6\rangle$	$\langle 2, 0.6, 0.3\rangle$	$\langle 3, 0.8, 0.1\rangle$

Table 27. Tabular form of $(\zeta, \rho, 5)$

$(\xi, \neg\rho, 5)$	$\neg\varepsilon_1$	$\neg\varepsilon_2$	$\neg\varepsilon_3$	$\neg\varepsilon_4$
ℓ_1	$\langle 1, 0.3, 0.5\rangle$	$\langle 2, 0.6, 0.2\rangle$	$\langle 0, 0.1, 0.3\rangle$	$\langle 2, 0.7, 0.1\rangle$
ℓ_2	$\langle 4, 0.8, 0.5\rangle$	$\langle 1, 0.3, 0.5\rangle$	$\langle 3, 0.3, 0.8\rangle$	$\langle 2, 0.2, 0.7\rangle$
ℓ_3	$\langle 3, 0.6, 0.5\rangle$	$\langle 2, 0.6, 0.4\rangle$	$\langle 2, 0.2, 0.7\rangle$	$\langle 0, 0.1, 0.4\rangle$
ℓ_4	$\langle 1, 0.4, 0.3\rangle$	$\langle 0, 0.0, 0.4\rangle$	$\langle 3, 0.3, 0.8\rangle$	$\langle 2, 0.3, 0.7\rangle$
ℓ_5	$\langle 1, 0.5, 0.3\rangle$	$\langle 3, 0.7, 0.5\rangle$	$\langle 1, 0.1, 0.5\rangle$	$\langle 1, 0.3, 0.4\rangle$

Table 28. Tabular form of $(\xi, \neg\rho, 5)$

$(\zeta, \rho, 5)$	ε_1	ε_2	ε_3	ε_4	$f_i = \sum_j z_{ij}$
ℓ_1	-0.60	0.00	0.20	-0.12	-0.52
ℓ_2	0.15	-0.33	0.00	0.40	0.22
ℓ_3	-0.35	0.09	0.32	0.24	0.30
ℓ_4	0.55	1.00	-0.35	0.12	1.32
ℓ_5	0.16	-0.35	0.27	0.63	0.71

Table 29. Score values of PFNs in $(\zeta, \rho, 5)$ and the corresponding calculations of $f_i = \sum_j z_{ij}$.

$(\xi, \neg\rho, 5)$	$\neg\varepsilon_1$	$\neg\varepsilon_2$	$\neg\varepsilon_3$	$\neg\varepsilon_4$	$g_i = \sum_j z_{ij}$
ℓ_1	-0.16	0.32	-0.08	0.48	0.56
ℓ_2	0.39	-0.16	-0.55	-0.45	-0.77
ℓ_3	0.11	0.20	-0.45	-0.15	-0.29
ℓ_4	0.07	-0.16	-0.55	-0.40	-1.04
ℓ_5	0.16	0.24	-0.24	-0.07	0.09

Table 30. Score values of PFNs in $(\xi, \neg\rho, 5)$ and the corresponding calculations of $g_i = \sum_j z_{ij}$.

$f_i = \sum_j z_{ij}$	$g_i = \sum_j z_{ij}$	$z_i = f_i - g_i$
$f_1 = -0.52$	$g_1 = 0.56$	$z_1 = -1.08$
$f_2 = 0.22$	$g_2 = -0.77$	$z_2 = 0.99$
$f_3 = 0.30$	$g_3 = -0.29$	$z_3 = 0.59$
$f_4 = 1.32$	$g_4 = -1.04$	$z_4 = 2.36$
$f_5 = 0.71$	$g_5 = 0.09$	$z_5 = 0.62$

Table 31. Final score table

membership enriched by bipolar information, providing a richer representation than classical crisp or traditional fuzzy values.

The computation of score values for each substructure (Tables 22 and 23) reveals the aggregated performance of each alternative on positive and negative fronts. Notably, company ℓ_4 achieves the highest aggregate positive score ($a_4 = 1.67$) while simultaneously showing a negative substructure score that reduces its penalty effect ($b_4 = -0.62$). This results in the highest net score $z_4 = 2.29$, marking ℓ_4 as the most sustainable option among the considered companies.

This outcome underscores the strength of the PFNBSS model in differentiating alternatives by synthesizing positive contributions and offsetting negative factors through a mathematically consistent mechanism. The net scoring approach effectively balances competing criteria, making it highly suitable for real-world MCDM problems where trade-offs are inevitable.

The methodology's transparency is enhanced by the explicit flowchart and algorithmic description, which facilitate replication and potential adaptation to other domains beyond manufacturing sustainability. Furthermore, the flexibility of the model to accommodate different evaluation scales and fuzzy parameters highlights its robustness.

In practical terms, the model empowers decision-makers in manufacturing sectors to systematically assess sustainability efforts, identify leading companies, and recognize areas for improvement. The bipolar fuzzy framework captures both the promise and challenges in sustainability, offering a more comprehensive decision basis than unipolar or crisp models.

In summary, the results affirm that the PFNBSS-based DM framework is an effective tool for sustainability evaluation, balancing complexity and interpretability while accommodating uncertainty and bipolar information inherent in expert assessments.

To further demonstrate the versatility of the PFNBSS model, a second example involving comparative sustainability risk assessment across diverse manufacturing sectors was presented. This example emphasizes the identification and evaluation of key risk factors alongside positive sustainability drivers, reflecting a more nuanced real-world decision context. Although the data for this example is synthetically generated and does not include a detailed quantitative scoring and benchmarking, as in Example 1, it effectively showcases the model's capacity to handle complex bipolar information in risk-focused sustainability evaluations. The inclusion of this case highlights the adaptability of the PFNBSS framework to varied industrial scenarios where balancing positive attributes and risk factors is critical for informed DM.

Assessment of Pythagorean Fuzzy N-Bipolar Soft Set Model

In this section, we evaluate the proposed PFNBSS model by discussing its strengths, comparing it with existing models, and identifying its limitations. The evaluation is conducted through both qualitative and quantitative analyses.

We begin by outlining the key advantages of the PFNBSS model, such as its capability to manage uncertainty using Pythagorean fuzzy membership functions, its support for multi-valued evaluations, and its incorporation of bipolarity. These features collectively contribute to its robustness in handling complex DM problems.

For the comparative analysis, we conduct a qualitative comparison focusing on structural and functional aspects, including membership type, evaluation methodology, and bipolarity consideration. This is followed by a quantitative comparison based on an illustrative example of sustainability evaluation in manufacturing industries, where the performance of the PFNBSS model is examined against existing models, namely FNBSS[35] and IFNBSS[36]. The comparison highlights the superior discrimination and expressive capability of the proposed model.

Additionally, we include a subsection analyzing the sensitivity of decision rankings to the fixed grading intervals used in mapping Pythagorean fuzzy values to evaluation grades. This analysis demonstrates the robustness of the model's rankings under the selected criteria.

Finally, we discuss the limitations of the PFNBSS model, which include computational complexity, scalability concerns, interpretability challenges, uncertainty management overhead, subjectivity in membership assignment, and parameter sensitivity. These insights serve as a foundation for identifying future directions to improve the model's practicality and effectiveness in real-world applications.

Strengths of the proposed model

The proposed PFNBSS model presents several key advantages over existing approaches:

- It integrates Pythagorean fuzzy membership, which provides a higher level of uncertainty handling compared to classical fuzzy and intuitionistic fuzzy models.
- Unlike many earlier models that operate on binary or single-valued evaluations, PFNBSS supports multi-valued evaluations, enabling richer and more realistic decision environments.
- The model fully incorporates bipolarity, allowing it to simultaneously handle both positive and negative aspects of information.
- It maintains parameterization support, aligning with SS theory's flexibility in dealing with varying sets of attributes or parameters.
- Among all the models reviewed, PFNBSS is the only one that combines all of these strengths – especially the integration of Pythagorean fuzzy logic with bipolar and multi-valued SS frameworks – making it particularly suitable for complex and nuanced DM scenarios.

Group	Approach	Membership Type	Membership Superiority	Parameterization Support	Evaluation Type	Evaluation Scale	Bipolar Capability
1. Classical Models	FS[1]	F	Low	Not Supported	Continuous	Single-Valued	Absent
	IFS[2]	IF	Medium	Not Supported	Continuous	Single-Valued	Absent
	PFS[3]	PF	High	Not Supported	Continuous	Single-Valued	Absent
2. Soft Sets	SS[7]	None	–	Supported	Discrete	Binary	Absent
	FSS[10]	F	Low	Supported	Continuous	Binary	Absent
	IFSS[11]	IF	Medium	Supported	Continuous	Binary	Absent
	PFSS[12]	PF	High	Supported	Continuous	Binary	Absent
3. Bipolar Soft Sets	BSS[20]	None	–	Supported	Discrete	Binary	Present
	FBSS[21]	F	Low	Supported	Continuous	Binary	Present
	IFBSS[22]	IF	Medium	Supported	Continuous	Binary	Present
	PFBSS[22]	PF	High	Supported	Continuous	Binary	Present
4. N-Soft Sets	NSS[25]	None	–	Supported	Discrete	multi-valued	Absent
	FNSS[26]	F	Low	Supported	Continuous	multi-valued	Absent
	IFNSS[27]	IF	Medium	Supported	Continuous	multi-valued	Absent
	PFNSS[30]	PF	High	Supported	Continuous	multi-valued	Absent
5. N-Bipolar Soft Sets	NBSS[33]	None	–	Supported	Discrete	multi-valued	Present
	FNBSS[35]	F	Low	Supported	Continuous	multi-valued	Present
	IFNBSS[36]	IF	Medium	Supported	Continuous	multi-valued	Present
	PFNBSS (Proposed)	PF	High	Supported	Continuous	multi-valued	Present

Table 32. Comparison of the PFNBSS model with relevant existing approaches.

Models	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	Ranking Order
FNBSS[35]	-0.10	0.90	-0.20	1.50	-0.20	$\ell_4 \succ \ell_2 \succ \ell_1 \succ \ell_5 = \ell_3$
IFNBSS[36]	-0.70	2.10	-0.90	2.50	-0.90	$\ell_4 \succ \ell_2 \succ \ell_1 \succ \ell_5 = \ell_3$
PFNBSS (Proposed)	-0.51	1.71	-0.65	2.29	-0.57	$\ell_4 \succ \ell_2 \succ \ell_1 \succ \ell_5 \succ \ell_3$

Table 33. Comparison between FNBSS[35], IFNBSS[36], and the proposed PFNBSS models, based on the scenario in Subsection 4.2.

Comparison of the proposed model with existing approaches

In this section, we compare the proposed PFNBSS model with existing approaches to evaluate its strengths and performance.

Qualitative comparison

In this subsection, we provide a qualitative comparison of the PFNBSS model with several existing approaches, including Classical Models, Soft Sets, Bipolar Soft Sets, N-Soft Sets, and N-Bipolar Soft Sets, and their related extensions. The comparison focuses on various factors such as membership type, membership superiority, parameterization support, evaluation type, evaluation scale, and bipolar capability. Table 32 summarizes the key characteristics of each approach, highlighting the advantages and differences of the PFNBSS model in these areas.

Quantitative comparison

In this subsection, we present a quantitative comparison of the proposed PFNBSS model against the existing FNBSS[35] and IFNBSS[36] models. The comparison utilizes the illustrative sustainability evaluation scenario discussed in Subsection 4.2. Table 33 displays the computed scores and resulting ranking orders for all models, while Figure 2 visually illustrates the differences in score distributions.

To evaluate model performance more comprehensively, Table 34 reports three metrics: (i) score spread (difference between the highest and lowest scores), (ii) rank distinction (presence or absence of tied alternatives), and (iii) consistency with expert judgment. These metrics help assess the discriminatory power and decision precision of each method.

Although IFNBSS shows the highest score spread numerically, it fails to distinguish between ℓ_5 and ℓ_3, resulting in tied ranks that limit its practical interpretability. A similar issue is seen with FNBSS. In contrast, the proposed PFNBSS model provides a complete ranking of all alternatives, avoiding ties while maintaining a strong score spread and preserving expert-preferred rankings.

It is also worth noting that Pythagorean fuzzy values are directly applied to the IFNBSS model–without adjusting for its admissibility condition–which may compromise its theoretical consistency. By contrast, the

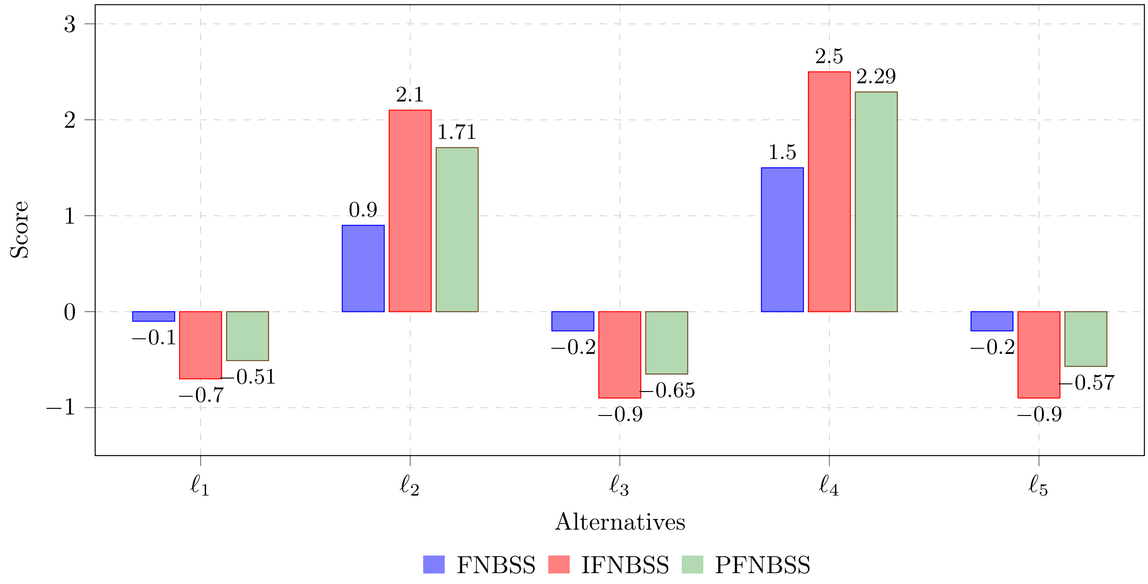

Fig. 2. Bar chart comparison of ℓ_i scores for FNBSS[35], IFNBSS[36], and the proposed PFNBSS models, based on the example in Subsection 4.2.

Model	Score Spread	Rank Distinction	Consistency with Expert Ranking
FNBSS[35]	1.70	Tied ranks ($\ell_5 = \ell_3$)	High
IFNBSS[36]	3.40	Tied ranks ($\ell_5 = \ell_3$)	High
PFNBSS (Proposed)	2.80	Full distinction (no ties)	High

Table 34. Comparison of Performance Metrics for FNBSS[35], IFNBSS[36], and the proposed PFNBSS models, based on the example in Subsection 4.2.

PFNBSS model adheres to the admissibility semantics of PFNs and integrates both positive and negative aspects with possibility weighting, offering a more coherent and discriminative evaluation in multi-criteria contexts.

Impact of grading intervals on ranking stability and sensitivity

The evaluation grades used in the PFNBSS model, as defined by specific intervals of squared membership and non-membership values (see Table 7), play a crucial role in mapping continuous fuzzy data into discrete assessment levels. These intervals effectively serve as parameters that influence how expert evaluations are categorized and aggregated.

It is important to note that modifications to these grading intervals can lead to changes in the categorization of fuzzy values, thereby affecting the computed scores and final rankings of alternatives. For example, a slight adjustment in the boundary between two consecutive grades can shift certain values into a different grade category, potentially altering the relative ordering of alternatives.

This phenomenon highlights the sensitivity of the model's outcomes to parameter selection, which is a significant aspect of DM robustness. Hence, an analysis of ranking stability and sensitivity to grading interval changes is essential for understanding the reliability and consistency of the PFNBSS model in practical applications.

While the current study employs fixed grading intervals based on well-established criteria, this approach demonstrates the model's capability to produce consistent and interpretable rankings under the selected evaluation scheme.

Challenges and limitations

Despite its notable advantages, the PFNBSS model has some limitations:

- **Computational complexity:** The incorporation of Pythagorean fuzzy membership and multi-valued evaluations increases processing time, especially for large datasets or real?time decision?making.
- **Scalability:** While the model is powerful for moderate-sized problems, applying it directly in large-scale industrial contexts may face challenges due to the computational and memory demands. Efficient algorithmic improvements and parallelization techniques are required to enhance practical deployment.
- **Interpretability concerns:** Multi-valued evaluations, while richer, may introduce ambiguity and make results harder to interpret for less?experienced users.

- **Uncertainty management overhead:** Integrating bipolarity with Pythagorean fuzzy logic demands sophisticated techniques to handle the higher degree of uncertainty, potentially complicating the decision process.
- **Subjectivity in membership assignment:** The process of assigning Pythagorean fuzzy membership and non-membership degrees may involve subjective judgments by experts, which can affect the consistency and reliability of the decision outcomes.
- **Parameter sensitivity:** Although parameterization is supported, accurate parameter selection can be challenging in dynamic environments where precise attribute information is unavailable or subject to change, further compounded by subjective expert inputs.

Concluding remarks and research outlook

This paper has presented a novel MCDM framework based on PFNBSSs, offering a flexible and expressive approach for evaluating complex decision problems involving multi-valued assessments and bipolarity under uncertainty. By integrating the strengths of PFSs, NSSs, and BSSs into a unified structure, the proposed model enables more nuanced representation and interpretation of expert evaluations.

We formally defined the PFNBSS structure, developed its algebraic operations, and applied the framework to two practical DM scenarios. The first example focused on evaluating the sustainability performance of manufacturing companies, while the second addressed comparative risk assessment in sustainable manufacturing contexts. Both applications demonstrated the model's ability to handle ambiguity, assess opposing dimensions of information, and differentiate between closely ranked alternatives through the incorporation of possibility degrees alongside membership and non-membership values.

Quantitative and qualitative comparisons confirmed that the PFNBSS model outperforms existing approaches such as FNBSS and IFNBSS in terms of information richness, decision granularity, and interpretability. These findings highlight the robustness and adaptability of the framework across diverse sustainability-driven evaluation tasks.

However, the PFNBSS model has some limitations, including increased computational complexity and scalability challenges when applied to large-scale problems, potential difficulties in interpreting multi-valued evaluations, uncertainty management overhead, and subjectivity involved in expert membership assignments. These challenges are discussed in detail in Section 6.4 (Challenges and Limitations). Addressing these issues through optimized algorithms, enhanced interpretability measures, and systematic parameter tuning forms an important direction for future research.

Future work will also focus on developing efficient computational strategies, such as parallel processing and approximate aggregation methods, to improve the model's scalability and practical deployment. Additionally, we plan to extend the PFNBSS framework to group DM settings, and q-rung orthopair systems–particularly Fermatean FNBSSs when $q = 3$–enabling more flexible modeling by relaxing traditional admissibility conditions on membership and non-membership degrees.

We anticipate that the proposed PFNBSS framework and its future extensions will serve as a powerful foundation for decision-support systems in various domains, including environmental sustainability, healthcare, supply chain risk assessment, and strategic planning.

Data availability

All data are included in the manuscript.

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References

1. Zadeh, L. A. Fuzzy sets. *Information and Control* **8**, 338–353 (1965).
2. Atanassov, K. T. Intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **20**, 87–96 (1986).
3. Yager, R. R. Pythagorean fuzzy subsets. In *2013 Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS)*, 57–61 (2013).
4. Akram, M. & Arshad, M. A novel trapezoidal bipolar fuzzy TOPSIS method for group decision-making. *Group Decis. Negot.* **28**, 565–584 (2019).
5. Zhang, X. & Xu, Z. Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets. *Int. J. Intell. Syst.* **29**(12), 1061–1078 (2014).
6. Akram, M., Dudek, W. A. & Dar, J. M. Pythagorean Dombi fuzzy aggregation operators with application in multi-criteria decision-making. *Int. J. Intell. Syst.* **34**(11), 3000–3019 (2019).
7. Molodtsov, D. Soft set theory-first results. *Comput. Math. Appl.* **37**, 19–31 (1999).
8. Maji, P. K., Biswas, R. & Roy, A. R. Soft set theory. *Comput. Math. Appl.* **45**, 555–562 (2003).
9. Ali, M. I., Feng, F., Liu, X., Min, W. K. & Shabir, M. On some new operations in soft set theory. *Comput. Math. Appl.* **57**, 1547–1553 (2009).
10. Maji, P. K., Biswas, R. & Roy, R. Fuzzy soft sets. *J. Fuzzy Math.* **9**, 589–602 (2001).
11. Maji, P. K., Biswas, R. & Roy, R. Intuitionistic fuzzy soft sets. *J. Fuzzy Math.* **9**, 677–692 (2001).
12. Peng, X., Yang, Y., Song, J. & Jiang, Y. Pythagorean fuzzy soft set and its application. *Computer Engineering* **41**(7), 224–229 (2015).
13. Alcantud, J. C. R., Khameneh, A. Z., Santos-García, G. & Akram, M. A systematic literature review of soft set theory. *Neural Comput. Appl.* **36**(16), 8951–8975 (2024).
14. Smarandache, F. Extension of soft set to hypersoft set and then to plithogenic hypersoft set. *Neutrosophic Sets Syst.* **22**, 168–170 (2018).
15. Musa, S. Y., Mohammed, R. A. & Asaad, B. A. N-hypersoft sets: An innovative extension of hypersoft sets and their applications. *Symmetry* **15**(9), 1795 (2023).
16. Mohammed, B. A., Dhumras, H., Shukla, V., Bajaj, R. K. & Al-Mekhlafi, Z. G. Innovative evaluation framework for consumer electronics-enabled intelligent transportation systems: Leveraging q-rung picture fuzzy hypersoft Schweizer-Sklar aggregation operators. *IEEE Open J. Comput. Soc.* **6**, 834–845 (2025).

17. Xue, X., Dhumras, H., Thakur, G., Bajaj, R. K. & Shukla, V. Schweizer-Sklar T-norm operators for picture fuzzy hypersoft sets: Advancing sustainable technology in social healthy environments. *Comput. Mater. Contin.* **84**(1), (2025).
18. Dhumras, H., Shukla, V. & Bajaj, R. K. (2025). Distance and similarity measures of q-rung picture fuzzy hypersoft set with application in pattern analysis. In: Chaturvedi, A., Roy, B.K., Tsaban, B. (eds) Mathematics and Logics in Computer Science. ICMLCS 2024. Algorithms for Intelligent Systems. Springer, Singapore.
19. Fujita, T. & Smarandache, F. An introduction to advanced soft set variants: SuperHyperSoft Sets, IndetermSuperHypersoft Sets, IndetermTreeSoft Sets, BiHyperSoft Sets, GraphicSoft Sets, and beyond. *Neutrosophic Sets Syst.* **82**, 817–843 (2025).
20. Shabir, M. & Naz, M. On bipolar soft sets. arXiv:1303.1344 (2013).
21. Naz, M. & Shabir, M. On fuzzy bipolar soft sets, their algebraic structures and applications. *J. Intell. Fuzzy Syst.* **26**, 1645–1656 (2014).
22. Akram, M. & Ali, G. Hybrid models for decision-making based on rough Pythagorean fuzzy bipolar soft information. *Granul. Comput.* **5**, 1–15 (2020).
23. Musa, S. Y. & Asaad, B. A. *Bipolar hypersoft sets. Mathematics* **9**, 1826 (2021).
24. Asaad, B. A., Musa, S. Y. & Ameen, Z. A. Fuzzy bipolar hypersoft sets: a novel approach for decision-making applications. *Math. Comput. Appl.* **29**, 50 (2024).
25. Fatimah, F., Rosadi, D., Hakim, R. B. & Alcantud, J. C. R. N-soft sets and their decision making algorithms. *Soft Comput.* **22**, 3829–3842 (2018).
26. Akram, M., Adeel, A. & Alcantud, J. C. R. Fuzzy N-soft sets: a novel model with applications. *J. Intell. Fuzzy Syst.* **35**, 4757–4771 (2018).
27. Akram, M., Ali, G. & Alcantud, J. C. R. New decision-making hybrid model: intuitionistic fuzzy N-soft rough sets. *Soft Comput.* **23**, 9853–9868 (2019).
28. Alcantud, J. C. R., Santos-García, G. & Akram, M. A novel methodology for multi-agent decision-making based on N-soft sets. *Soft Comput.* (2023).
29. Khan, M. J., Alcantud, J. C. R., Akram, M. & Ding, W. Separable N-soft sets: A tool for multinary descriptions with large-scale parameter sets. *Appl. Intell.* **55**(6), 561 (2025).
30. Zhang, H., Jia-Hua, D. & Yan, C. Multi-attribute group decision-making methods based on Pythagorean fuzzy N-soft sets. *IEEE Access* **8**, 62298–62309 (2020).
31. Riaz, M., Razzaq, A., Aslam, M. & Pamucar, D. M-parameterized N-soft topology-based TOPSIS approach for multi-attribute decision making. *Symmetry* **13**, 748 (2021).
32. Musa, S. Y. & Asaad, B. A. Bipolar M-parametrized N-soft sets: a gateway to informed decision-making. *J. Math. Comput. Sci.* **36**, 121–141 (2025).
33. Shabir, M. & Fatima, J. N-bipolar soft sets and their application in decision making. *Research Square* (2021).
34. Musa, S. Y., Alajlan, A. I., Asaad, B. A. & Ameen, Z. A. N-bipolar soft expert sets and their applications in robust multi-attribute group decision-making. *Mathematics* **13**, 530 (2025).
35. Musa, S. Y., Asaad, B. A., Alohali, H., Ameen, Z. A. & Alqahtani, M. H. Fuzzy N-bipolar soft sets for multi-criteria decision-making: Theory and application. *CMES-Comput. Model. Eng. Sci.* **143**, 911–943 (2025).
36. Musa, S. Y., Asaad, B. A., Alagal, W. & Ameen, Z. A. Novel multi-criteria decision-making approach for selecting the optimal artificial intelligence implementation in medical diagnosis using intuitionistic fuzzy N-bipolar soft sets. *Int. J. Comput. Intell. Syst.* (2025).
37. Musa, S. Y. N-bipolar hypersoft sets: Enhancing decision-making algorithms. *PLoS ONE* **19**, e0296396 (2024).
38. Musa, S. Y. & Asaad, B. A. A progressive approach to multi-criteria group decision-making: N-bipolar hypersoft topology perspective. *PLoS ONE* **19**, e0304016 (2024).
39. Paul, T. K., Jana, C. & Pal, M. Enhancing multi-attribute decision making with Pythagorean fuzzy Hamacher aggregation operators. *J. Ind. Intell.* **1**(1), 30–54 (2023).
40. El-Morsy, S. Stock portfolio optimization using Pythagorean fuzzy numbers. *J. Oper. Strateg. Analyt.* **1**(1), 8–13 (2023).
41. Chohan, M. S., Ashraf, S. & Dong, K. Enhanced forecasting of Alzheimer's disease progression using higher-order circular Pythagorean fuzzy time series. *Healthcraft Front.* **1**(1), 44–57 (2023).
42. Gul, R. & Tufail, F. Multi-attribute green supplier decision-making using picture fuzzy rough Schweizer-Sklar aggregation operators. *Int. J. Knowl. Innov. Stud.* **3**(2), 60–73 (2025).
43. Badi, I., Bouraima, M. B., Su, Q., Qiu, Y. J. & Wang, Q. P. Prioritization of poverty alleviation strategies in developing countries using the Fermatean fuzzy SWARA method. *Oppor. Chall. Sustain.* **4**(1), 33–40 (2025).
44. Garg, H. Linguistic Pythagorean fuzzy sets and its applications in multiattribute decision-making process. *Int. J. Intell. Syst.* **33**(6), 1234–1263 (2018).
45. Hussain, A., Ullah, K., Alshahrani, M. N., Yang, M.-S. & Pamucar, D. Novel Aczel-Alsina operators for Pythagorean fuzzy sets with application in multi-attribute decision making. *Symmetry* **14**(5), 940 (2022).
46. Mahmood, T., Ur Rehman, U., Ali, Z. & Haleemzai, I. Analysis of TOPSIS techniques based on bipolar complex fuzzy N-soft setting and their applications in decision-making problems. *CAAI Trans. Intell. Technol.* **8**(2), 478–499 (2023).
47. Mahmood, T., Ur Rehman, U., Shahab, S., Ali, Z. & Anjum, M. Decision-making by using TOPSIS techniques in the framework of bipolar complex intuitionistic fuzzy N-soft sets. *IEEE Access* **11**, 105677–105697 (2023).

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