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Einstein aggregation operators for multicriteria group decision making in uncertain environments using cubic picture fuzzy sets

Muhammad Naeem Khan Tanoli[1], Khadija Rafique[1], Zafar Mahmood[2]✉, Nesreen Sirelkhtam Elmki Abdalla[3], Ioan-Lucian Popa[4,5] & Abhinav Kumar[6,7,8]

This research presents an in-depth study of various operational laws, Einstein operations, and novel aggregation strategies for handling cubic picture fuzzy data. We developed and presented three new arithmetic averaging operators. These operators are called cubic picture fuzzy Einstein weighted averaging (CPFEWA), cubic picture fuzzy Einstein ordered weighted averaging (CPFEOWA), and cubic picture fuzzy Einstein hybrid weighted averaging (CPFEHWA). These operators have been designed to provide more precise and accurate calculations for arithmetic averaging. Particularly, the CPFEHWA operator extends the capabilities of both CPFEWA and CPFEOWA operators. To gain a better understanding, we thoroughly investigate the features of these operators and connect them to existing aggregate operators. We show how these innovative Einstein operators can be useful and expose their derived operators, such as CIFEWA, CFEWA, PFEWA, CIFEOWA, CFEOWA, PFEOWA, CIFEHWA, CFEHWA, and PFEHWA. We developed three properties of these operators as idempotency, monotonicity and boundedness. Furthermore, we show how the CPFEHWA operator may be used in multiple attribute decision-making (MADM) scenarios employing cubic picture fuzzy data. The new insights gained from this study are valuable as they offer innovative ways of collecting and interpreting cubic picture fuzzy data. This adds to the existing knowledge base, making it easier to understand and use in future research. We propose an informative set of tools for decision-making processes involving complex and unreliable data, presenting the CPFEWA, CPFEOWA, and CPFEHWA operators. A numerical example demonstrating the application of the CPFEHWA operator in a real-life setting is provided to demonstrate the effectiveness of our suggested concept. This example's results support the proposed methodology and illustrate its potential significance in practical applications.

Keywords Cubic picture fuzzy sets, Operational laws, Averaging aggregation operators

Abbreviations

CPFS	Cubic picture fuzzy set
CIFEWA	Cubic intuitionistic fuzzy Einstein weighted averaging
CFEWA	Cubic fuzzy Einstein weighted averaging
PFEWA	Picture fuzzy Einstein weighted averaging
CIFEOWA	Cubic intuitionistic fuzzy Einstein ordered weighted averaging
CFEOWA	Cubic fuzzy Einstein ordered weighted averaging
PFEOWA	Picture fuzzy Einstein ordered weighted averaging

[1]Department of Mathematics and Statistics, Hazara University, Mansehra, Pakistan. [2]College of Mechanical and Vehicle Engineering, Hunan University, Changsha, Hunan 410082, P. R. China. [3]Department of Mathematics, College of Science, King Khalid University, Abha, Saudi Arabia. [4]Department of Computing, Mathematics and Electronics, "1 December 1918" University of Alba Iulia, 510009 Alba Iulia, Romania. [5]Faculty of Mathematics and Computer Science, Transilvania University of Brasov, Iuliu Maniu Street 50, 500091 Brasov, Romania. [6]Department of Nuclear and Renewable Energy, Ural Federal University Named after the First President of Russia Boris Yeltsin, Ekaterinburg 620002, Russia. [7]Centre for Research Impact & Outcome, Chitkara University Institute of Engineering and Technology, Chitkara University, Rajpura 140401, Punjab, India. [8]Department of Mechanical Engineering and Renewable Energy, Technical Engineering College, The Islamic University, Najaf, Iraq. ✉email: zafarmaths222@gmail.com

CIFEHWA	Cubic intuitionistic fuzzy Einstein hybrid weighted averaging
CFEHWA	Cubic fuzzy Einstein hybrid weighted averaging
PFEHWA	Picture fuzzy Einstein hybrid weighted averaging
CPFEWA	Cubic picture fuzzy Einstein weighted averaging
CPFEOWA	Cubic picture fuzzy Einstein ordered weighted averaging
CPFEHWA	Cubic picture fuzzy Einstein hybrid weighted averaging
CPFNs	Cubic picture fuzzy numbers
MCGDM	Multi-criteria group decision-making

Literature review

Zadeh[1] had given the concept of fuzzy set, interval-value fuzzy sets (IVFSs)[2], the intuitionistic fuzzy sets (AIFSs) of Atanassov[3] and interval-valued the intuitionistic fuzzy sets (AIVIFSs)[4] are three fuzzy sets extensions. Anyway, as detected in (Zadeh)[5] and in[6,7], Atanassov's intuitionistic fuzzy sets (AIFS) are numerically equal to the IVFSs. The AIFS idea has been linked to the collection of intuitionistic fuzzy information. It's a complicated subject that demands an in-depth understanding of both intuitionistic fuzzy sets and aggregation operators[8]. After some time, the opportunity of AIFSs is one more outlined by sanctioning the non-membership and membership capability to assume interval-valued beside such appearances presenting the opportunity of interval-valued intuitionistic fuzzy sets (IVIFSs)[9]. Taking into consideration the purpose of establishing a decision consequence, the collection of intuitionistic fuzzy numbers (IFNs) is a significant milestone. For clarification, Xu[10] acquired several aggregation operators (aggregate), including the operators of the IFWA, IFOWA, and IFHA, and defined their characteristics. Apparently, there are some new geometric aggregation operators that have been proposed for IFNs. These operators, which include IFWG, IFOGA, and IFHG, were developed by Xu and Yager[11]. It's fascinating to see how these mathematical tools are being used in different fields and applications. Zeng and Su[12] investigated the development of an intuitionistic fuzzy ordered weighted distance (IFOWD) operator, as discussed in their research on the acquisition of IFOWD and associated with group decision-making on system selection. Their research has provided valuable insights into this important topic and has helped pave the way for future developments in this area. The IFOWD operator has many potential applications in a wide range of fields. Jun, Kim, and Yang[13] proposed the cubic set. Cubic sets, which have two different sorts of illustrations—by incorporating two separate factors, one for determining the degree of membership and the other for the non-membership degree, the concept of intuitionistic fuzzy sets extends and generalizes the traditional notion of fuzzy sets, allowing a more important representation of uncertainty and hesitancy in the membership assignment. The representation of the membership level as an interval in intuitionistic fuzzy sets allows for a more flexible and inclusive characterization of uncertainty, while considering the non-membership level as a standard fuzzy set enables a clear delineation of elements that do not belong to the set. In their pioneering work, Muhiuddin and Al-roqi[14] not only introduced the fundamental concepts of cubic soft sets, including internal and external cubic soft sets, but also extended the framework to encompass P-cubic and R-cubic soft subsets, along with operations such as R-union, R-intersection, P-union, and the cubic soft set complement. In addition to the core concepts, the study further considered various related features, and it proposes the introduction of present cubic soft BCK/BCI-algebras, integrating the innovative notion of cubic softer sets as a valuable tool for analyzing and characterizing these algebraic structures. According to the research conducted by Muhiuddin, Ahn, Kim, Su, and Jun[15], they not only introduce the novel concepts of stable cubic set, stable element, evaluative set, and stable degree but also conduct a comprehensive examination of their associated properties, paving the way for a deeper understanding of these interesting mathematical constructs. The research delves into the investigation of the stability and instability aspects concerning both internal and external cubic sets, shedding light on how these sets behave under certain conditions. Additionally, the study explores the stability and instability characteristics of cubic set P-union, R-union, P-intersection, and R-intersection, providing valuable insights into the behaviors and properties of these operations within the framework of cubic sets. Jun, Muhiuddin, Ozturk, and Roh[16] undertake a comprehensive investigation into the interrelationships among cubic soft-sub algebras, cubic soft sub-algebras, and cubic soft ideals, explaining the fundamental connections and paving the way for a deeper understanding of these algebraic structures. The descriptions of soft cubic ideals demonstrate a well-considered and systematic approach, providing clear justifications for the operations of R-union and R-intersection within the context of cubic soft ideals, enhancing the comprehension and practical applicability of these algebraic concepts. In their seminal work, Fahmi, Amin, Abdullah, and Ali[17] make a significant contribution to the field by introducing the application of Einstein aggregation operators within the framework of cubic fuzzy sets, offering a new perspective on handling aggregation and decision-making processes in uncertain and hesitant environments. In order to address the complexities of multi-criteria decision-making, Fahmi, Abdullah, and Amin proposed a trapezoidal linguistic cubic hesitant fuzzy TOPSIS methodology[18], which provides a robust and effective framework for handling decision problems involving uncertain and hesitant information in a linguistic context. In their comprehensive study, Fahmi, Abdullah, Amin, Ali, and Rahman[19] not only established the operating laws and bounds for triangular cubic fuzzy numbers but also proposed the development of a novel tool called the triangular cubic fuzzy hybrid aggregation (TCFHA) administrator. This innovative administrator enables the effective combination of individual decision-makers' specific fuzzy choice structures, resulting in an aggregate cubic fuzzy decision matrix, thus facilitating more informed and coherent decision-making processes in complex and uncertain situations.

In their important effort, Amin, Fahmi, Abdullah, Ali, Ahmad, and Ghani[20] introduced a series of novel operators, including the generalized triangular cubic linguistic hesitant fuzzy weighted geometric (GTCHFWG), GTCLHFOWA, GTCLHFOWG, GTCLHFHA, and GTCLHFHG operators, which offer useful tools for effectively handling linguistic hesitant fuzzy information and facilitating decision-making processes in various real-world applications. In their groundbreaking exploration, Fahmi, Abdullah, Amin, Ali, and Khan[21] present a comprehensive set of aggregation operators made-to-order for triangular cubic linguistic hesitant fuzzy sets. These operators contain the triangular cubic linguistic hesitant fuzzy (geometric) operator, triangular cubic linguistic hesitant fuzzy weighted geometric (TCLHFWG) operator, triangular cubic linguistic hesitant fuzzy ordered weighted geometric (TCHFOWG) operator, and triangular cubic linguistic hesitant fuzzy hybrid geometric (TCLHFHG) operator, offering a powerful toolkit to effectively manage and combine linguistic hesitant fuzzy information in decision-making processes and practical applications. In their revolutionary effort, Ashraf, Abdullah, and Qadir[22] proposed a novel framework for cubic picture fuzzy sets and introduced a range of associated operators, offering a fresh perspective on the representation and manipulation of uncertainty and vagueness in the context of picture fuzzy sets. In their comprehensive study, Coung and Pham[23] thoroughly investigate the fundamental fuzzy logic operators, including negations, conjunctions, disjunctions, and implications, within the domain of picture fuzzy sets, providing valuable insights into the manipulation and behavior of these operators in handling complex and imprecise information represented by picture fuzzy sets. In their formative investigation, Abdullah and Ashraf[24] introduce and elaborate on the concept of a cubic picture fuzzy set (CPFS), providing a comprehensive exploration of its fundamental operations and proposing innovative aggregation operators specifically designed to effectively aggregate and manage the information represented by CPFS in various decision-making and data analysis tasks. Tanoli et al.[25] established and formulated a set of aggregation operators within the context of complex cubic fuzzy sets, contributing to the advancement of decision-making and data processing methodologies in consequence involving intricate and uncertain information represented by complex cubic fuzzy sets. In their research, Jan, Barukab, Khan, Jun, and Khan[26] provide a comprehensive description of Hamacher aggregation operators, offering insights into the application and benefits of these operators in aggregating information and handling uncertainty in various decision-making and data fusion tasks. In their study, Mushtaq et al.[27] employed the Einstein operational laws to the realm of q-ROFHSS (quantum hesitant fuzzy soft sets) and successfully applied them to address decision-making problems, showcasing the flexibility and effectiveness of the Einstein operational laws in handling uncertainty and hesitation in quantum-based decision models. Abbasi, Ashraf, Hameed, and Eldin[28] introduced Pythagorean Einstein aggregation operators using an extended EDAS (Evaluation based on Distance from Average Solution) approach, demonstrating the applicability and value of these operators in resolving Complex Decision Aid Systems, thus providing a valuable contribution to decision-making methodologies in complicated and ambiguous environments. In their innovative study, Aslam and Fahmi[29] propose a novel and unique approach centered on the trapezoidal cubic linguistic uncertain fuzzy Einstein hybrid weighted averaging operator, offering a powerful and flexible tool for handling ambiguous and linguistic information, which can be valuable in decision-making processes and practical applications across various domains. Recently, several advanced aggregation operators have been introduced to handle uncertainty in fuzzy environments. For instance, Meher et al. developed Dombi weighted geometric operators for trapezoidal-valued intuitionistic fuzzy numbers in multi-attribute group decision-making[32], proposed Aczel–Alsina weighted geometric operators for the selection of e-learning platforms under trapezoidal-valued intuitionistic fuzzy information[33], and extended the Aczel–Alsina approach to interval-valued Fermatean fuzzy sets for group decision-making applications[34]. These studies highlight the growing importance of novel aggregation strategies to address complex decision-making problems under different fuzzy frameworks. In recent years, multi-criteria decision-making (MCDM) models and advanced fuzzy set extensions have been widely explored for solving complex real-world problems. Duško and Božanić[35] optimized military decision-making through an integrated FUCOM–EWAA–COPRAS-G framework, while Chohan et al.[36] applied higher-order circular Pythagorean fuzzy time series for forecasting Alzheimer's disease progression. Iqbal and Kalsoom[37] introduced advanced logarithmic aggregation operators to enhance decision-making under uncertainty, and Rahman[38] utilized complex polytopic fuzzy systems for COVID-19 vaccine selection. Similarly, Khan and Wang[39] proposed generalized Fermatean fuzzy aggregation operators, and Sarfarz and Božanić[40] applied Sugeno–Weber t-norm-based spherical fuzzy information for water recycling. Other studies have evaluated logistics performance using MCDM approaches[41], developed systematic literature review methods with multi-criteria analysis[42], improved industrial control using spherical hesitant fuzzy Yager aggregation[43], and enhanced ranking methods incorporating triangular fuzzy numbers[44]. Additionally, picture fuzzy linear programming has been investigated for addressing optimization challenges[45]. These works demonstrate the growing trend of integrating fuzzy logic and novel aggregation strategies into decision-making frameworks to handle complex, uncertain, and multi-dimensional problems.

In this study, we introduce cubic picture fuzzy numbers (CPFNs), a novel representation that combines interval-valued picture fuzzy numbers and generalized intuitionistic picture fuzzy numbers. Building upon Einstein's t-norm and t-conorm, we not only lay a solid foundation for the development of procedures to handle and manipulate cubic picture fuzzy numbers (CPFNs) but also introduce accuracy and score functions to evaluate and compare the two CPFNs, enabling a comprehensive and effective analysis of their performance in various decision-making and data processing situations. In this research, we put forth three novel aggregation operators, namely cubic picture fuzzy Einstein weighted averaging (CPFEWA), cubic picture fuzzy Einstein ordered weighted averaging (CPFEOWA), and cubic picture fuzzy Einstein hybrid weighted averaging (CPFEHWA). These operators are specifically designed to effectively aggregate a set of cubic picture fuzzy numbers (CPFNs), connecting the power of Einstein's t-norm and t-conorm, and providing valuable tools for usage and creating uncertain information represented by CPFNs in various decision-making and data analysis tasks.

Gap before establishing our proposed structure and important research contributions

The gap that existed before establishing Einstein's operations and operators like CPFEWA, CPFEOWA, and CPFEHWA in the environment of cubic picture fuzzy sets can be summarized as:

Traditional fuzzy sets were effective in handling uncertainty, but they had limitations when dealing with complex information and capturing decision-makers' preferences accurately. This created a gap in decision-making processes where complex and uncertain data needed to be processed more effectively. The introduction of cubic picture fuzzy sets bridged this gap by extending traditional fuzzy sets to handle complex information. However, there was still a need for advanced aggregation operators that could handle the complexities of cubic picture fuzzy sets while accurately capturing decision-makers' preferences and uncertainties. Einstein's operations and operators, such as CPFEWA, CPFEOWA, and CPFEHWA, were developed to fill this gap. These operators combine the benefits of cubic picture fuzzy sets with Einstein aggregation techniques, allowing for more accurate and flexible decision-making in complex and uncertain environments. They provide robust and comprehensive methods to aggregate information and capture decision-makers' preferences effectively. In summary, the gap before establishing Einstein's operations and operators in the environment of cubic picture fuzzy sets (CPFSs) was the lack of advanced aggregation techniques that could handle complex information and accurately capture decision-makers' preferences. Einstein's operators address this gap by providing effective and comprehensive aggregation methods in decision-making processes.

Motivations for the proposed structure

Cubic picture fuzzy sets (CPFS) and their operators, such as CPFEWA, CPFEOWA, and CPFEHWA, have the following motivations:

CPFS: Cubic picture fuzzy sets aim to represent and handle complex and uncertain information by extending traditional fuzzy sets. They provide a more flexible and accurate framework for decision-making in uncertain environments.

The Einstein operators provide a more comprehensive and informative framework for understanding cubic picture fuzzy sets compared to simple aggregation operators. Therefore, it is necessary to define these Einstein operators.

CPFEWA (Cubic Picture Fuzzy Einstein Weighted Averaging): CPFEWA combines the benefits of cubic picture fuzzy sets and Einstein aggregation to compute a weighted average. The motivation is to handle complex and uncertain information more effectively, resulting in accurate and flexible decision-making.

CPFEOWA (Cubic Picture Fuzzy Einstein Ordered Weighted Averaging): CPFEOWA extends the concept of ordered weighted averages to cubic picture fuzzy sets. It allows decision-makers to consider both the importance of criteria and their degree of optimism or pessimism. The aim is to capture decision-makers' preferences and uncertainties comprehensively.

CPFEHWA (Cubic Picture Fuzzy Einstein Hybrid Weighted Averaging): CPFEHWA is a hybrid aggregation operator that combines weighted and hybrid averages. It takes into account both the weights and the hybrid distances between cubic picture fuzzy sets. The motivation is to provide a robust and accurate aggregation approach by considering the impact of weights and hybrid distances. Moreover, the integration of cubic picture fuzzy sets into various decision support systems and artificial intelligence applications has shown promising results in enhancing the robustness and efficiency of these systems.

Novelty of research

We developed and presented three new arithmetic Einstein operations. This research introduces three novel Einstein-based aggregation operators – CPFEWA, CPFEOWA, and CPFEHWA – specifically designed for cubic picture fuzzy data. The CPFEHWA operator uniquely integrates the advantages of both CPFEWA and CPFEOWA, enhancing arithmetic averaging accuracy. The study establishes key mathematical properties (idempotency, monotonicity, boundedness) and demonstrates their effectiveness in multiple attribute decision-making (MADM). Additionally, several derived operators are formulated, expanding the framework of cubic picture fuzzy aggregation. A real-life numerical example validates the practical applicability and superiority of the proposed operators, offering an innovative and comprehensive toolset for handling complex and uncertain data in decision-making processes.

The sections of this piece of work are as follows.

In this paper, we embark on a comprehensive exploration of the intriguing world of fuzzy sets, cubic sets, cubic fuzzy sets, and cubic picture fuzzy sets. Sections "Prelimineries" serves as the bedrock of our journey, delving into the fundamental principles and definitions that underpin these concepts, setting the stage for deeper investigation. Sections "The operational laws under CPFSs" to onward are all novel concepts. Moving beyond the basics, in Section "The operational laws under CPFSs", we delve into the realm of arithmetic operations, exploring the operational laws governing these sets. This provides the necessary tools for more advanced exploration. Moreover, in Section "Einstein operations applied to CPFSs.", we introduce new operations inspired by Einstein's principles, adding a novel dimension to the study. Section 5 unveils the innovative concept of Einstein Operators, a trio comprising CPFEWA, CPFEOWA, and CPFEHWA. Through specific cases of these innovative Einstein Operators, we demonstrate their utility and unveil derived operators, including CIFEWA, CFEWA, PFEWA, CIFEOWA, CFEOWA, PFEOWA, CIFEHWA, CFEHWA, and PFEHWA. Sections "A method for generating decisions with several attributes using fuzzy information from CPFS" and "Explanatory example" shift our focus towards practical applications. We explore Multiple Attribute Decision Making (MADM) problems and illustrate their relevance through explanatory examples. Section Comparison analysis" offers an insightful comparative analysis, highlighting the advantages of our proposed techniques against existing methods, showcasing their potential contributions to the field. Lastly, in Section 9, we draw together the threads of our study to present a conclusive summary of our findings and their implications for the broader field of fuzzy set theory and operations.

Prelimineries

By utilizing procedures and operators that produce the triangle norm and co-norm, we will go through some fundamental concepts relating to fuzzy sets (FS), intuitionistic fuzzy sets (IFS), picture fuzzy sets (PFS), and cubic sets in this part. Think about additional well-known subjects that will be utilized in the ensuing investigation.

Definition 2.1 [1] Let z be a non-empty finite set. A fuzzy set A over in element t is defined as follows: $A = \{\langle t, P_A(t) | t \in Z\rangle\}$, where $P_A(t)$ represent the membership degree such that $P_A(t) \in [0,1]$.

Definition 2.2 [24] Consider Ω as a collection of entirely closed subintervals of [0,1] and $P_A = [P_{L_A}, P_{U_A}] \in \Omega$, where P_{L_A} and P_{U_A} are expressed as the left extreme and the right extreme, respectively. Let Z be the universe set that isn't empty. Then $A = \{\langle t, \xi_{P_A}(t) | t \in Z\rangle\}$, is presented as an interval-valued fuzzy set of Z, where ξ_{P_A} : Z → [0,1] is the degree of its membership in Z and $\xi_{P_A} = [\xi_{P_{LA}}, \xi_{P_{UA}}]$ is said to be an interval-valued fuzzy number.

Definition 2.3 [2] Let z be a non-empty finite set. An intuitionistic fuzzy set A over in element t is defined as follows $A = \{\langle t, P_A(t), N_A(t), | t \in Z\rangle\}$. where $P_A : Z \to [0,1]$ and $N_A : Z \to [0,1]$ are shown as the positive and negative membership degrees of t in Z, respectively. Also $P_{,A}$ and N_A fulfil the following requirement: $(\forall t \in Z)(0 \le P_A(t) + N_A(t) \le 1)$.

Definition 2.4 [12] Let z be a non-empty finite set. A cubic set A over in element t is defined as follows: $A = \{\langle t, \xi_A(t), e_A(t) | t \in Z\rangle\}$, where $\xi_A(t)$ is a fuzzy set with interval values in Z and $e_A(t)$ in Z is a fuzzy set. A cubic set is shortly denoted as $A = \langle \xi_A, e_A\rangle$.

Definition 2.5 [12] Let z be a non-empty finite set. Then the cubic set $A = \langle \xi_A, e_A\rangle$. is known as an internal cubic set if $\xi_A - (t) \le e_A(t) \le {\xi_A}^+(t)$ for all $t \in Z$.

Definition 2.6 [12] Let z be a non-empty finite set. Then the cubic set $A = \langle \xi_A, e_A\rangle$. is known as an external cubic set if $(e_A(t) \notin (\xi_A - (t), {\xi_A}^+(t))$ for all $t \in Z$.

Definition 2.7 [29] Let z be a non-empty finite set. A picture fuzzy set A over in element t is defined as follows: $A = \{\langle t, P_A(t), I_A(t), N_A(t), | t \in Z\rangle\}$, where $P_A : Z \to [0,1]$, $I_A : Z \to [0,1]$ and $N_A : Z \to [0,1]$ are the positive-, neutral-, and negative-membership degrees of t in Z are denoted in that order, symbolizing their respective degrees of membership or influence in the fuzzy set representation. Also P_A, I_A and N_A fulfil the following circumstance.

$$(\forall t \in Z)(0 \le P_A(t) + I_A(t) + N_A(t) \le 1)$$

Definition 2.8 [29] Let z be a non-empty finite set. An interval-valued picture fuzzy set A over in element t is defined as follows: $A = \{\langle t, \xi_{P_A}(t), \xi_{I_A}(t), \xi_{N_A}(t) | t \in Z\rangle\}$, where $\xi_{P_A} : Z \to \Omega, \xi_{I_A} : Z \to \Omega$ and $\xi_{N_A} : Z \to \Omega$ are represented by the positive-, neutral-, and negative-membership degrees of t in Z, respectively, in that order. Also ξ_{P_A}, ξ_{I_A} and ξ_{N_A} satisfy the following circumstances.

The fundamental ideas and their operators governing Cubic Picture Fuzzy Sets (CPFSs)

Definition 2.9.1 [21] Let z be a non-empty finite set.. A CPFS A over in element t is defined as follows:

$$A = \langle \xi_A, e_A\rangle = \left\{t, \langle\langle \xi_{P_A}(t), \xi_{I_A}(t), \xi_{N_A}(t)\rangle, \langle P_A(t), I_A(t), N_A(t)\rangle | t \in Z\right\},$$

where

$$\xi_A = \left\{\langle t, \xi_{P_A}(t) = [{\xi_{P_A}}^-(t), {\xi_{P_A}}^+(t)], \xi_{I_A}(t) = [{\xi_{I_A}}^-(t), {\xi_{I_A}}^+(t)], \xi_{N_A}(t) = [\xi_{N_A} - (t), {\xi_{N_A}}^+(t)] | t \in Z\rangle\right\}$$

is termed as a picture fuzzy set of Z with interval-valued and

$e_A = \{\langle t, P_A(t), I_A(t), N_A(t), | t \in Z\rangle\}$ is termed as a picture fuzzy set of Z. To keep it simple we represented the CPFS as $A = \langle \xi_A, e_A\rangle$ in Table 1.

The CPFS is restricted by the condition

$$[0, 0] \le \sup(\xi_{P_A}(t)) + Sup(\xi_{I_A}(t)) + \sup(\xi_{N_A}(t)) \le [1, 1] \text{ and } 0 \le P_A(t) + I_A(t) + N(t) \le 1$$

Z	ξ_{P_A}	ξ_{I_A}	ξ_{N_A}	e_A
t_1	[0.3,0.4]	[0.0,0.1]	[0.4,0.4]	(0.5,0.1,0.3)
t_2	[0.2,0.3]	[0.3,0.4]	[0.1,0.2]	(0.4,0.4,0.1)
t_3	[0.1,0.3]	[0.3,0.3]	[0.2,0.3]	(0.2,0.1,0.6)

Table 1. Cubic picture fuzzy set $A = \langle \xi_A, e_A\rangle$.

Example 2.1 Suppose $Z = \{t_1, t_2, t_3\}$ is the universe set. Then the CPFS $A = \langle \xi_A, e_A \rangle$ is represented in the following table.

Definition 2.9.2 [21] Let z be a non-empty finite set. A CPFS $A = \langle \xi_A, e_A \rangle$ over an element t is defined as follows:

(1) Positive-internal if $\xi_{P_A}{-}(t) \leq P_A(t) \leq \xi_{P_A}{}^{+}(t) \forall\, t \in Z$.
(2) Neutral-internal if $\xi_{I_A}{-}(t) \leq I_A(t) \leq \xi_{I_A}{}^{+}(t), \forall\, t \in Z$.
(3) Nagative-internal if $\xi_{N_A}{-}(t) \leq N_A(t) \leq \xi_{N_A}{}^{+}(t), \forall\, t \in Z$.

If the CPFS $A = \langle \xi_A, e_A \rangle$ in Z is a satisfied condition $(1) - (3)$ then the CPFS is known as an internal CPFS in Z.

Example 2.2. Suppose $Z = \{t_1, t_2, t_3\}$ is the non-empty universe set. Then the internal CPFS $A = \langle \xi_A, e_A \rangle$ is represented as the following Table 2.

Definition 2.9.3 [21] Let z be a non-empty finite set. Then the CPFS $A = \langle \xi_A, e_A \rangle$ of Z is referred to as

(1) Positive-external if $P_A(t) \notin (\xi_{P_A}{-}(t), \xi_{P_A}{}^{+}(t)), \forall\, t \in Z$.
(2) Neutral-external if.$I_A(t) \notin (\xi_{I_A}{-}(t), \xi_{I_A}{}^{+}(t)), \forall\, t \in Z$.
(3) Negative-external if $N_A(t) \notin (\xi_{N_A}{-}(t), \xi_{N_A}{}^{+}(t)), \forall\, t \in Z$.

If the CPFS set $A = \langle \xi_A, e_A \rangle$ in Z is satisfied conditions, $(1) - (3)$ then the CPFS is represented as an external CPFS in Z.

Definition 2.9.4 [23]: Let $A = \langle \xi_A, e_A \rangle$ and $B = \langle \xi_B, e_B \rangle$ be two CPFSs, then equality is defined as

$$A = B \Longleftrightarrow \xi_A = \xi_B \text{ and } e_A = e_B.$$

Definition 2.9.5 [23] Let $A = \langle \xi_A, e_A \rangle$ and $B = \langle \xi_B, e_B \rangle$ be two CPFSs, then P-order is defined as

$$A \subseteq_P B \iff \xi_A \subseteq_P \xi_B \text{ and } e_A \leq e_B.$$

Definition 2.9.6 [23] Let $A = \langle \xi_A, e_A \rangle$ and $B = \langle \xi_B, e_B \rangle$ be two CPFSs, then R-order is defined as

$$A \subseteq_R B \iff \xi_A \subseteq_R \xi_B \text{ and } e_A \geq e_B$$

Definition 2.9.7 [23] Let $A_j = \langle \xi_{A_j}, e_{A_j} \rangle$ be a family of CPFSs of Z, then P-order union is defined as

$$\bigcup_{j \in N}^{P} A_j = \left\langle \bigcup_{j \in N} (\xi_{A_j}), \bigvee_{j \in N} (e_{A_j}) \right\rangle$$
$$= \left\{ t, \left\langle \bigcup_{j \in N} \xi_{P_{A_j}}, \bigvee_{j \in N} (P_{A_j}(t)) \right\rangle, \left\langle \bigcup_{j \in N} \xi_{I_{A_j}}(t), \bigvee_{j \in N} (I_{A_j}(t)) \right\rangle \left\langle \bigcup_{j \in N} \xi_{N_{A_j}}(t), \bigvee_{j \in N} (N_{A_j}(t)) \right\rangle | t \epsilon Z \right\}$$

Definition 2.9.8 [23] Let $A_j = \langle \xi_{A_j}, e_{A_j} \rangle$ be a family of CPFSs of Z, then R-order union is defined as

$$\bigcup_{j \in N}^{P} A_j = \left\langle \bigcup_{j \in N} (\xi_{A_j}), \bigwedge_{j \in N} (e_{A_j}) \right\rangle$$
$$= \left\{ t, \left\langle \bigcup_{j \in N} \xi_{P_{A_j}}, \bigwedge_{j \in N} (P_{A_j}(t)) \right\rangle, \left\langle \bigcup_{j \in N} \xi_{I_{A_j}}(t), \bigwedge_{j \in N} (I_{A_j}(t)) \right\rangle \left\langle \bigcup_{j \in N} \xi_{N_{A_j}}(t), \bigwedge_{j \in N} (N_{A_j}(t)) \right\rangle | t \epsilon Z \right\}$$

Definition 2.9.9 [23] Let $A_j = \langle \xi_{A_j}, e_{A_j} \rangle$ be a family of CPFSs of Z, then P-order intersection is defined as

Z	ξ_{P_A}	ξ_{I_A}	ξ_{N_A}	e_A
t_1	[0.33,0.40]	[0.23,0.31]	[0.11,0.17]	(0.36,0.28,0.14)
t_2	[0.21,0.32]	[0.33,0.41]	[0.13,0.23]	(0.24,0.37,0.19)
t_3	[0.14,0.25]	[0.30,0.39]	[0.29,0.37]	(0.20,0.35,0.33)

Table 2. Internal cubic picture fuzzy set $A = \langle \xi_A, e_A \rangle$.

$$\bigcap_{j\in N} A_j = \left\langle \bigcap_{j\in N}\left(\xi_{A_j}\right), \bigwedge_{j\in N}\left(e_{A_j}\right)\right\rangle$$
$$= \left\{t, \left\langle \bigcap_{j\in N} \xi_{P_{A_j}}, \bigwedge_{j\in N}\left(P_{A_j}(t)\right)\right\rangle, \left\langle \bigcap_{j\in N} \xi_{I_{A_j}}(t), \bigwedge_{j\in N}\left(I_{A_j}(t)\right)\right\rangle \left\langle \bigcap_{j\in N} \xi_{N_{A_j}}(t), \bigwedge_{j\in N}\left(N_{A_j}(t)\right)\right\rangle |t\epsilon Z\right\}$$

Definition 2.9.10 [23] Let $A_j = \langle \xi_{A_j}, e_{A_j}\rangle$ be a family of CPFSs of Z, then R-order intersection is defined as

$$\bigcap_{j\in N} A_j = \left\langle \bigcap_{j\in N}\left(\xi_{A_j}\right), \bigvee_{j\in N}\left(e_{A_j}\right)\right\rangle$$
$$= \left\{t, \left\langle \bigcap_{j\in N} \xi_{P_{A_j}}, \bigvee_{j\in N}\left(P_{A_j}(t)\right)\right\rangle, \left\langle \bigcap_{j\in N} \xi_{I_{A_j}}(t), \bigvee_{j\in N}\left(I_{A_j}(t)\right)\right\rangle \left\langle \bigcap_{j\in N} \xi_{N_{A_j}}(t), \bigvee_{j\in N}\left(N_{A_j}(t)\right)\right\rangle |t\epsilon Z\right\}$$

Definition 2.9.11 [23] Given a CPFS A with parameters $A = \langle \xi_A, e_A\rangle$, the scoring function S(A) allows evaluating the system's performance, the accuracy function H(A) provides a measure of its correctness, and the membership uncertainty index T(A) as well as the hesitation uncertainty index G(A) offer insights into the system's uncertainty levels.

Score function:

$$Sc(A) = \frac{\left(\xi_{P_A}- + \xi_{P_A}{}^{+} - \xi_{I_A}- - \xi_{I_A}{}^{+} - \xi_{N_A}- + \xi_{N_A}{}^{+}\right) + \left(P_A + 1 - I_A + 1 + N_A\right)}{6}$$
$$Sc(A) = \frac{2 + \xi_{P_A}- + \xi_{P_A}{}^{+} - \xi_{I_A}- - \xi_{I_A}{}^{+} - \xi_{N_A}- + \xi_{N_A}{}^{+} + P_A - I_A + N_A}{6}$$

Accuracy function:

$Ac(A) = H(A) = \frac{\left(\xi_{P_A}- + \xi_{P_A}{}^{+} + \xi_{N_A}- + \xi_{N_A}{}^{+}\right) + \left(P_A + N_A\right)}{6}$

Membership uncertainty index $\boldsymbol{T}(\boldsymbol{A})$:

$$T(A) = \left(\xi_{P_A}{}^{+} + \xi_{I_A}{}^{+} + \xi_{N_A}{}^{+} + P_A + I_A + N_A\right) - \left(\xi_{P_A}- + \xi_{I_A}- + \xi_{N_A}-\right)$$

Hesitation uncertainty index:

$$G(A) = \left(\xi_{P_A}{}^{+} + \xi_{I_A}{}^{+} + \xi_{N_A}{}^{+} + \xi_{P_A}- + \xi_{I_A}- + \xi_{N_A}-\right) - \left(P_A + I_A + N_A\right)$$

The operational laws under CPFSs

In this section, we deliberated several basic operations like sum, product, scalar and exponential multiplication.

Consider two CPFSs

$$A = \left\{\left\langle \left[\xi_{P_A}-, \xi_{P_A}{}^{+}\right], \left[\xi_{I_A}-, \xi_{I_A}{}^{+}\right], \left[\xi_{N_A}-, \xi_{N_A}{}^{+}\right]\right\rangle, \langle P_A, I_A, N_A\rangle\right\}$$
$$B = \left\{\left\langle \left[\xi_{P_B}-, \xi_{P_B}{}^{+}\right], \left[\xi_{I_B}-, \xi_{I_B}{}^{+}\right], \left[\xi_{N_B}-, \xi_{N_B}{}^{+}\right]\right\rangle, \langle P_B, I_B, N_B\rangle\right\}$$

and for any scalar $k > 0$

The sum of two CPFSs is defined as

$$A \oplus B = \left[\left\langle \begin{array}{c} \left[\xi_{P_A}{}^{-} + \xi_{P_B}{}^{-} - \xi_{P_A}{}^{-}.\xi_{P_B}{}^{-}, \xi_{P_A}{}^{+} + \xi_{P_B}{}^{+} - \xi_{P_A}{}^{+}.\xi_{P_B}{}^{+}\right] \\ \left[\xi_{I_A}{}^{-} + \xi_{I_B}{}^{-} - \xi_{I_A}{}^{-}.\xi_{I_B}{}^{-}, \xi_{I_A}{}^{+} + \xi_{I_B}{}^{+} - \xi_{I_A}{}^{+}.\xi_{I_B}{}^{+}\right] \\ \left[\xi_{N_A}{}^{-}.\xi_{N_B}{}^{-}, \xi_{N_A}{}^{+}.\xi_{N_B}{}^{+}\right] \\ \langle P_A.P_B, I_A.I_B, N_A + N_B - N_A.N_B\rangle \end{array}\right\rangle\right]$$

The product of two CPFSs is defined as

$$A \otimes B = \left[\left\langle \begin{array}{c} \left[\xi_{P_A}{}^{-}.\xi_{P_B}{}^{-}, \xi_{P_A}{}^{+}.\xi_{P_B}{}^{+}\right] \\ \left[\xi_{I_A}{}^{-}.\xi_{I_B}{}^{-}, \xi_{I_A}{}^{+}.\xi_{I_B}{}^{+}\right] \\ \left[\begin{array}{c} \xi_{N_A}{}^{-} + \xi_{N_B}{}^{-} - \xi_{N_A}{}^{-}.\xi_{N_B}{}^{-}, \\ \xi_{N_A}{}^{+} + \xi_{N_B}{}^{+} - \xi_{N_A}{}^{+}.\xi_{N_B}{}^{+} \end{array}\right] \\ \left\langle \begin{array}{c} P_A + P_B - P_A.P_B, \\ I_A + I_B - I_A.I_B, \\ N_A.N_B \end{array}\right\rangle \end{array}\right\rangle\right]$$

The scalar multiplication is defined as

$$KA=\left[\left\langle\begin{array}{c}\left[(1-\left(1-\xi_{P_A}{}^{-}\right)^k,(1-\left(1-\xi_{P_A}{}^{+}\right)^k\right]\\\left[(1-\left(1-\xi_{I_A}{}^{-}\right)^k,(1-\left(1-\xi_{I_A}{}^{+}\right)^k\right]\\\left[(\xi_{N_A}{}^{-})^k,(\xi_{N_A}{}^{+})^k\right]\\\left\langle(P_A)^k,(I_A)^k,(1-(1-N_A)^k\right\rangle\end{array}\right\rangle\right]$$

The exponential multiplication is defined as

$$A^K=\left[\left\langle\begin{array}{c}\left[\left(\xi_{P_A}{}^{-}\right)^k,\left(\xi_{P_A}{}^{+}\right)^k\right]\\\left[\left(\xi_{I_A}{}^{-}\right)^k,\left(\xi_{I_A}{}^{+}\right)^k\right]\\\left[(1-\left(1-\xi_{N_A}{}^{-}\right)^k,(1-\left(1-\xi_{N_A}{}^{+}\right)^k\right]\\\left\langle 1-(1-P_A)^k,1-(1-I_A)^k,(N_A)^k\right\rangle\end{array}\right\rangle\right]$$

Einstein operations applied to CPFSs

This section contains, we expressed t-norm T, t-conorm and we define operation on cubic picture fuzzy set. t-norm T as

$$T_E\left(X,Y\right)=\frac{X+Y}{1+(1-X)(1-Y)}$$

Additionally, its t-conorm S is written as

$$S_E\left(X,Y\right)=\frac{X+Y}{1+XY}$$

Definition 4.1 let us suppose that

$A=\{\langle[\xi_{P_A}-(t),\xi_{P_A}{}^{+}(t)],\ [\xi_{I_A}-(t),\ \xi_{I_A}{}^{+}(t)],[\xi_{N_A}-,(t)\ \xi_{N_A}{}^{+}(t)]\rangle,\langle P_A(t),I_A(t),N_A(t)\rangle|t\in Z\}$ be the cubic picture fuzzy set (CPFS). Then we have

$$A=\left\{\langle\left[\xi_{P_A}-,\xi_{P_A}{}^{+}\right],\left[\xi_{I_A}-,\xi_{I_A}{}^{+}\right],\left[\xi_{N_A}-,\xi_{N_A}{}^{+}\right]\rangle,\langle P_A,I_A,N_A\rangle\right\}$$
$$A^c=\left\{\langle P_A,I_A,N_A\rangle,\left\langle[\xi_{N_A}-,\xi_{N_A}{}^{+}],[\xi_{I_A}-,\xi_{I_A}{}^{+}],[\xi_{P_A}-,\xi_{P_A}{}^{+}]\right\rangle\right\}$$

For two cubic picture fuzzy sets' Einstein sum, product, scalar multiplication, and exponential multiplication

Let

$$A=\left\{\left\langle\left[\xi_{P_A}-,\xi_{P_A}{}^{+}\right],\left[\xi_{I_A}-,\xi_{I_A}{}^{+}\right],\left[\xi_{N_A}-,\xi_{N_A}{}^{+}\right]\right\rangle,\langle P_A,I_A,N_A\rangle\right\}$$
$$B=\left\{\left\langle\left[\xi_{P_B}-,\xi_{P_B}{}^{+}\right],\left[\xi_{I_B}-,\xi_{I_B}{}^{+}\right],\left[\xi_{N_B}-,\xi_{N_B}{}^{+}\right]\right\rangle,\langle P_B,I_B,N_B\rangle\right\}$$

be two CPFSs and for scalar $k>0$, we have

(i). The Einstein sum of two CPFSs is defined

$$A\oplus_E B=\left[\left\langle\begin{array}{c}\left[\frac{\xi_{P_A}{}^{-}+\xi_{P_B}{}^{-}}{1+\xi_{P_A}{}^{-}.\xi_{P_B}{}^{-}},\frac{\xi_{P_A}{}^{+}+\xi_{P_B}{}^{+}}{1+\xi_{P_A}{}^{+}.\xi_{P_B}{}^{+}}\right]\\\left[\frac{\xi_{I_A}{}^{-}+\xi_{I_B}{}^{-}}{1+\xi_{I_A}{}^{-}.\xi_{I_B}{}^{-}},\frac{\xi_{I_A}{}^{+}+\xi_{I_B}{}^{+}}{1+\xi_{I_A}{}^{+}.\xi_{I_B}{}^{+}}\right]\\\left[\frac{\xi_{N_A}{}^{-}.\xi_{N_B}{}^{-}}{1+\left(1-\xi_{N_A}{}^{-}\right).\left(1-\xi_{N_B}{}^{-}\right)},\frac{\xi_{N_A}{}^{+}.\xi_{N_B}{}^{+}}{1+(1-\xi_{N_A}{}^{+}).\left(1-\xi_{N_B}{}^{+}\right)}\right]\\\left\langle\frac{P_A.P_B}{1+(1-P_A)(1-P_B)},\frac{I_A.I_B}{1+(1-I_A(1-I_B)},\frac{N_A+N_B}{1+N_A.N_B}\right\rangle\end{array}\right\rangle\right]$$

(ii). The Einstein product of two CPFSs is defined as

$$A\otimes_E B=\left[\left\langle\begin{array}{c}\left[\frac{\xi_{P_A}{}^{-}.\xi_{P_B}{}^{-}}{1+\left(1-\xi_{P_A}{}^{-}\right).\left(1-\xi_{P_B}{}^{-}\right)},\frac{\xi_{P_A}{}^{+}.\xi_{P_B}{}^{+}}{1+\left(1-\xi_{P_A}{}^{+}\right).\left(1-\xi_{P_B}{}^{+}\right)}\right]\\\left[\frac{\xi_{I_A}{}^{-}\xi_{I_B}{}^{-}}{1+(1-\xi_{I_A}{}^{-}).(1-\xi_{I_B}{}^{-})},\frac{\xi_{I_A}{}^{+}.\xi_{I_B}{}^{+}}{1+(1-\xi_{I_A}{}^{+}).(1-\xi_{I_B}{}^{+})}\right]\\\left[\frac{\xi_{N_A}{}^{+}+\xi_{N_B}{}^{+}}{1+\left(\xi_{N_A}{}^{-}\right).\left(\xi_{N_B}{}^{-}\right)},\frac{\xi_{N_A}{}^{+}+\xi_{N_B}{}^{+}}{1+(\xi_{N_A}{}^{+}).\left(\xi_{N_B}{}^{+}\right)}\right]\\\left\langle\frac{P_A+P_B}{1+(P_A)(P_B)},\frac{I_A+I_B}{1+I_A.I_B},\frac{N_A.N_B}{1+(1-N_A).(1-N_B)}\right\rangle\end{array}\right\rangle\right]$$

(iii). The Einstein scalar multiplication of two CPFSs is defined as

$$K_E A = \begin{bmatrix} \left\langle \begin{matrix} \left[\frac{\left(1+\xi_{P_A}{}^-\right)^k-\left(1-\xi_{P_A}{}^-\right)^k}{\left(1+\xi_{P_A}{}^-\right)^k+\left(1-\xi_{P_A}{}^-\right)^k}, \frac{\left(1+\xi_{P_A}{}^+\right)^k-\left(1-\xi_{P_A}{}^+\right)^k}{\left(1+\xi_{P_A}{}^+\right)^k+\left(1-\xi_{P_A}{}^+\right)^k}\right] \\ \left[\frac{\left(1+\xi_{P_A}{}^-\right)^k-\left(1-\xi_{P_A}{}^-\right)^k}{\left(1+\xi_{P_A}{}^-\right)^k+\left(1-\xi_{P_A}{}^-\right)^k}, \frac{\left(1+\xi_{P_A}{}^+\right)^k-\left(1-\xi_{P_A}{}^+\right)^k}{\left(1+\xi_{P_A}{}^+\right)^k+\left(1-\xi_{P_A}{}^+\right)^k}\right] \end{matrix} \right\rangle \\ \left[\frac{2\left(\xi_{N_A}{}^-\right)^k}{\left(2-\xi_{N_A}{}^-\right)^k+\left(\xi_{N_A}{}^-\right)^k}, \frac{2\left(\xi_{N_A}{}^+\right)^k}{\left(2-\xi_{N_A}{}^+\right)^k+\left(\xi_{N_A}{}^+\right)^k}\right] \\ \left\langle \frac{2(P_A)^k}{(2-P_A)^k+(P_A)^k}, \frac{2(I_A)^k}{(2-I_A)^k+(I_A)^k}, \frac{(1+N_A)^k-(1-N_A)^k}{(1+N_A)^k+(1-N_A)^k} \right\rangle \end{bmatrix}$$

(iv). The Einstein exponential multiplication of CPFS is defined as

$$A^{E^K} = \begin{bmatrix} \left\langle \begin{matrix} \left[\frac{2\left(\xi_{P_A}{}^-\right)^k}{\left(2-\xi_{P_A}{}^-\right)^k+\left(\xi_{P_A}{}^-\right)^k}, \frac{2\left(\xi_{P_A}{}^+\right)^k}{\left(2-\xi_{P_A}{}^+\right)^k+\left(\xi_{P_A}{}^+\right)^k}\right] \\ \left[\frac{2\left(\xi_{I_A}{}^-\right)^k}{\left(2-\xi_{I_A}{}^-\right)^k+\left(\xi_{I_A}{}^-\right)^k}, \frac{2\left(\xi_{I_A}{}^+\right)^k}{\left(2-\xi_{I_A}{}^+\right)^k+\left(\xi_{I_A}{}^+\right)^k}\right] \end{matrix} \right\rangle \\ \left[\frac{\left(1+\xi_{N_A}{}^-\right)^k-\left(1-\xi_{N_A}{}^-\right)^k}{\left(1+\xi_{N_A}{}^-\right)^k+\left(1-\xi_{N_A}{}^-\right)^k}, \frac{\left(1+\xi_{N_A}{}^+\right)^k-\left(1-\xi_{N_A}{}^+\right)^k}{\left(1+\xi_{N_A}{}^+\right)^k+\left(1-\xi_{N_A}{}^+\right)^k}\right] \\ \left\langle \frac{(1+P_A)^k-(1-P_A)^k}{(1+P_A)^k+(1-P_A)^k}, \frac{(1+I_A)^k-(1-I_A)^k}{(1+I_A)^k+(1-I_A)^k}, \frac{2(N_A)^k}{(2-N_A)^k+(N_A)^k} \right\rangle \end{bmatrix}$$

Einstein's operators

"Einstein operations" is a term that is sometimes used to refer to a family of mathematical operations that are commonly used in the field of fuzzy sets. These operations were named after Albert Einstein, who was one of the first physicists to extensively use tensors in his work on the theory of relativity. Einstein operations are an essential tool for working with fuzzy set, and they are used extensively in MADM problems.

This section provides the cubic picture fuzzy Einstein weighted averaging (CPFEWA), CPFEOWA and CPFEHWA operators for aggregating cubic picture fuzzy data. (CPFEWA), CPFEOWA and CPFEHWA operators. Each operator is described along with its properties and corresponding examples.

Operator for the cubic picture fuzzy Einstein weighted averaging (CPFEWA)

Let $A = \langle \xi_A(t), e_A(t) \rangle \; t \in Z$ be any assortment CPFNs in L_{CPFN} and $w = (w_1, w_2, \ldots w_n)^T$ is a weighted vector of $A_j (j = 1,2,3, \ldots, n)$ such that $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$ then mapping is the CPFEWA operator of dimension n.

$CPFEWA : L^n{}_{CPFN} \to L_{CPFN}$ and

$$CPFEWA\,(A_1, A_2, \ldots, A_n) = w_1 A_1 \oplus w_2 A_2 \oplus \ldots \oplus w_n A_n = \sum_{k=1}^{n} w_k A_k$$

Now, let $A_j = \left\{ \left\langle \left[\xi^-{}_{P_{A_j}}, \xi^+{}_{P_{A_j}} \right], P_{A_j} \right\rangle, \left\langle \left[\xi^-{}_{I_{A_j}}, \xi^+{}_{I_{A_j}} \right], I_{A_j} \right\rangle, \left\langle \left[\xi^-{}_{N_{A_j}}, \xi^+{}_{N_{A_j}} \right], N_{A_j} \right\rangle \right\}$ be a family of CPFNs, $w = (w_1, w_2, \ldots w_n)^T$ be a vector with weights $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$ CPFEWA is the name given to it if it satisfies.

$$CPFEWA\left(A_1, A_2, \ldots, A_n\right) =$$
$$\left[\begin{array}{c}\left\langle\left[\begin{array}{c}\frac{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{-}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{-}\right)^{w_j}},\\ \frac{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{+}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{+}\right)^{w_j}}\end{array}\right], \frac{2\prod_{j=1}^{n}\left(P_{A_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-P_{A_j}\right)^{w_j}+\prod_{j=1}^{n}\left(P_{A_j}\right)^{w_j}}\right\rangle\\ \left\langle\left[\begin{array}{c}\frac{\prod_{j=1}^{n}\left(1+\xi_{I_{A_j}}{}^{-}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{I_{A_j}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{I_{A_j}}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{I_{A_j}}{}^{-}\right)^{w_j}},\\ \frac{\prod_{j=1}^{n}\left(1+\xi_{I_{A_j}}{}^{+}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{I_{A_j}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{I_{A_j}}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{I_{A_j}}{}^{+}\right)^{w_j}}\end{array}\right], \frac{2\prod_{j=1}^{n}\left(I_{A_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-I_{A_j}\right)^{w_j}+\prod_{j=1}^{n}\left(I_{A_j}\right)^{w_j}}\right\rangle\\ \left\langle\left[\begin{array}{c}\frac{2\prod_{j=1}^{n}\left(\xi_{N_{A_j}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\xi_{N_{A_j}}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(\xi_{N_{A_j}}{}^{-}\right)^{w_j}},\\ \frac{2\prod_{j=1}^{n}\left(\xi_{N_{A_j}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\xi_{N_{A_j}}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(\xi_{N_{A_j}}{}^{+}\right)^{w_j}}\end{array}\right], \frac{\prod_{j=1}^{n}\left(1+N_{A_j}\right)^{w_j}-\prod_{j=1}^{n}\left(1-N_{A_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+N_{A_j}\right)^{w_j}+\prod_{j=1}^{n}\left(1-N_{A_j}\right)^{w_j}}\right\rangle\end{array}\right]$$

Theorem 5.1.1 Let $A_j = \left\{\left\langle\left[\xi^{-}{}_{P_{A_j}}, \xi^{+}{}_{P_{A_j}}\right], \left[\xi^{-}{}_{I_{A_j}}, \xi^{+}{}_{I_{A_j}}\right], \left[\xi^{-}{}_{N_{A_j}}, \xi^{+}{}_{N_{A_j}}\right]\right\rangle, \left\langle P_{A_j}, I_{A_j}, N_{A_j}\right\rangle\right\}$

be a family of CPFNs, $w = \left(w_1, w_2, \ldots w_n\right)^T$ be a vector with weights $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$ then it is called CPFEWA if it satisfies

$$CPFEWA\left(A_1, A_2, \ldots, A_n\right)$$
$$= \left[\begin{array}{c}\left\langle\left[\frac{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{-}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{-}\right)^{w_j}}, \frac{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{+}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{A_j}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{+}\right)^{w_j}}\right]\right.\\ \left[\frac{\prod_{j=1}^{n}\left(1+\xi_{I_{A_j}}{}^{-}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{I_{A_j}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{I_{A_j}}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{I_{A_j}}{}^{-}\right)^{w_j}}, \frac{\prod_{j=1}^{n}\left(1+\xi_{I_{A_j}}{}^{+}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{I_{A_j}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{I_{A_j}}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{I_{A_j}}{}^{+}\right)^{w_j}}\right]\\ \left.\left[\frac{2\prod_{j=1}^{n}\left(\xi_{N_{A_j}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\xi_{N_{A_j}}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(\xi_{N_{A_j}}{}^{-}\right)^{w_j}}, \frac{2\prod_{j=1}^{n}\left(\xi_{N_{A_j}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\xi_{N_{A_j}}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(\xi_{N_{A_j}}{}^{+}\right)^{w_j}}\right]\right\rangle\\ \left\langle\frac{2\prod_{j=1}^{n}\left(P_{A_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-P_{A_j}\right)^{w_j}+\prod_{j=1}^{n}\left(P_{A_j}\right)^{w_j}}, \frac{2\prod_{j=1}^{n}\left(I_{A_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-I_{A_j}\right)^{w_j}+\prod_{j=1}^{n}\left(I_{A_j}\right)^{w_j}}, \frac{\prod_{j=1}^{n}\left(1+N_{A_j}\right)^{w_j}-\prod_{j=1}^{n}\left(1-N_{A_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+N_{A_j}\right)^{w_j}+\prod_{j=1}^{n}\left(1-N_{A_j}\right)^{w_j}}\right\rangle\end{array}\right]$$

Proof (1). First it is true for $n = 2$

$$w_1 A_1 = \left[\begin{array}{c}\left\langle\left[\frac{\left(1+\xi_{P_{A_1}}{}^{-}\right)^{w_1}-\left(1-\xi_{P_{A_1}}{}^{-}\right)^{w_1}}{\left(1+\xi_{P_{A_1}}{}^{-}\right)^{w_1}+\left(1-\xi_{P_{A_1}}{}^{-}\right)^{w_1}}, \frac{\left(1+\xi_{P_{A_1}}{}^{+}\right)^{w_1}-\left(1-\xi_{P_{A_1}}{}^{+}\right)^{w_1}}{\left(1+\xi_{P_{A_1}}{}^{+}\right)^{w_1}+\left(1-\xi_{P_{A_1}}{}^{+}\right)^{w_1}}\right]\right.\\ \left[\frac{\left(1+\xi_{I_{A_1}}{}^{-}\right)^{w_1}-\left(1-\xi_{I_{A_1}}{}^{-}\right)^{w_1}}{\left(1+\xi_{I_{A_1}}{}^{-}\right)^{w_1}+\left(1-\xi_{I_{A_1}}{}^{-}\right)^{w_1}}, \frac{\left(1+\xi_{I_{A_1}}{}^{+}\right)^{w_1}-\left(1-\xi_{I_{A_1}}{}^{+}\right)^{w_1}}{\left(1+\xi_{I_{A_1}}{}^{+}\right)^{w_1}+\left(1-\xi_{I_{A_1}}{}^{+}\right)^{w_1}}\right]\\ \left.\left[\frac{2\left(\xi_{N_{A_1}}{}^{-}\right)^{w_1}}{\left(2-\xi_{N_{A_1}}{}^{-}\right)^{w_1}+\left(\xi_{N_{A_1}}{}^{-}\right)^{w_1}}, \frac{2\left(\xi_{N_{A_1}}{}^{+}\right)^{w_1}}{\left(2-\xi_{N_{A_1}}{}^{+}\right)^{w_1}+\left(\xi_{N_{A_1}}{}^{+}\right)^{w_1}}\right]\right\rangle\\ \left\langle\frac{2\left(P_{A_1}\right)^{w_1}}{\left(2-P_{A_1}\right)^{w_1}+\left(P_{A_1}\right)^{w_1}}, \frac{2\left(I_{A_1}\right)^{w_1}}{\left(2-I_{A_1}\right)^{w_1}+\left(I_{A_1}\right)^{w_1,}}, \frac{\left(1+N_{A_1}\right)^{w_1}-\left(1-N_{A_1}\right)^{w_1}}{\left(1+N_{A_1}\right)^{w_1}+\left(1-N_{A_1}\right)^{w_1}}\right\rangle\end{array}\right]$$

$$w_2A_2 = \left[\begin{array}{c} \left\langle \left[\frac{\left(1+\xi_{P_{A_2}}{}^{-}\right)^{w_2}-\left(1-\xi_{P_{A_2}}{}^{-}\right)^{w_2}}{\left(1+\xi_{P_{A_2}}{}^{-}\right)^{w_2}+\left(1-\xi_{P_{A_2}}{}^{-}\right)^{w_2}}, \frac{\left(1+\xi_{P_{A_2}}{}^{+}\right)^{w_2}-\left(1-\xi_{P_{A_1}}{}^{+}\right)^{w_2}}{\left(1+\xi_{P_{A_2}}{}^{+}\right)^{w_2}+\left(1-\xi_{P_{A_2}}{}^{+}\right)^{w_2}}\right] \right. \\ \left[\frac{\left(1+\xi_{I_{A_2}}{}^{-}\right)^{w_2}-\left(1-\xi_{I_{A_2}}{}^{-}\right)^{w_2}}{\left(1+\xi_{I_{A_2}}{}^{-}\right)^{w_2}+\left(1-\xi_{I_{A_2}}{}^{-}\right)^{w_2}}, \frac{\left(1+\xi_{I_{A_2}}{}^{+}\right)^{w_2}-\left(1-\xi_{I_{A_2}}{}^{+}\right)^{w_2}}{\left(1+\xi_{I_{A_2}}{}^{+}\right)^{w_2}+\left(1-\xi_{I_{A_2}}{}^{+}\right)^{w_2}}\right] \\ \left. \left[\frac{2\left(\xi_{N_{A_2}}{}^{-}\right)^{w_2}}{\left(2-\xi_{N_{A_2}}{}^{-}\right)^{w_2}+\left(\xi_{N_{A_2}}{}^{-}\right)^{w_2}}, \frac{2\left(\xi_{N_{A_2}}{}^{+}\right)^{w_2}}{\left(2-\xi_{N_{A_2}}{}^{+}\right)^{w_2}+\left(\xi_{N_{A_2}}{}^{+}\right)^{w_2}}\right] \right\rangle \\ \left\langle \frac{2\left(P_{A_2}\right)^{w_2}}{\left(2-P_{A_2}\right)^{w_2}+\left(P_{A_2}\right)^{w_2}}, \frac{2\left(I_{A_2}\right)^{w_2}}{\left(2-I_{A_2}\right)^{w_2}+\left(I_{A_2}\right)^{w_2}}, \frac{\left(1+N_{A_2}\right)^{w_2}-\left(1-N_{A_2}\right)^{w_2}}{\left(1+N_{A_2}\right)^{w_2}+\left(1-N_{A_2}\right)^{w_2}} \right\rangle \end{array}\right]$$

$$
\begin{aligned}
&w_1A_1+w_2A_2\\
&=\left[\begin{array}{c}
\left\langle\begin{array}{c}
\left[\frac{\left\{\frac{\left(1+\xi_{P_{A_1}}^{-}\right)^{w_1}-\left(1-\xi_{P_{A_1}}^{-}\right)^{w_1}}{\left(1+\xi_{P_{A_1}}^{-}\right)^{w_1}+\left(1-\xi_{P_{A_1}}^{-}\right)^{w_1}}\right\}+\left\{\frac{\left(1+\xi_{P_{A_2}}^{-}\right)^{w_2}-\left(1-\xi_{P_{A_2}}^{-}\right)^{w_2}}{\left(1+\xi_{P_{A_2}}^{-}\right)^{w_2}+\left(1-\xi_{P_{A_2}}^{-}\right)^{w_2}}\right\}}{1+\left\{\frac{\left(1+\xi_{P_{A_1}}^{-}\right)^{w_1}-\left(1-\xi_{P_{A_1}}^{-}\right)^{w_1}}{\left(1+\xi_{P_{A_1}}^{-}\right)^{w_1}+\left(1-\xi_{P_{A_1}}^{-}\right)^{w_1}}\right\}\left\{\frac{\left(1+\xi_{P_{A_2}}^{-}\right)^{w_2}-\left(1-\xi_{P_{A_2}}^{-}\right)^{w_2}}{\left(1+\xi_{P_{A_2}}^{-}\right)^{w_2}+\left(1-\xi_{P_{A_2}}^{-}\right)^{w_2}}\right\}},\;
\frac{\left\{\frac{\left(1+\xi_{P_{A_1}}^{+}\right)^{w_1}-\left(1-\xi_{P_{A_1}}^{+}\right)^{w_1}}{\left(1+\xi_{P_{A_1}}^{+}\right)^{w_1}+\left(1-\xi_{P_{A_1}}^{+}\right)^{w_1}}\right\}+\left\{\frac{\left(1+\xi_{P_{A_2}}^{+}\right)^{w_2}-\left(1-\xi_{P_{A_2}}^{+}\right)^{w_2}}{\left(1+\xi_{P_{A_2}}^{+}\right)^{w_2}+\left(1-\xi_{P_{A_2}}^{+}\right)^{w_2}}\right\}}{1+\left\{\frac{\left(1+\xi_{P_{A_1}}^{+}\right)^{w_1}-\left(1-\xi_{P_{A_1}}^{+}\right)^{w_1}}{\left(1+\xi_{P_{A_1}}^{+}\right)^{w_1}+\left(1-\xi_{P_{A_1}}^{+}\right)^{w_1}}\right\}\left\{\frac{\left(1+\xi_{P_{A_2}}^{+}\right)^{w_2}-\left(1-\xi_{P_{A_2}}^{+}\right)^{w_2}}{\left(1+\xi_{P_{A_2}}^{+}\right)^{w_2}+\left(1-\xi_{P_{A_2}}^{+}\right)^{w_2}}\right\}}\right],\\
\left[\frac{\left\{\frac{\left(1+\xi_{I_{A_1}}^{-}\right)^{w_1}-\left(1-\xi_{I_{A_1}}^{-}\right)^{w_1}}{\left(1+\xi_{I_{A_1}}^{-}\right)^{w_1}+\left(1-\xi_{I_{A_1}}^{-}\right)^{w_1}}\right\}+\left\{\frac{\left(1+\xi_{I_{A_2}}^{-}\right)^{w_2}-\left(1-\xi_{I_{A_2}}^{-}\right)^{w_2}}{\left(1+\xi_{I_{A_2}}^{-}\right)^{w_2}+\left(1-\xi_{I_{A_2}}^{-}\right)^{w_2}}\right\}}{1+\left\{\frac{\left(1+\xi_{I_{A_1}}^{-}\right)^{w_1}-\left(1-\xi_{I_{A_1}}^{-}\right)^{w_1}}{\left(1+\xi_{I_{A_1}}^{-}\right)^{w_1}+\left(1-\xi_{I_{A_1}}^{-}\right)^{w_1}}\right\}\left\{\frac{\left(1+\xi_{I_{A_2}}^{-}\right)^{w_2}-\left(1-\xi_{I_{A_2}}^{-}\right)^{w_2}}{\left(1+\xi_{I_{A_2}}^{-}\right)^{w_2}+\left(1-\xi_{I_{A_2}}^{-}\right)^{w_2}}\right\}},\;
\frac{\left\{\frac{\left(1+\xi_{I_{A_1}}^{+}\right)^{w_1}-\left(1-\xi_{I_{A_1}}^{+}\right)^{w_1}}{\left(1+\xi_{I_{A_1}}^{+}\right)^{w_1}+\left(1-\xi_{I_{A_1}}^{+}\right)^{w_1}}\right\}+\left\{\frac{\left(1+\xi_{I_{A_2}}^{+}\right)^{w_2}-\left(1-\xi_{I_{A_2}}^{+}\right)^{w_2}}{\left(1+\xi_{I_{A_2}}^{+}\right)^{w_2}+\left(1-\xi_{I_{A_2}}^{+}\right)^{w_2}}\right\}}{1+\left\{\frac{\left(1+\xi_{I_{A_1}}^{+}\right)^{w_1}-\left(1-\xi_{I_{A_1}}^{+}\right)^{w_1}}{\left(1+\xi_{I_{A_1}}^{+}\right)^{w_1}+\left(1-\xi_{I_{A_1}}^{+}\right)^{w_1}}\right\}\left\{\frac{\left(1+\xi_{I_{A_2}}^{+}\right)^{w_2}-\left(1-\xi_{I_{A_2}}^{+}\right)^{w_2}}{\left(1+\xi_{I_{A_2}}^{+}\right)^{w_2}+\left(1-\xi_{I_{A_2}}^{+}\right)^{w_2}}\right\}}\right],\\
\left[\frac{\left\{\frac{2\left(\xi_{N_{A_1}}^{-}\right)^{w_1}}{\left(2-\xi_{N_{A_1}}^{-}\right)^{w_1}+\left(\xi_{N_{A_1}}^{-}\right)^{w_1}}\right\}\left\{\frac{2\left(\xi_{N_{A_2}}^{-}\right)^{w_2}}{\left(2-\xi_{N_{A_2}}^{-}\right)^{w_2}+\left(\xi_{N_{A_2}}^{-}\right)^{w_2}}\right\}}{1+\left\{1-\frac{2\left(\xi_{N_{A_1}}^{-}\right)^{w_1}}{\left(2-\xi_{N_{A_1}}^{-}\right)^{w_1}+\left(\xi_{N_{A_1}}^{-}\right)^{w_1}}\right\}\left\{1-\frac{2\left(\xi_{N_{A_2}}^{-}\right)^{w_2}}{\left(2-\xi_{N_{A_2}}^{-}\right)^{w_2}+\left(\xi_{N_{A_2}}^{-}\right)^{w_2}}\right\}},\;
\frac{\left\{\frac{2\left(\xi_{N_{A_1}}^{+}\right)^{w_1}}{\left(2-\xi_{N_{A_1}}^{+}\right)^{w_1}+\left(\xi_{N_{A_1}}^{+}\right)^{w_1}}\right\}\left\{\frac{2\left(\xi_{N_{A_2}}^{-}\right)^{w_2}}{\left(2-\xi_{N_{A_2}}^{+}\right)^{w_2}+\left(\xi_{N_{A_2}}^{+}\right)^{w_2}}\right\}}{1+\left\{1-\frac{2\left(\xi_{N_{A_1}}^{+}\right)^{w_1}}{\left(2\xi_{N_{A_1}}^{+}\right)^{w_1}+\left(\xi_{N_{A_1}}^{+}\right)^{w_1}}\right\}\left\{1-\frac{2\left(\xi_{N_{A_2}}^{-}\right)^{w_2}}{\left(2-\xi_{N_{A_2}}^{+}\right)^{w_2}+\left(\xi_{N_{A_2}}^{+}\right)^{w_2}}\right\}}\right]
\end{array}\right\rangle\\
\left\langle\begin{array}{c}
\frac{\left\{\frac{2\left(P_{A_1}\right)^{w_1}}{\left(2-P_{A_1}\right)^{w_1}+\left(P_{A_1}\right)^{w_1}}\right\}\left\{\frac{2\left(P_{A_2}\right)^{w_2}}{\left(2-P_{A_2}\right)^{w_2}+\left(P_{A_2}\right)^{w_2}}\right\}}{1+\left\{1-\frac{2\left(P_{A_1}\right)^{w_1}}{\left(2-P_{A_1}\right)^{w_1}+\left(P_{A_1}\right)^{w_1}}\right\}\left\{1-\frac{2\left(P_{A_2}\right)^{w_2}}{\left(2-P_{A_2}\right)^{w_2}+\left(P_{A_2}\right)^{w_2}}\right\}},\\
\frac{\left\{\frac{2\left(I_{A_1}\right)^{w_1}}{\left(2-I_{A_1}\right)^{w_1}+\left(I_{A_1}\right)^{w_1}}\right\}\left\{\frac{2\left(I_{A_2}\right)^{w_2}}{\left(2-I_{A_2}\right)^{w_2}+\left(I_{A_2}\right)^{w_2}}\right\}}{1+\left\{1-\frac{2\left(I_{A_1}\right)^{w_1}}{\left(2-I_{A_1}\right)^{w_1}+\left(I_{A_1}\right)^{w_1}}\right\}\left\{1-\frac{2\left(I_{A_2}\right)^{w_2}}{\left(2-I_{A_2}\right)^{w_2}+\left(I_{A_2}\right)^{w_2}}\right\}}\\
\frac{\left\{\frac{\left(1+N_{A_1}\right)^{w_1}-\left(1-N_{A_1}\right)^{w_1}}{\left(1+N_{A_1}\right)^{w_1}+\left(1-N_{A_1}\right)^{w_1}}\right\}+\left\{\frac{\left(1+N_{A_2}\right)^{w_2}-\left(1-N_{A_2}\right)^{w_2}}{\left(1+N_{A_2}\right)^{w_2}+\left(1-N_{A_2}\right)^{w_2}}\right\}}{1+\left\{\frac{\left(1+N_{A_1}\right)^{w_1}-\left(1-N_{A_1}\right)^{w_1}}{\left(1+N_{A_1}\right)^{w_1}+\left(1-N_{A_1}\right)^{w_1}}\right\}\left\{\frac{\left(1+N_{A_2}\right)^{w_2}-\left(1-N_{A_2}\right)^{w_2}}{\left(1+N_{A_2}\right)^{w_2}+\left(1-N_{A_2}\right)^{w_2}}\right\}}
\end{array}\right\rangle
\end{array}\right]
\end{aligned}
$$

$$\left[\begin{array}{c}\left\langle\begin{array}{c}\left[\dfrac{\{(1+\xi_{P_{A_1}}^-)^{w_1}-(1-\xi_{P_{A_1}}^-)^{w_1}\}\{(1+\xi_{P_{A_2}}^-)^{w_2}+(1-\xi_{P_{A_2}}^-)^{w_2}\}+\{(1+\xi_{P_{A_1}}^-)^{w_1}+(1-\xi_{P_{A_1}}^-)^{w_1}\}\{(1+\xi_{P_{A_2}}^-)^{w_2}-(1-\xi_{P_{A_2}}^-)^{w_2}\}}{\{(1+\xi_{P_{A_1}}^-)^{w_1}+(1-\xi_{P_{A_1}}^-)^{w_1}\}\{(1+\xi_{P_{A_2}}^-)^{w_2}+(1-\xi_{P_{A_2}}^-)^{w_2}\}+\{(1+\xi_{P_{A_1}}^-)^{w_1}-(1-\xi_{P_{A_1}}^-)^{w_1}\}\{(1+\xi_{P_{A_2}}^-)^{w_2}-(1-\xi_{P_{A_2}}^-)^{w_2}\}}, \dfrac{\{(1+\xi_{P_{A_1}}^+)^{w_1}-(1-\xi_{P_{A_1}}^+)^{w_1}\}\{(1+\xi_{P_{A_2}}^+)^{w_2}+(1-\xi_{P_{A_2}}^+)^{w_2}\}+\{(1+\xi_{P_{A_1}}^+)^{w_1}+(1-\xi_{P_{A_1}}^+)^{w_1}\}\{(1+\xi_{P_{A_2}}^+)^{w_2}-(1-\xi_{P_{A_2}}^+)^{w_2}\}}{\{(1+\xi_{P_{A_1}}^+)^{w_1}+(1-\xi_{P_{A_1}}^+)^{w_1}\}\{(1+\xi_{P_{A_2}}^+)^{w_2}+(1-\xi_{P_{A_2}}^+)^{w_2}\}+\{(1+\xi_{P_{A_1}}^+)^{w_1}-(1-\xi_{P_{A_1}}^+)^{w_1}\}\{(1+\xi_{P_{A_2}}^+)^{w_2}-(1-\xi_{P_{A_2}}^+)^{w_2}\}}\right]\\ \left[\dfrac{\{(1+\xi_{I_{A_1}}^-)^{w_1}-(1-\xi_{I_{A_1}}^-)^{w_1}\}\{(1+\xi_{I_{A_2}}^-)^{w_2}+(1-\xi_{I_{A_2}}^-)^{w_2}\}+\{(1+\xi_{I_{A_1}}^-)^{w_1}+(1-\xi_{I_{A_1}}^-)^{w_1}\}\{(1+\xi_{I_{A_2}}^-)^{w_2}-(1-\xi_{I_{A_2}}^-)^{w_2}\}}{\{(1+\xi_{I_{A_1}}^-)^{w_1}+(1-\xi_{I_{A_1}}^-)^{w_1}\}\{(1+\xi_{I_{A_2}}^-)^{w_2}+(1-\xi_{I_{A_2}}^-)^{w_2}\}+\{(1+\xi_{I_{A_1}}^-)^{w_1}-(1-\xi_{I_{A_1}}^-)^{w_1}\}\{(1+\xi_{I_{A_2}}^-)^{w_2}-(1-\xi_{I_{A_2}}^-)^{w_2}\}}, \dfrac{\{(1+\xi_{I_{A_1}}^+)^{w_1}-(1-\xi_{I_{A_1}}^+)^{w_1}\}\{(1+\xi_{I_{A_2}}^+)^{w_2}+(1-\xi_{I_{A_2}}^+)^{w_2}\}+\{(1+\xi_{I_{A_1}}^+)^{w_1}+(1-\xi_{I_{A_1}}^+)^{w_1}\}\{(1+\xi_{I_{A_2}}^+)^{w_2}-(1-\xi_{I_{A_2}}^+)^{w_2}\}}{\{(1+\xi_{I_{A_1}}^+)^{w_1}+(1-\xi_{I_{A_1}}^+)^{w_1}\}\{(1+\xi_{I_{A_2}}^+)^{w_2}+(1-\xi_{I_{A_2}}^+)^{w_2}\}+\{(1+\xi_{I_{A_1}}^+)^{w_1}-(1-\xi_{I_{A_1}}^+)^{w_1}\}\{(1+\xi_{I_{A_2}}^+)^{w_2}-(1-\xi_{I_{A_2}}^+)^{w_2}\}}\right]\\ \left[\dfrac{\{2(\xi_{N_{A_1}}^-)^{w_1}\}\{2(\xi_{N_{A_2}}^-)^{w_2}\}}{\{(2-\xi_{N_{A_1}}^-)^{w_1}+(\xi_{N_{A_1}}^-)^{w_1}\}\{2-\xi_{N_{A_2}}^-)^{w_2}+(\xi_{N_{A_2}}^-)^{w_2}\}+\{(2-\xi_{N_{A_1}}^-)^{w_1}-(\xi_{N_{A_1}}^-)^{w_1}\}\{2-\xi_{N_{A_2}}^-)^{w_2}-(\xi_{N_{A_2}}^-)^{w_2}\}}, \dfrac{\{2(\xi_{N_{A_1}}^+)^{w_1}\}\{2(\xi_{N_{A_2}}^+)^{w_2}\}}{\{(2-\xi_{N_{A_1}}^+)^{w_1}+(\xi_{N_{A_1}}^+)^{w_1}\}\{2-\xi_{N_{A_2}}^+)^{w_2}+(\xi_{N_{A_2}}^+)^{w_2}\}+\{(2-\xi_{N_{A_1}}^+)^{w_1}-(\xi_{N_{A_1}}^+)^{w_1}\}\{2-\xi_{N_{A_2}}^+)^{w_2}-(\xi_{N_{A_2}}^+)^{w_2}\}}\right]\end{array}\right\rangle\\ \left\langle\begin{array}{c}\dfrac{\{2(P_{A_1})^{w_1}\}\{2(P_{A_2})^{w_2}\}}{\{(2-P_{A_1})^{w_1}+(P_{A_1})^{w_1}\}\{2-P_{A_2})^{w_2}+(P_{A_2})^{w_2}\}+\{(2-P_{A_1})^{w_1}-(P_{A_1})^{w_1}\}\{2-P_{A_2})^{w_2}-(P_{A_2})^{w_2}\}},\\ \dfrac{\{2(I_{A_1})^{w_1}\}\{2(I_{A_2})^{w_2}\}}{\{(2-I_{A_1})^{w_1}+(I_{A_1})^{w_1}\}\{(2-I_{A_2})^{w_2}+(I_{A_2})^{w_2}\}+\{(2-I_{A_1})^{w_1}-(I_{A_1})^{w_1}\}\{(2-I_{A_2})^{w_2}-(I_{A_2})^{w_2}\}}\\ \dfrac{\{(1+N_{A_1})^{w_1}-(1-N_{A_1})^{w_1}\}\{(1+N_{A_2})^{w_2}+(1-N_{A_2})^{w_2}\}+\{(1+N_{A_1})^{w_1}+(1-N_{A_1})^{w_1}\}\{(1+N_{A_2})^{w_2}-(1-N_{A_2})^{w_2}\}}{\{(1+N_{A_1})^{w_1}+(1-N_{A_1})^{w_1}\}\{(1+N_{A_2})^{w_2}+(1-N_{A_2})^{w_2}\}+\{(1+N_{A_1})^{w_1}-(1-N_{A_1})^{w_1}\}\{(1+N_{A_2})^{w_2}-(1-N_{A_2})^{w_2}\}}\end{array}\right\rangle\end{array}\right]$$

(2) When n = k we have

$$CPFEWA\left(A_1, A_2, \ldots, A_k\right)$$

$$= \left[\begin{array}{c}\left\langle\begin{array}{c}\left[\dfrac{\prod_{j=1}^{k}\left(1+\xi_{P_{A_j}}^-\right)^{w_j}-\prod_{j=1}^{k}\left(1-\xi_{P_{A_j}}^-\right)^{w_j}}{\prod_{j=1}^{k}\left(1+\xi_{P_{A_j}}^-\right)^{w_j}+\prod_{j=1}^{k}\left(1-\xi_{P_{A_j}}^-\right)^{w_j}}, \dfrac{\prod_{j=1}^{k}\left(1+\xi_{P_{A_j}}^+\right)^{w_j}-\prod_{j=1}^{k}\left(1-\xi_{A_j}^+\right)^{w_j}}{\prod_{j=1}^{k}\left(1+\xi_{P_{A_j}}^+\right)^{w_j}+\prod_{j=1}^{k}\left(1-\xi_{P_{A_j}}^+\right)^{w_j}}\right]\\ \left[\dfrac{\prod_{j=1}^{k}\left(1+\xi_{I_{A_j}}^-\right)^{w_j}-\prod_{j=1}^{k}\left(1-\xi_{I_{A_j}}^-\right)^{w_j}}{\prod_{j=1}^{k}\left(1+\xi_{I_{A_j}}^-\right)^{w_j}+\prod_{j=1}^{k}\left(1-\xi_{I_{A_j}}^-\right)^{w_j}}, \dfrac{\prod_{j=1}^{k}\left(1+\xi_{I_{A_j}}^+\right)^{w_j}-\prod_{j=1}^{k}\left(1-\xi_{I_{A_j}}^+\right)^{w_j}}{\prod_{j=1}^{k}\left(1+\xi_{I_{A_j}}^+\right)^{w_j}+\prod_{j=1}^{k}\left(1-\xi_{I_{A_j}}^+\right)^{w_j}}\right]\\ \left[\dfrac{2\prod_{j=1}^{k}\left(\xi_{N_{A_j}}^-\right)^{w_j}}{\prod_{j=1}^{k}\left(2-\xi_{N_{A_j}}^-\right)^{w_j}+\prod_{j=1}^{k}\left(\xi_{N_{A_j}}^-\right)^{w_j}}, \dfrac{2\prod_{j=1}^{k}\left(\xi_{N_{A_j}}^+\right)^{w_j}}{\prod_{j=1}^{k}\left(2-\xi_{N_{A_j}}^+\right)^{w_j}+\prod_{j=1}^{k}\left(\xi_{N_{A_j}}^+\right)^{w_j}}\right]\end{array}\right\rangle\\ \left\langle\begin{array}{c}\dfrac{2\prod_{j=1}^{k}\left(P_{A_j}\right)^{w_j}}{\prod_{j=1}^{k}\left(2-P_{A_j}\right)^{w_j}+\prod_{j=1}^{k}\left(P_{A_j}\right)^{w_j}}, \dfrac{2\prod_{j=1}^{k}\left(I_{A_j}\right)^{w_j}}{\prod_{j=1}^{k}\left(2-I_{A_j}\right)^{w_j}+\prod_{j=1}^{k}\left(I_{A_j}\right)^{w_j}},\\ \dfrac{\prod_{j=1}^{k}\left(1+N_{A_j}\right)^{w_j}-\prod_{j=1}^{k}\left(1-N_{A_j}\right)^{w_j}}{\prod_{j=1}^{k}\left(1+N_{A_j}\right)^{w_j}+\prod_{j=1}^{k}\left(1-N_{A_j}\right)^{w_j}}\end{array}\right\rangle\end{array}\right]$$

(3) When $n = k + 1$, we have

$$CPFEWA\Big(A_1, A_2, \ldots, A_{k+1}\Big)$$

$$= \left[\begin{array}{c} \left\langle \left[\frac{\prod_{j=1}^{k}\left(1+\xi_{P_{A_j}}{}^{-}\right)^{w_j}-\prod_{j=1}^{k}\left(1-\xi_{P_{A_j}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{k}\left(1+\xi_{P_{A_j}}{}^{-}\right)^{w_j}+\prod_{j=1}^{k}\left(1-\xi_{P_{A_j}}{}^{-}\right)^{w_j}}, \frac{\prod_{j=1}^{k}\left(1+\xi_{P_{A_j}}{}^{+}\right)^{w_j}-\prod_{j=1}^{k}\left(1-\xi_{A_j}{}^{+}\right)^{w_j}}{\prod_{j=1}^{k}\left(1+\xi_{P_{A_j}}{}^{+}\right)^{w_j}+\prod_{j=1}^{k}\left(1-\xi_{P_{A_j}}{}^{+}\right)^{w_j}}\right] \right. \\ \left[\frac{\prod_{j=1}^{k}\left(1+\xi_{I_{A_j}}{}^{-}\right)^{w_j}-\prod_{j=1}^{k}\left(1-\xi_{I_{A_j}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{k}\left(1+\xi_{I_{A_j}}{}^{-}\right)^{w_j}+\prod_{j=1}^{k}\left(1-\xi_{I_{A_j}}{}^{-}\right)^{w_j}}, \frac{\prod_{j=1}^{k}\left(1+\xi_{I_{A_j}}{}^{+}\right)^{w_j}-\prod_{j=1}^{k}\left(1-\xi_{I_{A_j}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{k}\left(1+\xi_{I_{A_j}}{}^{+}\right)^{w_j}+\prod_{j=1}^{k}\left(1-\xi_{I_{A_j}}{}^{+}\right)^{w_j}}\right] \\ \left. \left[\frac{2\prod_{j=1}^{k}\left(\xi_{N_{A_j}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{k}\left(2-\xi_{N_{A_j}}{}^{-}\right)^{w_j}+\prod_{j=1}^{k}\left(\xi_{N_{A_j}}{}^{-}\right)^{w_j}}, \frac{2\prod_{j=1}^{k}\left(\xi_{N_{A_j}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{k}\left(2-\xi_{N_{A_j}}{}^{+}\right)^{w_j}+\prod_{j=1}^{k}\left(\xi_{N_{A_j}}{}^{+}\right)^{w_j}}\right] \right\rangle \\ \left\langle \frac{2\prod_{j=1}^{k}\left(P_{A_j}\right)^{w_j}}{\prod_{j=1}^{k}\left(2-P_{A_j}\right)^{w_j}+\prod_{j=1}^{k}\left(P_{A_j}\right)^{w_j}}, \frac{2\prod_{j=1}^{k}\left(I_{A_j}\right)^{w_j}}{\prod_{j=1}^{k}\left(2-I_{A_j}\right)^{w_j}+\prod_{j=1}^{k}\left(I_{A_j}\right)^{w_j}}, \right. \\ \left. \frac{\prod_{j=1}^{k}\left(1+N_{A_j}\right)^{w_j}-\prod_{j=1}^{k}\left(1-N_{A_j}\right)^{w_j}}{\prod_{j=1}^{k}\left(1+N_{A_j}\right)^{w_j}+\prod_{j=1}^{k}\left(1-N_{A_j}\right)^{w_j}} \right\rangle \end{array}\right]$$

$$\oplus$$

$$\left[\begin{array}{c} \left\langle \left[\frac{\left(1+\xi_{P_{A_{k+1}}}{}^{-}\right)^{w_{k+1}}-\left(1-\xi_{P_{A_{k+1}}}{}^{-}\right)^{w_{k+1}}}{\left(1+\xi_{P_{A_{k+1}}}{}^{-}\right)^{w_{k+1}}+\left(1-\xi_{P_{A_{k+1}}}{}^{-}\right)^{w_{k+1}}}, \frac{\left(1+\xi_{P_{A_{k+1}}}{}^{+}\right)^{w_{k+1}}-\left(1-\xi_{P_{A_{k+1}}}{}^{+}\right)^{w_{k+1}}}{\left(1+\xi_{P_{A_{k+1}}}{}^{+}\right)^{w_{k+1}}+\left(1-\xi_{P_{A_{k+1}}}{}^{+}\right)^{w_{k+1}}}\right] \right. \\ \left[\frac{\left(1+\xi_{I_{A_{k+1}}}{}^{-}\right)^{w_{k+1}}-\left(1-\xi_{I_{A_{k+1}}}{}^{-}\right)^{w_{k+1}}}{\left(1+\xi_{I_{A_{k+1}}}{}^{-}\right)^{w_{k+1}}+\left(1-\xi_{I_{A_{k+1}}}{}^{-}\right)^{w_{k+1}}}, \frac{\left(1+\xi_{I_{A_{k+1}}}{}^{+}\right)^{w_{k+1}}-\left(1-\xi_{I_{A_{k+1}}}{}^{+}\right)^{w_{k+1}}}{\left(1+\xi_{I_{A_{k+1}}}{}^{+}\right)^{w_{k+1}}+\left(1-\xi_{I_{A_{k+1}}}{}^{+}\right)^{w_{k+1}}}\right] \\ \left. \left[\frac{2\left(\xi_{N_{A_{k+1}}}{}^{-}\right)^{w_{k+1}}}{\left(2-\xi_{N_{A_{k+1}}}{}^{-}\right)^{w_{k+1}}+\left(\xi_{N_{A_{k+1}}}{}^{-}\right)^{w_{k+1}}}, \frac{2\left(\xi_{N_{A_{k+1}}}{}^{+}\right)^{w_{k+1}}}{\left(2-\xi_{N_{A_{k+1}}}{}^{+}\right)^{w_{k+1}}+\left(\xi_{N_{A_{k+1}}}{}^{+}\right)^{w_{k+1}}}\right] \right\rangle \\ \left\langle \frac{2\left(P_{A_{k+1}}\right)^{w_{k+1}}}{\left(2-P_{A_{k+1}}\right)^{w_{k+1}}+\left(P_{A_{k+1}}\right)^{w_{k+1}}}, \frac{2\left(I_{A_{k+1}}\right)^{w_{k+1}}}{\left(2-I_{A_{k+1}}\right)^{w_{k+1}}+\left(I_{A_{k+1}}\right)^{w_{k+1}}}, \right. \\ \left. \frac{\left(1+N_{A_{k+1}}\right)^{w_{k+1}}-\left(1-N_{A_{k+1}}\right)^{w_{k+1}}}{\left(1+N_{A_{k+1}}\right)^{w_{k+1}}+\left(1-N_{A_{k+1}}\right)^{w_{k+1}}} \right\rangle \end{array}\right]$$

$$
\begin{aligned}
&CPFEWA\left(A_1, A_2, \ldots, A_{k+1}\right) \\
&= \left\langle
\left[
\frac{\left\{\frac{\prod_{j=1}^{k}\left(1+\xi_{P_{A_j}}^{-}\right)^{w_j}-\prod_{j=1}^{k}\left(1-\xi_{P_{A_j}}^{-}\right)^{w_j}}{\prod_{j=1}^{k}\left(1+\xi_{P_{A_j}}^{-}\right)^{w_j}+\prod_{j=1}^{k}\left(1-\xi_{P_{A_j}}^{-}\right)^{w_j}}\right\}+\left\{\frac{\left(1+\xi_{P_{A_{k+1}}}^{-}\right)^{w_{k+1}}-\left(1-\xi_{P_{A_{k+1}}}^{-}\right)^{w_{k+1}}}{\left(1+\xi_{P_{A_{k+1}}}^{-}\right)^{w_{k+1}}+\left(1-\xi_{P_{A_{k+1}}}^{-}\right)^{w_{k+1}}}\right\}}{1+\left\{\frac{\prod_{j=1}^{k}\left(1+\xi_{P_{A_j}}^{-}\right)^{w_j}-\prod_{j=1}^{k}\left(1-\xi_{P_{A_j}}^{-}\right)^{w_j}}{\prod_{j=1}^{k}\left(1+\xi_{P_{A_j}}^{-}\right)^{w_j}+\prod_{j=1}^{k}\left(1-\xi_{P_{A_j}}^{-}\right)^{w_j}}\right\}\left\{\frac{\left(1+\xi_{P_{A_{k+1}}}^{-}\right)^{w_{k+1}}-\left(1-\xi_{P_{A_{k+1}}}^{-}\right)^{w_{k+1}}}{\left(1+\xi_{P_{A_{k+1}}}^{-}\right)^{w_{k+1}}+\left(1-\xi_{P_{A_{k+1}}}^{-}\right)^{w_{k+1}}}\right\}},
\frac{\left\{\frac{\prod_{j=1}^{k}\left(1+\xi_{P_{A_j}}^{+}\right)^{w_j}-\prod_{j=1}^{k}\left(1-\xi_{P_{A_j}}^{+}\right)^{w_j}}{\prod_{j=1}^{k}\left(1+\xi_{P_{A_j}}^{+}\right)^{w_j}+\prod_{j=1}^{k}\left(1-\xi_{P_{A_j}}^{+}\right)^{w_j}}\right\}+\left\{\frac{\left(1+\xi_{P_{A_{k+1}}}^{+}\right)^{w_{k+1}}-\left(1-\xi_{P_{A_{k+1}}}^{+}\right)^{w_{k+1}}}{\left(1+\xi_{P_{A_{k+1}}}^{+}\right)^{w_{k+1}}+\left(1-\xi_{P_{A_{k+1}}}^{+}\right)^{w_{k+1}}}\right\}}{1+\left\{\frac{\prod_{j=1}^{k}\left(1+\xi_{P_{A_j}}^{+}\right)^{w_j}-\prod_{j=1}^{k}\left(1-\xi_{P_{A_j}}^{+}\right)^{w_j}}{\prod_{j=1}^{k}\left(1+\xi_{P_{A_j}}^{+}\right)^{w_j}+\prod_{j=1}^{k}\left(1-\xi_{P_{A_j}}^{+}\right)^{w_j}}\right\}\left\{\frac{\left(1+\xi_{P_{A_{k+1}}}^{+}\right)^{w_{k+1}}-\left(1-\xi_{P_{A_{k+1}}}^{+}\right)^{w_{k+1}}}{\left(1+\xi_{P_{A_{k+1}}}^{+}\right)^{w_{k+1}}+\left(1-\xi_{P_{A_{k+1}}}^{+}\right)^{w_{k+1}}}\right\}}
\right] \right.\\
&\left[
\frac{\left\{\frac{\prod_{j=1}^{k}\left(1+\xi_{I_{A_j}}^{-}\right)^{w_j}-\prod_{j=1}^{k}\left(1-\xi_{I_{A_j}}^{-}\right)^{w_j}}{\prod_{j=1}^{k}\left(1+\xi_{I_{A_j}}^{-}\right)^{w_j}+\prod_{j=1}^{k}\left(1-\xi_{I_{A_j}}^{-}\right)^{w_j}}\right\}+\left\{\frac{\left(1+\xi_{I_{A_{k+1}}}^{-}\right)^{w_{k+1}}-\left(1-\xi_{I_{A_{k+1}}}^{-}\right)^{w_{k+1}}}{\left(1+\xi_{I_{A_{k+1}}}^{-}\right)^{w_{k+1}}+\left(1-\xi_{I_{A_{k+1}}}^{-}\right)^{w_{k+1}}}\right\}}{1+\left\{\frac{\prod_{j=1}^{k}\left(1+\xi_{I_{A_j}}^{-}\right)^{w_j}-\prod_{j=1}^{k}\left(1-\xi_{I_{A_j}}^{-}\right)^{w_j}}{\prod_{j=1}^{k}\left(1+\xi_{I_{A_j}}^{-}\right)^{w_j}+\prod_{j=1}^{k}\left(1-\xi_{I_{A_j}}^{-}\right)^{w_j}}\right\}\left\{\frac{\left(1+\xi_{I_{A_{k+1}}}^{-}\right)^{w_{k+1}}-\left(1-\xi_{I_{A_{k+1}}}^{-}\right)^{w_{k+1}}}{\left(1+\xi_{I_{A_{k+1}}}^{-}\right)^{w_{k+1}}+\left(1-\xi_{I_{A_{k+1}}}^{-}\right)^{w_{k+1}}}\right\}},
\frac{\left\{\frac{\prod_{j=1}^{k}\left(1+\xi_{I_{A_j}}^{+}\right)^{w_j}-\prod_{j=1}^{k}\left(1-\xi_{I_{A_j}}^{+}\right)^{w_j}}{\prod_{j=1}^{k}\left(1+\xi_{I_{A_j}}^{+}\right)^{w_j}+\prod_{j=1}^{k}\left(1-\xi_{I_{A_j}}^{+}\right)^{w_j}}\right\}+\left\{\frac{\left(1+\xi_{I_{A_{k+1}}}^{+}\right)^{w_{k+1}}-\left(1-\xi_{I_{A_{k+1}}}^{+}\right)^{w_{k+1}}}{\left(1+\xi_{I_{A_{k+1}}}^{+}\right)^{w_{k+1}}+\left(1-\xi_{I_{A_{k+1}}}^{+}\right)^{w_{k+1}}}\right\}}{1+\left\{\frac{\prod_{j=1}^{k}\left(1+\xi_{I_{A_j}}^{+}\right)^{w_j}-\prod_{j=1}^{k}\left(1-\xi_{I_{A_j}}^{+}\right)^{w_j}}{\prod_{j=1}^{k}\left(1+\xi_{I_{A_j}}^{+}\right)^{w_j}+\prod_{j=1}^{k}\left(1-\xi_{I_{A_j}}^{+}\right)^{w_j}}\right\}\left\{\frac{\left(1+\xi_{I_{A_{k+1}}}^{+}\right)^{w_{k+1}}-\left(1-\xi_{I_{A_{k+1}}}^{+}\right)^{w_{k+1}}}{\left(1+\xi_{I_{A_{k+1}}}^{+}\right)^{w_{k+1}}+\left(1-\xi_{I_{A_{k+1}}}^{+}\right)^{w_{k+1}}}\right\}}
\right]\\
&\left[
\frac{\left\{\frac{2\prod_{j=1}^{k}\left(\xi_{N_{A_j}}^{-}\right)^{w_j}}{\prod_{j=1}^{k}\left(2-\xi_{N_{A_j}}^{-}\right)^{w_j}+\prod_{j=1}^{k}\left(\xi_{N_{A_j}}^{-}\right)^{w_j}}\right\}\left\{\frac{2\left(\xi_{N_{A_{k+1}}}^{-}\right)^{w_{k+1}}}{\left(2-\xi_{N_{A_{k+1}}}^{-}\right)^{w_{k+1}}+\left(\xi_{N_{A_{k+1}}}^{-}\right)^{w_{k+1}}}\right\}}{1+\left\{1-\frac{2\prod_{j=1}^{k}\left(\xi_{N_{A_j}}^{-}\right)^{w_j}}{\prod_{j=1}^{k}\left(2-\xi_{N_{A_j}}^{-}\right)^{w_j}+\prod_{j=1}^{k}\left(\xi_{N_{A_j}}^{-}\right)^{w_j}}\right\}\left\{1-\frac{2\left(\xi_{N_{A_{k+1}}}^{-}\right)^{w_{k+1}}}{\left(2-\xi_{N_{A_{k+1}}}^{-}\right)^{w_{k+1}}+\left(\xi_{N_{A_{k+1}}}^{-}\right)^{w_{k+1}}}\right\}},
\frac{\left\{\frac{2\prod_{j=1}^{k}\left(\xi_{N_{A_j}}^{+}\right)^{w_j}}{\prod_{j=1}^{k}\left(2-\xi_{N_{A_j}}^{+}\right)^{w_j}+\prod_{j=1}^{k}\left(\xi_{N_{A_j}}^{+}\right)^{w_j}}\right\}\left\{\frac{2\left(\xi_{N_{A_{k+1}}}^{+}\right)^{w_{k+1}}}{\left(2-\xi_{N_{A_{k+1}}}^{+}\right)^{w_{k+1}}+\left(\xi_{N_{A_{k+1}}}^{+}\right)^{w_{k+1}}}\right\}}{1+\left\{1-\frac{2\prod_{j=1}^{k}\left(\xi_{N_{A_j}}^{+}\right)^{w_j}}{\prod_{j=1}^{k}\left(2-\xi_{N_{A_j}}^{+}\right)^{w_j}+\prod_{j=1}^{k}\left(\xi_{N_{A_j}}^{+}\right)^{w_j}}\right\}\left\{1-\frac{2\left(\xi_{N_{A_{k+1}}}^{+}\right)^{w_{k+1}}}{\left(2-\xi_{N_{A_{k+1}}}^{+}\right)^{w_{k+1}}+\left(\xi_{N_{A_{k+1}}}^{+}\right)^{w_{k+1}}}\right\}}
\right]\Bigg\rangle,\\
&\Bigg\langle
\frac{\left\{\frac{2\prod_{j=1}^{k}\left(P_{A_j}\right)^{w_j}}{\prod_{j=1}^{k}\left(2-P_{A_j}\right)^{w_j}+\prod_{j=1}^{k}\left(P_{A_j}\right)^{w_j}}\right\}\left\{\frac{2\left(P_{A_{k+1}}\right)^{w_{k+1}}}{\left(2-P_{A_{k+1}}\right)^{w_{k+1}}+\left(P_{A_{k+1}}\right)^{w_{k+1}}}\right\}}{1+\left\{1-\frac{2\prod_{j=1}^{k}\left(P_{A_j}\right)^{w_j}}{\prod_{j=1}^{k}\left(2-P_{A_j}\right)^{w_j}+\prod_{j=1}^{k}\left(P_{A_j}\right)^{w_j}}\right\}\left\{1-\frac{2\left(P_{A_{k+1}}\right)^{w_{k+1}}}{\left(2-P_{A_{k+1}}\right)^{w_{k+1}}+\left(P_{A_{k+1}}\right)^{w_{k+1}}}\right\}},
\frac{\left\{\frac{2\prod_{j=1}^{k}\left(I_{A_j}\right)^{w_j}}{\prod_{j=1}^{k}\left(2-I_{A_j}\right)^{w_j}+\prod_{j=1}^{k}\left(I_{A_j}\right)^{w_j}}\right\}\left\{\frac{2\left(I_{A_{k+1}}\right)^{w_{k+1}}}{\left(2-I_{A_{k+1}}\right)^{w_{k+1}}+\left(I_{A_{k+1}}\right)^{w_{k+1}}}\right\}}{1+\left\{1-\frac{2\prod_{j=1}^{k}\left(I_{A_j}\right)^{w_j}}{\prod_{j=1}^{k}\left(2-I_{A_j}\right)^{w_j}+\prod_{j=1}^{k}\left(I_{A_j}\right)^{w_j}}\right\}\left\{1-\frac{2\left(I_{A_{k+1}}\right)^{w_{k+1}}}{\left(2-I_{A_{k+1}}\right)^{w_{k+1}}+\left(I_{A_{k+1}}\right)^{w_{k+1}}}\right\}},\\
&\frac{\left\{\frac{\prod_{j=1}^{k}\left(1+N_{A_j}\right)^{w_j}-\prod_{j=1}^{k}\left(1-N_{A_j}\right)^{w_j}}{\prod_{j=1}^{k}\left(1+N_{A_j}\right)^{w_j}+\prod_{j=1}^{k}\left(1-N_{A_j}\right)^{w_j}}\right\}+\left\{\frac{\left(1+N_{A_{k+1}}\right)^{w_{k+1}}-\left(1-N_{A_{k+1}}\right)^{w_{k+1}}}{\left(1+N_{A_{k+1}}\right)^{w_{k+1}}+\left(1-N_{A_{k+1}}\right)^{w_{k+1}}}\right\}}{1+\left\{\frac{\prod_{j=1}^{k}\left(1+N_{A_j}\right)^{w_j}-\prod_{j=1}^{k}\left(1-N_{A_j}\right)^{w_j}}{\prod_{j=1}^{k}\left(1+N_{A_j}\right)^{w_j}+\prod_{j=1}^{k}\left(1-N_{A_j}\right)^{w_j}}\right\}\left\{\frac{\left(1+N_{A_{k+1}}\right)^{w_{k+1}}-\left(1-N_{A_{k+1}}\right)^{w_{k+1}}}{\left(1+N_{A_{k+1}}\right)^{w_{k+1}}+\left(1-N_{A_{k+1}}\right)^{w_{k+1}}}\right\}}
\Bigg\rangle
\end{aligned}
$$

$$CPFEWA\left(A_1, A_2, \ldots, A_{k+1}\right)$$

$$= \left[\left\langle \left[\frac{\left\{\prod_{j=1}^{k}\left(1+\xi_{P_{A_j}}{}^{-}\right)^{w_j} - \prod_{j=1}^{k}\left(1-\xi_{P_{A_j}}{}^{-}\right)^{w_j}\right\}\left\{\left(1+\xi_{P_{A_{k+1}}}{}^{-}\right)^{w_{k+1}} + \left(1-\xi_{P_{A_{k+1}}}{}^{-}\right)^{w_{k+1}}\right\} + \left\{\prod_{j=1}^{k}\left(1+\xi_{P_{A_j}}{}^{-}\right)^{w_j} + \prod_{j=1}^{k}\left(1-\xi_{P_{A_j}}{}^{-}\right)^{w_j}\right\}\left\{\left(1+\xi_{P_{A_{k+1}}}{}^{-}\right)^{w_{k+1}} - \left(1-\xi_{P_{A_{k+1}}}{}^{-}\right)^{w_{k+1}}\right\}}{\left\{\prod_{j=1}^{k}\left(1+\xi_{P_{A_j}}{}^{-}\right)^{w_j} + \prod_{j=1}^{k}\left(1-\xi_{P_{A_j}}{}^{-}\right)^{w_j}\right\}\left\{\left(1+\xi_{P_{A_{k+1}}}{}^{-}\right)^{w_{k+1}} + \left(1-\xi_{P_{A_{k+1}}}{}^{-}\right)^{w_{k+1}}\right\} + \left\{\prod_{j=1}^{k}\left(1+\xi_{P_{A_j}}{}^{-}\right)^{w_j} - \prod_{j=1}^{k}\left(1-\xi_{P_{A_j}}{}^{-}\right)^{w_j}\right\}\left\{\left(1+\xi_{P_{A_{k+1}}}{}^{-}\right)^{w_{k+1}} - \left(1-\xi_{P_{A_{k+1}}}{}^{-}\right)^{w_{k+1}}\right\}}, \frac{\left\{\prod_{j=1}^{k}\left(1+\xi_{P_{A_j}}{}^{+}\right)^{w_j} - \prod_{j=1}^{k}\left(1-\xi_{P_{A_j}}{}^{+}\right)^{w_j}\right\}\left\{\left(1+\xi_{P_{A_{k+1}}}{}^{+}\right)^{w_{k+1}} + \left(1-\xi_{P_{A_{k+1}}}{}^{+}\right)^{w_{k+1}}\right\} + \left\{\prod_{j=1}^{k}\left(1+\xi_{P_{A_j}}{}^{+}\right)^{w_j} + \prod_{j=1}^{k}\left(1-\xi_{P_{A_j}}{}^{+}\right)^{w_j}\right\}\left\{\left(1+\xi_{P_{A_{k+1}}}{}^{+}\right)^{w_{k+1}} - \left(1-\xi_{P_{A_{k+1}}}{}^{+}\right)^{w_{k+1}}\right\}}{\left\{\prod_{j=1}^{k}\left(1+\xi_{P_{A_j}}{}^{+}\right)^{w_j} + \prod_{j=1}^{k}\left(1-\xi_{P_{A_j}}{}^{+}\right)^{w_j}\right\}\left\{\left(1+\xi_{P_{A_{k+1}}}{}^{+}\right)^{w_{k+1}} + \left(1-\xi_{P_{A_{k+1}}}{}^{+}\right)^{w_{k+1}}\right\} + \left\{\prod_{j=1}^{k}\left(1+\xi_{P_{A_j}}{}^{+}\right)^{w_j} - \prod_{j=1}^{k}\left(1-\xi_{P_{A_j}}{}^{+}\right)^{w_j}\right\}\left\{\left(1+\xi_{P_{A_{k+1}}}{}^{+}\right)^{w_{k+1}} - \left(1-\xi_{P_{A_{k+1}}}{}^{+}\right)^{w_{k+1}}\right\}} \right] \right.\right.$$

$$\left[\frac{\left\{\prod_{j=1}^{k}\left(1+\xi_{I_{A_j}}{}^{-}\right)^{w_j} - \prod_{j=1}^{k}\left(1-\xi_{I_{A_j}}{}^{-}\right)^{w_j}\right\}\left\{\left(1+\xi_{I_{A_{k+1}}}{}^{-}\right)^{w_{k+1}} + \left(1-\xi_{I_{A_{k+1}}}{}^{-}\right)^{w_{k+1}}\right\} + \left\{\prod_{j=1}^{k}\left(1+\xi_{I_{A_j}}{}^{-}\right)^{w_j} + \prod_{j=1}^{k}\left(1-\xi_{I_{A_j}}{}^{-}\right)^{w_j}\right\}\left\{\left(1+\xi_{I_{A_{k+1}}}{}^{-}\right)^{w_{k+1}} - \left(1-\xi_{I_{A_{k+1}}}{}^{-}\right)^{w_{k+1}}\right\}}{\left\{\prod_{j=1}^{k}\left(1+\xi_{I_{A_j}}{}^{-}\right)^{w_j} + \prod_{j=1}^{k}\left(1-\xi_{I_{A_j}}{}^{-}\right)^{w_j}\right\}\left\{\left(1+\xi_{I_{A_{k+1}}}{}^{-}\right)^{w_{k+1}} + \left(1-\xi_{I_{A_{k+1}}}{}^{-}\right)^{w_{k+1}}\right\} + \left\{\prod_{j=1}^{k}\left(1+\xi_{I_{A_j}}{}^{-}\right)^{w_j} - \prod_{j=1}^{k}\left(1-\xi_{I_{A_j}}{}^{-}\right)^{w_j}\right\}\left\{\left(1+\xi_{I_{A_{k+1}}}{}^{-}\right)^{w_{k+1}} - \left(1-\xi_{I_{A_{k+1}}}{}^{-}\right)^{w_{k+1}}\right\}}, \frac{\left\{\prod_{j=1}^{k}\left(1+\xi_{I_{A_j}}{}^{+}\right)^{w_j} - \prod_{j=1}^{k}\left(1-\xi_{I_{A_j}}{}^{+}\right)^{w_j}\right\}\left\{\left(1+\xi_{I_{A_{k+1}}}{}^{+}\right)^{w_{k+1}} + \left(1-\xi_{I_{A_{k+1}}}{}^{+}\right)^{w_{k+1}}\right\} + \left\{\prod_{j=1}^{k}\left(1+\xi_{I_{A_j}}{}^{+}\right)^{w_j} + \prod_{j=1}^{k}\left(1-\xi_{I_{A_j}}{}^{+}\right)^{w_j}\right\}\left\{\left(1+\xi_{I_{A_{k+1}}}{}^{+}\right)^{w_{k+1}} - \left(1-\xi_{I_{A_{k+1}}}{}^{+}\right)^{w_{k+1}}\right\}}{\left\{\prod_{j=1}^{k}\left(1+\xi_{I_{A_j}}{}^{+}\right)^{w_j} + \prod_{j=1}^{k}\left(1-\xi_{I_{A_j}}{}^{+}\right)^{w_j}\right\}\left\{\left(1+\xi_{I_{A_{k+1}}}{}^{+}\right)^{w_{k+1}} + \left(1-\xi_{I_{A_{k+1}}}{}^{+}\right)^{w_{k+1}}\right\} + \left\{\prod_{j=1}^{k}\left(1+\xi_{I_{A_j}}{}^{+}\right)^{w_j} - \prod_{j=1}^{k}\left(1-\xi_{I_{A_j}}{}^{+}\right)^{w_j}\right\}\left\{\left(1+\xi_{I_{A_{k+1}}}{}^{+}\right)^{w_{k+1}} - \left(1-\xi_{I_{A_{k+1}}}{}^{+}\right)^{w_{k+1}}\right\}} \right]$$

$$\left.\left[\frac{\left\{2\prod_{j=1}^{k}\left(\xi_{N_{A_j}}{}^{-}\right)^{w_j}\right\}\left\{2\left(\xi_{N_{A_{k+1}}}{}^{-}\right)^{w_{k+1}}\right\}}{\left\{\prod_{j=1}^{k}\left(2-\xi_{N_{A_j}}{}^{-}\right)^{w_j} + \prod_{j=1}^{k}\left(\xi_{N_{A_j}}{}^{-}\right)^{w_j}\right\}\left\{\left(2-\xi_{N_{A_2}}{}^{-}\right)^{w_2} + \left(\xi_{N_{A_2}}{}^{-}\right)^{w_2}\right\} + \left\{\prod_{j=1}^{k}\left(2-\xi_{N_{A_j}}{}^{-}\right)^{w_j} - \prod_{j=1}^{k}\left(\xi_{N_{A_j}}{}^{-}\right)^{w_j}\right\}\left\{\left(2-\xi_{N_{A_2}}{}^{-}\right)^{w_2} - \left(\xi_{N_{A_2}}{}^{-}\right)^{w_2}\right\}}, \frac{\left\{2\prod_{j=1}^{k}\left(\xi_{N_{A_j}}{}^{+}\right)^{w_j}\right\}\left\{2\left(\xi_{N_{A_{k+1}}}{}^{+}\right)^{w_{k+1}}\right\}}{\left\{\prod_{j=1}^{k}\left(2-\xi_{N_{A_j}}{}^{+}\right)^{w_j} + \prod_{j=1}^{k}\left(\xi_{N_{A_j}}{}^{+}\right)^{w_j}\right\}\left\{\left(2-\xi_{N_{A_2}}{}^{+}\right)^{w_2} + \left(\xi_{N_{A_2}}{}^{+}\right)^{w_2}\right\} + \left\{\prod_{j=1}^{k}\left(2-\xi_{N_{A_j}}{}^{+}\right)^{w_j} - \prod_{j=1}^{k}\left(\xi_{N_{A_j}}{}^{+}\right)^{w_j}\right\}\left\{\left(2-\xi_{N_{A_2}}{}^{+}\right)^{w_2} - \left(\xi_{N_{A_2}}{}^{+}\right)^{w_2}\right\}} \right]\right\rangle,$$

$$\left\langle \frac{\left\{2\prod_{j=1}^{k}\left(P_{A_j}\right)^{w_j}\right\}\left\{2\left(P_{A_{k+1}}\right)^{w_{k+1}}\right\}}{\left\{\prod_{j=1}^{k}\left(2-P_{A_j}\right)^{w_j} + \prod_{j=1}^{k}\left(P_{A_j}\right)^{w_j}\right\}\left\{\left(2-P_{A_{k+1}}\right)^{w_{k+1}} + \left(P_{A_{k+1}}\right)^{w_{k+1}}\right\} + \left\{\prod_{j=1}^{k}\left(2-P_{A_j}\right)^{w_j} - \prod_{j=1}^{k}\left(P_{A_j}\right)^{w_j}\right\}\left\{\left(2-P_{A_{k+1}}\right)^{w_{k+1}} - \left(P_{A_{k+1}}\right)^{w_{k+1}}\right\}}, \right.$$

$$\frac{\left\{2\prod_{j=1}^{k}\left(I_{A_j}\right)^{w_j}\right\}\left\{2\left(I_{A_{k+1}}\right)^{w_{k+1}}\right\}}{\left\{\prod_{j=1}^{k}\left(2-I_{A_j}\right)^{w_j} + \prod_{j=1}^{k}\left(I_{A_j}\right)^{w_j}\right\}\left\{\left(2-I_{A_{k+1}}\right)^{w_{k+1}} + \left(I_{A_{k+1}}\right)^{w_{k+1}}\right\} + \left\{\prod_{j=1}^{k}\left(2-I_{A_j}\right)^{w_j} - \prod_{j=1}^{k}\left(I_{A_j}\right)^{w_j}\right\}\left\{\left(2-I_{A_{k+1}}\right)^{w_{k+1}} - \left(I_{A_{k+1}}\right)^{w_{k+1}}\right\}}$$

$$\left.\left. \frac{\left\{\prod_{j=1}^{k}\left(1+N_{A_j}\right)^{w_j} - \prod_{j=1}^{k}\left(1-N_{A_j}\right)^{w_j}\right\}\left\{\left(1+N_{A_{k+1}}\right)^{w_{k+1}} + \left(1-N_{A_{k+1}}\right)^{w_{k+1}}\right\} + \left\{\prod_{j=1}^{k}\left(1+N_{A_j}\right)^{w_j} + \prod_{j=1}^{k}\left(1-N_{A_j}\right)^{w_j}\right\}\left\{\left(1+N_{A_{k+1}}\right)^{w_{k+1}} - \left(1-N_{A_{k+1}}\right)^{w_{k+1}}\right\}}{\left\{\prod_{j=1}^{k}\left(1+N_{A_j}\right)^{w_j} + \prod_{j=1}^{k}\left(1-N_{A_j}\right)^{w_j}\right\}\left\{\left(1+N_{A_{k+1}}\right)^{w_{k+1}} + \left(1-N_{A_{k+1}}\right)^{w_{k+1}}\right\} + \left\{\prod_{j=1}^{k}\left(1+N_{A_j}\right)^{w_j} - \prod_{j=1}^{k}\left(1-N_{A_j}\right)^{w_j}\right\}\left\{\left(1+N_{A_{k+1}}\right)^{w_{k+1}} - \left(1-N_{A_{k+1}}\right)^{w_{k+1}}\right\}} \right\rangle \right]$$

It is true for $n=k+1$.

Due to (1), (2), and (3), the following conclusion can be drawn:

$$CPFEWA\left(A_1,A_2,\ldots,A_n\right)=$$
$$\left[\begin{array}{c}\left\langle\left[\begin{array}{c}\frac{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{-}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{-}\right)^{w_j}},\\ \frac{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{+}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{+}\right)^{w_j}}\end{array}\right],\frac{2\prod_{j=1}^{n}\left(P_{A_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-P_{A_j}\right)^{w_j}+\prod_{j=1}^{n}\left(P_{A_j}\right)^{w_j}}\right\rangle\\ \left\langle\left[\begin{array}{c}\frac{\prod_{j=1}^{n}\left(1+\xi_{I_{A_j}}{}^{-}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{I_{A_j}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{I_{A_j}}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{I_{A_j}}{}^{-}\right)^{w_j}},\\ \frac{\prod_{j=1}^{n}\left(1+\xi_{I_{A_j}}{}^{+}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{I_{A_j}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{I_{A_j}}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{I_{A_j}}{}^{+}\right)^{w_j}}\end{array}\right],\frac{2\prod_{j=1}^{n}\left(I_{A_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-I_{A_j}\right)^{w_j}+\prod_{j=1}^{n}\left(I_{A_j}\right)^{w_j}}\right\rangle\\ \left\langle\left[\begin{array}{c}\frac{2\prod_{j=1}^{n}\left(\xi_{N_{A_j}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\xi_{N_{A_j}}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(\xi_{N_{A_j}}{}^{-}\right)^{w_j}},\\ \frac{2\prod_{j=1}^{n}\left(\xi_{N_{A_j}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\xi_{N_{A_j}}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(\xi_{N_{A_j}}{}^{+}\right)^{w_j}}\end{array}\right],\frac{\prod_{j=1}^{n}\left(1+N_{A_j}\right)^{w_j}-\prod_{j=1}^{n}\left(1-N_{A_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+N_{A_j}\right)^{w_j}+\prod_{j=1}^{n}\left(1-N_{A_j}\right)^{w_j}}\right\rangle\end{array}\right]$$

This can also be written as

$$CPFEWA\left(A_1,A_2,\ldots,A_n\right)=$$
$$\left[\begin{array}{c}\left\langle\left[\begin{array}{c}\frac{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{-}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{-}\right)^{w_j}},\\ \frac{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{+}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{+}\right)^{w_j}}\end{array}\right],\frac{2\prod_{j=1}^{n}\left(P_{A_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-P_{A_j}\right)^{w_j}+\prod_{j=1}^{n}\left(P_{A_j}\right)^{w_j}}\right\rangle\\ \left\langle\left[\begin{array}{c}\frac{\prod_{j=1}^{n}\left(1+\xi_{I_{A_j}}{}^{-}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{I_{A_j}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{I_{A_j}}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{I_{A_j}}{}^{-}\right)^{w_j}},\\ \frac{\prod_{j=1}^{n}\left(1+\xi_{I_{A_j}}{}^{+}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{I_{A_j}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{I_{A_j}}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{I_{A_j}}{}^{+}\right)^{w_j}}\end{array}\right],\frac{2\prod_{j=1}^{n}\left(I_{A_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-I_{A_j}\right)^{w_j}+\prod_{j=1}^{n}\left(I_{A_j}\right)^{w_j}}\right\rangle\\ \left\langle\left[\begin{array}{c}\frac{2\prod_{j=1}^{n}\left(\xi_{N_{A_j}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\xi_{N_{A_j}}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(\xi_{N_{A_j}}{}^{-}\right)^{w_j}},\\ \frac{2\prod_{j=1}^{n}\left(\xi_{N_{A_j}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\xi_{N_{A_j}}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(\xi_{N_{A_j}}{}^{+}\right)^{w_j}}\end{array}\right],\frac{\prod_{j=1}^{n}\left(1+N_{A_j}\right)^{w_j}-\prod_{j=1}^{n}\left(1-N_{A_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+N_{A_j}\right)^{w_j}+\prod_{j=1}^{n}\left(1-N_{A_j}\right)^{w_j}}\right\rangle\end{array}\right]$$

Theorem 5.1.2 **(Idempotency)** Let

$$A_j=\left\{\left\langle\left[\xi^{-}{}_{P_{A_j}},\xi^{+}{}_{P_{A_j}}\right],P_{A_j}\right\rangle,\left\langle\left[\xi^{-}{}_{I_{A_j}},\xi^{+}{}_{I_{A_j}}\right],I_{A_j}\right\rangle,\left\langle\left[\xi^{-}{}_{N_{A_j}},\xi^{+}{}_{N_{A_j}}\right],N_{A_j}\right\rangle\right\}$$

be a collection of CPFNs If all $A_j=1,2,3,\ldots,n)$ are the same, i.e.

$$A_j=\mathrm{A}=\left\{\left\langle\left[\xi^{-}{}_{P_A},\xi^{+}{}_{P_A}\right],P_A\right\rangle,\left\langle\left[\xi^{-}{}_{I_A},\xi^{+}{}_{I_A}\right],I_A\right\rangle,\left\langle\left[\xi^{-}{}_{N_A},\xi^{+}{}_{N_A}\right],N_A\right\rangle\right\}$$

$\forall j$, then

$$\mathrm{CPFEWA}\left(A_1,A_2,\ldots,A_n\right)=A$$

Proof Theorem 5.1.1 says that we have

$$CPFEWA(A_1,A_2,\ldots,A_n)=$$

$$\left[\begin{array}{l}\left\langle\left[\begin{array}{l}\frac{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{-}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{-}\right)^{w_j}},\\ \frac{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{+}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{+}\right)^{w_j}}\end{array}\right],\frac{2\prod_{j=1}^{n}\left(P_{A_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-P_{A_j}\right)^{w_j}+\prod_{j=1}^{n}\left(P_{A_j}\right)^{w_j}}\right\rangle\\ \left\langle\left[\begin{array}{l}\frac{\prod_{j=1}^{n}\left(1+\xi_{I_{A_j}}{}^{-}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{I_{A_j}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{I_{A_j}}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{I_{A_j}}{}^{-}\right)^{w_j}},\\ \frac{\prod_{j=1}^{n}\left(1+\xi_{I_{A_j}}{}^{+}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{I_{A_j}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{I_{A_j}}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{I_{A_j}}{}^{+}\right)^{w_j}}\end{array}\right],\frac{2\prod_{j=1}^{n}\left(I_{A_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-I_{A_j}\right)^{w_j}+\prod_{j=1}^{n}\left(I_{A_j}\right)^{w_j}}\right\rangle\\ \left\langle\left[\begin{array}{l}\frac{2\prod_{j=1}^{n}\left(\xi_{N_{A_j}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\xi_{N_{A_j}}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(\xi_{N_{A_j}}{}^{-}\right)^{w_j}},\\ \frac{2\prod_{j=1}^{n}\left(\xi_{N_{A_j}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\xi_{N_{A_j}}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(\xi_{N_{A_j}}{}^{+}\right)^{w_j}}\end{array}\right],\frac{\prod_{j=1}^{n}\left(1+N_{A_j}\right)^{w_j}-\prod_{j=1}^{n}\left(1-N_{A_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+N_{A_j}\right)^{w_j}+\prod_{j=1}^{n}\left(1-N_{A_j}\right)^{w_j}}\right\rangle\end{array}\right]$$

$$CPFEWA\big(A_1,A_2,\ldots,A_n\big)=$$

$$\left[\begin{array}{l}\left\langle\left[\begin{array}{l}\frac{\prod_{j=1}^{n}\left(1+\xi_{P_A}{}^{-}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{P_A}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{P_A}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{P_A}{}^{-}\right)^{w_j}},\\ \frac{\prod_{j=1}^{n}\left(1+\xi_{P_A}{}^{+}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{P_A}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{P_A}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{P_A}{}^{+}\right)^{w_j}}\end{array}\right],\frac{2\prod_{j=1}^{n}\left(P_A\right)^{w_j}}{\prod_{j=1}^{n}\left(2-P_A\right)^{w_j}+\prod_{j=1}^{n}\left(P_A\right)^{w_j}}\right\rangle\\ \left\langle\left[\begin{array}{l}\frac{\prod_{j=1}^{n}\left(1+\xi_{I_A}{}^{-}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{I_A}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{I_A}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{I_A}{}^{-}\right)^{w_j}},\\ \frac{\prod_{j=1}^{n}\left(1+\xi_{I_A}{}^{+}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{I_A}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{I_A}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{I_A}{}^{+}\right)^{w_j}}\end{array}\right],\frac{2\prod_{j=1}^{n}\left(I_A\right)^{w_j}}{\prod_{j=1}^{n}\left(2-I_A\right)^{w_j}+\prod_{j=1}^{n}\left(I_A\right)^{w_j}}\right\rangle\\ \left\langle\left[\begin{array}{l}\frac{2\prod_{j=1}^{n}\left(\xi_{N_A}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\xi_{N_A}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(\xi_{N_A}{}^{-}\right)^{w_j}},\\ \frac{2\prod_{j=1}^{n}\left(\xi_{N_A}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\xi_{N_A}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(\xi_{N_A}{}^{+}\right)^{w_j}}\end{array}\right],\frac{\prod_{j=1}^{n}\left(1+N_A\right)^{w_j}-\prod_{j=1}^{n}\left(1-N_A\right)^{w_j}}{\prod_{j=1}^{n}\left(1+N_A\right)^{w_j}+\prod_{j=1}^{n}\left(1-N_A\right)^{w_j}}\right\rangle\end{array}\right]$$

$$CPFEWA(A_1,A_2,\ldots,A_n)=$$

$$\left[\begin{array}{l}\left\langle\left[\frac{\left(1+\xi_{P_A}{}^{-}\right)-\left(1-\xi_{P_A}{}^{-}\right)}{\left(1+\xi_{P_A}{}^{-}\right)+\left(1-\xi_{P_A}{}^{-}\right)},\frac{\left(1+\xi_{P_A}{}^{+}\right)-\left(1-\xi_{P_A}{}^{+}\right)}{\left(1+\xi_{P_A}{}^{+}\right)+\left(1-\xi_{P_A}{}^{+}\right)}\right],\frac{2(P_A)}{(2-P_A)+(P_A)}\right\rangle\\ \left\langle\left[\frac{\left(1+\xi_{I_A}-\right)-\left(1-\xi_{I_A}-\right)}{\left(1+\xi_{I_A}-\right)+\left(1-\xi_{I_A}-\right)},\frac{\left(1+\xi_{I_A}{}^{+}\right)-\left(1-\xi_{I_A}{}^{+}\right)}{\left(1+\xi_{I_A}{}^{+}\right)+\left(1-\xi_{I_A}{}^{+}\right)}\right],\frac{2(I_A)}{(2-I_A)+(I_A)}\right\rangle\\ \left\langle\left[\frac{2\left(\xi_{N_A}-\right)}{\left(2-\xi_{N_A}-\right)+\left(\xi_{N_A}-\right)},\frac{2\left(\xi_{N_A}{}^{+}\right)}{\left(2-\xi_{N_A}{}^{+}\right)+\left(\xi_{N_A}{}^{+}\right)}\right],\frac{(1+N_A)-(1-N_A)}{(1+N_A)+(1-N_A)}\right\rangle\end{array}\right]$$

$$CPFEWA(A_1,A_2,\ldots,A_n)=\left\{\langle\left[\xi^-{}_{P_A},\xi^+{}_{P_A}\right],P_A\rangle,\langle\left[\xi^-{}_{I_A},\xi^+{}_{I_A}\right],I_A\rangle,\langle\left[\xi^-{}_{N_A},\xi^+{}_{N_A}\right],N_A\rangle\right\}$$

$$CPFEWA(A_1,A_2,\ldots,A_n)=\mathrm{A}$$

Theorem 5.1.3 **(Boundedness)** Let

$$A_j=\left\{\left\langle\left[\xi^-{}_{P_{A_j}},\xi^+{}_{P_{A_j}}\right],P_{A_j}\right\rangle,\left\langle\left[\xi^-{}_{I_{A_j}},\xi^+{}_{I_{A_j}}\right],I_{A_j}\right\rangle,\left\langle\left[\xi^-{}_{N_{A_j}},\xi^+{}_{N_{A_j}}\right],N_{A_j}\right\rangle\right\}$$

be a collection of CPFNs if all $A_j(j=1,2,3,\ldots,n)$ and let

$$A^- = \left\{ \begin{array}{c} \left\langle \left[\min_j \left(\xi^-_{P_{A_j}} \right), \min_j \left(\xi^+{}_{P_{A_j}} \right) \right], \max_j (P_{A_j}) \right\rangle, \\ \left\langle \left[\min_j \left(\xi^-_{I_{A_j}} \right), \min_j (\xi^+{}_{I_{A_j}}) \right], \max_j (I_{A_j}) \right\rangle, \\ \left\langle \left[[\min_j \left(\xi^-_{N_{A_j}} \right), \min_j \left(\xi^+{}_{N_{A_j}} \right) \right], \max_j \left(N_{A_j} \right) \right\rangle \end{array} \right\}$$

$$A^+ = \left\{ \begin{array}{c} \left\langle \left[\underset{j}{max} \left(\xi_{P_{A_j}}{}^+ \right), \underset{j}{max} \left(\xi_{P_{A_j}}{}^+ \right) \right], \underset{j}{min} \left(P_{A_j} \right) \right\rangle, \\ \left\langle \left[\underset{j}{max} \left(\xi_{I_{A_j}}{}^- \right), \underset{j}{max} \left(\xi_{I_{A_j}}{}^+ \right) \right], \underset{j}{min} \left(I_{A_j} \right) \right\rangle, \\ \left\langle \left[\underset{j}{min} \left(\xi_{N_{A_j}}{}^+ \right), \underset{j}{min} \left(\xi_{N_{A_j}}{}^+ \right) \right], \underset{j}{max} \left(N_{A_j} \right) \right\rangle \end{array} \right\}$$

Then $A^- \leq \text{CPFEWA}\,(A_1, A_2, \ldots, A_n) \leq A^+$

Proof Theorem 5.1.1 says that we have

$$CPFEWA\,(A_1, A_2, \ldots, A_n) =$$

$$\left[\begin{array}{c} \left\langle \left[\begin{array}{c} \frac{\prod_{j=1}^n \left(1+\xi_{P_{A_j}}{}^-\right)^{w_j} - \prod_{j=1}^n \left(1-\xi_{P_{A_j}}{}^-\right)^{w_j}}{\prod_{j=1}^n \left(1+\xi_{P_{A_j}}{}^-\right)^{w_j} + \prod_{j=1}^n \left(1-\xi_{P_{A_j}}{}^-\right)^{w_j}}, \\ \frac{\prod_{j=1}^n \left(1+\xi_{P_{A_j}}{}^+\right)^{w_j} - \prod_{j=1}^n \left(1-\xi_{P_{A_j}}{}^+\right)^{w_j}}{\prod_{j=1}^n \left(1+\xi_{P_{A_j}}{}^+\right)^{w_j} + \prod_{j=1}^n \left(1-\xi_{P_{A_j}}{}^+\right)^{w_j}} \end{array} \right], \frac{2\prod_{j=1}^n (P_{A_j})^{w_j}}{\prod_{j=1}^n (2-P_{A_j})^{w_j} + \prod_{j=1}^n (P_{A_j})^{w_j}} \right\rangle \\ \left\langle \left[\begin{array}{c} \frac{\prod_{j=1}^n \left(1+\xi_{I_{A_j}}{}^-\right)^{w_j} - \prod_{j=1}^n \left(1-\xi_{I_{A_j}}{}^-\right)^{w_j}}{\prod_{j=1}^n \left(1+\xi_{I_{A_j}}{}^-\right)^{w_j} + \prod_{j=1}^n \left(1-\xi_{I_{A_j}}{}^-\right)^{w_j}}, \\ \frac{\prod_{j=1}^n (1+\xi_{I_{A_j}}{}^+)^{w_j} - \prod_{j=1}^n (1-\xi_{I_{A_j}}{}^+)^{w_j}}{\prod_{j=1}^n (1+\xi_{I_{A_j}}{}^+)^{w_j} + \prod_{j=1}^n (1-\xi_{I_{A_j}}{}^+)^{w_j}} \end{array} \right], \frac{2\prod_{j=1}^n (I_{A_j})^{w_j}}{\prod_{j=1}^n (2-I_{A_j})^{w_j} + \prod_{j=1}^n (I_{A_j})^{w_j}} \right\rangle \\ \left\langle \left[\begin{array}{c} \frac{2\prod_{j=1}^n \left(\xi_{N_{A_j}}{}^-\right)^{w_j}}{\prod_{j=1}^n \left(2-\xi_{N_{A_j}}{}^-\right)^{w_j} + \prod_{j=1}^n \left(\xi_{N_{A_j}}{}^-\right)^{w_j}}, \\ \frac{2\prod_{j=1}^n (\xi_{N_{A_j}}{}^+)^{w_j}}{\prod_{j=1}^n (2-\xi_{N_{A_j}}{}^+)^{w_j} + \prod_{j=1}^n (\xi_{N_{A_j}}{}^+)^{w_j}} \end{array} \right], \frac{\prod_{j=1}^n (1+N_{A_j})^{w_j} - \prod_{j=1}^n \left(1-N_{A_j}\right)^{w_j}}{\prod_{j=1}^n (1+N_{A_j})^{w_j} + \prod_{j=1}^n (1-N_{A_j})^{w_j}} \right\rangle \end{array} \right]$$

Firstly, for degree of **membership**

Since $\left(\underset{j}{min} \left(\xi_{P_{A_j}}{}^- \right) \right) \leq \left(\xi_{P_{A_j}}{}^- \right) \leq \left(\max_j \left(\xi_{P_{A_j}}{}^- \right) \right)$

$$\left(\prod_{j=1}^n \left(1 + \xi_{P_{A_j}}{}^-\right)^{w_j} \right) \geq \left(\prod_{j=1}^n \left(1 - \max_j (\xi^-_{P_{A_j}})\right)^{w_j} \right) = \left(1 - \max_j \left(\xi^-_{P_{A_j}} \right) \right)$$

Then $\left(\frac{\prod_{j=1}^n \left(1+\xi_{P_{A_j}}{}^-\right)^{w_j} - \prod_{j=1}^n \left(1-\xi_{P_{A_j}}{}^-\right)^{w_j}}{\prod_{j=1}^n \left(1+\xi_{P_{A_j}}{}^-\right)^{w_j} + \prod_{j=1}^n \left(1-\xi_{P_{A_j}}{}^-\right)^{w_j}} \right) \geq \left(\underset{j}{min} \left(\xi_{P_{A_j}}{}^- \right) \right)$

And

$$\left(\underset{j}{min} \left(\xi_{P_{A_j}}{}^+ \right) \right) \leq \left(\frac{\prod_{j=1}^n \left(1 + \xi_{P_{A_j}}{}^+\right)^{w_j} - \prod_{j=1}^n \left(1 - \xi_{P_{A_j}}{}^+\right)^{w_j}}{\prod_{j=1}^n \left(1 + \xi_{P_{A_j}}{}^+\right)^{w_j} + \prod_{j=1}^n \left(1 - \xi_{P_{A_j}}{}^+\right)^{w_j}} \right) \leq \left(\max_j \left(\xi_{P_{A_j}}{}^+ \right) \right)$$

Now, since $\left(\min_j \left(P_{A_j} \right) \right) \leq \left(P_{A_j} \right) \leq \left(\max_j \left(P_{A_j} \right) \right)$

We have

$$\left(2\prod_{j=1}^{n}\left(P_{A_j}\right)^{w_j}\right) \geq \left(2\prod_{j=1}^{n}\left(\max_j\left(P_{A_j}\right)\right)^{w_j}\right) = \left(2\max_j\left(P_{A_j}\right)\right)$$

Then

$$\left(\frac{2\prod_{j=1}^{n}\left(P_{A_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-P_{A_j}\right)^{w_j}+\prod_{j=1}^{n}\left(P_{A_j}\right)^{w_j}}\right) \geq \left(\min_j\left(P_{A_j}\right)\right)$$

Since $\left(\underset{j}{min}\left(\xi_{P_{A_j}}{}^{-}\right)\right) \leq \left(\xi_{P_{A_j}}{}^{-}\right) \leq \left(\underset{j}{max}\left(\xi_{P_{A_j}}{}^{-}\right)\right)$,

$$\left(\underset{j}{min}\left(\xi_{P_{A_j}}{}^{+}\right)\right) \leq \left(\xi_{P_{A_j}}{}^{+}\right) \leq \left(\underset{j}{max}\left(\xi_{P_{A_j}}{}^{+}\right)\right),$$

$$\Rightarrow \left(\min_j\left(\xi_{P_{A_j}}\right)\right) \leq \left(\xi_{P_{A_j}}\right) \leq \left(\max_j\left(\xi_{P_{A_j}}\right)\right)$$

Secondly, for degree of **indeterminacy**

Since $\left(\underset{j}{min}\left(\xi_{I_{A_j}}{}^{-}\right)\right) \leq \left(\xi_{I_{A_j}}{}^{-}\right) \leq \left(\max_j\left(\xi_{I_{A_j}}{}^{-}\right)\right)$

$$\left(\prod_{j=1}^{n}\left(1+\xi_{I_{A_j}}{}^{-}\right)^{w_j}\right) \geq \left(\prod_{j=1}^{n}\left(1-\max_j(\xi_{I_{A_j}}{}^{-})\right)^{w_j}\right) = \left(1-\max_j(\xi_{I_{A_j}}{}^{-})\right)$$

Then $\left(\frac{\prod_{j=1}^{n}\left(1+\xi_{I_{A_j}}{}^{-}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{I_{A_j}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{I_{A_j}}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{I_{A_j}}{}^{-}\right)^{w_j}}\right) \geq \left(\underset{j}{min}\left(\xi_{I_{A_j}}{}^{-}\right)\right)$

And

$$\left(\underset{j}{min}\left(\xi_{I_{A_j}}{}^{+}\right)\right) \leq \left(\frac{\prod_{j=1}^{n}\left(1+\xi_{I_{A_j}}{}^{+}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{I_{A_j}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{I_{A_j}}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{I_{A_j}}{}^{+}\right)^{w_j}}\right) \leq \left(\max_j\left(\xi_{I_{A_j}}{}^{+}\right)\right)$$

Now, since $\left(\min_j\left(P_{A_j}\right)\right) \leq \left(P_{A_j}\right) \leq \left(\max_j\left(P_{A_j}\right)\right)$

We have

$$\left(2\prod_{j=1}^{n}\left(P_{A_j}\right)^{w_j}\right) \geq \left(2\prod_{j=1}^{n}\left(\max_j\left(P_{A_j}\right)\right)^{w_j}\right) = \left(2\max_j\left(P_{A_j}\right)\right)$$

Then

$$\left(\frac{2\prod_{j=1}^{n}\left(P_{A_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-P_{A_j}\right)^{w_j}+\prod_{j=1}^{n}\left(P_{A_j}\right)^{w_j}}\right) \geq \left(\min_j\left(P_{A_j}\right)\right)$$

Since $\left(\underset{j}{min}\left(\xi_{I_{A_j}}{}^{-}\right)\right) \leq \left(\xi_{I_{A_j}}{}^{-}\right) \leq \left(\underset{j}{max}\left(\xi_{I_{A_j}}{}^{-}\right)\right)$,

$$\left(\underset{j}{min}\left(\xi_{I_{A_j}}{}^{+}\right)\right) \leq \left(\xi_{I_{A_j}}{}^{+}\right) \leq \left(\underset{j}{max}\left(\xi_{I_{A_j}}{}^{+}\right)\right),$$

$$\Rightarrow \left(\min_j\left(\xi_{I_{A_j}}\right)\right) \leq \left(\xi_{I_{A_j}}\right) \leq \left(\max_j\left(\xi_{I_{A_j}}\right)\right)$$

Also, for degree of **non-membership**

Since $\left(\min_j(\xi^-{}_{N_{A_j}})\right) \leq \left(\xi_{N_{A_j}}{}^-\right) \leq \left(\max_j(\xi^-{}_{N_{A_j}})\right)$
We have

$$\left(2\prod_{j=1}^{n}\left(\xi^-{}_{N_{A_j}}\right)^{w_j}\right) \geq \left(2\prod_{j=1}^{n}\left(\max_j\left(\xi^-{}_{N_{A_j}}\right)\right)^{w_j}\right) = \left(2\max_j\left(\xi^-{}_{N_{A_j}}\right)\right)$$

Then

$$\left(\frac{2\prod_{j=1}^{n}\left(\xi^-{}_{N_{A_j}}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\xi^-{}_{N_{A_j}}\right)^{w_j}+\prod_{j=1}^{n}\left(\xi^-{}_{N_{A_j}}\right)^{w_j}}\right) \geq \left(\min_j\left(\xi^-{}_{N_{A_j}}\right)\right)$$

$$\left(\min_j\left(\xi^+{}_{N_{A_j}}\right)\right) \leq \left(\frac{2\prod_{j=1}^{n}\left(\xi_{N_{A_j}}{}^+\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\xi_{N_{A_j}}{}^+\right)^{w_j}+\prod_{j=1}^{n}\left(\xi_{N_{A_j}}{}^+\right)^{w_j}}\right) \leq \left(\max_j\left(\xi^+{}_{N_{A_j}}\right)\right)$$

Since $\left(\min_j\left(N_{A_j}\right)\right) \leq \left(N_{A_j}\right) \leq \left(\max_j\left(N_{A_j}\right)\right)$

$$\left(\prod_{j=1}^{n}\left(1+N_{A_j}\right)^{w_j}\right) \geq \left(\prod_{j=1}^{n}\left(1-\max_j(N_{A_j})\right)^{w_j}\right) = \left(1-\max_j\left(N_{A_j}\right)\right)$$

Then $\left(\frac{\prod_{j=1}^{n}\left(1+N_{A_j}\right)^{w_j}-\prod_{j=1}^{n}\left(1-N_{A_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+N_{A_j}\right)^{w_j}+\prod_{j=1}^{n}\left(1-N_{A_j}\right)^{w_j}}\right) \geq \left(\min_j\left(N_{A_j}\right)\right)$

$$A^- \leq CPFEWA(A_1, A_2, \ldots, A_n) \leq A^+$$

Theorem 5.1.4 **(Monotonicity)** Let

$$A_j = \left\{\left\langle\left[\xi^-{}_{P_{A_j}}, \xi^+{}_{P_{A_j}}\right], P_{A_j}\right\rangle, \left\langle\left[\xi^-{}_{I_{A_j}}, \xi^+{}_{I_{A_j}}\right], I_{A_j}\right\rangle, \left\langle\left[\xi^-{}_{N_{A_j}}, \xi^+{}_{N_{A_j}}\right], N_{A_j}\right\rangle\right\}$$

and

$$A^*{}_j = \left\{\left\langle\left[\xi^-{}_{P_{A^*{}_j}}, \xi^+{}_{P_{A^*{}_j}}\right], P_{A^*{}_j}\right\rangle, \left\langle\left[\xi^-{}_{I_{A^*{}_j}}, \xi^+{}_{I_{A^*{}_j}}\right], I_{A^*{}_j}\right\rangle, \left\langle\left[\xi^-{}_{N_{A^*{}_j}}, \xi^+{}_{N_{A^*{}_j}}\right], N_{A^*{}_j}\right\rangle\right\}$$

be a collection of two CPFNs if all $A_j(j = 1,2,3,\ldots,n)$

If $\xi^-{}_{P_{A_j}} \leq \xi^-{}_{P_{A^*{}_j}}, \xi^+{}_{P_{A_j}} \leq \xi^+{}_{P_{A^*{}_j}}, P_{A_j} \leq P_{A^*{}_j}$ and $\xi^-{}_{I_{A_j}} \leq \xi^-{}_{I_{A^*{}_j}}, \xi^+{}_{I_{A_j}} \leq \xi^+{}_{I_{A^*{}_j}}$, $I_{A_j} \leq I_{A^*{}_j}$ also $\xi^-{}_{N_{A_j}} \leq \xi^-{}_{N_{A^*{}_j}}, \xi^+{}_{N_{A_j}} \leq \xi^+{}_{N_{A^*{}_j}}$, $N_{A_j} \leq N_{A^*{}_j}$ for all j then CPFEWA $(A_1, A_2, \ldots, A_n) \leq$ CPFEWA $(A^*{}_1, A^*{}_2, \ldots, A^*{}_n)$

Proof Using the result of theorem 5.1.1, we have

$$CPFEWA\left(A_1, A_2, \ldots, A_n\right) = \left[\begin{array}{l} \left\langle \left[\begin{array}{l} \frac{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{-}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{-}\right)^{w_j}}, \\ \frac{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{+}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{+}\right)^{w_j}} \end{array}\right], \frac{2\prod_{j=1}^{n}\left(P_{A_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-P_{A_j}\right)^{w_j}+\prod_{j=1}^{n}\left(P_{A_j}\right)^{w_j}} \right\rangle \\ \left\langle \left[\begin{array}{l} \frac{\prod_{j=1}^{n}\left(1+\xi_{I_{A_j}}{}^{-}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{I_{A_j}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{I_{A_j}}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{I_{A_j}}{}^{-}\right)^{w_j}}, \\ \frac{\prod_{j=1}^{n}\left(1+\xi_{I_{A_j}}{}^{+}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{I_{A_j}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{I_{A_j}}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{I_{A_j}}{}^{+}\right)^{w_j}} \end{array}\right], \frac{2\prod_{j=1}^{n}\left(I_{A_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-I_{A_j}\right)^{w_j}+\prod_{j=1}^{n}\left(I_{A_j}\right)^{w_j}} \right\rangle \\ \left\langle \left[\begin{array}{l} \frac{2\prod_{j=1}^{n}\left(\xi_{N_{A_j}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\xi_{N_{A_j}}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(\xi_{N_{A_j}}{}^{-}\right)^{w_j}}, \\ \frac{2\prod_{j=1}^{n}\left(\xi_{N_{A_j}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\xi_{N_{A_j}}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(\xi_{N_{A_j}}{}^{+}\right)^{w_j}} \end{array}\right], \frac{\prod_{j=1}^{n}\left(1+N_{A_j}\right)^{w_j}-\prod_{j=1}^{n}\left(1-N_{A_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+N_{A_j}\right)^{w_j}+\prod_{j=1}^{n}\left(1-N_{A_j}\right)^{w_j}} \right\rangle \end{array}\right]$$

Firstly, for degree of **membership**

Since $\xi^{-}{}_{P_{A_j}} \leq \xi^{-}{}_{P_{A^*{}_j}}, \xi^{+}{}_{P_{A_j}} \leq \xi^{+}{}_{P_{A^*{}_j}}$ for all j then

$$\left(\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{-}\right)^{w_j}\right) \geq \left(\prod_{j=1}^{n}\left(1-\xi^{-}{}_{P_{A^*{}_j}}\right)^{w_j}\right)$$

$$\left(\frac{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{-}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{-}\right)^{w_j}}\right) \leq \left(\frac{\prod_{j=1}^{n}\left(1+\xi^{-}{}_{P_{A^*{}_j}}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi^{-}{}_{P_{A^*{}_j}}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi^{-}{}_{P_{A^*{}_j}}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi^{-}{}_{P_{A^*{}_j}}\right)^{w_j}}\right)$$

And

$$\left(\frac{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{+}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{+}\right)^{w_j}}\right) \leq \left(\frac{\prod_{j=1}^{n}\left(1+\xi^{+}{}_{P_{A^*{}_j}}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi^{+}{}_{P_{A^*{}_j}}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi^{+}{}_{P_{A^*{}_j}}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi^{+}{}_{P_{A^*{}_j}}\right)^{w_j}}\right)$$

Since $P_{A_j} \leq P_{A^*{}_j}$ for all j then

$$\left(2\prod_{j=1}^{n}\left(P_{A_j}\right)^{w_j}\right) \geq \left(2\prod_{j=1}^{n}\left(P_{A^*{}_j}\right)^{w_j}\right)$$

$$\left(\frac{2\prod_{j=1}^{n}\left(P_{A_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-P_{A_j}\right)^{w_j}+\prod_{j=1}^{n}\left(P_{A_j}\right)^{w_j}}\right) \leq \left(\frac{2\prod_{j=1}^{n}\left(P_{A^*{}_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-P_{A^*{}_j}\right)^{w_j}+\prod_{j=1}^{n}\left(P_{A^*{}_j}\right)^{w_j}}\right)$$

Secondly, for degree of **neutral (indeterminacy)**

Since $\xi^{-}{}_{I_{A_j}} \leq \xi^{-}{}_{I_{A^*{}_j}}, \xi^{+}{}_{I_{A_j}} \leq \xi^{+}{}_{I_{A^*{}_j}}$ for all j then

$$\left(\prod_{j=1}^{n}\left(1-\xi_{I_{A_j}}{}^{-}\right)^{w_j}\right) \geq \left(\prod_{j=1}^{n}\left(1-\xi^{-}{}_{I_{A^*{}_j}}\right)^{w_j}\right)$$

$$\left(\frac{\prod_{j=1}^{n}\left(1+\xi_{I_{A_j}}{}^{-}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{I_{A_j}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{I_{A_j}}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{I_{A_j}}{}^{-}\right)^{w_j}}\right) \leq \left(\frac{\prod_{j=1}^{n}\left(1+\xi^{-}{}_{I_{A^*{}_j}}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi^{-}{}_{I_{A^*{}_j}}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi^{-}{}_{I_{A^*{}_j}}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi^{-}{}_{I_{A^*{}_j}}\right)^{w_j}}\right)$$

And

$$\left(\frac{\prod_{j=1}^{n}\left(1+\xi_{I_{A_j}}{}^{+}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{I_{A_j}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{I_{A_j}}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{I_{A_j}}{}^{+}\right)^{w_j}}\right)\leq\left(\frac{\prod_{j=1}^{n}\left(1+\xi^{+}{}_{I_{A^*}{}_j}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi^{+}{}_{I_{A^*}{}_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi^{+}{}_{I_{A^*}{}_j}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi^{+}{}_{I_{A^*}{}_j}\right)^{w_j}}\right)$$

Since $I_{A_j} \leq I_{A^*{}_j}$ for all j then

$$\left(2\prod_{j=1}^{n}\left(I_{A_j}\right)^{w_j}\right)\geq\left(2\prod_{j=1}^{n}\left(I_{A^*{}_j}\right)^{w_j}\right)$$

$$\left(\frac{2\prod_{j=1}^{n}\left(I_{A_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-I_{A_j}\right)^{w_j}+\prod_{j=1}^{n}\left(I_{A_j}\right)^{w_j}}\right)\leq\left(\frac{2\prod_{j=1}^{n}\left(I_{A^*{}_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-I_{A^*{}_j}\right)^{w_j}+\prod_{j=1}^{n}\left(I_{A^*{}_j}\right)^{w_j}}\right)$$

Also, for degree of **non-membership**

Since $\xi^{-}{}_{N_{A_j}} \leq \xi^{-}{}_{N_{A^*{}_j}}, \xi^{+}{}_{N_{A_j}} \leq \xi^{+}{}_{N_{A^*{}_j}}$ for all j then

We have

$$\left(2\prod_{j=1}^{n}\left(\xi_{N_{A_j}}{}^{-}\right)^{w_j}\right)\geq\left(2\prod_{j=1}^{n}\left(\xi_{N_{A^*{}_j}}{}^{-}\right)^{w_j}\right)$$

$$\left(\frac{2\prod_{j=1}^{n}\left(\xi_{N_{A_j}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\xi_{N_{A_j}}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(\xi_{N_{A_j}}{}^{-}\right)^{w_j}}\right)\leq\left(\frac{2\prod_{j=1}^{n}\left(\xi_{N_{A^*{}_j}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\xi_{N_{A^*{}_j}}{}^{-s}\right)^{w_j}+\prod_{j=1}^{n}\left(\xi_{N_{A^*{}_j}}\right)^{w_j}}\right)$$

And

$$\left(\frac{2\prod_{j=1}^{n}\left(\xi_{N_{A_j}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\xi_{N_{A_j}}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(\xi_{N_{A_j}}{}^{+}\right)^{w_j}}\right)\leq\left(\frac{2\prod_{j=1}^{n}\left(\xi_{N_{A^*{}_j}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\xi_{N_{A^*{}_j}}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(\xi_{N_{A^*{}_j}}{}^{+}\right)^{w_j}}\right)$$

Since $N_{A_j} \leq N_{A^*{}_j}$ for all j then

We have

$$\left(\prod_{j=1}^{n}(1-N_{A_j})^{w_j}\right)\geq\left(\prod_{j=1}^{n}(1-N_{A^*{}_j})^{w_j}\right)$$

$$\left(\frac{\prod_{j=1}^{n}\left(1+N_{A_j}\right)^{w_j}-\prod_{j=1}^{n}\left(1-N_{A_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+N_{A_j}\right)^{w_j}+\prod_{j=1}^{n}\left(1-N_{A_j}\right)^{w_j}}\right)\leq\left(\frac{\prod_{j=1}^{n}\left(1+N_{A^*{}_j}\right)^{w_j}-\prod_{j=1}^{n}\left(1-N_{A^*{}_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+N_{A^*{}_j}\right)^{w_j}+\prod_{j=1}^{n}\left(1-N_{A^*{}_j}\right)^{w_j}}\right)$$

Hence we get

$$\text{CPFEWA}\left(A_1, A_2, \ldots, A_n\right) \leq \text{CPFEWA}\left(A^*{}_1, A^*{}_2, \ldots, A^*{}_n\right)$$

Specific cases regarding CPFEWA Operator

For the CPFEWA operator i.e

$$CPFEWA\left(A_1, A_2, \ldots, A_n\right) = \left[\begin{array}{c} \left\langle \left[\begin{array}{c} \frac{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{-}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{-}\right)^{w_j}}, \\ \frac{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{+}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{+}\right)^{w_j}} \end{array}\right], \frac{2\prod_{j=1}^{n}\left(P_{A_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-P_{A_j}\right)^{w_j}+\prod_{j=1}^{n}\left(P_{A_j}\right)^{w_j}} \right\rangle \\ \left\langle \left[\begin{array}{c} \frac{\prod_{j=1}^{n}\left(1+\xi_{I_{A_j}}{}^{-}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{I_{A_j}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{I_{A_j}}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{I_{A_j}}{}^{-}\right)^{w_j}}, \\ \frac{\prod_{j=1}^{n}\left(1+\xi_{I_{A_j}}{}^{+}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{I_{A_j}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{I_{A_j}}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{I_{A_j}}{}^{+}\right)^{w_j}} \end{array}\right], \frac{2\prod_{j=1}^{n}\left(I_{A_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-I_{A_j}\right)^{w_j}+\prod_{j=1}^{n}\left(I_{A_j}\right)^{w_j}} \right\rangle \\ \left\langle \left[\begin{array}{c} \frac{2\prod_{j=1}^{n}\left(\xi_{N_{A_j}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\xi_{N_{A_j}}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(\xi_{N_{A_j}}{}^{-}\right)^{w_j}}, \\ \frac{2\prod_{j=1}^{n}\left(\xi_{N_{A_j}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\xi_{N_{A_j}}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(\xi_{N_{A_j}}{}^{+}\right)^{w_j}} \end{array}\right], \frac{\prod_{j=1}^{n}\left(1+N_{A_j}\right)^{w_j}-\prod_{j=1}^{n}\left(1-N_{A_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+N_{A_j}\right)^{w_j}+\prod_{j=1}^{n}\left(1-N_{A_j}\right)^{w_j}} \right\rangle \end{array}\right] \quad (4)$$

We have the following cases:

Case-1: If $\xi_{I_{A_j}}{}^{-} = \xi_{I_{A_j}}{}^{+} = I_{A_j} = 0$ in (4), then we get the following operator

$$\left[\begin{array}{c} \left\langle \left[\begin{array}{c} \frac{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{-}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{-}\right)^{w_j}}, \\ \frac{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{+}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{+}\right)^{w_j}} \end{array}\right], \frac{2\prod_{j=1}^{n}\left(P_{A_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-P_{A_j}\right)^{w_j}+\prod_{j=1}^{n}\left(P_{A_j}\right)^{w_j}} \right\rangle \\ \left\langle \left[\begin{array}{c} \frac{2\prod_{j=1}^{n}\left(\xi_{N_{A_j}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\xi_{N_{A_j}}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(\xi_{N_{A_j}}{}^{-}\right)^{w_j}}, \\ \frac{2\prod_{j=1}^{n}\left(\xi_{N_{A_j}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\xi_{N_{A_j}}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(\xi_{N_{A_j}}{}^{+}\right)^{w_j}} \end{array}\right], \frac{\prod_{j=1}^{n}\left(1+N_{A_j}\right)^{w_j}-\prod_{j=1}^{n}\left(1-N_{A_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+N_{A_j}\right)^{w_j}+\prod_{j=1}^{n}\left(1-N_{A_j}\right)^{w_j}} \right\rangle \end{array}\right] = CIFEWA\left(A_1, A_2, \ldots, A_n\right)$$

(The cubic intuitionistic Fuzzy Einstein weighted averaging operator)

Case-2: If $\xi_{I_{A_j}}{}^{-} = \xi_{I_{A_j}}{}^{+} = I_{A_j} = 0$ and $\xi_{N_{A_j}}{}^{-} = \xi_{N_{A_j}}{}^{+} = N_{A_j} = 0$ in (4), then we obtained the following operator

$$\left[\left\langle \left[\begin{array}{c} \frac{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{-}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{-}\right)^{w_j}}, \\ \frac{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{+}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{P_{A_j}}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{P_{A_j}}{}^{+}\right)^{w_j}} \end{array}\right], \frac{2\prod_{j=1}^{n}\left(P_{A_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-P_{A_j}\right)^{w_j}+\prod_{j=1}^{n}\left(P_{A_j}\right)^{w_j}} \right\rangle \right] = CFEWA\left(A_1, A_2, \ldots, A_n\right)$$

(Operator for cubic fuzzy Einstein weighted averaging)

Case-3: If $\xi_{P_{A_j}}{}^{-} = \xi_{P_{A_j}}{}^{+} = \xi_{A_j} \neq 0$, $\xi_{I_{A_j}}{}^{-} = \xi_{I_{A_j}}{}^{+} = \xi_{N_{A_j}}{}^{-} = \xi_{N_{A_j}}{}^{+} = P_{A_j} = 0$ in (4), then we get the following operator

$$\left[\begin{array}{c} \frac{\prod_{j=1}^{n}\left(1+\xi_{A_j}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{A_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{A_j}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{A_j}\right)^{w_j}}, \\ \frac{2\prod_{j=1}^{n}\left(I_{A_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-I_{A_j}\right)^{w_j}+\prod_{j=1}^{n}\left(I_{A_j}\right)^{w_j}}, \\ \frac{2\prod_{j=1}^{n}\left(N_{A_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-N_{A_j}\right)^{w_j}+\prod_{j=1}^{n}\left(N_{A_j}\right)^{w_j}} \end{array}\right] = PFEWA\left(A_1, A_2, \ldots, A_n\right)$$

(Picture fuzzy Einstein weighted averaging operator)

Einstein ordered weighted averaging operator for a cubic picture fuzzy set (CPFEOWA)

Let $A = \langle \xi_{A_k}, e_{A_k} \rangle\ k \in N$ be any collection CPFNs in L_{CPFN}. A CPFEOWA operator of dimension n is a mapping that takes n-dimensional Cubic picture fuzzy sets

$CPFEOWA : L^{n}{}_{CPFN} \to L_{CPFN}$ is associated with vector $w = (w_1, w_2, \ldots w_n)^T$ is a vector $A_j (j = 1,2,3,\ldots,n)$ with weights $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$

$$CPFEOWA\left(A_1, A_2, \ldots, A_n\right) = w_1 A_{\alpha(1)} \oplus w_2 A_{\alpha(2)} \oplus \cdots \oplus w_n A_{\alpha(n)} = \sum_{j=1}^{n} w_j A_{\alpha(j)}$$

where $\alpha(1), \alpha(2), \ldots, \alpha(n)$. Is a permutation $(1,2,3,..,n)$ of such that $A_{\alpha(1)} \leq A_{\alpha(j-1)}$ for all $j = 1,2,3,..,n$.

Now, Let

$$A_j = \left\{ \left\langle \left[\xi^{-}{}_{P_{A_{\alpha(j)}}}, \xi^{+}{}_{P_{A_{\alpha(j)}}} \right], P_{A_{\alpha(j)}} \right\rangle, \left\langle \left[\xi^{-}{}_{I_{A_{\alpha(j)}}}, \xi^{+}{}_{I_{A_{\alpha(j)}}} \right], I_{A_{\alpha(j)}} \right\rangle, \left\langle \left[\xi^{-}{}_{N_{A_{\alpha(j)}}}, \xi^{+}{}_{N_{A_{\alpha(j)}}} \right], N_{A_{\alpha(j)}} \right\rangle \right\}$$

be a family CPFNs, $w = (w_1, w_2, \ldots w_n)^T$ be a vector with weights $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$ If it meets the requirements, it is known as CPFEOWA.

$CPFEOWA\left(A_1, A_2, \ldots, A_n\right) =$

$$\left[\begin{array}{l} \left\langle \left[\begin{array}{c} \frac{\prod_{j=1}^{n}\left(1+\xi_{P_{A_{\alpha(j)}}}{}^{-}\right)^{w_j} - \prod_{j=1}^{n}\left(1-\xi_{P_{A_{\alpha(j)}}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{P_{A_{\alpha(j)}}}{}^{-}\right)^{w_j} + \prod_{j=1}^{n}\left(1-\xi_{P_{A_{\alpha(j)}}}{}^{-}\right)^{w_j}}, \\ \frac{\prod_{j=1}^{n}\left(1+\xi_{P_{A_{\alpha(j)}}}{}^{+}\right)^{w_j} - \prod_{j=1}^{n}\left(1-\xi_{P_{A_{\alpha(j)}}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{P_{A_{\alpha(j)}}}{}^{+}\right)^{w_j} + \prod_{j=1}^{n}\left(1-\xi_{P_{A_{\alpha(j)}}}{}^{+}\right)^{w_j}} \end{array} \right], \frac{2\prod_{j=1}^{n}\left(P_{A_{\alpha(j)}}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-P_{A_{\alpha(j)}}\right)^{w_j} + \prod_{j=1}^{n}\left(P_{A_{\alpha(j)}}\right)^{w_j}} \right\rangle \\ \left\langle \left[\begin{array}{c} \frac{\prod_{j=1}^{n}\left(1+\xi_{I_{A_{\alpha(j)}}}{}^{-}\right)^{w_j} - \prod_{j=1}^{n}\left(1-\xi_{I_{A_{\alpha(j)}}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{I_{A_{\alpha(j)}}}{}^{-}\right)^{w_j} + \prod_{j=1}^{n}\left(1-\xi_{I_{A_{\alpha(j)}}}{}^{-}\right)^{w_j}}, \\ \frac{\prod_{j=1}^{n}\left(1+\xi_{I_{A_{\alpha(j)}}}{}^{+}\right)^{w_j} - \prod_{j=1}^{n}\left(1-\xi_{I_{A_{\alpha(j)}}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{I_{A_{\alpha(j)}}}{}^{+}\right)^{w_j} + \prod_{j=1}^{n}\left(1-\xi_{I_{A_{\alpha(j)}}}{}^{+}\right)^{w_j}} \end{array} \right], \frac{2\prod_{j=1}^{n}\left(I_{A_{\alpha(j)}}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-I_{A_{\alpha(j)}}\right)^{w_j} + \prod_{j=1}^{n}\left(I_{A_{\alpha(j)}}\right)^{w_j}} \right\rangle \\ \left\langle \left[\begin{array}{c} \frac{2\prod_{j=1}^{n}\left(\xi_{N_{A_{\alpha(j)}}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\xi_{N_{A_{\alpha(j)}}}{}^{-}\right)^{w_j} + \prod_{j=1}^{n}\left(\xi_{N_{A_{\alpha(j)}}}{}^{-}\right)^{w_j}}, \\ \frac{2\prod_{j=1}^{n}\left(\xi_{N_{A_{\alpha(j)}}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\xi_{N_{A_{\alpha(j)}}}{}^{+}\right)^{w_j} + \prod_{j=1}^{n}\left(\xi_{N_{A_{\alpha(j)}}}{}^{+}\right)^{w_j}} \end{array} \right], \frac{\prod_{j=1}^{n}\left(1+N_{A_{\alpha(j)}}\right)^{w_j} - \prod_{j=1}^{n}\left(1-N_{A_{\alpha(j)}}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+N_{A_{\alpha(j)}}\right)^{w_j} + \prod_{j=1}^{n}\left(1-N_{A_{\alpha(j)}}\right)^{w_j}} \right\rangle \end{array} \right]$$

Theorem 5.2.1 Let

$$A_j = \left\{ \left\langle \left[\xi^{-}{}_{P_{A_j}}, \xi^{+}{}_{P_{A_j}} \right], P_{A_j} \right\rangle, \left\langle \left[\xi^{-}{}_{I_{A_j}}, \xi^{+}{}_{I_{A_j}} \right], I_{A_j} \right\rangle, \left\langle \left[\xi^{-}{}_{N_{A_j}}, \xi^{+}{}_{N_{A_j}} \right], N_{A_j} \right\rangle \right\}$$

be a family of CPFNs, $w = (w_1, w_2, \ldots w_n)^T$ be a vector with weights $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$ If it meets the requirements, it is known as CPFEOWA.

$$CPFEOWA(A_1, A_2, \ldots, A_n) = \left[\begin{array}{l} \left\langle \left[\begin{array}{l} \frac{\prod_{j=1}^{n}\left(1+\xi_{P_{A_{\alpha(j)}}}{}^{-}\right)^{w_j} - \prod_{j=1}^{n}\left(1-\xi_{P_{A_{\alpha(j)}}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{P_{A_{\alpha(j)}}}{}^{-}\right)^{w_j} + \prod_{j=1}^{n}\left(1-\xi_{P_{A_{\alpha(j)}}}{}^{-}\right)^{w_j}}, \\ \frac{\prod_{j=1}^{n}\left(1+\xi_{P_{A_{\alpha(j)}}}{}^{+}\right)^{w_j} - \prod_{j=1}^{n}\left(1-\xi_{P_{A_{\alpha(j)}}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{P_{A_{\alpha(j)}}}{}^{+}\right)^{w_j} + \prod_{j=1}^{n}\left(1-\xi_{P_{A_{\alpha(j)}}}{}^{+}\right)^{w_j}} \end{array}\right], \frac{2\prod_{j=1}^{n}(P_{A_{\alpha(j)}})^{w_j}}{\prod_{j=1}^{n}(2-P_{A_{\alpha(j)}})^{w_j} + \prod_{j=1}^{n}(P_{A_{\alpha(j)}})^{w_j}} \right\rangle \\ \left\langle \left[\begin{array}{l} \frac{\prod_{j=1}^{n}\left(1+\xi_{I_{A_{\alpha(j)}}}{}^{-}\right)^{w_j} - \prod_{j=1}^{n}\left(1-\xi_{I_{A_{\alpha(j)}}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{I_{A_{\alpha(j)}}}{}^{-}\right)^{w_j} + \prod_{j=1}^{n}\left(1-\xi_{I_{A_{\alpha(j)}}}{}^{-}\right)^{w_j}}, \\ \frac{\prod_{j=1}^{n}(1+\xi_{I_{A_{\alpha(j)}}}{}^{+})^{w_j} - \prod_{j=1}^{n}(1-\xi_{I_{A_{\alpha(j)}}}{}^{+})^{w_j}}{\prod_{j=1}^{n}(1+\xi_{I_{A_{\alpha(j)}}}{}^{+})^{w_j} + \prod_{j=1}^{n}(1-\xi_{I_{A_{\alpha(j)}}}{}^{+})^{w_j}} \end{array}\right], \frac{2\prod_{j=1}^{n}(I_{A_{\alpha(j)}})^{w_j}}{\prod_{j=1}^{n}(2-I_{A_{\alpha(j)}})^{w_j} + \prod_{j=1}^{n}(I_{A_{\alpha(j)}})^{w_j}} \right\rangle \\ \left\langle \left[\begin{array}{l} \frac{2\prod_{j=1}^{n}\left(\xi_{N_{A_{\alpha(j)}}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\xi_{N_{A_{\alpha(j)}}}{}^{-}\right)^{w_j} + \prod_{j=1}^{n}\left(\xi_{N_{A_{\alpha(j)}}}{}^{-}\right)^{w_j}}, \\ \frac{2\prod_{j=1}^{n}(\xi_{N_{A_{\alpha(j)}}}{}^{+})^{w_j}}{\prod_{j=1}^{n}(2-\xi_{N_{A_{\alpha(j)}}}{}^{+})^{w_j} + \prod_{j=1}^{n}(\xi_{N_{A_{\alpha(j)}}}{}^{+})^{w_j}} \end{array}\right], \frac{\prod_{j=1}^{n}(1+N_{A_{\alpha(j)}})^{w_j} - \prod_{j=1}^{n}\left(1-N_{A_{\alpha(j)}}\right)^{w_j}}{\prod_{j=1}^{n}(1+N_{A_{\alpha(j)}})^{w_j} + \prod_{j=1}^{n}(1-N_{A_{\alpha(j)}})^{w_j}} \right\rangle \end{array}\right]$$

Proof Theorem 5.1.1 serves as an analogue to the proof.

Theorem 5.2.2 (Idempotency) Let

$$A_j = \left\{\langle\left[\xi^{-}{}_{P_{A_j}}, \xi^{+}{}_{P_{A_j}}\right], P_{A_j}\rangle, \langle\left[\xi^{-}{}_{I_{A_j}}, \xi^{+}{}_{I_{A_j}}\right], I_{A_j}\rangle s, \langle\left[\xi^{-}{}_{N_{A_j}}, \xi^{+}{}_{N_{A_j}}\right], N_{A_j}\rangle\right\}$$

be a collection of CPFNs if all $A_j = 1,2,3,\ldots,n)$ are equal i.e.

$$A_j = \mathrm{A} = \left\{\left\langle\left[\xi^{-}{}_{P_A}, \xi^{+}{}_{P_A}\right], P_A\right\rangle, \left\langle\left[\xi^{-}{}_{I_A}, \xi^{+}{}_{I_A}\right], I_A\right\rangle, \left\langle\left[\xi^{-}{}_{N_A}, \xi^{+}{}_{N_A}\right], N_A\right\rangle\right\} \forall j,$$

Then

$$\text{CPFEOWA}\left(A_{\alpha(1)}, A_{\alpha(2)}, \ldots, A_{\alpha(n)}\right) = A$$

Proof The proof is analogous to theorem 5.1.2.

Theorem 5.2.3 **(Boundedness)** Let

$$A_j = \left\{\left\langle\left[\xi^{-}{}_{P_{A_j}}, \xi^{+}{}_{P_{A_j}}\right], P_{A_j}\right\rangle, \left\langle\left[\xi^{-}{}_{I_{A_j}}, \xi^{+}{}_{I_{A_j}}\right], I_{A_j}\right\rangle, \left\langle\left[\xi^{-}{}_{N_{A_j}}, \xi^{+}{}_{N_{A_j}}\right], N_{A_j}\right\rangle\right\}$$

be a collection of CPFNs if all $A_j = 1,2,3,\ldots,n)$ and let

$$A^{-} = \left\{\begin{array}{l} \left\langle\left[\min_j\left(\xi^{-}{}_{P_{A_j}}\right), \min_j\left(\xi^{+}{}_{P_{A_j}}\right)\right], \max_j\left(P_{A_j}\right)\right\rangle, \\ \left\langle\left[\min_j\left(\xi^{-}{}_{I_{A_j}}\right), \min_j\left(\xi^{+}{}_{I_{A_j}}\right)\right], \max_j\left(I_{A_j}\right)\right\rangle, \\ \left\langle\left[[\min_j\left(\xi^{-}{}_{N_{A_j}}\right), \min_j\left(\xi^{+}{}_{N_{A_j}}\right)\right], \max_j(N_{A_j})\right\rangle \end{array}\right\}$$

$$A^{+} = \left\{\begin{array}{l} \left\langle\left[\mathop{max}_{j}\left(\xi_{P_{A_j}}{}^{+}\right), \mathop{max}_{j}\left(\xi_{P_{A_j}}{}^{+}\right)\right], \mathop{min}_{j}\left(P_{A_j}\right)\right\rangle, \\ \left\langle\left[\mathop{max}_{j}\left(\xi_{I_{A_j}}{}^{-}\right), \mathop{max}_{j}\left(\xi_{I_{A_j}}{}^{+}\right)\right], \mathop{min}_{j}\left(I_{A_j}\right)\right\rangle, \\ \left\langle\left[\mathop{min}_{j}\left(\xi_{N_{A_j}}{}^{+}\right), \mathop{min}_{j}\left(\xi_{N_{A_j}}{}^{+}\right)\right], \mathop{max}_{j}\left(N_{A_j}\right)\right\rangle \end{array}\right\}$$

Then $A^{-} \leq \text{CPFEOWA}\left(A_{\alpha(1)}, A_{\alpha(2)}, \ldots, A_{\alpha(n)}\right) \leq A^{+}$

Proof Theorem 5.1.3 serves as an analogue to the proof.

Theorem 5.2.4 (Monotonicity) Let

$$A_j = \left\{\left\langle\left[\xi^-{}_{P_{A_j}}, \xi^+{}_{P_{A_j}}\right], P_{A_j}\right\rangle, \left\langle\left[\xi^-{}_{I_{A_j}}, \xi^+{}_{I_{A_j}}\right], I_{A_j}\right\rangle, \left\langle\left[\xi^-{}_{N_{A_j}}, \xi^+{}_{N_{A_j}}\right], N_{A_j}\right\rangle\right\}$$

and

$$A^*{}_j = \left\{\left\langle\left[\xi^-{}_{P_{A^*{}_j}}, \xi^+{}_{P_{A^*{}_j}}\right], P_{A^*{}_j}\right\rangle, \left\langle\left[\xi^-{}_{I_{A^*{}_j}}, \xi^+{}_{I_{A^*{}_j}}\right], I_{A^*{}_j}\right\rangle, \left\langle\left[\xi^-{}_{N_{A^*{}_j}}, \xi^+{}_{N_{A^*{}_j}}\right], N_{A^*{}_j}\right\rangle\right\}$$

be a collection of two CPFNs if all $A_j = 1,2,3,\ldots,n)$

If $\xi^-{}_{P_{A_j}} \le \xi^-{}_{P_{A^*{}_j}}, \xi^+{}_{P_{A_j}} \le \xi^+{}_{P_{A^*{}_j}}$, $P_{A_j} \le P_{A^*{}_j}$ and $\xi^-{}_{I_{A_j}} \le \xi^-{}_{I_{A^*{}_j}}, \xi^+{}_{I_{A_j}} \le \xi^+{}_{I_{A^*{}_j}}$, $I_{A_j} \le I_{A^*{}_j}$ also $\xi^-{}_{N_{A_j}} \le \xi^-{}_{N_{A^*{}_j}}, \xi^+{}_{N_{A_j}} \le \xi^+{}_{N_{A^*{}_j}}$, $N_{A_j} \le N_{A^*{}_j}$ for all j then $\text{CPFEOWA}\left(A_{\alpha(1)}, A_{\alpha(2)}, \ldots, A_{\alpha(n)}\right) \le \text{CPFEOWA}\left(A^*{}_{\alpha(1)}, A^*{}_{\alpha(2)}, \ldots, A^*{}_{\alpha(n)}\right)$

Proof The proof is analogous to theorem 5.1.3.

Specific cases regarding CPFEOWA Operator
For the CPFEOWA operator i.e.

$$CPFEOWA(A_1, A_2, \ldots, A_n) = \left[\begin{array}{l} \left\langle\left[\begin{array}{c}\frac{\prod_{j=1}^n\left(1+\xi_{P_{A_{\alpha(j)}}}{}^-\right)^{w_j}-\prod_{j=1}^n\left(1-\xi_{P_{A_{\alpha(j)}}}{}^-\right)^{w_j}}{\prod_{j=1}^n\left(1+\xi_{P_{A_{\alpha(j)}}}{}^-\right)^{w_j}+\prod_{j=1}^n\left(1-\xi_{P_{A_{\alpha(j)}}}{}^-\right)^{w_j}}, \\ \frac{\prod_{j=1}^n\left(1+\xi_{P_{A_{\alpha(j)}}}{}^+\right)^{w_j}-\prod_{j=1}^n\left(1-\xi_{P_{A_{\alpha(j)}}}{}^+\right)^{w_j}}{\prod_{j=1}^n\left(1+\xi_{P_{A_{\alpha(j)}}}{}^+\right)^{w_j}+\prod_{j=1}^n\left(1-\xi_{P_{A_{\alpha(j)}}}{}^+\right)^{w_j}}\end{array}\right], \frac{2\prod_{j=1}^n(P_{A_{\alpha(j)}})^{w_j}}{\prod_{j=1}^n(2-P_{A_{\alpha(j)}})^{w_j}+\prod_{j=1}^n(P_{A_{\alpha(j)}})^{w_j}}\right\rangle \\ \left\langle\left[\begin{array}{c}\frac{\prod_{j=1}^n\left(1+\xi_{I_{A_{\alpha(j)}}}{}^-\right)^{w_j}-\prod_{j=1}^n\left(1-\xi_{I_{A_{\alpha(j)}}}{}^-\right)^{w_j}}{\prod_{j=1}^n\left(1+\xi_{I_{A_{\alpha(j)}}}{}^-\right)^{w_j}+\prod_{j=1}^n\left(1-\xi_{I_{A_{\alpha(j)}}}{}^-\right)^{w_j}}, \\ \frac{\prod_{j=1}^n(1+\xi_{I_{A_{\alpha(j)}}}{}^+)^{w_j}-\prod_{j=1}^n(1-\xi_{I_{A_{\alpha(j)}}}{}^+)^{w_j}}{\prod_{j=1}^n(1+\xi_{I_{A_{\alpha(j)}}}{}^+)^{w_j}+\prod_{j=1}^n(1-\xi_{I_{A_{\alpha(j)}}}{}^+)^{w_j}}\end{array}\right], \frac{2\prod_{j=1}^n(I_{A_{\alpha(j)}})^{w_j}}{\prod_{j=1}^n(2-I_{A_{\alpha(j)}})^{w_j}+\prod_{j=1}^n(I_{A_{\alpha(j)}})^{w_j}}\right\rangle \\ \left\langle\left[\begin{array}{c}\frac{2\prod_{j=1}^n\left(\xi_{N_{A_{\alpha(j)}}}{}^-\right)^{w_j}}{\prod_{j=1}^n\left(2-\xi_{N_{A_{\alpha(j)}}}{}^-\right)^{w_j}+\prod_{j=1}^n\left(\xi_{N_{A_{\alpha(j)}}}{}^-\right)^{w_j}}, \\ \frac{2\prod_{j=1}^n(\xi_{N_{A_{\alpha(j)}}}{}^+)^{w_j}}{\prod_{j=1}^n(2-\xi_{N_{A_{\alpha(j)}}}{}^+)^{w_j}+\prod_{j=1}^n(\xi_{N_{A_{\alpha(j)}}}{}^+)^{w_j}}\end{array}\right], \frac{\prod_{j=1}^n(1+N_{A_{\alpha(j)}})^{w_j}-\prod_{j=1}^n\left(1-N_{A_{\alpha(j)}}\right)^{w_j}}{\prod_{j=1}^n(1+N_{A_{\alpha(j)}})^{w_j}+\prod_{j=1}^n(1-N_{A_{\alpha(j)}})^{w_j}}\right\rangle \end{array}\right] \tag{5}$$

The following conditions exist:

Case-1: If $\xi_{I_{A_{\alpha(j)}}}{}^- = \xi_{I_{A_{\alpha(j)}}}{}^+ = I_{A_{\alpha(j)}} = 0$ in (5), then we get the following operator

$$\left[\begin{array}{l} \left\langle\left[\begin{array}{c}\frac{\prod_{j=1}^n\left(1+\xi_{P_{A_{\alpha(j)}}}{}^-\right)^{w_j}-\prod_{j=1}^n\left(1-\xi_{P_{A_{\alpha(j)}}}{}^-\right)^{w_j}}{\prod_{j=1}^n\left(1+\xi_{P_{A_{\alpha(j)}}}{}^-\right)^{w_j}+\prod_{j=1}^n\left(1-\xi_{P_{A_{\alpha(j)}}}{}^-\right)^{w_j}}, \\ \frac{\prod_{j=1}^n\left(1+\xi_{P_{A_{\alpha(j)}}}{}^+\right)^{w_j}-\prod_{j=1}^n\left(1-\xi_{P_{A_{\alpha(j)}}}{}^+\right)^{w_j}}{\prod_{j=1}^n\left(1+\xi_{P_{A_{\alpha(j)}}}{}^+\right)^{w_j}+\prod_{j=1}^n\left(1-\xi_{P_{A_{\alpha(j)}}}{}^+\right)^{w_j}}\end{array}\right], \frac{2\prod_{j=1}^n(P_{A_{\alpha(j)}})^{w_j}}{\prod_{j=1}^n(2-P_{A_{\alpha(j)}})^{w_j}+\prod_{j=1}^n(P_{A_{\alpha(j)}})^{w_j}}\right\rangle \\ \left\langle\left[\begin{array}{c}\frac{2\prod_{j=1}^n\left(\xi_{N_{A_{\alpha(j)}}}{}^-\right)^{w_j}}{\prod_{j=1}^n\left(2-\xi_{N_{A_{\alpha(j)}}}{}^-\right)^{w_j}+\prod_{j=1}^n\left(\xi_{N_{A_{\alpha(j)}}}{}^-\right)^{w_j}}, \\ \frac{2\prod_{j=1}^n(\xi_{N_{A_{\alpha(j)}}}{}^+)^{w_j}}{\prod_{j=1}^n(2-\xi_{N_{A_{\alpha(j)}}}{}^+)^{w_j}+\prod_{j=1}^n(\xi_{N_{A_{\alpha(j)}}}{}^+)^{w_j}}\end{array}\right], \frac{\prod_{j=1}^n(1+N_{A_{\alpha(j)}})^{w_j}-\prod_{j=1}^n\left(1-N_{A_{\alpha(j)}}\right)^{w_j}}{\prod_{j=1}^n(1+N_{A_{\alpha(j)}})^{w_j}+\prod_{j=1}^n(1-N_{A_{\alpha(j)}})^{w_j}}\right\rangle \end{array}\right] = CIFEOWA(A_1, A_2, \ldots, A_n)$$

("Cubic intuitionistic fuzzy Einstein ordered Weighted averaging operator")

Case-2: If $\xi_{I_{A_{\alpha(j)}}}{}^- = \xi_{I_{A_{\alpha(j)}}}{}^+ = I_{A_{\alpha(j)}} = 0$ and $\xi_{N_{A_{\alpha(j)}}}{}^- = \xi_{N_{A_{\alpha(j)}}}{}^+ = N_{A_{\alpha(j)}} = 0$ in (5), then we get the following operator

$$\left[\left\langle\left[\begin{array}{c}\frac{\prod_{j=1}^n\left(1+\xi_{P_{A_{\alpha(j)}}}{}^-\right)^{w_j}-\prod_{j=1}^n\left(1-\xi_{P_{A_{\alpha(j)}}}{}^-\right)^{w_j}}{\prod_{j=1}^n\left(1+\xi_{P_{A_{\alpha(j)}}}{}^-\right)^{w_j}+\prod_{j=1}^n\left(1-\xi_{P_{A_{\alpha(j)}}}{}^-\right)^{w_j}}, \\ \frac{\prod_{j=1}^n\left(1+\xi_{P_{A_{\alpha(j)}}}{}^+\right)^{w_j}-\prod_{j=1}^n\left(1-\xi_{P_{A_{\alpha(j)}}}{}^+\right)^{w_j}}{\prod_{j=1}^n\left(1+\xi_{P_{A_{\alpha(j)}}}{}^+\right)^{w_j}+\prod_{j=1}^n\left(1-\xi_{P_{A_{\alpha(j)}}}{}^+\right)^{w_j}}\end{array}\right], \frac{2\prod_{j=1}^n\left(P_{A_{\alpha(j)}}\right)^{w_j}}{\prod_{j=1}^n\left(2-P_{A_{\alpha(j)}}\right)^{w_j}+\prod_{j=1}^n\left(P_{A_{\alpha(j)}}\right)^{w_j}}\right\rangle\right] = CFEOWA(A_1, A_2, \ldots, A_n)$$

("cubic fuzzy Einstein-ordered weighted averaging operator")

Case-3: If

$$\xi_{P_{A_{\alpha(j)}}}{}^{-} = \xi_{P_{A_{\alpha(j)}}}{}^{+} = \xi_{A_{\alpha(j)}} \neq 0, \xi_{I_{A_{\alpha(j)}}}{}^{-} = \xi_{I_{A_{\alpha(j)}}}{}^{+} = \xi_{N_{A_{\alpha(j)}}}{}^{-} = \xi_{N_{A_{\alpha(j)}}}{}^{+} = P_{A_{\alpha(j)}} = 0$$

in (5), then we get the following operator

$$\begin{bmatrix} \frac{\prod_{j=1}^{n}\left(1+\xi_{A_{\alpha(j)}}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{A_{\alpha(j)}}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{A_{\alpha(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{A_{\alpha(j)}}\right)^{w_j}}, \\ \frac{2\prod_{j=1}^{n}\left(I_{A_{\alpha(j)}}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-I_{A_{\alpha(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(I_{A_{\alpha(j)}}\right)^{w_j}}, \\ \frac{2\prod_{j=1}^{n}\left(N_{A_{\alpha(j)}}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-N_{A_{\alpha(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(N_{A_{\alpha(j)}}\right)^{w_j}} \end{bmatrix} = PFEOWA\left(A_1, A_2, \ldots, A_n\right)$$

("Picture fuzzy Einstein-ordered weighted averaging operator")

Cubic picture fuzzy Einstein hybrid weighted averaging operator (CPFEHWA)

Let $A = \langle \xi_{A_k}, e_{A_k} \rangle \; k \in N$ be any collection CPFNs in L_{CPFN}. A CPFEHWA operator of dimension n is a mapping

$CPFEHWA : L^{n}{}_{CPFN} \rightarrow L_{CPFN}$ is associated with vector $w = (w_1, w_2, \ldots w_n)^T$ is a vector of $A_j (j = 1,2,3,\ldots,n)$ with weights $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$

$$CPFEHWA\left(A_1, A_2, \ldots, A_n\right) = w_1 A_{\delta(1)} \oplus w_2 A_{\delta(2)} \oplus \cdots \oplus w_n A_{\delta(n)} = \sum_{j=1}^{n} w_j A_{\delta(j)}$$

Where $\delta(1), \delta(2), \ldots, \delta(n)$. Is a permutation of $(1,2,3,..,n)$ such that $A_{\delta(1)} \leq A_{\delta(j-1)}$ for all $j = 1,2,3,..,n$.

Now, let

$$A_j = \left\{ \left\langle \left[\xi^{-}{}_{P_{A_{\delta(j)}}}, \xi^{+}{}_{P_{A_{\delta(j)}}} \right], P_{A_{\delta(j)}} \right\rangle, \left\langle \left[\xi^{-}{}_{I_{A_{\delta(j)}}}, \xi^{+}{}_{I_{A_{\delta(j)}}} \right], I_{A_{\delta(j)}} \right\rangle, \left\langle \left[\xi^{-}{}_{N_{A_{\delta(j)}}}, \xi^{+}{}_{N_{A_{\delta(j)}}} \right], N_{A_{\delta(j)}} \right\rangle \right\}$$

be a family of CPFNs, $w = (w_1, w_2, \ldots w_n)^T$ be a vector with weights $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$. If it meets the requirements, it is referred to as CPFEHWA

$$CPFEHWA\left(A_{\delta(1)}, A_{\delta(2)}, \ldots, A_{\delta(n)}\right) =$$

$$\begin{bmatrix} \left\langle \begin{bmatrix} \frac{\prod_{j=1}^{n}\left(1+\xi_{P_{A_{\delta(j)}}}{}^{-}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{P_{A_{\delta(j)}}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{P_{A_{\delta(j)}}}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{P_{A_{\delta(j)}}}{}^{-}\right)^{w_j}}, \\ \frac{\prod_{j=1}^{n}\left(1+\xi_{P_{A_{\delta(j)}}}{}^{+}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{P_{A_{\delta(j)}}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{P_{A_{\delta(j)}}}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{P_{A_{\delta(j)}}}{}^{+}\right)^{w_j}} \end{bmatrix}, \frac{2\prod_{j=1}^{n}\left(P_{A_{\delta(j)}}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-P_{A_{\delta(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(P_{A_{\delta(j)}}\right)^{w_j}} \right\rangle \\ \left\langle \begin{bmatrix} \frac{\prod_{j=1}^{n}\left(1+\xi_{I_{A_{\delta(j)}}}{}^{-}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{I_{A_{\delta(j)}}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{I_{A_{\delta(j)}}}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{I_{A_{\delta(j)}}}{}^{-}\right)^{w_j}}, \\ \frac{\prod_{j=1}^{n}\left(1+\xi_{I_{A_{\delta(j)}}}{}^{+}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{I_{A_{\delta(j)}}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{I_{A_{\delta(j)}}}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{I_{A_{\delta(j)}}}{}^{+}\right)^{w_j}} \end{bmatrix}, \frac{2\prod_{j=1}^{n}\left(I_{A_{\delta(j)}}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-I_{A_{\delta(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(I_{A_{\delta(j)}}\right)^{w_j}} \right\rangle \\ \left\langle \begin{bmatrix} \frac{2\prod_{j=1}^{n}\left(\xi_{N_{A_{\delta(j)}}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\xi_{N_{A_{\delta(j)}}}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(\xi_{N_{A_{\delta(j)}}}{}^{-}\right)^{w_j}}, \\ \frac{2\prod_{j=1}^{n}\left(\xi_{N_{A_{\delta(j)}}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\xi_{N_{A_{\delta(j)}}}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(\xi_{N_{A_{\delta(j)}}}{}^{+}\right)^{w_j}} \end{bmatrix}, \frac{\prod_{j=1}^{n}\left(1+N_{A_{\delta(j)}}\right)^{w_j}-\prod_{j=1}^{n}\left(1-N_{A_{\delta(j)}}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+N_{A_{\delta(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(1-N_{A_{\delta(j)}}\right)^{w_j}} \right\rangle \end{bmatrix}$$

Where $\delta(j)$, the jth greatest value as determined by the overall order of $A_{\delta(1)} \geq A_{\delta(2)} \geq \cdots \geq A_{\delta(n)}$ where $A_{\delta(j)}$ has the jth highest weighted value.

Theorem 5.3.1 Let

$$A_j = \left\{\left\langle\left[\xi^-{}_{P_{A_j}}, \xi^+{}_{P_{A_j}}\right], P_{A_j}\right\rangle, \left\langle\left[\xi^-{}_{I_{A_j}}, \xi^+{}_{I_{A_j}}\right], I_{A_j}\right\rangle, \left\langle\left[\xi^-{}_{N_{A_j}}, \xi^+{}_{N_{A_j}}\right], N_{A_j}\right\rangle\right\}$$

be a family of CPFNs, $w = (w_1, w_2, \ldots w_n)^T$ be a vector with weights $w_j \in [0,1]$ and $\sum_{j=1}^n w_j = 1$ if it meets the criteria, it is known as CPFEHWA.

$$CPFEHWA\left(A_{\delta(1)}, A_{\delta(2)}, \ldots, A_{\delta(n)}\right) =$$

$$\left[\begin{array}{l}
\left\langle\left[\begin{array}{c}\frac{\prod_{j=1}^n\left(1+\xi_{P_{A_{\delta(j)}}}{}^-\right)^{w_j}-\prod_{j=1}^n\left(1-\xi_{P_{A_{\delta(j)}}}{}^-\right)^{w_j}}{\prod_{j=1}^n\left(1+\xi_{P_{A_{\delta(j)}}}{}^-\right)^{w_j}+\prod_{j=1}^n\left(1-\xi_{P_{A_{\delta(j)}}}{}^-\right)^{w_j}}, \\ \frac{\prod_{j=1}^n\left(1+\xi_{P_{A_{\delta(j)}}}{}^+\right)^{w_j}-\prod_{j=1}^n\left(1-\xi_{P_{A_{\delta(j)}}}{}^+\right)^{w_j}}{\prod_{j=1}^n\left(1+\xi_{P_{A_{\delta(j)}}}{}^+\right)^{w_j}+\prod_{j=1}^n\left(1-\xi_{P_{A_{\delta(j)}}}{}^+\right)^{w_j}}\end{array}\right], \frac{2\prod_{j=1}^n\left(P_{A_{\delta(j)}}\right)^{w_j}}{\prod_{j=1}^n\left(2-P_{A_{\delta(j)}}\right)^{w_j}+\prod_{j=1}^n\left(P_{A_{\delta(j)}}\right)^{w_j}}\right\rangle \\
\left\langle\left[\begin{array}{c}\frac{\prod_{j=1}^n\left(1+\xi_{I_{A_{\delta(j)}}}{}^-\right)^{w_j}-\prod_{j=1}^n\left(1-\xi_{I_{A_{\delta(j)}}}{}^-\right)^{w_j}}{\prod_{j=1}^n\left(1+\xi_{I_{A_{\delta(j)}}}{}^-\right)^{w_j}+\prod_{j=1}^n\left(1-\xi_{I_{A_{\delta(j)}}}{}^-\right)^{w_j}}, \\ \frac{\prod_{j=1}^n\left(1+\xi_{I_{A_{\delta(j)}}}{}^+\right)^{w_j}-\prod_{j=1}^n\left(1-\xi_{I_{A_{\delta(j)}}}{}^+\right)^{w_j}}{\prod_{j=1}^n\left(1+\xi_{I_{A_{\delta(j)}}}{}^+\right)^{w_j}+\prod_{j=1}^n\left(1-\xi_{I_{A_{\delta(j)}}}{}^+\right)^{w_j}}\end{array}\right], \frac{2\prod_{j=1}^n\left(I_{A_{\delta(j)}}\right)^{w_j}}{\prod_{j=1}^n\left(2-I_{A_{\delta(j)}}\right)^{w_j}+\prod_{j=1}^n\left(I_{A_{\delta(j)}}\right)^{w_j}}\right\rangle \\
\left\langle\left[\begin{array}{c}\frac{2\prod_{j=1}^n\left(\xi_{N_{A_{\delta(j)}}}{}^-\right)^{w_j}}{\prod_{j=1}^n\left(2-\xi_{N_{A_{\delta(j)}}}{}^-\right)^{w_j}+\prod_{j=1}^n\left(\xi_{N_{A_{\delta(j)}}}{}^-\right)^{w_j}}, \\ \frac{2\prod_{j=1}^n\left(\xi_{N_{A_{\delta(j)}}}{}^+\right)^{w_j}}{\prod_{j=1}^n\left(2-\xi_{N_{A_{\delta(j)}}}{}^+\right)^{w_j}+\prod_{j=1}^n\left(\xi_{N_{A_{\delta(j)}}}{}^+\right)^{w_j}}\end{array}\right], \frac{\prod_{j=1}^n\left(1+N_{A_{\delta(j)}}\right)^{w_j}-\prod_{j=1}^n\left(1-N_{A_{\delta(j)}}\right)^{w_j}}{\prod_{j=1}^n\left(1+N_{A_{\delta(j)}}\right)^{w_j}+\prod_{j=1}^n\left(1-N_{A_{\delta(j)}}\right)^{w_j}}\right\rangle
\end{array}\right]$$

Proof The evidence is consistent with Theorem 5.1.1

Theorem 5.3.2 (Idempotency) Let

$$A_j = \left\{\left\langle\left[\xi^-{}_{P_{A_j}}, \xi^+{}_{P_{A_j}}\right], P_{A_j}\right\rangle, \left\langle\left[\xi^-{}_{I_{A_j}}, \xi^+{}_{I_{A_j}}\right], I_{A_j}\right\rangle, \left\langle\left[\xi^-{}_{N_{A_j}}, \xi^+{}_{N_{A_j}}\right], N_{A_j}\right\rangle\right\}$$

be a collection of CPFNs if all are equal i.e.
$A_j = \mathrm{A} = \left\{\left\langle\left[\xi^-{}_{P_A}, \xi^+{}_{P_A}\right], P_A\right\rangle, \left\langle\left[\xi^-{}_{I_A}, \xi^+{}_{I_A}\right], I_A\right\rangle, \left\langle\left[\xi^-{}_{N_A}, \xi^+{}_{N_A}\right], N_A\right\rangle\right\} \forall j$, Then

$$\mathrm{CPFEHWA}\left(A_{\delta(1)}, A_{\delta(2)}, \ldots, A_{\delta(n)}\right) = A$$

Proof The evidence is consistent with Theorem 5.1.2.

Theorem 5.3.3 (Boundedness) Let

$$A_j = \left\{\left\langle\left[\xi^-{}_{P_{A_j}}, \xi^+{}_{P_{A_j}}\right], P_{A_j}\right\rangle, \left\langle\left[\xi^-{}_{I_{A_j}}, \xi^+{}_{I_{A_j}}\right], I_{A_j}\right\rangle, \left\langle\left[\xi^-{}_{N_{A_j}}, \xi^+{}_{N_{A_j}}\right], N_{A_j}\right\rangle\right\}$$

be a collection of CPFNs if all
$A_j = 1,2,3,\ldots,n)$ and let

$$A^- = \left\{\begin{array}{c}\left\langle\left[\min_j\left(\xi^-{}_{P_{A_j}}\right), \min_j\left(\xi^+{}_{P_{A_j}}\right)\right], \max_j(P_{A_j})\right\rangle, \\ \left\langle\left[\min_j\left(\xi^-{}_{I_{A_j}}\right), \min_j\left(\xi^+{}_{I_{A_j}}\right)\right], \max_j(I_{A_j})\right\rangle, \\ \left\langle\left[[\min_j\left(\xi^-{}_{N_{A_j}}\right), \min_j\left(\xi^+{}_{N_{A_j}}\right)\right], \max_j\left(N_{A_j}\right)\right\rangle\end{array}\right\}$$

$$A^{+}=\left\{\begin{array}{c}\left\langle\left[\underset{j}{max}\left(\xi_{P_{A_j}}{}^{+}\right),\underset{j}{max}\left(\xi_{P_{A_j}}{}^{+}\right)\right],\underset{j}{min}\left(P_{A_j}\right)\right\rangle,\\ \left\langle\left[\underset{j}{max}\left(\xi_{I_{A_j}}{}^{-}\right),\underset{j}{max}\left(\xi_{I_{A_j}}{}^{+}\right)\right],\underset{j}{min}\left(I_{A_j}\right)\right\rangle,\\ \left\langle\left[\underset{j}{min}\left(\xi_{N_{A_j}}{}^{+}\right),\underset{j}{min}\left(\xi_{N_{A_j}}{}^{+}\right)\right],\underset{j}{max}\left(N_{A_j}\right)\right\rangle\end{array}\right\}$$

Then $A^{-}\leq \text{CPFEHWA}\left(A_{\delta(1)},A_{\delta(2)},\ldots,A_{\delta(n)}\right)\leq A^{+}$

Proof The evidence is consistent with Theorem 5.1.3.

Theorem 5.3.4 **(Monotonicity)** Let

$$A_j=\left\{\left\langle\left[\xi^{-}{}_{P_{A_j}},\xi^{+}{}_{P_{A_j}}\right],P_{A_j}\right\rangle,\left\langle\left[\xi^{-}{}_{I_{A_j}},\xi^{+}{}_{I_{A_j}}\right],I_{A_j}\right\rangle,\left\langle\left[\xi^{-}{}_{N_{A_j}},\xi^{+}{}_{N_{A_j}}\right],N_{A_j}\right\rangle\right\}$$

and

$$A^{*}{}_{j}=\left\{\left\langle\left[\xi^{-}{}_{P_{A^{*}{}_{j}}},\xi^{+}{}_{P_{A^{*}{}_{j}}}\right],P_{A^{*}{}_{j}}\right\rangle,\left\langle\left[\xi^{-}{}_{I_{A^{*}{}_{j}}},\xi^{+}{}_{I_{A^{*}{}_{j}}}\right],I_{A^{*}{}_{j}}\right\rangle,\left\langle\left[\xi^{-}{}_{N_{A^{*}{}_{j}}},\xi^{+}{}_{N_{A^{*}{}_{j}}}\right],N_{A^{*}{}_{j}}\right\rangle\right\}$$

be a collection of two CPFNs if all $A_j=1,2,3,\ldots,n)$

If $\xi^{-}{}_{P_{A_j}}\leq\xi^{-}{}_{P_{A^{*}{}_{j}}},\xi^{+}{}_{P_{A_j}}\leq\xi^{+}{}_{P_{A^{*}{}_{j}}}$, $P_{A_j}\leq P_{A^{*}{}_{j}}$ and $\xi^{-}{}_{I_{A_j}}\leq\xi^{-}{}_{I_{A^{*}{}_{j}}},\xi^{+}{}_{I_{A_j}}\leq\xi^{+}{}_{I_{A^{*}{}_{j}}}$, $I_{A_j}\leq I_{A^{*}{}_{j}}$ also $\xi^{-}{}_{N_{A_j}}\leq\xi^{-}{}_{N_{A^{*}{}_{j}}},\xi^{+}{}_{N_{A_j}}\leq\xi^{+}{}_{N_{A^{*}{}_{j}}}$, $N_{A_j}\leq N_{A^{*}{}_{j}}$ for all j then $\text{CPFEHWA}\left(A_{\delta(1)},A_{\delta(2)},\ldots,A_{\delta(n)}\right)\leq\text{CPFEHWA}\left(A^{*}{}_{\delta(1)},A^{*}{}_{\delta(2)},\ldots,A^{*}{}_{\delta(n)}\right)$

Proof The evidence is consistent with Theorem 5.1.4.

Specific cases regarding CPFEHWA operator

For the CPFEHWA operator i.e.

$$\text{CPFEHWA}\left(A_{\delta(1)},A_{\delta(2)},\ldots,A_{\delta(n)}\right)$$

$$=\left[\begin{array}{c}\left\langle\left[\begin{array}{c}\frac{\prod_{j=1}^{n}\left(1+\xi_{P_{A_{\delta(j)}}}{}^{-}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{P_{A_{\delta(j)}}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{P_{A_{\delta(j)}}}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{P_{A_{\delta(j)}}}{}^{-}\right)^{w_j}},\\ \frac{\prod_{j=1}^{n}\left(1+\xi_{P_{A_{\delta(j)}}}{}^{+}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{P_{A_{\delta(j)}}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{P_{A_{\delta(j)}}}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{P_{A_{\delta(j)}}}{}^{+}\right)^{w_j}}\end{array}\right],\frac{2\prod_{j=1}^{n}\left(P_{A_{\delta(j)}}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-P_{A_{\delta(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(P_{A_{\delta(j)}}\right)^{w_j}}\right\rangle\\ \left\langle\left[\begin{array}{c}\frac{\prod_{j=1}^{n}\left(1+\xi_{I_{A_{\delta(j)}}}{}^{-}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{I_{A_{\delta(j)}}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{I_{A_{\delta(j)}}}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{I_{A_{\delta(j)}}}{}^{-}\right)^{w_j}},\\ \frac{\prod_{j=1}^{n}\left(1+\xi_{I_{A_{\delta(j)}}}{}^{+}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{I_{A_{\delta(j)}}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{I_{A_{\delta(j)}}}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{I_{A_{\delta(j)}}}{}^{+}\right)^{w_j}}\end{array}\right],\frac{2\prod_{j=1}^{n}\left(I_{A_{\delta(j)}}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-I_{A_{\delta(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(I_{A_{\delta(j)}}\right)^{w_j}}\right\rangle\\ \left\langle\left[\begin{array}{c}\frac{2\prod_{j=1}^{n}\left(\xi_{N_{A_{\delta(j)}}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\xi_{N_{A_{\delta(j)}}}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(\xi_{N_{A_{\delta(j)}}}{}^{-}\right)^{w_j}},\\ \frac{2\prod_{j=1}^{n}\left(\xi_{N_{A_{\delta(j)}}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\xi_{N_{A_{\delta(j)}}}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(\xi_{N_{A_{\delta(j)}}}{}^{+}\right)^{w_j}}\end{array}\right],\frac{\prod_{j=1}^{n}\left(1+N_{A_{\delta(j)}}\right)^{w_j}-\prod_{j=1}^{n}\left(1-N_{A_{\delta(j)}}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+N_{A_{\delta(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(1-N_{A_{\delta(j)}}\right)^{w_j}}\right\rangle\end{array}\right] \tag{6}$$

The following situations exist:

Case-1: If $\xi_{I_{A_{\delta(j)}}}{}^{-}=\xi_{I_{A_{\delta(j)}}}{}^{+}=I_{A_{\delta(j)}}=0$ in (6), then we get the following operator

$$\left[\begin{array}{l}\left\langle\left[\begin{array}{l}\frac{\prod_{j=1}^{n}\left(1+\xi_{P_{A_{\delta(j)}}}{}^{-}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{P_{A_{\delta(j)}}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{P_{A_{\delta(j)}}}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{P_{A_{\delta(j)}}}{}^{-}\right)^{w_j}},\\ \frac{\prod_{j=1}^{n}\left(1+\xi_{P_{A_{\delta(j)}}}{}^{+}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{P_{A_{\delta(j)}}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{P_{A_{\delta(j)}}}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{P_{A_{\delta(j)}}}{}^{+}\right)^{w_j}}\end{array}\right],\frac{2\prod_{j=1}^{n}\left(P_{A_{\delta(j)}}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-P_{A_{\delta(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(P_{A_{\delta(j)}}\right)^{w_j}}\right\rangle\\ \left\langle\left[\begin{array}{l}\frac{2\prod_{j=1}^{n}\left(\xi_{N_{A_{\delta(j)}}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\xi_{N_{A_{\delta(j)}}}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(\xi_{N_{A_{\delta(j)}}}{}^{-}\right)^{w_j}},\\ \frac{2\prod_{j=1}^{n}\left(\xi_{N_{A_{\delta(j)}}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\xi_{N_{A_{\delta(j)}}}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(\xi_{N_{A_{\delta(j)}}}{}^{+}\right)^{w_j}}\end{array}\right],\frac{\prod_{j=1}^{n}\left(1+N_{A_{\delta(j)}}\right)^{w_j}-\prod_{j=1}^{n}\left(1-N_{A_{\delta(j)}}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+N_{A_{\delta(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(1-N_{A_{\delta(j)}}\right)^{w_j}}\right\rangle\end{array}\right]=CIFEHWA\left(A_{\delta(1)},A_{\delta(2)},\ldots,A_{\delta(n)}\right)$$

("Cubic intuitionistic fuzzy Einstein hybrid Weighted averaging operator")

Case-2: If $\xi_{I_{A_{\delta(j)}}}{}^{-}=\xi_{I_{A_{\delta(j)}}}{}^{+}=I_{A_{\delta(j)}}=0$ and $\xi_{N_{A_{\delta(j)}}}{}^{-}=\xi_{N_{A_{\delta(j)}}}{}^{+}=N_{A_{\delta(j)}}=0$ in (6), then we get the following operator

$$\left[\left\langle\left[\begin{array}{l}\frac{\prod_{j=1}^{n}\left(1+\xi_{P_{A_{\delta(j)}}}{}^{-}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{P_{A_{\delta(j)}}}{}^{-}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{P_{A_{\delta(j)}}}{}^{-}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{P_{A_{\delta(j)}}}{}^{-}\right)^{w_j}},\\ \frac{\prod_{j=1}^{n}\left(1+\xi_{P_{A_{\delta(j)}}}{}^{+}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{P_{A_{\delta(j)}}}{}^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{P_{A_{\delta(j)}}}{}^{+}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{P_{A_{\delta(j)}}}{}^{+}\right)^{w_j}}\end{array}\right],\frac{2\prod_{j=1}^{n}\left(P_{A_{\delta(j)}}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-P_{A_{\delta(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(P_{A_{\delta(j)}}\right)^{w_j}}\right\rangle\right]=CFEHWA\left(A_1,A_2,\ldots,A_n\right)$$

(cubic fuzzy Einstein Hybrid weighted averaging operator)

Case-3: If

$$\begin{aligned}\xi_{P_{A_{\delta(j)}}}{}^{-}=\xi_{P_{A_{\delta(j)}}}{}^{+}=\xi_{A_{\delta(j)}}\neq 0,\xi_{I_{A_{\delta(j)}}}{}^{-}=\xi_{I_{A_{\delta(j)}}}{}^{+}\\ =\xi_{N_{A_{\delta(j)}}}{}^{-}=\xi_{N_{A_{\delta(j)}}}{}^{+}=P_{A_{\delta(j)}}=0\end{aligned}$$

in (6), then we get the following operator

$$\left[\begin{array}{l}\frac{\prod_{j=1}^{n}\left(1+\xi_{A_{\delta(j)}}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\xi_{A_{\delta(j)}}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\xi_{A_{\delta(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\xi_{A_{\delta(j)}}\right)^{w_j}},\\ \frac{2\prod_{j=1}^{n}\left(I_{A_{\delta(j)}}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-I_{A_{\delta(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(I_{A_{\delta(j)}}\right)^{w_j}},\\ \frac{2\prod_{j=1}^{n}\left(N_{A_{\delta(j)}}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-N_{A_{\delta(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(N_{A_{\delta(j)}}\right)^{w_j}}\end{array}\right]=PFEHWA\left(A_1,A_2,\ldots,A_n\right)$$

("Picture fuzzy Einstein hybrid Weighted averaging operator")

A method for generating decisions with several attributes using fuzzy information from CPFS

MADM stands for a multi-attribute decision-making technique that is used to estimate and rank alternatives based on multiple criteria or attributes. MADM techniques are particularly useful when there are several alternatives to choose from and when each alternative has multiple attributes that need to be considered.

The general process of MADM typically entails the following steps:

Identify the decision problem and the relevant attributes: To begin, one must define the decision problem and identify the attributes that are relevant to the decision. For example, if you are choosing between different job offers, the attributes might include salary, benefits, location, work environment, and opportunities for advancement.

Determine the weights of the attributes: Once the relevant attributes have been recognized, the next step is to conclude the weight of each attribute. This can be done by using a variety of methods, such as direct rating, pairwise comparison, or analytic hierarchy process (AHP).

Evaluate the alternatives: After the weights of the attributes have been determined, the next step is to evaluate each alternative on each attribute. This can be done by using a rating scale or a numerical score.

Determine the overall rating for each option. To establish the overall score for each alternative, the scores for each attribute are multiplied by the weights of the attributes and then added together.

Select the best alternative: The best alternative is then determined by having the highest overall score.

Additionally, other popular MADM approaches such as the Additive Weighting (SAW) method and the Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) have been widely used in various decision-making scenarios. These techniques differ in their approach to weighting the attributes and evaluating the alternatives, but all aim to provide a structured and systematic way of making decisions when faced with multiple, complex alternatives.

A MADM technique's objective is to choose a compromise solution from all workable alternatives based on collective traits.

Suppose $Z=\{z_1,z_2,\ldots,z_n\}$ is said to be a distinct set of possibilities and $G=\{g_1,g_2,\ldots,g_n\}$ be the set of attribute. Now consider cubic picture fuzzy numbers (CPFNs) are referred to as the ratings given by a decision-making process for options $Z_i(i=1{,}2,\ldots,n)$ on attributes $g_j(j=1{,}2,\ldots,m)$ in L_{CPFN}: $A=\{\langle \xi_A(t), e_A(t) | t\in Z\rangle\}$ where $\xi_A(t)$ is interval-value PFS that the attribute g_j is satisfied by the options Z_i and the picture fuzzy set $e_A(t)$ is the set of alternatives Z_i that do not satisfy the attribute g_j. Consequently, the cubic picture fuzzy decision matrix represents the many attribute decision-making problem.

$$D=(A_{ij})_{m*n}=\langle \xi_{A_{ij}}, e_{A_{ij}}\rangle$$

We now assume the role of the CPFEHWA operator to address the MADM technique, which involves the actions that follow. Figure 1 illustrates the roadmap of the proposed decision-making technique, providing a clear overview of the methodological framework. It should be cited in the main text to guide readers through the structure of the approach.

Algorithm

Let $B_{ij}=\{B_1,B_2,\ldots,B_n\}$ be the collection of n alternatives and $G=\{g_1,g_2,\ldots,g_m\}$ be m attributes or criteria subject to the weight vector $w=(w_1,w_2,\ldots,w_n)^T$ such that and $\sum_{j=1}^{n} w_j=1$. These are the steps in this process.

Step-1: In this stage, the normalized CPF decision matrix is obtained. We have the following normalization formula

$E^k=({B_{ij}}^k)_{n*1}$ That is $B_{ij}=\{\langle[\xi^-{}_{N_{A_j}},\xi^+{}_{N_{A_j}}],N_{A_j}\rangle,\langle[\xi^-{}_{I_{A_j}},\xi^+{}_{I_{A_j}}],I_{A_j}\rangle,\langle[\xi^-{}_{P_{A_j}},\xi^+{}_{P_{A_j}}],P_{A_j}\rangle\}$

Step-2: We employ the operator of CPFEHWA for aggregating all of the rating values $B_{ij}\,(j=1{,}2,\ldots,m)$ on the ith line and get all $CPFEHWA(B_{i1},B_{i2},\ldots,B_{im})$, rating values $(i=1{,}2,3,\ldots,n)$ where $w=(w_1,w_2,\ldots,w_n)^T$ is the attribute weight vector $g_j\,(j=1{,}2,\ldots,m)$ such that $w_j\in[0{,}1]\,,(j=1{,}2,\ldots,m)$ and $\sum_{j=1}^{n} w_j=1.w=(w_1,w_2,\ldots,w_n)^T$ is the associated vector of CPFEHWA operator, such that $w_j\in[0{,}1]\,, j=1{,}2,\ldots,n$ and $\sum_{j=1}^{n} w_j=1$

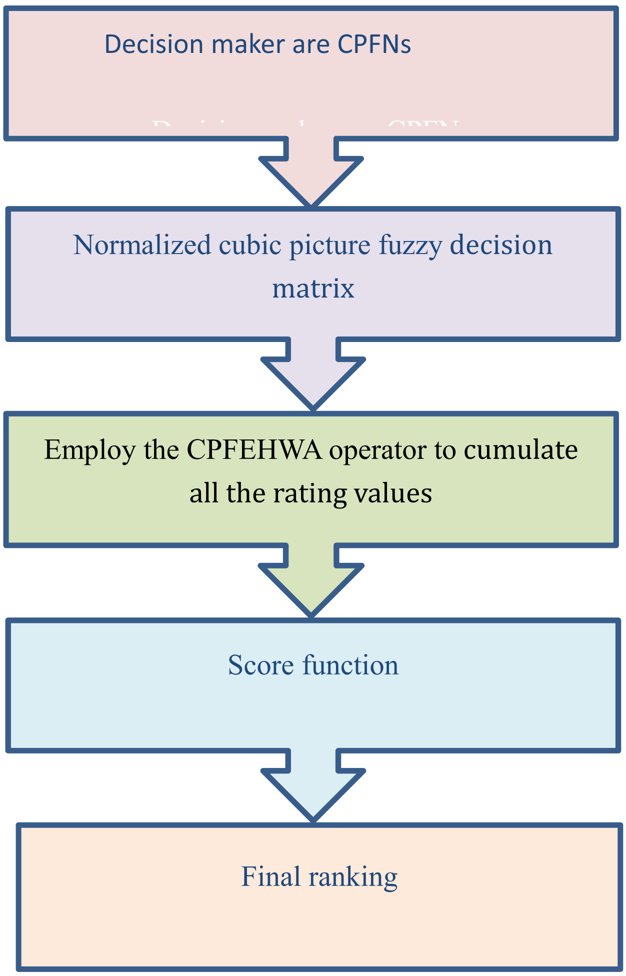

Fig. 1. Description of scoring functions, normalized CPF decision matrix, and CPF decision matrix.

Linguistic terms	CPFVs
Extremely prominent (EP)	$\begin{bmatrix} \langle [0.83, 0.95], 0.78 \rangle \\ \langle [0.03, 0.04], 0.01 \rangle \\ \langle [0.01, 0.05], 0.20 \rangle \end{bmatrix}$
Extremely squat (ES)	$\begin{bmatrix} \langle [0.11, 0.31], 0.35 \rangle \\ \langle [0.07, 0.13], 0.17 \rangle \\ \langle [0.12, 0.14], 0.20 \rangle \end{bmatrix}$
Extremely (E)	$\begin{bmatrix} \langle [0.25, 0.41], 0.45 \rangle \\ \langle [0.07, 0.30], 0.35 \rangle \\ \langle [0.09, 0.19], 0.02 \rangle \end{bmatrix}$
Medium squat (MS)	$\begin{bmatrix} \langle [0.33, 0.67], 0.37 \rangle \\ \langle [0.04, 0.22], 0.17 \rangle \\ \langle [0.01, 0.34], 0.47 \rangle \end{bmatrix}$
Medium (M)	$\begin{bmatrix} \langle [0.40, 0.51], 0.50 \rangle \\ \langle [0.20, 0.31], 0.12 \rangle \\ \langle [0.11, 0.21], 0.13 \rangle \end{bmatrix}$
Medium prominent (MP)	$\begin{bmatrix} \langle [0.51, 0.72], 0.63 \rangle \\ \langle [0.18, 0.32], 0.27 \rangle \\ \langle [0.14, 0.16], 0.19 \rangle \end{bmatrix}$
Prominent (P)	$\begin{bmatrix} \langle [0.71, 0.81], 0.52 \rangle \\ \langle [0.06, 0.08], 0.21 \rangle \\ \langle [0.07, 0.09], 0.16 \rangle \end{bmatrix}$

Table 3. The cubic picture has fuzzy values in linguistic terms.

	C_1	C_2	C_3	C_4
A_1	EP	S	EP	ES
A_2	ES	EP	ES	S
A_3	S	ES	S	MS
A_4	EP	MS	MS	EP

Table 4. Decision matrix.

To combine all of the rating values $B_{ij}\ (j = 1,2,\ldots,m)$ on the ith line, we use the CPFEHWA operator. We obtain all $CPFEHWA(B_{i1}, B_{i2},\ldots,B_{im})$, rating values $(i = 1,2,3,\ldots,n)$ where $w = (w_1, w_2,\ldots,w_n)^T$ is the attribute weight vector $g_j\ (j = 1,2,\ldots,m)$, where $w_j \in [0,1], (j = 1,2,\ldots,m)$, and $\sum_{j=1}^{n} w_j = 1.w = (w_1, w_2,\ldots,w_n)^T$ is the associated vector of the CPFEHWA operator, and $w_j \in [0,1], j = 1,2,\ldots,n$ and $\sum_{j=1}^{n} w_j = 1$ are its properties.

Step-3 In ascending order, we rate the overall rating values $\beta_i (i = 1,2,\ldots,n)$ and the full alternative $z_i (i = 1,2,\ldots,n)$ by $\beta_i (i = 1,2,\ldots,n)$.

Step-4 Finally, we determined which alternative had the lowest aggregate rating value was the best.

Explanatory example

The incentive outline purpose depends on the analysis that was instructed for a stimulus/moving procedure Mr. Ali is the sales manager whose agents require new laptops because their current ones are outdated and insufficient for meeting their job requirements. To summarize, consider that adding RAM to outdated systems is not cheap and that the company's philosophy is to buy rather than lease. of a doubly finished traveler ship to operate in Islamabad, rather than in London to reduce the exploration time, especially over following seaway movement. Alternatives for incentive outlines are Table 3.

Tables 4, 5 and 6 are presented with a list of alternatives

$$A = \{A_1, A_2, A_3, A_4\}$$

A_1 :Traditional propeller as well as a high wheel,
A_2 : Drive,
A_3 : Cyclodal propeller,
A_4 : Old-fashioned.

The decision is based on one goal and four topic traits, which are as follows:

C_1 : The cost of speculation,
C_2 : Working expenses, which include dealing with,

	C_1	C_2	C_3	C_4
A_1	P	S	EP	MP
A_2	MP	P	S	S
A_3	S	MP	P	MS
A_4	EP	MS	MS	P

Table 5. Decision matrix.

	C_1	C_2	C_3	C_4
A_1	MS	S	EP	MS
A_2	MS	ES	ES	M
A_3	S	M	M	MS
A_4	EP	MS	MS	EP

Table 6. Decision matrix.

	C_1	C_2	C_3	C_4
A_1	$\begin{bmatrix} \langle[0.83, 0.95], 0.78\rangle \\ \langle[0.03, 0.04], 0.01\rangle \\ \langle[0.01, 0.05], 0.20\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.25, 0.41], 0.45\rangle \\ \langle[0.07, 0.30], 0.35\rangle \\ \langle[0.09, 0.19], 0.02\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.83, 0.95], 0.78\rangle \\ \langle[0.03, 0.04], 0.01\rangle \\ \langle[0.01, 0.05], 0.20\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.11, 0.31], 0.35\rangle \\ \langle[0.07, 0.13], 0.17\rangle \\ \langle[0.12, 0.14], 0.20\rangle \end{bmatrix}$
A_2	$\begin{bmatrix} \langle[0.11, 0.31], 0.35\rangle \\ \langle[0.07, 0.13], 0.17\rangle \\ \langle[0.12, 0.14], 0.20\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.83, 0.95], 0.78\rangle \\ \langle[0.03, 0.04], 0.01\rangle \\ \langle[0.01, 0.05], 0.20\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.11, 0.31], 0.35\rangle \\ \langle[0.07, 0.13], 0.17\rangle \\ \langle[0.12, 0.14], 0.20\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.25, 0.41], 0.45\rangle \\ \langle[0.07, 0.30], 0.35\rangle \\ \langle[0.09, 0.19], 0.02\rangle \end{bmatrix}$
A_3	$\begin{bmatrix} \langle[0.25, 0.41], 0.45\rangle \\ \langle[0.07, 0.30], 0.35\rangle \\ \langle[0.09, 0.19], 0.02\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.11, 0.31], 0.35\rangle \\ \langle[0.07, 0.13], 0.17\rangle \\ \langle[0.12, 0.14], 0.20\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.25, 0.41], 0.45\rangle \\ \langle[0.07, 0.30], 0.35\rangle \\ \langle[0.09, 0.19], 0.02\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.33, 0.67], 0.37\rangle \\ \langle[0.04, 0.22], 0.17\rangle \\ \langle[0.01, 0.34], 0.47\rangle \end{bmatrix}$
A_4	$\begin{bmatrix} \langle[0.83, 0.95], 0.78\rangle \\ \langle[0.03, 0.04], 0.01\rangle \\ \langle[0.01, 0.05], 0.20\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.33, 0.67], 0.37\rangle \\ \langle[0.04, 0.22], 0.17\rangle \\ \langle[0.01, 0.34], 0.47\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.33, 0.67], 0.37\rangle \\ \langle[0.04, 0.22], 0.17\rangle \\ \langle[0.01, 0.34], 0.47\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.83, 0.95], 0.78\rangle \\ \langle[0.03, 0.04], 0.01\rangle \\ \langle[0.01, 0.05], 0.20\rangle \end{bmatrix}$

Table 7. Expert decision matrix.

C_3 : Mobility,
C_4 : Movement and disorder

The weight vector (property) is w=(0.24,0.35,0.41). The cubic picture fuzzy MADM task is to choose an appropriate drive (to movement) framework among three options.

Assume that, as shown in Table 3, the decision-maker uses linguistic concepts to evaluate different estimates in terms of the distinct characteristics independently. Tables 4, 5 and 6 show the connection between the linguistic concepts and comparing CPFNs in L_{CPFN}.

In Tables 7, 8 and 9, we present the expert decision matrix.

Step-1. It created the normalized cubic picture fuzzy decision matrix.

The normalization formula shown below is provided.

$$E^k = (\beta^k{}_{ij})_{n*1}$$
$$B_{ij} = \left\{\left\langle[\xi^-{}_{N_{A_j}}, \xi^+{}_{N_{A_j}}], N_{A_j}\right\rangle, \left\langle[\xi^-{}_{I_{A_j}}, \xi^+{}_{I_{A_j}}], I_{A_j}\right\rangle, \left\langle[\xi^-{}_{P_{A_j}}, \xi^+{}_{P_{A_j}}], P_{A_j}\right\rangle\right\}.$$

Tables 10, 11 and 12 include the normalized cubic picture choice matrix.

Utilize the CPFEHWA operator on normalized cubic picture fuzzy decision-making (from Tables 10, 11, 12).

A complete solution can be seen in Appendix "D".

After Utilizing the CPFEHWA operator on normalized cubic picture fuzzy decision-making (from Tables 10, 11, 12), In Table 13, we find the results that follow.

The generalized cubic picture fuzzy decision matrix is then obtained.

$$E = (\beta_{ij})_{n*m}$$

Step-2. The general rating value β_{ij} associated with the alternative z_i is created by summing all of the rating values $\beta_{ij}(j = 1,2,\ldots,m)$ of the ith line using the CPFEHWA operator, which is given in Table 13.

	C_1	C_2	C_3	C_4
A_1	$\langle[0.71, 0.81], 0.52\rangle$ $\langle[0.06, 0.08], 0.21\rangle$ $\langle[0.07, 0.09], 0.16\rangle$	$\langle[0.25, 0.41], 0.45\rangle$ $\langle[0.07, 0.30], 0.35\rangle$ $\langle[0.09, 0.19], 0.02\rangle$	$\langle[0.83, 0.95], 0.78\rangle$ $\langle[0.03, 0.04], 0.01\rangle$ $\langle[0.01, 0.05], 0.20\rangle$	$\langle[0.51, 0.72], 0.63\rangle$ $\langle[0.18, 0.32], 0.27\rangle$ $\langle[0.14, 0.16], 0.19\rangle$
A_2	$\langle[0.51, 0.72], 0.63\rangle$ $\langle[0.18, 0.32], 0.27\rangle$ $\langle[0.14, 0.16], 0.19\rangle$	$\langle[0.71, 0.81], 0.52\rangle$ $\langle[0.06, 0.08], 0.21\rangle$ $\langle[0.07, 0.09], 0.16\rangle$	$\langle[0.11, 0.31], 0.35\rangle$ $\langle[0.07, 0.13], 0.17\rangle$ $\langle[0.12, 0.14], 0.20\rangle$	$\langle[0.25, 0.41], 0.45\rangle$ $\langle[0.07, 0.30], 0.35\rangle$ $\langle[0.09, 0.19], 0.02\rangle$
A_3	$\langle[0.25, 0.41], 0.45\rangle$ $\langle[0.07, 0.30], 0.35\rangle$ $\langle[0.09, 0.19], 0.02\rangle$	$\langle[0.51, 0.72], 0.63\rangle$ $\langle[0.18, 0.32], 0.27\rangle$ $\langle[0.14, 0.16], 0.19\rangle$	$\langle[0.71, 0.81], 0.52\rangle$ $\langle[0.06, 0.08], 0.21\rangle$ $\langle[0.07, 0.09], 0.16\rangle$	$\langle[0.33, 0.67], 0.37\rangle$ $\langle[0.04, 0.22], 0.17\rangle$ $\langle[0.01, 0.34], 0.47\rangle$
A_4	$\langle[0.83, 0.95], 0.78\rangle$ $\langle[0.03, 0.04], 0.01\rangle$ $\langle[0.01, 0.05], 0.20\rangle$	$\langle[0.33, 0.67], 0.37\rangle$ $\langle[0.04, 0.22], 0.17\rangle$ $\langle[0.01, 0.34], 0.47\rangle$	$\langle[0.33, 0.67], 0.37\rangle$ $\langle[0.04, 0.22], 0.17\rangle$ $\langle[0.01, 0.34], 0.47\rangle$	$\langle[0.71, 0.81], 0.52\rangle$ $\langle[0.06, 0.08], 0.21\rangle$ $\langle[0.07, 0.09], 0.16\rangle$

Table 8. Expert decision matrix.

	C_1	C_2	C_3	C_4
A_1	$\langle[0.33, 0.67], 0.37\rangle$ $\langle[0.04, 0.22], 0.17\rangle$ $\langle[0.01, 0.34], 0.47\rangle$	$\langle[0.25, 0.41], 0.45\rangle$ $\langle[0.07, 0.30], 0.35\rangle$ $\langle[0.09, 0.19], 0.02\rangle$	$\langle[0.83, 0.95], 0.78\rangle$ $\langle[0.03, 0.04], 0.01\rangle$ $\langle[0.01, 0.05], 0.20\rangle$	$\langle[0.33, 0.67], 0.37\rangle$ $\langle[0.04, 0.22], 0.17\rangle$ $\langle[0.01, 0.34], 0.47\rangle$
A_2	$\langle[0.33, 0.67], 0.37\rangle$ $\langle[0.04, 0.22], 0.17\rangle$ $\langle[0.01, 0.34], 0.47\rangle$	$\langle[0.11, 0.31], 0.35\rangle$ $\langle[0.07, 0.13], 0.17\rangle$ $\langle[0.12, 0.14], 0.20\rangle$	$\langle[0.11, 0.31], 0.35\rangle$ $\langle[0.07, 0.13], 0.17\rangle$ $\langle[0.12, 0.14], 0.20\rangle$	$\langle[0.40, 0.51], 0.50\rangle$ $\langle[0.20, 0.31], 0.12\rangle$ $\langle[0.11, 0.21], 0.13\rangle$
A_3	$\langle[0.25, 0.41], 0.45\rangle$ $\langle[0.07, 0.30], 0.35\rangle$ $\langle[0.09, 0.19], 0.02\rangle$	$\langle[0.40, 0.51], 0.50\rangle$ $\langle[0.20, 0.31], 0.12\rangle$ $\langle[0.11, 0.21], 0.13\rangle$	$\langle[0.40, 0.51], 0.50\rangle$ $\langle[0.20, 0.31], 0.12\rangle$ $\langle[0.11, 0.21], 0.13\rangle$	$\langle[0.33, 0.67], 0.37\rangle$ $\langle[0.04, 0.22], 0.17\rangle$ $\langle[0.01, 0.34], 0.47\rangle$
A_4	$\langle[0.83, 0.95], 0.78\rangle$ $\langle[0.03, 0.04], 0.01\rangle$ $\langle[0.01, 0.05], 0.20\rangle$	$\langle[0.33, 0.67], 0.37\rangle$ $\langle[0.04, 0.22], 0.17\rangle$ $\langle[0.01, 0.34], 0.47\rangle$	$\langle[0.33, 0.67], 0.37\rangle$ $\langle[0.04, 0.22], 0.17\rangle$ $\langle[0.01, 0.34], 0.47\rangle$	$\langle[0.83, 0.95], 0.78\rangle$ $\langle[0.03, 0.04], 0.01\rangle$ $\langle[0.01, 0.05], 0.20\rangle$

Table 9. Expert decision matrix.

	C_1	C_2	C_3	C_4
A_1	$\langle[0.01, 0.05], 0.20\rangle$ $\langle[0.03, 0.04], 0.01\rangle$ $\langle[0.83, 0.95], 0.78\rangle$	$\langle[0.09, 0.19], 0.02\rangle$ $\langle[0.07, 0.30], 0.35\rangle$ $\langle[0.25, 0.41], 0.45\rangle$	$\langle[0.01, 0.05], 0.20\rangle$ $\langle[0.03, 0.04], 0.01\rangle$ $\langle[0.83, 0.95], 0.78\rangle$	$\langle[0.12, 0.14], 0.20\rangle$ $\langle[0.07, 0.13], 0.17\rangle$ $\langle[0.11, 0.31], 0.35\rangle$
A_2	$\langle[0.12, 0.14], 0.20\rangle$ $\langle[0.07, 0.13], 0.17\rangle$ $\langle[0.11, 0.31], 0.35\rangle$	$\langle[0.01, 0.05], 0.20\rangle$ $\langle[0.03, 0.04], 0.01\rangle$ $\langle[0.83, 0.95], 0.78\rangle$	$\langle[0.12, 0.14], 0.20\rangle$ $\langle[0.07, 0.13], 0.17\rangle$ $\langle[0.11, 0.31], 0.35\rangle$	$\langle[0.09, 0.19], 0.02\rangle$ $\langle[0.07, 0.30], 0.35\rangle$ $\langle[0.25, 0.41], 0.45\rangle$
A_3	$\langle[0.09, 0.19], 0.02\rangle$ $\langle[0.07, 0.30], 0.35\rangle$ $\langle[0.25, 0.41], 0.45\rangle$	$\langle[0.01, 0.05], 0.20\rangle$ $\langle[0.03, 0.04], 0.01\rangle$ $\langle[0.83, 0.95], 0.78\rangle$	$\langle[0.09, 0.19], 0.02\rangle$ $\langle[0.07, 0.30], 0.35\rangle$ $\langle[0.25, 0.41], 0.45\rangle$	$\langle[0.01, 0.34], 0.47\rangle$ $\langle[0.04, 0.22], 0.17\rangle$ $\langle[0.33, 0.67], 0.37\rangle$
A_4	$\langle[0.01, 0.05], 0.20\rangle$ $\langle[0.03, 0.04], 0.01\rangle$ $\langle[0.83, 0.95], 0.78\rangle$	$\langle[0.01, 0.34], 0.47\rangle$ $\langle[0.04, 0.22], 0.17\rangle$ $\langle[0.33, 0.67], 0.37\rangle$	$\langle[0.01, 0.34], 0.47\rangle$ $\langle[0.04, 0.22], 0.17\rangle$ $\langle[0.33, 0.67], 0.37\rangle$	$\langle[0.01, 0.05], 0.20\rangle$ $\langle[0.03, 0.04], 0.01\rangle$ $\langle[0.83, 0.95], 0.78\rangle$

Table 10. Normalized cubic picture decision matrix.

$$w = (0.155, 0.266, 0.579)$$

Table 14 contains the overall rating value β_{ij}

Step-3. Sort the request by preference. Apply the method (CPFEWA) to rank the general rating values $\beta_i(i = 1,2,\ldots,n)$ and one or more alternatives.

$$Sc(A_1) = 0.1775, \quad Sc(A_2) = 0.2127, \quad Sc(A_3) = 0.2598, \quad Sc(A_4) = 0.2092$$

Step-4. All options are ranked in order of preference. Ranking determined by the scoring function S(z_i) is as follows:

$A_3 > A_2 > A_4 > A_1$. Each of the alternatives is ranked in order $A_3 > A_2 > A_4 > A_1$ and the best selection is A_3. A_3 is the best option out of all the possibilities, followed by $A_3 > A_2 > A_4 > A_1$.

The graph of the above alternatives is given as follows.

	C_1	C_2	C_3	C_4
A_1	$\begin{bmatrix} \langle[0.07,0.09],0.16\rangle \\ \langle[0.06,0.08],0.21\rangle \\ \langle[0.71,0.81],0.52\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.09,0.19],0.02\rangle \\ \langle[0.07,0.30],0.35\rangle \\ \langle[0.25,0.41],0.45\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.01,0.05],0.20\rangle \\ \langle[0.03,0.04],0.01\rangle \\ \langle[0.83,0.95],0.78\rangle \end{bmatrix}$	$\begin{bmatrix} [0.14,0.16],0.19 \\ \langle[0.18,0.32],0.27\rangle \\ \langle[0.51,0.72],0.63\rangle \end{bmatrix}$
A_2	$\begin{bmatrix} [0.14,0.16],0.19 \\ \langle[0.18,0.32],0.27\rangle \\ \langle[0.51,0.72],0.63\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.07,0.09],0.16\rangle \\ \langle[0.06,0.08],0.21\rangle \\ \langle[0.71,0.81],0.52\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.12,0.14],0.20\rangle \\ \langle[0.07,0.13],0.17\rangle \\ \langle[0.11,0.31],0.35\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.09,0.19],0.02\rangle \\ \langle[0.07,0.30],0.35\rangle \\ \langle[0.25,0.41],0.45\rangle \end{bmatrix}$
A_3	$\begin{bmatrix} \langle[0.09,0.19],0.02\rangle \\ \langle[0.07,0.30],0.35\rangle \\ \langle[0.25,0.41],0.45\rangle \end{bmatrix}$	$\begin{bmatrix} [0.14,0.16],0.19 \\ \langle[0.18,0.32],0.27\rangle \\ \langle[0.51,0.72],0.63\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.07,0.09],0.16\rangle \\ \langle[0.06,0.08],0.21\rangle \\ \langle[0.71,0.81],0.52\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.01,0.34],0.47\rangle \\ \langle[0.04,0.22],0.17\rangle \\ \langle[0.33,0.67],0.37\rangle \end{bmatrix}$
A_4	$\begin{bmatrix} \langle[0.01,0.05],0.20\rangle \\ \langle[0.03,0.04],0.01\rangle \\ \langle[0.83,0.95],0.78\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.01,0.34],0.47\rangle \\ \langle[0.04,0.22],0.17\rangle \\ \langle[0.33,0.67],0.37\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.01,0.34],0.47\rangle \\ \langle[0.04,0.22],0.17\rangle \\ \langle[0.33,0.67],0.37\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.07,0.09],0.16\rangle \\ \langle[0.06,0.08],0.21\rangle \\ \langle[0.71,0.81],0.52\rangle \end{bmatrix}$

Table 11. Normalized cubic picture decision matrix.

	C_1	C_2	C_3	C_4
A_1	$\begin{bmatrix} \langle[0.01,0.34],0.47\rangle \\ \langle[0.04,0.22],0.17\rangle \\ \langle[0.33,0.67],0.37\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.09,0.19],0.02\rangle \\ \langle[0.07,0.30],0.35\rangle \\ \langle[0.25,0.41],0.45\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.01,0.05],0.20\rangle \\ \langle[0.03,0.04],0.01\rangle \\ \langle[0.83,0.95],0.78\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.01,0.34],0.47\rangle \\ \langle[0.04,0.22],0.17\rangle \\ \langle[0.33,0.67],0.37\rangle \end{bmatrix}$
A_2	$\begin{bmatrix} \langle[0.01,0.34],0.47\rangle \\ \langle[0.04,0.22],0.17\rangle \\ \langle[0.33,0.67],0.37\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.12,0.14],0.20\rangle \\ \langle[0.07,0.13],0.17\rangle \\ \langle[0.11,0.31],0.35\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.12,0.14],0.20\rangle \\ \langle[0.07,0.13],0.17\rangle \\ \langle[0.11,0.31],0.35\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.11,0.21],0.13\rangle \\ \langle[0.20,0.31],0.12\rangle \\ \langle[0.40,0.51],0.50\rangle \end{bmatrix}$
A_3	$\begin{bmatrix} \langle[0.09,0.19],0.02\rangle \\ \langle[0.07,0.30],0.35\rangle \\ \langle[0.25,0.41],0.45\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.11,0.21],0.13\rangle \\ \langle[0.20,0.31],0.12\rangle \\ \langle[0.40,0.51],0.50\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.11,0.21],0.13\rangle \\ \langle[0.20,0.31],0.12\rangle \\ \langle[0.40,0.51],0.50\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.01,0.34],0.47\rangle \\ \langle[0.04,0.22],0.17\rangle \\ \langle[0.33,0.67],0.37\rangle \end{bmatrix}$
A_4	$\begin{bmatrix} \langle[0.01,0.05],0.20\rangle \\ \langle[0.03,0.04],0.01\rangle \\ \langle[0.83,0.95],0.78\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.01,0.34],0.47\rangle \\ \langle[0.04,0.22],0.17\rangle \\ \langle[0.33,0.67],0.37\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.01,0.34],0.47\rangle \\ \langle[0.04,0.22],0.17\rangle \\ \langle[0.33,0.67],0.37\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.01,0.05],0.20\rangle \\ \langle[0.03,0.04],0.01\rangle \\ \langle[0.83,0.95],0.78\rangle \end{bmatrix}$

Table 12. Normalized cubic picture decision matrix.

	C_1	C_2	C_3	C_4
A_1	$\begin{bmatrix} \langle[0.0427,0.0558],0.0026\rangle \\ \langle[0.0725,0.0839],0.0010\rangle \\ \langle[0.5677,0.6700],0.8968\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.0987,0.0996],0.0373\rangle \\ \langle[0.0659,0.0884],0.0024\rangle \\ \langle[0.5463,0.6288],0.7325\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.1138,0.2373],0.062\rangle \\ \langle[0.0794,0.0854],0.0190\rangle \\ \langle[0.4642,0.4894],0.5291\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.0762,0.0974],0.0094\rangle \\ \langle[0.0548,0.0733],0.0008\rangle \\ \langle[0.6433,0.8778],0.8808\rangle \end{bmatrix}$
A_2	$\begin{bmatrix} \langle[0.0427,0.0559],0.0025\rangle \\ \langle[0.0725,0.0840],0.0014\rangle \\ \langle[0.6727,0.8786],0.8966\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.1264,0.1623],0.0315\rangle \\ \langle[0.1301,0.1837],0.0176\rangle \\ \langle[0.6720,0.6849],0.7516\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.1136,0.1678],0.0311\rangle \\ \langle[0.0794,0.0854],0.0125\rangle \\ \langle[0.5680,0.5801],0.5291\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.0982,0.0996],0.0075\rangle \\ \langle[0.0686,0.0730],0.0205\rangle \\ \langle[0.6293,0.8530],0.7149\rangle \end{bmatrix}$
A_3	$\begin{bmatrix} \langle[0.0512,0.0760],0.0171\rangle \\ \langle[0.0388,0.0483],0.0010\rangle \\ \langle[0.6373,0.6669],0.8554\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.1138,0.1378],0.0265\rangle \\ \langle[0.0796,0.0854],0.0081\rangle \\ \langle[0.5023,0.5694],0.6792\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.1138,0.1378],0.0219\rangle \\ \langle[0.0794,0.0854],0.0080\rangle \\ \langle[0.5029,0.5694],0.5291\rangle \end{bmatrix}$	$\begin{bmatrix} \langle[0.0764,0.0873],0.0143\rangle \\ \langle[0.0548,0.0632],0.0002\rangle \\ \langle[0.6404,0.6533],0.8808\rangle \end{bmatrix}$

Table 13. Utilize the CPFEHWA operator.

A_1	$\langle[0.0132,0.0297],0.000056\rangle\langle[0.0225,0.0234],0.0000044\rangle\langle[0.8187,0.8202],0.9335\rangle$
A_2	$\langle[0.0350,0.0510],0.0015\rangle\langle[0.0305,0.0399],0.00010\rangle\langle[0.7899,0.8524],0.8028\rangle$
A_3	$\langle[0.0354,0.0500],0.00047\rangle\langle[0.0247,0.0188],0.00029\rangle\langle[0.7561,0.8661],0.5928\rangle$
A_4	$\langle[0.0271,0.0371],0.0098\rangle\langle[0.0190,0.0203],0.0124\rangle\langle[0.2290,0.3283],0.8663\rangle$

Table 14. General rating values.

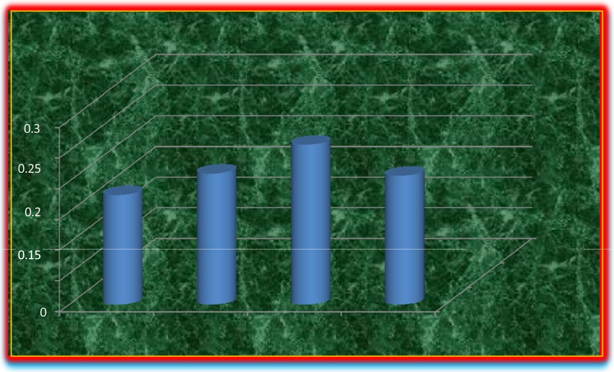

Fig. 2. Ranking the above alternatives.

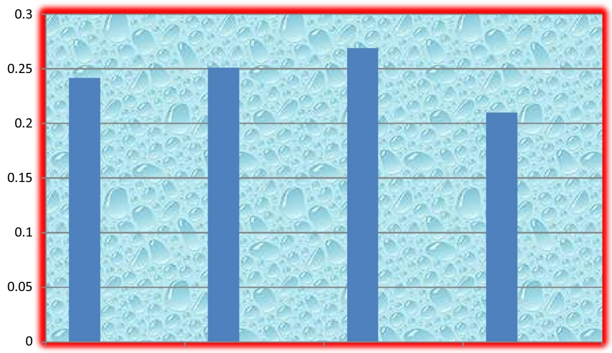

Fig. 3. Ranking of alternatives.

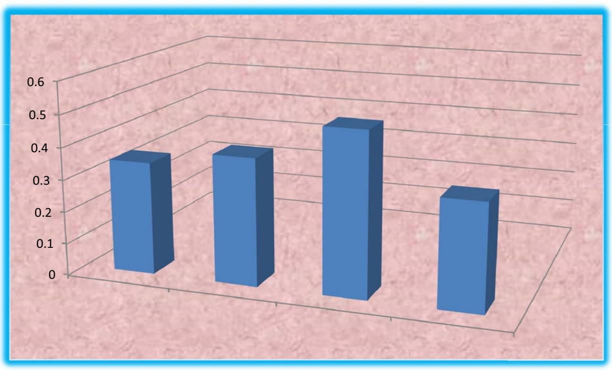

Fig. 4. Ranking of alternatives.

Comparison analysis

Figures 2, 3, 4 and 5 in the following present a ranking comparison using our proposed novel technique, highlighting its effectiveness.

Cubic fuzzy number comparison with the present MCDM approach

Step 1[17]: By the information used to make the decision provided by the cubic fuzzy number decision matrix $E^k = ({B_{ij}}^k)_{n*1}$.

Step 2 involves aggregating all combining all of the decision matrices into one with cubic fuzzy ratings using the cubic fuzzy Einstein operator.

In **step 3**, we utilize the score function to analyze the order of ranking of the alternatives. $\mathrm{Sc}(A_1) = 0.2413$, $\mathrm{Sc}(A_2) = 0.2507$, $\mathrm{Sc}(A_3) = 0.2688$, $\mathrm{Sc}(A_4) = 0.2097$.

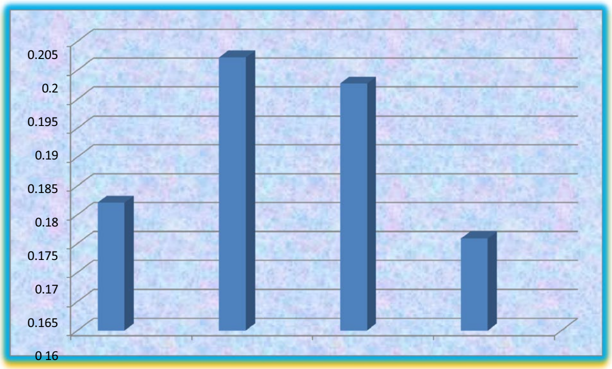

Fig. 5. Ranking of alternatives.

Methods	Ranking
Cubic picture fuzzy Einstein information (the proposed method)	$A_3 > A_2 > A_4 > A_1$
Cubic fuzzy Einstein information[17]	$A_3 > A_2 > A_1 > A_4$
Picture fuzzy weighted average information[24]	$A_3 > A_2 > A_1 > A_4$
Trapezoidal cubic fuzzy information[31]	$A_2 > A_3 > A_1 > A_4$

Table 15. Analysis of comparison with already-used techniques.

Step 4. Order all of the options. A_i $(i = 1,2,3,4)$ based on the scores $\mathrm{S}(r_i)$ of all r_i : values: $A_3 > A_2 > A_1 > A_4$ and hence the more suitable alternative is A_3.

The graph of the above alternatives is given as the following.

Analyzing comparisons with the current MCDM approach using picture fuzzy numbers

[24] **Step 1:** Using the decision data from the cubic fuzzy number decision matrix $R^{\sim} = (r_{ij}{}^{\sim})_{n*m}$

Step 2: Using the picture's fuzzy operator, combines all of the decision matrices into a single collective decision matrix with cubic fuzzy ratings.

In **step 3**, we utilize the score function to analyze the order of ranking of the alternatives.

$$Sc(A_1) = 0.350, \quad Sc(A_2) = 0.394, \quad Sc(A_3) = 0.504, \quad Sc(A_4) = 0.329,$$

Step 4. Rank the entire options $A_i(i = 1,2,3,4)$ based on the scores $\mathrm{S}(r_i)$ of all values r_i : $A_3 > A_2 > A_1 > A_4$ and hence the better appropriate choice is A_3.

The graph of the above alternatives is given as follows:

Analyzing comparisons with the current MCDM approach using the trapezoidal cubic fuzzy number

[31] **Step 1:** Using the trapezoidal cubic fuzzy number decision matrix's decision information $E^k = (B_{ij}{}^k)_{n*1}$

Step 2 involves aggregating all of combines all of the decision matrices into a single collective decision matrix using the trapezoidal cubic fuzzy Einstein operator.

In **step 3**, we employ the score function to analyze the order of ranking of the alternatives.

$$Sc(A_1) = 0.1771, \quad Sc(A_2) = 0.2022, \quad Sc(A_3) = 0.1977, \quad Sc(A_4) = 0.1709.$$

Step 4. Rank entire options $A_i(i = 1,2,3,4)$ based on the scores $\mathrm{S}(r_i)$ of all values r_i : $A_2 > A_3 > A_1 > A_4$ and hence the better appropriate choice is A_2.

The graph of the above alternatives is given as the following.

The Table 15 mentioned the comparison analysis with existing methods. Which introduced our technique is more appropriate due to Einstein operators.

CPFEWA vs. Existing Operators: CPFEWA combines cubic picture fuzzy reasoning with Einstein operations to provide improved handling of uncertainty and element interrelationships as compared to existing weighted averaging.

It outperforms classic fuzzy weighted averaging in numerous decision-making contexts by capturing deeper relationships between input elements.

CPFEOWA vs. Existing Operators: By combining ordered weighting and cubic picture fuzzy reasoning, CPFEOWA offers greater adaptability and flexibility for determining the relative value of elements as compared to existing ordered weighted averaging.

Perform better than traditional fuzzy-ordered weighted averaging when dealing with input pieces with varying degrees of significance.

CPFEHWA vs. Existing Operators: CPFEHWA effectively combines hybrid weighting and cubic picture fuzzy reasoning to produce improved aggregate outcomes in multi-criteria decision-making. By taking into account both weights and order-based relevance simultaneously, superior fuzzy hybrid weighted averaging produces results that are more accurate as compared to others(existing).

Sensitivity of proposed aggregation operators

The Einstein averaging aggregation operator serves as a powerful tool in cubic picture fuzzy environments, combining membership and non-membership grades in a balanced way that reflects both optimistic and pessimistic perspectives.

When comparing the sensitivity of cubic picture fuzzy Einstein averaging aggregation operators (CPFEAAO) with other operators, several key aspects must be considered:

Weighting schemes

Different aggregation operators use various weighting methods to combine input values. CPFEAAO applies a balanced scheme, whereas other operators may emphasize certain inputs differently, leading to varied outcomes.

Mathematical properties

Aggregation operators are influenced by properties such as boundedness, monotonicity, and idempotence. CPFEAAO maintains specific mathematical characteristics, which may differ from those of other operators and affect the final results.

Result interpretability

The clarity of aggregated outcomes is crucial in cubic picture fuzzy decision-making. CPFEAAO provides a structured and interpretable framework, while other operators may yield results with varying degrees of interpretability.

Application sensitivity

The effectiveness of an operator often depends on the problem domain. CPFEAAO may perform better in some scenarios, while alternative operators could be more suitable for others.

The sensitivity of cubic picture fuzzy Einstein averaging aggregation operators is influenced by the weighting scheme, mathematical properties, interpretability, and application-specific requirements. A comparative analysis of different operators within the context of a specific problem is essential to understand their impact on decision-making in cubic picture fuzzy environments.

Limitations

The computation of CPF Einstein averaging aggregation operators involves multiple iterations and calculations, which can result in high computational complexity. As the number of input elements increases, the computational burden also grows, making it less efficient for large-scale data sets.

The interpretation and communication of the aggregated results can be challenging. The operators often produce complex and abstract outputs, which may be difficult to explain or understand for non-experts. Interpreting the meaning and implications of the aggregated values can be a limitation when it comes to practical applications or decision-making processes. It's important to note that while cubic picture fuzzy Einstein averaging aggregation operators have their limitations, they may still be useful in certain contexts and provide valuable insights when appropriately applied.

Figure 6 below presents the roadmap of the entire manuscript, designed to help readers easily understand the overall structure and flow.

Conclusion

Subsequently, in conclusion, this study addressed the thought of cubic picture fuzzy numbers as well as their simple operational laws and Einstein operations. For aggregating cubic picture fuzzy data, we have introduced three arithmetic averaging operators: cubic picture fuzzy Einstein weighted averaging (CPFEWA), CPFEOWA, and CPFEHWA. The CPFEHWA operator simplifies both the CPFEWA and CPFEOWA operators. The connection between the proposed operators and the current aggregate operators has been defined, and we have emphasized the key features of these operators. We developed three properties of these operators as idempotency, monotonicity and boundedness. We demonstrate the usefulness of these novel Einstein operators and reveal their derived operators, such as CIFEWA, CFEWA, PFEWA, CIFEOWA, CFEOWA, PFEOWA, and CIFEHWA, CFEHWA, and PFEHWA. Through a numerical example, we have shown how the CPFEHWA operator can be used in MADM with fuzzy substances. Furthermore, because of their varying backgrounds, circumstances, and levels of information, experts in group decision-making problems frequently have diverse opinions. The developed operators can be used in a variety of applications, such as information fusion, data mining, pattern recognition, the cubic linguistic fuzzy Vikor technique, and the Atlantic Hierarchy approach. Overall, the findings of this study contribute to the understanding and practical implementation of cubic picture fuzzy numbers and their associated operators. We will define and establish some more geometric aggregation operators in the context of the Einstein field like cubic picture fuzzy Einstein weighted geometric (CPFEWG), CPFEOWG, and CPFEHWG.

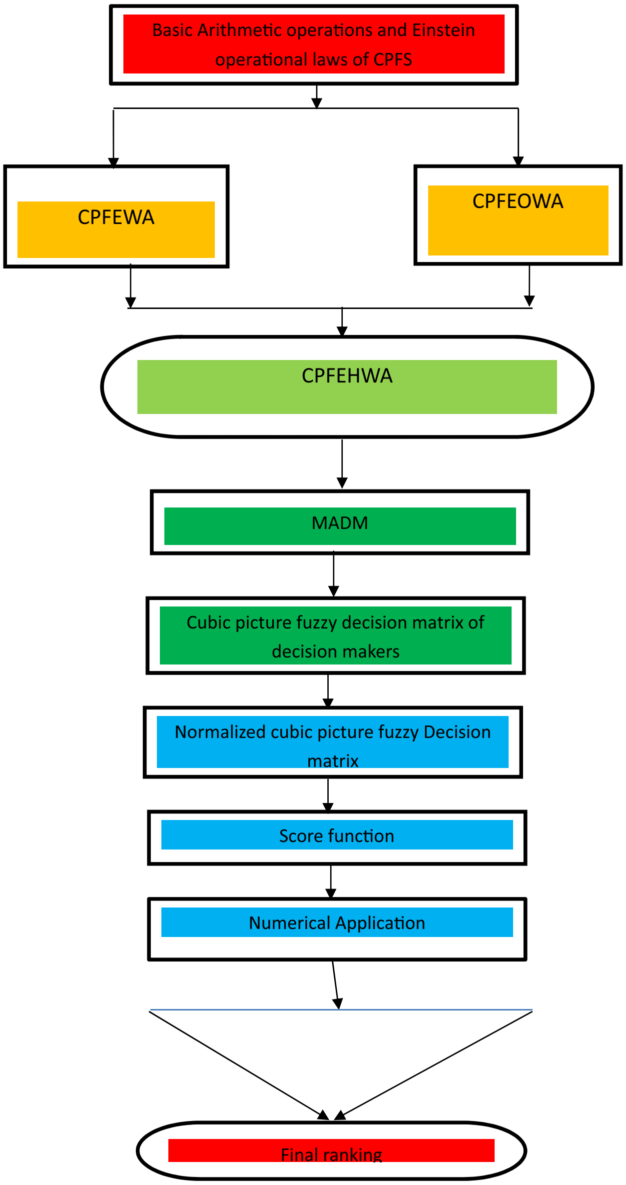

Fig. 6.. We provide the reader with a better understanding of the paper's major theme by displaying the paper's flow chart.

Appendix A

Example 5.1.1 Suppose that $A_1 = \langle \xi_{A_1}, e_{A_1} \rangle$, $A_2 = \langle \xi_{A_2}, e_{A_2} \rangle$ and $A_3 = \langle \xi_{A_3}, e_{A_3} \rangle$ are cubic picture fuzzy sets

$$A_1 = \{\langle [0.32{,}0.53], 0.48\rangle, \langle [0.09{,}0.15], 0.13\rangle, \langle [0.22{,}0.31], 0.36\rangle\}$$
$$A_2 = \{\langle [0.29, 0.43], 0.37\rangle, \langle [0.24, 0.38], 0.29\rangle, \langle [0.06, 0.18], 0.33\rangle\}$$
$$A_3 = \{\langle [0.22{,}0.40], 0.44\rangle, \langle [0.10{,}0.12], 0.23\rangle, \langle [0.28{,}0.39], 0.31\rangle\}$$

And $(0.3, 0.3, 0.4)^T$ are weighted vectors of $A_j\ (j = 1{,}2,3)$, utilize $CPFEWA$ operator.

Solution

For $A_j (j = 1{,}2, 3)$ we have

$$CPFEWA(A_1, A_2, A_3)$$

$$= \left[\begin{array}{c} \left\langle \left[\begin{array}{c} \frac{(1+0.32)^{0.3}(1+0.29)^{0.3}(1+0.22)^{0.3}-(1-0.32)^{0.3}(1-0.29)^{0.3}(1-0.22)^{0.3}}{(1+0.32)^{0.3}(1+0.29)^{0.3}(1+0.22)^{0.3}+(1-0.32)^{0.3}(1-0.29)^{0.3}(1-0.22)^{0.3}}, \\ \frac{(1+0.53)^{0.3}(1+0.43)^{0.3}(1+0.40)^{0.3}-(1-0.53)^{0.3}(1-0.43)^{0.3}(1-0.40)^{0.3}}{(1+0.53)^{0.3}(1+0.43)^{0.3}(1+0.40)^{0.3}+(1-0.53)^{0.3}(1-0.43)^{0.3}(1-0.40)^{0.3}} \end{array} \right] \right. \\ \left[\begin{array}{c} \frac{(1+0.09)^{0.3}(1+0.24)^{0.3}(1+0.10)^{0.3}-(1-0.09)^{0.3}(1-0.24)^{0.3}(1-0.10)^{0.3}}{(1+0.09)^{0.3}(1+0.24)^{0.3}(1+0.10)^{0.3}+(1-0.09)^{0.3}(1-0.24)^{0.3}(1-0.10)^{0.3}}, \\ \frac{(1+0.15)^{0.3}(1+0.38)^{0.3}(1+0.12)^{0.3}-(1-0.15)^{0.3}(1-0.38)^{0.3}(1-0.12)^{0.3}}{(1+0.15)^{0.3}(1+0.38)^{0.3}(1+0.12)^{0.3}+(1-0.15)^{0.3}(1-0.38)^{0.3}(1-0.12)^{0.3}} \end{array} \right] \\ \left. \left[\begin{array}{c} \frac{2(0.22)^{0.3}(0.06)^{0.3}(0.28)^{0.3}}{(2-0.22)^{0.3}(2-0.06)^{0.3}(2-0.28)^{0.3}+(0.22)^{0.3}(0.06)^{0.3}(0.28)^{0.3}}, \\ \frac{2(0.31)^{0.3}(0.18)^{0.3}(0.39)^{0.3}}{(2-0.31)^{0.3}(2-0.18)^{0.3}(2-0.39)^{0.3}+(0.31)^{0.3}(0.18)^{0.3}(0.39)^{0.3}} \end{array} \right] \right\rangle \\ \left\langle \frac{2(0.48)^{0.4}(0.37)^{04}(0.44)^{0.4}}{(2-0.48)^{0.4}(2-0.37)^{0.4}(2-0.44)^{0.4}+(0.48)^{0.4}(0.37)^{04}(0.44)^{0.4}}, \right. \\ \frac{2(0.13)^{0.4}(0.29)^{04}(0.23)^{0.4}}{(2-0.13)^{0.4}(2-0.29)^{0.4}(2-0.23)^{0.4}+(0.13)^{0.4}(0.29)^{04}(0.23)^{0.4}}, \\ \left. \frac{(1+0.36)^{0.4}(1+0.33)^{0.4}(1+0.31)^{0.4}-(1-0.36)^{0.4}(1-0.33)^{0.4}(1-0.31)^{0.4}}{(1+0.36)^{0.4}(1+0.33)^{0.4}(1+0.31)^{0.4}+(1-0.36)^{0.4}(1-0.33)^{0.4}(1-0.31)^{0.4}} \right\rangle \end{array} \right]$$

$$CPFEWA\left(A_1, A_2, A_3\right) = \{\langle[0.2507, 0.4155]\,,\ [0.1298, 0.1988]\,, [0.1968, 0.3280]\rangle, \langle 0.3471, 0.1392, 0.3936\rangle\}$$

$$CPFEWA\left(A_1, A_2, A_3\right) = \left[\begin{array}{c} \langle[0.2507, 0.4155]\,, 0.3471,\rangle \\ \langle[0.1298, 0.1988]\,, 0.1392,\rangle \\ \langle[0.1968, 0.3280]\,, 0.3936\rangle \end{array} \right]$$

Appendix B

Example 5.2.1 Suppose that $A_1 = \langle \xi_{A_1}, e_{A_1} \rangle$, $A_2 = \langle \xi_{A_2}, e_{A_2} \rangle$ and $A_3 = \langle \xi_{A_3}, e_{A_3} \rangle$ are cubic picture fuzzy Einstein sets

$$A_1 = \{\langle[0.32{,}0.53]\,, [0.09{,}0.15]\,, [0.22{,}0.31]\rangle, \langle 0.48{,}0.13{,}0.36\rangle\}$$
$$A_2 = \{\langle[0.29{,}0.43], [0.24{,}0.38], [0.06{,}0.18]\rangle, \langle 0.37{,}0.29{,}0.33\rangle\}$$
$$A_3 = \{\langle[0.22{,}0.40], [0.10{,}0.12], [0.28{,}0.39]\rangle, \langle 0.44{,}0.23{,}0.31\rangle\}$$

And $(0.3, 0.3, 0.4)^T$ are weighted vectors of $A_j(j = 1{,}2, 3)$ then find CPFOWA operator.
Solution

Now we calculate the score values of A_j $(j = 1{,}2, 3)$

$$Sc\left(A_1\right) = \frac{(2 + 0.32 + 0.53 - 0.09 - 0.15 - 0.22 + 0.31) + (0.48 - 0.13 - 0.36)}{6}$$
$$Sc\left(A_1\right) = 0.44833$$
$$Sc\left(A_2\right) = \frac{(2 + 0.29 + 0.43 - 0.24 - 0.38 - 0.06 + 0.18) + (0.37 - 0.29 - 0.33)}{6}$$
$$Sc\left(A_2\right) = 0.3283$$
$$Sc\left(A_3\right) = \frac{(2 + 0.22 + 0.40 - 0.10 - 0.12 - 0.28 + 0.31) + (0.44 - 0.23 - 0.31)}{6}$$
$$Sc\left(A_3\right) = 0.38833$$

Since $Sc\left(A_1\right) > Sc\left(A_3\right) > Sc\left(A_2\right)$

$$\left(A_1\right) > \left(A_3\right) > \left(A_2\right)$$

Hence

$A_{\propto(1)} = A_1 = \{\langle[0.32{,}0.53], [0.09{,}0.15], [0.22{,}0.31]\rangle, \langle 0.48{,}0.13{,}0.36\rangle\}$
$A_{\propto(2)} = A_3 = \{\langle[0.22{,}0.40], [0.10{,}0.12], [0.28{,}0.39]\rangle, \langle 0.44{,}0.23{,}0.31\rangle\}$
$A_{\propto(3)} = A_2 = \{\langle[0.29{,}0.43], [0.24{,}0.38], [0.06{,}0.18]\rangle, \langle 0.37{,}0.29{,}0.33\rangle\}$
For $A_j(j = 1{,}2, 3)$ we have

$$CPFEOWA(A_{\propto(1)},A_{\propto(2)},A_{\propto(3)}) = \begin{bmatrix} \left\langle \begin{bmatrix} \frac{(1+0.32)^{0.3}(1+0.22)^{0.3}(1+0.29)^{0.3}-(1-0.32)^{0.3}(1-0.22)^{0.3}(1-0.29)^{0.3}}{(1+0.32)^{0.3}(1+0.22)^{0.3}(1+0.29)^{0.3}+(1-0.32)^{0.3}(1-0.22)^{0.3}(1-0.29)^{0.3}}, \\ \frac{(1+0.53)^{0.3}(1+0.40)^{0.3}(1+0.43)^{0.3}-(1-0.53)^{0.3}(1-0.40)^{0.3}(1-0.43)^{0.3}}{(1+0.53)^{0.3}(1+0.40)^{0.3}(1+0.43)^{0.3}+(1-0.53)^{0.3}(1-0.40)^{0.3}(1-0.43)^{0.3}} \end{bmatrix} \right. \\ \begin{bmatrix} \frac{(1+0.09)^{0.3}(1+0.10)^{0.3}(1+0.24)^{0.3}-(1-0.09)^{0.3}(1-0.10)^{0.3}(1-0.24)^{0.3}}{(1+0.09)^{0.3}(1+0.10)^{0.3}(1+0.24)^{0.3}+(1-0.09)^{0.3}(1-0.10)^{0.3}(1-0.24)^{0.3}}, \\ \frac{(1+0.15)^{0.3}(1+0.12)^{0.3}(1+0.38)^{0.3}-(1-0.15)^{0.3}(1-0.12)^{0.3}(1-0.38)^{0.3}}{(1+0.15)^{0.3}(1+0.12)^{0.3}(1+0.38)^{0.3}+(1-0.15)^{0.3}(1-0.12)^{0.3}(1-0.38)^{0.3}} \end{bmatrix} \\ \left. \begin{bmatrix} \frac{2(0.22)^{0.3}(0.28)^{0.3}(0.06)^{0.3}}{(2-0.22)^{0.3}(2-0.28)^{0.3}(2-0.06)^{0.3}+(0.22)^{0.3}(0.28)^{0.3}(0.06)^{0.3}}, \\ \frac{2(0.31)^{0.3}(0.39)^{0.3}(0.18)^{0.3}}{(2-0.31)^{0.3}(2-0.39)^{0.3}(2-0.18)^{0.3}+(0.31)^{0.3}(0.39)^{0.3}(0.18)^{0.3}} \end{bmatrix} \right\rangle \\ \left\langle \frac{2(0.48)^{0.4}(0.44)^{0.4}(0.37)^{04}}{(2-0.48)^{0.4}(2-0.44)^{0.4}(2-0.37)^{0.4}+(0.48)^{0.4}(0.44)^{0.4}(0.37)^{04}}, \right. \\ \frac{2(0.13)^{0.4}(0.23)^{0.4}(0.29)^{04}}{(2-0.13)^{0.4}(2-0.23)^{0.4}(2-0.29)^{0.4}+(0.13)^{0.4}(0.23)^{0.4}(0.29)^{04}}, \\ \left. \frac{(1+0.36)^{0.4}(1+0.31)^{0.4}(1+0.33)^{0.4}-(1-0.36)^{0.4}(1-0.31)^{0.4}(1-0.33)^{0.4}}{(1+0.36)^{0.4}(1+0.31)^{0.4}(1+0.33)^{0.4}+(1-0.36)^{0.4}(1-0.31)^{0.4}(1-0.33)^{0.4}} \right\rangle \end{bmatrix}$$

$$CPFEOWA\left(A_{\propto(1)},A_{\propto(2)},A_{\propto(3)}\right) = \{\langle[0.2507,0.4155],[0.1298,0.1988],[0.1968,0.3281]\rangle,\langle 0.3471,0.1392,0.3936\rangle\}$$

$$CPFEOWA\left(A_{\propto(1)},A_{\propto(2)},A_{\propto(3)}\right) = \begin{bmatrix} \langle[0.2507,0.4155],0.3471\rangle \\ \langle[0.1298,0.1988],0.1392\rangle \\ \langle[0.1968,0.3281],0.3936\rangle \end{bmatrix}$$

Appendix C

Example 5.3.1 Suppose that $A_1 = \langle \xi_{A_1}, e_{A_1}\rangle$, $A_2 = \langle \xi_{A_2}, e_{A_2}\rangle$ and $A_3 = \langle \xi_{A_3}, e_{A_3}\rangle$ are cubic picture fuzzy sets defined as

$$A_1 = \{\langle[0.32,0.53],[0.09,0.15],[0.22,0.31]\rangle,\langle 0.48,0.0.13,0.36\rangle\}$$
$$A_2 = \{\langle[0.19,0.23],[0.17,0.33],[0.39,0.42]\rangle,\langle 0.52,0.08,0.38\rangle\}$$
$$A_3 = \{\langle[0.29,0.43],[0.11,0.14],[0.06,0.08]\rangle,\langle 0.32,0.14,0.09\rangle\}$$

$\tau = (0.220,0.335,0.445)^T$ Is a vector of $A_j\ (j = 1,2,3)$. To aggregate the provided data using the CPFEHWA operator, we must first establish the weight vector $w = (0.3, 0.3, 0.4)^T$ associated with a vector of $A'_j (j = 1,2,3)$ Then find CPFEHWA operator.

Solution

$$A'_1 = \begin{bmatrix} \left\langle \begin{bmatrix} \frac{\left(1+\xi_{P_{A_1}}^-\right)^{n\times\tau_1}-\left(1-\xi_{P_{A_1}}^-\right)^{n\times\tau_1}}{\left(1+\xi_{P_{A_1}}^-\right)^{n\times\tau_1}+\left(1-\xi_{P_{A_1}}^-\right)^{n\times\tau_1}}, \frac{\left(1+\xi_{P_{A_1}}^+\right)^{n\times\tau_1}-\left(1-\xi_{P_{A_1}}^+\right)^{n\times\tau_1}}{\left(1+\xi_{P_{A_1}}^+\right)^{n\times\tau_1}+\left(1-\xi_{P_{A_1}}^+\right)^{n\times\tau_1}} \end{bmatrix} \right. \\ \begin{bmatrix} \frac{\left(1+\xi_{I_{A_1}}^-\right)^{n\times\tau_1}-\left(1-\xi_{I_{A_1}}^-\right)^{n\times\tau_1}}{\left(1+\xi_{I_{A_1}}^-\right)^{n\times\tau_1}+\left(1-\xi_{I_{A_1}}^-\right)^{n\times\tau_1}}, \frac{\left(1+\xi_{I_{A_1}}^+\right)^{n\times\tau_1}-\left(1-\xi_{I_{A_1}}^+\right)^{n\times\tau_1}}{\left(1+\xi_{I_{A_1}}^+\right)^{n\times\tau_1}+\left(1-\xi_{I_{A_1}}^+\right)^{n\times\tau_1}} \end{bmatrix} \\ \left. \begin{bmatrix} \frac{2\left(\xi_{N_{A_1}}^-\right)^{n\times\tau_1}}{\left(2-\xi_{N_{A_1}}^-\right)^{n\times\tau_1}+\left(\xi_{N_{A_1}}^-\right)^{n\times\tau_1}}, \frac{2\left(\xi_{N_{A_1}}^+\right)^{n\times\tau_1}}{\left(2-\xi_{N_{A_1}}^+\right)^{n\times\tau_1}+\left(\xi_{N_{A_1}}^+\right)^{n\times\tau_1}} \end{bmatrix} \right\rangle \\ \left\langle \frac{2\left(P_{A_1}\right)^{n\times\tau_1}}{\left(2-P_{A_1}\right)^{n\times\tau_1}+\left(P_{A_1}\right)^{n\times\tau_1}}, \frac{2\left(I_{A_1}\right)^{n\times\tau_1}}{\left(2-I_{A_1}\right)^{n\times\tau_1}+\left(I_{A_1}\right)^{n\times\tau_1}}, \frac{\left(1+N_{A_1}\right)^{n\times\tau_1}-\left(1-N_{A_1}\right)^{n\times\tau_1}}{\left(1+N_{A_1}\right)^{n\times\tau_1}+\left(1-N_{A_1}\right)^{n\times\tau_1}} \right\rangle \end{bmatrix}$$

$$A'_1 = \begin{bmatrix} \left\langle \begin{bmatrix} \frac{(1+0.32)^{3\times0.220}-(1-0.32)^{3\times0.220}}{(1+0.32)^{3\times0.220}+(1-0.32)^{3\times0.220}}, \frac{(1+0.53)^{3\times0.220}-(1-0.53)^{3\times0.220}}{(1+0.53)^{3\times0.220}+(1-0.53)^{3\times0.220}} \end{bmatrix} \right. \\ \begin{bmatrix} \frac{(1+0.09)^{3\times0.220}-(1-0.09)^{3\times0.220}}{(1+0.09)^{3\times0.220}+(1-0.09)^{3\times0.220}}, \frac{(1+0.15)^{3\times0.220}-(1-0.15)^{3\times0.220}}{(1+0.15)^{3\times0.220}+(1-0.15)^{3\times0.220}} \end{bmatrix} \\ \left. \begin{bmatrix} \frac{2(0.22)^{3\times0.220}}{(2-0.22)^{3\times0.220}+(0.22)^{3\times0.220}}, \frac{2(0.31)^{3\times0.220}}{(2-0.31)^{3\times0.220}+(0.31)^{3\times0.220}} \end{bmatrix} \right\rangle \\ \left\langle \frac{2(0.48)^{3\times0.220}}{(2-0.48)^{3\times0.220}+(0.48)^{3\times0.220}}, \frac{2(0.13)^{3\times0.220}}{(2-0.13)^{3\times0.220}+(0.13)^{3\times0.220}}, \frac{(1+0.36)^{3\times0.220}-(1-0.36)^{3\times0.220}}{(1+0.36)^{3\times0.220}+(1-0.36)^{3\times0.220}} \right\rangle \end{bmatrix}$$

$A'_1 = \{\langle[0.2154{,}0.3709], [0.0594{,}0.0994], [0.4020{,}0.4922]\rangle, \langle[0.6369{,}0.2936{,}0.2437]\rangle\}.$

$$A'_2 = \begin{bmatrix} \left\langle \left[\frac{\left(1+\xi_{P_{A_2}}{}^{-}\right)^{n\times\tau_2}-\left(1-\xi_{P_{A_2}}{}^{-}\right)^{n\times\tau_2}}{\left(1+\xi_{P_{A_2}}{}^{-}\right)^{n\times\tau_2}+\left(1-\xi_{P_{A_2}}{}^{-}\right)^{n\times\tau_2}}, \frac{\left(1+\xi_{P_{A_2}}{}^{+}\right)^{n\times\tau_2}-\left(1-\xi_{P_2}{}^{+}\right)^{n\times\tau_2}}{\left(1+\xi_{P_{A_2}}{}^{+}\right)^{n\times\tau_2}+\left(1-\xi_{P_{A_2}}{}^{+}\right)^{n\times\tau_2}}\right] \right. \\ \left[\frac{\left(1+\xi_{I_{A_2}}{}^{-}\right)^{n\times\tau_2}-\left(1-\xi_{I_{A_2}}{}^{-}\right)^{n\times\tau_2}}{\left(1+\xi_{I_{A_2}}{}^{-}\right)^{n\times\tau_2}+\left(1-\xi_{I_{A_2}}{}^{-}\right)^{n\times\tau_2}}, \frac{\left(1+\xi_{I_{A_2}}{}^{+}\right)^{n\times\tau_2}-\left(1-\xi_{I_{A_2}}{}^{+}\right)^{n\times\tau_2}}{\left(1+\xi_{I_{A_2}}{}^{+}\right)^{n\times\tau_2}+\left(1-\xi_{I_{A_2}}{}^{+}\right)^{n\times\tau_2}}\right] \\ \left. \left[\frac{2\left(\xi_{N_{A_2}}{}^{-}\right)^{n\times\tau_2}}{\left(2-\xi_{N_{A_2}}{}^{-}\right)^{n\times\tau_2}+\left(\xi_{N_{A_2}}{}^{-}\right)^{n\times\tau_2}}, \frac{2\left(\xi_{N_{A_2}}{}^{+}\right)^{n\times\tau_2}}{\left(2-\xi_{N_{A_2}}{}^{+}\right)^{n\times\tau_2}+\left(\xi_{N_{A_2}}{}^{+}\right)^{n\times\tau_2}}\right] \right\rangle \\ \left\langle \frac{2\left(P_{A_2}\right)^{n\times\tau_2}}{\left(2-P_{A_2}\right)^{n\times\tau_2}+\left(P_{A_2}\right)^{n\times\tau_2}}, \frac{2\left(I_{A_2}\right)^{n\times\tau_2}}{\left(2-I_{A_2}\right)^{n\times\tau_2}+\left(I_{A_2}\right)^{n\times\tau_2}}, \frac{\left(1+N_{A_2}\right)^{n\times\tau_2}-\left(1-N_{A_2}\right)^{n\times\tau_2}}{\left(1+N_{A_2}\right)^{n\times\tau_2}+\left(1-N_{A_2}\right)^{n\times\tau_2}} \right\rangle \end{bmatrix}$$

$$A'_2 = \begin{bmatrix} \left\langle \left[\frac{(1+0.19)^{3\times0.335}-(1-0.19)^{3\times0.335}}{(1+0.19)^{3\times0.335}+(1-0.19)^{3\times0.335}}, \frac{(1+0.23)^{3\times0.335}-(1-0.23)^{3\times0.335}}{(1+0.23)^{3\times0.335}+(1-0.23)^{3\times0.335}}\right] \right. \\ \left[\frac{(1+0.17)^{3\times0.335}-(1-0.17)^{3\times0.335}}{(1+0.17)^{3\times0.335}+(1-0.17)^{3\times0.335}}, \frac{(1+0.33)^{3\times0.335}-(1-0.33)^{3\times0.335}}{(1+0.33)^{3\times0.335}+(1-0.33)^{3\times0.335}}\right] \\ \left. \left[\frac{2(0.39)^{3\times0.335}}{(2-0.39)^{3\times0.335}+(0.39)^{3\times0.335}}, \frac{2(0.42)^{3\times0.335}}{(2-0.42)^{3\times0.335}+(0.42)^{3\times0.335}}\right] \right\rangle \\ \left\langle \frac{2(0.52)^{3\times0.335}}{(2-0.52)^{3\times0.335}+(0.52)^{3\times0.335}}, \frac{2(0.08)^{3\times0.335}}{(2-0.08)^{3\times0.335}+(0.08)^{3\times0.335}}, \frac{(1+0.38)^{3\times0.335}-(1-0.38)^{3\times0.335}}{(1+0.38)^{3\times0.335}+(1-0.38)^{3\times0.335}} \right\rangle \end{bmatrix}$$

$A'_2 = \{\langle[0.1909{,}0.2311], [0.1708{,}0.3315], [0.3877{,}0.4178]\rangle, \langle[0.5179{,}0.0787{,}0.3817]\rangle\}.$

$$A'_3 = \begin{bmatrix} \left\langle \left[\frac{\left(1+\xi_{P_{A_3}}{}^{-}\right)^{n\times\tau_3}-\left(1-\xi_{P_{A_3}}{}^{-}\right)^{n\times\tau_3}}{\left(1+\xi_{P_{A_3}}{}^{-}\right)^{n\times\tau_3}+\left(1-\xi_{P_{A_3}}{}^{-}\right)^{n\times\tau_3}}, \frac{\left(1+\xi_{P_{A_3}}{}^{+}\right)^{n\times\tau_3}-\left(1-\xi_{P_3}{}^{+}\right)^{n\times\tau_3}}{\left(1+\xi_{P_{A_3}}{}^{+}\right)^{n\times\tau_3}+\left(1-\xi_{P_{A_3}}{}^{+}\right)^{n\times\tau_3}}\right] \right. \\ \left[\frac{\left(1+\xi_{I_{A_3}}{}^{-}\right)^{n\times\tau_3}-\left(1-\xi_{I_{A_3}}{}^{-}\right)^{n\times\tau_3}}{\left(1+\xi_{I_{A_3}}{}^{-}\right)^{n\times\tau_3}+\left(1-\xi_{I_{A_3}}{}^{-}\right)^{n\times\tau_3}}, \frac{\left(1+\xi_{I_{A_3}}{}^{+}\right)^{n\times\tau_3}-\left(1-\xi_{I_{A_3}}{}^{+}\right)^{n\times\tau_3}}{\left(1+\xi_{I_{A_3}}{}^{+}\right)^{n\times\tau_3}+\left(1-\xi_{I_{A_3}}{}^{+}\right)^{n\times\tau_3}}\right] \\ \left. \left[\frac{2\left(\xi_{N_{A_3}}{}^{-}\right)^{n\times\tau_3}}{\left(2-\xi_{N_{A_3}}{}^{-}\right)^{n\times\tau_3}+\left(\xi_{N_{A_3}}{}^{-}\right)^{n\times\tau_3}}, \frac{2\left(\xi_{N_{A_3}}{}^{+}\right)^{n\times\tau_3}}{\left(2-\xi_{N_{A_3}}{}^{+}\right)^{n\times\tau_3}+\left(\xi_{N_3}{}^{+}\right)^{n\times\tau_3}}\right] \right\rangle \\ \left\langle \frac{2\left(P_{A_3}\right)^{n\times\tau_3}}{\left(2-P_{A_3}\right)^{n\times\tau_3}+\left(P_{A_3}\right)^{n\times\tau_3}}, \frac{2\left(I_{A_3}\right)^{n\times\tau_3}}{\left(2-I_{A_3}\right)^{n\times\tau_3}+\left(I_{A_3}\right)^{n\times\tau_3}}, \frac{\left(1+N_{A_3}\right)^{n\times\tau_3}-\left(1-N_{A_3}\right)^{n\times\tau_3}}{\left(1+N_{A_3}\right)^{n\times\tau_3}+\left(1-N_{A_3}\right)^{n\times\tau_3}} \right\rangle \end{bmatrix}$$

$$A'_3 = \begin{bmatrix} \left\langle \left[\frac{(1+0.29)^{3\times0.445}-(1-0.29)^{3\times0.445}}{(1+0.29)^{3\times0.445}+(1-0.29)^{3\times0.445}}, \frac{(1+0.43)^{3\times0.445}-(1-0.43)^{3\times0.445}}{(1+0.43)^{3\times0.445}+(1-0.43)^{3\times0.445}}\right] \right. \\ \left[\frac{(1+0.11)^{3\times0.445}-(1-0.11)^{3\times0.445}}{(1+0.11)^{3\times0.445}+(1-0.11)^{3\times0.445}}, \frac{(1+0.14)^{3\times0.445}-(1-0.14)^{3\times0.445}}{(1+0.14)^{3\times0.445}+(1-0.14)^{3\times0.445}}\right] \\ \left. \left[\frac{2(0.06)^{3\times0.445}}{(2-0.06)^{3\times0.4455}+(0.06)^{3\times0.445}}, \frac{2(0.08)^{3\times0.445}}{(2-0.08)^{3\times0.4455}+(0.08)^{3\times0.445}}\right] \right\rangle \\ \left\langle \frac{2(0.32)^{3\times0.445}}{(2-0.32)^{3\times0.4455}+(0.32)^{3\times0.445}}, \frac{2(0.14)^{3\times0.445}}{(2-0.14)^{3\times0.4455}+(0.14)^{3\times0.445}}, \frac{(1+0.09)^{3\times0.445}-(1-0.09)^{3\times0.445}}{(1+0.09)^{3\times0.445}+(1-0.09)^{3\times0.445}} \right\rangle \end{bmatrix}$$

$A'_3 = \{\langle[0.3787{,}0.5469], [0.1463{,}0.1859], [0.0191{,}0.0283]\rangle, \langle[0.1970{,}0.0613{,}0.1198]\rangle\}.$

We now evaluate the score values of A_j $(j = 1,2,3)$

$A'_1 = \{\langle[0.2154{,}0.3709], [0.0594{,}0.0994], [0.4020{,}0.4922]\rangle, \langle[0.6369{,}0.2936{,}0.2437]\rangle\}.$

$$sc(A'_1) = \frac{(2+0.2154+0.3709-0.0594-0.0994-0.4020+0.4922)+(0.6369-0.2936-0.2437)}{6}$$
$$sc(A'_1) = 0.4362$$
$$sc(A'_2) = \frac{(2+0.1909+0.2311-0.1708-0.3315-0.3877+0.4178)+(0.5179-0.0787-0.3817)}{6}$$
$$sc(A'_2) = 0.4617$$
$$sc(A'_3) = \frac{(2+0.3787+0.5469-0.1463-0.1859-0.0191+0.0283)+(0.1970-0.0613-0.1198)}{6}$$
$$sc(A'_3) = 0.4763$$

Since

$$sc(A'_3) > sc(A'_2) > sc(A'_1)$$
$$A'_3 > A'_2 > A'_1$$

hence

$A_{\delta(1)} = A'_3 = \{\langle[0.3787{,}0.5469], [0.1463{,}0.1859], [0.0191{,}0.0283]\rangle, \langle[0.1970{,}0.0613{,}0.1198]\rangle\}$

$A_{\delta(2)} = A'_2 = \{\langle[0.1909{,}0.2311], [0.1708{,}0.3315], [0.3877{,}0.4178]\rangle, \langle[0.5179{,}0.0787{,}0.3817]\rangle\}$

$A_{\delta(3)} = A'_1 = \{\langle[0.2154{,}0.3709], [0.0594{,}0.0994], [0.4020{,}0.4922]\rangle, \langle[0.6369{,}0.2936{,}0.2437]\rangle\}$.

For $A_j (j = 1{,}2{,}3)$ we have

$$CPFEHWA(A_{\delta(1)}, A_{\delta(2)}, A_{\delta(3)}) =$$

$$\left[\begin{array}{c} \left\langle \left[\begin{array}{c} \frac{(1+0.3787)^{0.3}(1+0.1909)^{0.3}(1+0.2154)^{0.3}-(1-0.3787)^{0.3}(1-0.1909)^{0.3}(1-0.2154)^{0.3}}{(1+0.3787)^{0.3}(1+0.1909)^{0.3}(1+0.2154)^{0.3}+(1-0.3787)^{0.3}(1-0.1909)^{0.3}(1-0.2154)^{0.3}}, \\ \frac{(1+0.5469)^{0.3}(1+0.2311)^{0.3}(1+0.3709)^{0.3}-(1-0.5469)^{0.3}(1-0.2311)^{0.3}(1-0.3709)^{0.3}}{(1+0.5469)^{0.3}(1+0.2311)^{0.3}(1+0.3709)^{0.3}+(1-0.5469)^{0.3}(1-0.2311)^{0.3}(1-0.3709)^{0.3}} \end{array}\right] \right. \\ \left[\begin{array}{c} \frac{(1+0.1463)^{0.3}(1+0.1708)^{0.3}(1+0.0594)^{0.3}-(1-0.1463)^{0.3}(1-0.1708)^{0.3}(1-0.0594)^{0.3}}{(1+0.1463)^{0.3}(1+0.1708)^{0.3}(1+0.0594)^{0.3}+(1-0.1463)^{0.3}(1-0.1708)^{0.3}(1-0.0594)^{0.3}}, \\ \frac{(1+0.1859)^{0.3}(1+0.3315)^{0.3}(1+0.0994)^{0.3}-(1-0.1859)^{0.3}(1-0.3315)^{0.3}(1-0.0994)^{0.3}}{(1+0.1859)^{0.3}(1+0.3315)^{0.3}(1+0.0994)^{0.3}+(1-0.1859)^{0.3}(1-0.3315)^{0.3}(1-0.0994)^{0.3}} \end{array}\right] \\ \left. \left[\begin{array}{c} \frac{2(0.0191)^{0.3}(0.3877)^{0.3}(0.4020)^{0.3}}{(2-0.0191)^{0.3}(2-0.3877)^{0.3}(2-0.4020)^{0.3}+(0.0191)^{0.3}(0.3877)^{0.3}(0.4020)^{0.3}}, \\ \frac{2(0.0283)^{0.3}(0.4178)^{0.3}(0.4922)^{0.3}}{(2-0.0283)^{0.3}(2-0.4178)^{0.3}(2-0.4922)^{0.3}+(0.0283)^{0.3}(0.4178)^{0.3}(0.4922)^{0.3}} \end{array}\right] \right\rangle \\ \left\langle \frac{2(0.1970)^{04}(0.5179)^{0.4}(0.6369)^{0.4}}{((2-0.1970)^{04}(2-0.5179)^{0.4}(2-0.6369)^{0.4}+(0.1970)^{04}(0.5179)^{0.4}(0.6369)^{0.4}}, \right. \\ \frac{2(0.0613)^{04}(0.0787)^{0.4}(0.2936)^{0.4}}{((2-0.0613)^{04}(2-0.0787)^{0.4}(2-0.2936)^{0.4}+(0.0613)^{04}(0.0787)^{0.4}(0.2936)^{0.4}}, \\ \left. \frac{(1+0.1198)^{0.4}(1+0.3817)^{0.4}(1+0.2437)^{0.4}-(1-0.1198)^{0.4}(1-0.3817)^{0.4}(1-0.2437)^{0.4}}{(1+0.1198)^{0.4}(1+0.3817)^{0.4}(1+0.2437)^{0.4}+(1-0.1198)^{0.4}(1-0.3817)^{0.4}(1-0.2437)^{0.4}} \right\rangle \end{array}\right]$$

$$CPFEHWA\left(A_{\delta(1)}, A_{\delta(2)}, A_{\delta(3)}\right) = \{\langle[0.2384{,}0.3553], [0.1133{,}0.1874], [0.1934{,}0.2366]\rangle, \langle 0.3330, 0.0666, 0.2989\rangle\}$$

$$CPFEHWA\left(A_{\delta(1)}, A_{\delta(2)}, A_{\delta(3)}\right) = \left[\begin{array}{c} \langle[0.2384{,}0.3553], 0.3330\rangle, \\ \langle[0.1133{,}0.1874], 0.0666\rangle, \\ \langle[0.1934{,}0.2366], 0.2989\rangle \end{array}\right]$$

Appendix D

Utilize the CPFEHWA operator on normalized cubic picture fuzzy decision-making (from Tables 8, 9, 10).

First of all, in column-1 of Table 8, we have

$$A_1 = \left[\begin{array}{c} \langle[0.01, 0.05], 0.20\rangle \\ \langle[0.03, 0.04], 0.01\rangle \\ \langle[0.83, 0.95], 0.78\rangle \end{array}\right], \quad A_2 = \left[\begin{array}{c} \langle[0.12, 0.14], 0.20\rangle \\ \langle[0.07, 0.13], 0.17\rangle \\ \langle[0.11, 0.31], 0.35\rangle \end{array}\right],$$

$$A_3 = \left[\begin{array}{c} \langle[0.09, 0.19], 0.02\rangle \\ \langle[0.07, 0.30], 0.35\rangle \\ \langle[0.25, 0.41], 0.45\rangle \end{array}\right], \quad A_4 = \left[\begin{array}{c} \langle[0.01, 0.05], 0.20\rangle \\ \langle[0.03, 0.04], 0.01\rangle \\ \langle[0.83, 0.95], 0.78\rangle \end{array}\right]$$

$\tau = (0.001{,}0.220{,}0.335{,}0.445)^T$ Is the A_j's weight vector $(j = 1{,}2{,}3{,}4)$. If we want to aggregate the provided data using the CPFEHWA operator, first, we have to find the weight vector., which is $w = (0.155, 0.266, 0.579)^T$ connected to the vector of $A'_j (j = 1{,}2{,}3{,}4)$, where n=4.

$$A'_1=\begin{bmatrix}\left\langle\left[\frac{\left(1+\xi_{P_{A_1}}{}^-\right)^{n\times\tau_1}-\left(1-\xi_{P_{A_1}}{}^-\right)^{n\times\tau_1}}{\left(1+\xi_{P_{A_1}}{}^-\right)^{n\times\tau_1}+\left(1-\xi_{P_{A_1}}{}^-\right)^{n\times\tau_1}},\frac{\left(1+\xi_{P_{A_1}}{}^+\right)^{n\times\tau_1}-\left(1-\xi_{P_{A_1}}{}^+\right)^{n\times\tau_1}}{\left(1+\xi_{P_{A_1}}{}^+\right)^{n\times\tau_1}+\left(1-\xi_{P_{A_1}}{}^+\right)^{n\times\tau_1}}\right],\frac{2\left(P_{A_1}\right)^{n\times\tau_1}}{(2-P_{A_1})^{n\times\tau_1}+\left(P_{A_1}\right)^{n\times\tau_1}}\right\rangle\\ \left\langle\left[\frac{\left(1+\xi_{I_{A_1}}{}^-\right)^{n\times\tau_1}-\left(1-\xi_{I_{A_1}}{}^-\right)^{n\times\tau_1}}{\left(1+\xi_{I_{A_1}}{}^-\right)^{n\times\tau_1}+\left(1-\xi_{I_{A_1}}{}^-\right)^{n\times\tau_1}},\frac{\left(1+\xi_{I_{A_1}}{}^+\right)^{n\times\tau_1}-\left(1-\xi_{I_{A_1}}{}^+\right)^{n\times\tau_1}}{\left(1+\xi_{I_{A_1}}{}^+\right)^{n\times\tau_1}+\left(1-\xi_{I_{A_1}}{}^+\right)^{n\times\tau_1}}\right],\frac{2\left(I_{A_1}\right)^{n\times\tau_1}}{(2-I_{A_1})^{n\times\tau_1}+\left(I_{A_1}\right)^{n\times\tau_1}}\right\rangle\\ \left\langle\left[\frac{2\left(\xi_{N_{A_1}}{}^-\right)^{n\times\tau_1}}{(2-\xi_{N_{A_1}}{}^-)^{n\times\tau_1}+\left(\xi_{N_{A_1}}{}^-\right)^{n\times\tau_1}},\frac{2\left(\xi_{N_{A_1}}{}^+\right)^{n\times\tau_1}}{(2-\xi_{N_{A_1}}{}^+)^{n\times\tau_1}+\left(\xi_{N_{A_1}}{}^+\right)^{n\times\tau_1}}\right],\frac{\left(1+N_{A_1}\right)^{n\times\tau_1}-\left(1-N_{A_1}\right)^{n\times\tau_1}}{\left(1+N_{A_1}\right)^{n\times\tau_1}+\left(1-N_{A_1}\right)^{n\times\tau_1}}\right\rangle\end{bmatrix}$$

$$A'_1=\langle[0.00004,0.0002],0.9937\rangle,\langle[0.00012,0.00016],0.9817\rangle,\langle[0.9993,0.9997],0.0042\rangle$$

$$A'_2=\begin{bmatrix}\left\langle\left[\frac{\left(1+\xi_{P_{A_2}}{}^-\right)^{n\times\tau_2}-\left(1-\xi_{P_{A_2}}{}^-\right)^{n\times\tau_2}}{\left(1+\xi_{P_{A_2}}{}^-\right)^{n\times\tau_2}+\left(1-\xi_{P_{A_2}}{}^-\right)^{n\times\tau_2}},\frac{\left(1+\xi_{P_{A_2}}{}^+\right)^{n\times\tau_2}-(1-\xi_{P_2}{}^+)^{n\times\tau_2}}{\left(1+\xi_{P_{A_2}}{}^+\right)^{n\times\tau_2}+\left(1-\xi_{P_{A_2}}{}^+\right)^{n\times\tau_2}}\right],\frac{2\left(P_{A_2}\right)^{n\times\tau_2}}{(2-P_{A_2})^{n\times\tau_2}+\left(P_{A_2}\right)^{n\times\tau_2}}\right\rangle\\ \left\langle\left[\frac{\left(1+\xi_{I_{A_2}}{}^-\right)^{n\times\tau_2}-\left(1-\xi_{I_{A_2}}{}^-\right)^{n\times\tau_2}}{\left(1+\xi_{I_{A_2}}{}^-\right)^{n\times\tau_2}+\left(1-\xi_{I_{A_2}}{}^-\right)^{n\times\tau_2}},\frac{\left(1+\xi_{I_{A_2}}{}^+\right)^{n\times\tau_2}-\left(1-\xi_{I_{A_2}}{}^+\right)^{n\times\tau_2}}{\left(1+\xi_{I_{A_2}}{}^+\right)^{n\times\tau_2}+\left(1-\xi_{I_{A_2}}{}^+\right)^{n\times\tau_2}}\right],\frac{2\left(I_{A_2}\right)^{n\times\tau_2}}{(2-I_{A_2})^{n\times\tau_2}+\left(I_{A_2}\right)^{n\times\tau_2}}\right\rangle\\ \left\langle\left[\frac{2\left(\xi_{N_{A_2}}{}^-\right)^{n\times\tau_2}}{(2-\xi_{N_{A_2}}{}^-)^{n\times\tau_2}+\left(\xi_{N_{A_2}}{}^-\right)^{n\times\tau_2}},\frac{2\left(\xi_{N_{A_2}}{}^+\right)^{n\times\tau_2}}{(2-\xi_{N_{A_2}}{}^+)^{n\times\tau_2}+\left(\xi_{N_{A_2}}{}^+\right)^{n\times\tau_2}}\right],\frac{\left(1+N_{A_2}\right)^{n\times\tau_2}-\left(1-N_{A_2}\right)^{n\times\tau_2}}{\left(1+N_{A_2}\right)^{n\times\tau_2}+\left(1-N_{A_2}\right)^{n\times\tau_2}}\right\rangle\end{bmatrix}$$

$$A'_2=\langle[0.1057,0.1234],0.2431\rangle,\langle[0.0616,0.1145],0.2106\rangle,\langle[0.1435,0.3586],0.3109\rangle$$

$$A'_3=\begin{bmatrix}\left\langle\left[\frac{\left(1+\xi_{P_{A_3}}{}^-\right)^{n\times\tau_3}-\left(1-\xi_{P_{A_3}}{}^-\right)^{n\times\tau_3}}{\left(1+\xi_{P_{A_3}}{}^-\right)^{n\times\tau_3}+\left(1-\xi_{P_{A_3}}{}^-\right)^{n\times\tau_3}},\frac{\left(1+\xi_{P_{A_3}}{}^+\right)^{n\times\tau_3}-(1-\xi_{P_3}{}^+)^{n\times\tau_3}}{\left(1+\xi_{P_{A_3}}{}^+\right)^{n\times\tau_3}+\left(1-\xi_{P_{A_3}}{}^+\right)^{n\times\tau_3}}\right],\frac{2\left(P_{A_3}\right)^{n\times\tau_3}}{(2-P_{A_3})^{n\times\tau_3}+\left(P_{A_3}\right)^{n\times\tau_3}}\right\rangle\\ \left\langle\left[\frac{\left(1+\xi_{I_{A_3}}{}^-\right)^{n\times\tau_3}-\left(1-\xi_{I_{A_3}}{}^-\right)^{n\times\tau_3}}{\left(1+\xi_{I_{A_3}}{}^-\right)^{n\times\tau_3}+\left(1-\xi_{I_{A_3}}{}^-\right)^{n\times\tau_3}},\frac{\left(1+\xi_{I_{A_3}}{}^+\right)^{n\times\tau_3}-\left(1-\xi_{I_{A_3}}{}^+\right)^{n\times\tau_3}}{\left(1+\xi_{I_{A_3}}{}^+\right)^{n\times\tau_3}+\left(1-\xi_{I_{A_3}}{}^+\right)^{n\times\tau_3}}\right],\frac{2\left(I_{A_3}\right)^{n\times\tau_3}}{(2-I_{A_3})^{n\times\tau_3}+\left(I_{A_3}\right)^{n\times\tau_3}}\right\rangle\\ \left\langle\left[\frac{2\left(\xi_{N_{A_3}}{}^-\right)^{n\times\tau_3}}{(2-\xi_{N_{A_3}}{}^-)^{n\times\tau_3}+\left(\xi_{N_{A_3}}{}^-\right)^{n\times\tau_3}},\frac{2\left(\xi_{N_{A_3}}{}^+\right)^{n\times\tau_3}}{(2-\xi_{N_{A_3}}{}^+)^{n\times\tau_3}+(\xi_{N_3}{}^+)^{n\times\tau_3}}\right],\frac{\left(1+N_{A_3}\right)^{n\times\tau_3}-\left(1-N_{A_3}\right)^{n\times\tau_3}}{\left(1+N_{A_3}\right)^{n\times\tau_3}+\left(1-N_{A_3}\right)^{n\times\tau_3}}\right\rangle\end{bmatrix}$$

$$A'_3=\langle[0.1203,0.2522],0.0053\rangle,\langle[0.0937,0.3925],0.2382\rangle,\langle[0.1538,0.2914],0.5713\rangle$$

$$A'_4=\begin{bmatrix}\left\langle\left[\frac{\left(1+\xi_{P_{A_4}}{}^-\right)^{n\times\tau_4}-\left(1-\xi_{P_{A_4}}{}^-\right)^{n\times\tau_4}}{\left(1+\xi_{P_{A_4}}{}^-\right)^{n\times\tau_4}+\left(1-\xi_{P_{A_4}}{}^-\right)^{n\times\tau_4}},\frac{\left(1+\xi_{P_{A_4}}{}^+\right)^{n\times\tau_4}-\left(1-\xi_{P_{A_4}}{}^+\right)^{n\times\tau_4}}{\left(1+\xi_{P_{A_4}}{}^+\right)^{n\times\tau_4}+\left(1-\xi_{P_{A_4}}{}^+\right)^{n\times\tau_4}}\right],\frac{2\left(P_{A_4}\right)^{n\times\tau_4}}{\left(2-P_{A_4}\right)^{n\times\tau_4}+\left(P_{A_4}\right)^{n\times\tau_4}}\right\rangle\\ \left\langle\left[\frac{\left(1+\xi_{I_{A_4}}{}^-\right)^{n\times\tau_4}-\left(1-\xi_{I_{A_4}}{}^-\right)^{n\times\tau_4}}{\left(1+\xi_{I_{A_4}}{}^-\right)^{n\times\tau_4}+\left(1-\xi_{I_{A_4}}{}^-\right)^{n\times\tau_4}},\frac{\left(1+\xi_{I_{A_4}}{}^+\right)^{n\times\tau_4}-\left(1-\xi_{I_{A_4}}{}^+\right)^{n\times\tau_4}}{\left(1+\xi_{I_{A_4}}{}^+\right)^{n\times\tau_4}+\left(1-\xi_{I_{A_4}}{}^+\right)^{n\times\tau_4}}\right],\frac{2\left(I_{A_4}\right)^{n\times\tau_4}}{\left(2-I_{A_4}\right)^{n\times\tau_4}+\left(I_{A_4}\right)^{n\times\tau_4}}\right\rangle\\ \left\langle\left[\frac{2\left(\xi_{N_{A_4}}{}^-\right)^{n\times\tau_4}}{\left(2-\xi_{N_{A_4}}{}^-\right)^{n\times\tau_4}+\left(\xi_{N_{A_4}}{}^-\right)^{n\times\tau_4}},\frac{2\left(\xi_{N_{A_4}}{}^+\right)^{n\times\tau_4}}{\left(2-\xi_{N_{A_4}}{}^+\right)^{n\times\tau_4}+\left(\xi_{N_4}{}^+\right)^{n\times\tau_4}}\right],\frac{\left(1+N_{A_4}\right)^{n\times\tau_4}-\left(1-N_{A_4}\right)^{n\times\tau_4}}{\left(1+N_{A_4}\right)^{n\times\tau_4}+\left(1-N_{A_4}\right)^{n\times\tau_4}}\right\rangle\end{bmatrix}$$

$$A'_4=\langle[0.0178,0.0888],0.555\rangle,\langle[0.0534,0.0711],0.000275\rangle,\langle[0.4826,0.5552],0.9527\rangle$$

Now we compute the A_j score values (j=1,2,3,4).

$$sc(A'_1)=0.3360,sc(A'_2)=0.4352,sc(A'_3)=0.3373,sc(A'_4)=0.5936$$

$$\text{Since } sc(A'_4)>sc(A'_2)>sc(A'_3)>sc(A'_1)A'_4>A'_2>A'_3>A'_1$$

$$A'_4>A'_2>A'_3>A'_1$$

Hence

$$A_{\delta(1)}=A'_4=\langle[0.0178,0.0888],0.555\rangle,\langle[0.0534,0.0711],0.000275\rangle,\langle[0.4826,0.5552],0.9527\rangle$$
$$A_{\delta(2)}=A'_2=\langle[0.1057,0.1234],0.2431\rangle,\langle[0.0616,0.1145],0.2106\rangle,\langle[0.1435,0.3586],0.3109\rangle$$
$$A_{\delta(3)}=A'_3=\langle[0.1203,0.2522],0.0053\rangle,\langle[0.0937,0.3925],0.2382\rangle,\langle[0.1538,0.2914],0.5713\rangle$$
$$A_{\delta(4)}=A'_1=\langle[0.00004,0.0002],0.9937\rangle,\langle[0.00012,0.00016],0.9817\rangle,\langle[0.9993,0.9997],0.0042\rangle$$

For $A_j(j=1,2,3)$ we have

$$CPFEHWA(A_{\delta(1)},A_{\delta(2)},A_{\delta(3)},A_{\delta(4)})=\langle[0.0427,0.558],0.0026\rangle,\langle[0.725,0.0839],0.0010\rangle,\langle[0.5677,0.6700],0.8968\rangle$$

In column-2 of Table 8, we have

$$A'_1 = \langle[0.00036{,}0.000769], 0.9961\rangle, \langle[0.00028{,}0.0012], 0.9973\rangle, \langle[0.9946{,}0.9968], 0.0019\rangle$$

$$A'_2 = \langle[0.0088{,}0.040], 0.2431\rangle, \langle[0.0264{,}0.0352], 0.0174\rangle, \langle[0.8877{,}0.9988], 0.7259\rangle$$

$$A'_3 = \langle[0.1602{,}0.1866], 0.1147\rangle, \langle[0.0934{,}0.1734], 0.0925\rangle, \langle[0.0518{,}0.2037], 0.4540\rangle$$

$$A'_4 = \langle[0.0178{,}0.5582], 0.2260\rangle, \langle[0.0711{,}0.3783], 0.0418\rangle, \langle[0.1292{,}0.3728], 0.5989\rangle$$

Now we compute the A_j score values (j=1,2,3,4).

$$sc({A'}_1) = 0.3337, sc({A'}_2) = 0.4720, sc({A'}_3) = 0.4513, sc({A'}_4) = 0.5255$$

Since $sc({A'}_4) > sc({A'}_2) > sc({A'}_3) > sc({A'}_1)$

$$A'_4 > A'_2 > A'_3 > A'_1$$

hence $A_{\delta(1)} = A'_4 = \langle[0.0178{,}0.5582], 0.2260\rangle, \langle[0.0711{,}0.3783], 0.0418\rangle, \langle[0.1292{,}0.3728], 0.5989\rangle$

$$A_{\delta(2)} = A'_2 = \langle[0.0088{,}0.040], 0.2431\rangle, \langle[0.0264{,}0.0352], 0.0174\rangle, \langle[0.8877{,}0.9988], 0.7259\rangle$$

$$A_{\delta(3)} = A'_3 = \langle[0.1602{,}0.1866], 0.1147\rangle, \langle[0.0934{,}0.1734], 0.0925\rangle, \langle[0.0518{,}0.2037], 0.4540\rangle$$

$$A_{\delta(4)} = A'_1 = \langle[0.00036{,}0.000769], 0.9961\rangle, \langle[0.00028{,}0.0012], , 0.9973\rangle, \langle[0.9946{,}0.9968], 0.0019\rangle$$

For $A_j (j = 1{,}2,3{,}4)$ we have

$CPFEHWA(A_{\delta(1)}, A_{\delta(2)}, A_{\delta(3)}, A_{\delta(4)}) = \langle[0.0987{,}0.0996], 0.00373\rangle, \langle[0.659{,}0.0884], 0.0024\rangle, \langle[0.5463, 0.6288], 0.7325\rangle.$

In column-3 of Table 8, we have

$$A'_1 = \langle[0.00004{,}0.0002], 0.9937\rangle, \langle[0.00016{,}0.0012], 0.9817\rangle, \langle[0.9993{,}0.9997], 0.0042\rangle$$

$$A'_2 = \langle[0.1057{,}0.1234], 0.2431\rangle, \langle[0.0616{,}0.1145], 0.2106\rangle, \langle[0.1435{,}0.3586], 0.3109\rangle$$

$$A'_3 = \langle[0.1203{,}0.2522], 0.0053\rangle, \langle[0.0937{,}0.3925], 0.2382\rangle, \langle[0.1538{,}0.2914], 0.5713\rangle$$

$$A'_4 = \langle[0.0178{,}0.5582], 0.2260\rangle, \langle[0.0711{,}0.3783], 0.0418\rangle, \langle[0.1292{,}0.3728], 0.5989\rangle$$

Now we compute the A_j score values (j=1,2,3,4).

$$sc(A'_1) = 0.3359, sc(A'_2) = 0.4352, sc(A'_3) = 0.3937, sc(A'_4) = 0.5255$$

Since $sc({A'}_4) > sc({A'}_2) > sc({A'}_3) > sc({A'}_1)$

$$A'_4 > A'_2 > A'_3 > A'_1$$

Hence

$$A_{\delta(1)} = A'_4 = \langle[0.0178{,}0.5582], 0.2260\rangle, \langle[0.0711{,}0.3783], 0.0418\rangle, \langle[0.1292, 0.3728], 0.5989\rangle$$
$$A_{\delta(2)} = A'_2 = \langle[0.1057{,}0.1234], 0.2431\rangle, \langle[0.0616{,}0.1145], 0.2106\rangle, \langle[0.1435{,}0.3586], 0.3109\rangle$$
$$A_{\delta(3)} = A'_3 = \langle[0.1203{,}0.2522], 0.0053\rangle, \langle[0.0937{,}0.3925], 0.2382\rangle, \langle[0.1538{,}0.2914], 0.5713\rangle$$
$$A_{\delta(4)} = A'_1 = \langle[0.00004{,}0.0002], 0.9937\rangle, \langle[0.00016{,}0.0012], 0.9817\rangle, \langle[0.9993{,}0.9997], 0.0042\rangle$$

For $A_j (j = 1{,}2, 3{,}4)$ we have

$CPFEHWA(A_{\delta(1)}, A_{\delta(2)}, A_{\delta(3)}, A_{\delta(4)}) = \langle[0.1138, 0.2373]\ , 0.062\rangle, \langle[0.659{,}0.0884], 0.0024\rangle, \langle[0.5463, 0.6288], 0.7325\rangle.$

In column-4 of Table 8, we have

$$A'_1 = \langle[0.00048{,}0.00056], 0.9937\rangle, \langle[0.00028{,}0.0052], 0.9930\rangle, \langle[0.9912{,}0.9955], 0.0015\rangle$$

$$A'_2 = \langle[0.0792{,}0.1677], 0.0320\rangle, \langle[0.0616{,}0.2628], 0.3996\rangle, \langle[0.2962{,}0.4605], 0.4024\rangle$$

$$A'_3 = \langle[0.0134{,}0.4418], 0.3458\rangle, \langle[0.0536{,}0.2910], 0.0925\rangle, \langle[0.2208{,}0.5281], 0.4781\rangle$$

$$A'_4 = \langle[0.0178{,}0.0888], 0.555\rangle, \langle[0.0534{,}0.0711], 0.000275\rangle, \langle[0.4826{,}0.5552], 0.9527\rangle$$

Now we compute the A_j score values (j=1,2,3,4).

$$sc({A'}_1) = 0.3336, sc({A'}_2) = 0.3531, sc({A'}_3) = 0.5248, sc({A'}_4) = 0.5936$$

Since $sc({A'}_4) > sc({A'}_3) > sc({A'}_2) > sc({A'}_1)$

$$ {A'}_4 > {A'}_3 > {A'}_2 > {A'}_1 $$

Hence

$$ A_{\delta(1)} = {A'}_4 = \langle [0.0178{,}0.0888], 0.555 \rangle, \langle [0.0534{,}0.0711], 0.000275 \rangle, \langle [0.4826{,}0.5552], 0.9527 \rangle $$
$$ A_{\delta(2)} = {A'}_3 = \langle [0.0134{,}0.4418], 0.3458 \rangle, \langle [0.0536{,}0.2910], 0.0925 \rangle, \langle [0.2208{,}0.5281], 0.4781 \rangle $$
$$ A_{\delta(3)} = {A'}_2 = \langle [0.0792{,}0.1677], 0.0320 \rangle, \langle [0.0616{,}0.2628], 0.3996 \rangle, \langle [0.2962{,}0.4605], 0.4024 \rangle $$
$$ A_{\delta(4)} = {A'}_1 = \langle [0.00048{,}0.00056], 0.9937 \rangle, \langle [0.00028{,}0.0052], 0.9930 \rangle, \langle [0.9912{,}0.9955], 0.0015 \rangle $$

For $A_j (j = 1{,}2,3{,}4)$ we have

$$ CPFEHWA(A_{\delta(1)}, A_{\delta(2)}, A_{\delta(3)}, A_{\delta(4)}) = \langle [0.0762{,}0.0974], 0.0094 \rangle, \langle [0.0548{,}0.0733], 0.000894 \rangle, \langle [0.6433, 0.8778], 0.8808 \rangle. $$

In column-1 of Table 9, we have

$$ {A'}_1 = \langle [0.00028{,}0.00036], 0.9909 \rangle, \langle [0.00024{,}0.0032], 0.9939 \rangle, \langle [0.9992{,}0.9988], 0.0023 \rangle $$
$$ {A'}_2 = \langle [0.1211{,}0.1234], 0.2323 \rangle, \langle [0.1588{,}0.2838], 0.3172 \rangle, \langle [0.5611{,}0.7730], 0.2658 \rangle $$
$$ {A'}_3 = \langle [0.1203{,}0.2522], 0.0053 \rangle, \langle [0.0937{,}0.3925], 0.2382 \rangle, \langle [0.1538{,}0.2914], 0.5713 \rangle $$
$$ {A'}_4 = \langle [0.0178{,}0.0888], 0.555 \rangle, \langle [0.0534{,}0.0711], 0.000275 \rangle, \langle [0.4826{,}0.5552], 0.9527 \rangle $$

N Now we compute the A_j score values (j=1,2,3,4).

$$ sc({A'}_1) = 0.3326,\ sc({A'}_2) = 0.3657, sc({A'}_3) = 0.3937, sc({A'}_4) = 0.5936 $$

Since $sc({A'}_4) > sc({A'}_3) > sc({A'}_2) > sc({A'}_1)$

$$ {A'}_4 > {A'}_3 > {A'}_2 > {A'}_1 $$

hence $A_{\delta(1)} = {A'}_4 = \langle [0.0178{,}0.0888], 0.555 \rangle, \langle [0.0534{,}0.0711], 0.000275 \rangle, \langle [0.4826{,}0.5552], 0.9527 \rangle$

$$ A_{\delta(2)} = {A'}_3 = \langle [0.1203{,}0.2522], 0.0053 \rangle, \langle [0.0937{,}0.3925], 0.2382 \rangle, \langle [0.1538{,}0.2914], 0.5713 \rangle $$

$$ A_{\delta(3)} = {A'}_2 = \langle [0.1211{,}0.1234], 0.2323 \rangle, \langle [0.1588{,}0.2838], 0.3172 \rangle, \langle [0.5611{,}0.7730], 0.2658 \rangle $$

$$ A_{\delta(4)} = {A'}_1 = \langle [0.00028{,}0.00036], 0.9909 \rangle, \langle [0.00024{,}0.0032], 0.9939 \rangle, \langle [0.9992{,}0.9988], 0.0023 \rangle $$

For $A_j (j = 1{,}2,3{,}4)$ we have

$$ CPFEHWA(A_{\delta(1)}, A_{\delta(2)}, A_{\delta(3)}, A_{\delta(4)}) = \langle [0.0427{,}0.0559], 0.0025 \rangle, \langle [0.0725{,}0.0840], 0.0014 \rangle, \langle [0.6767, 0.8786]\ , 0.8966 \rangle $$

In column-2 of Table 9, we have

$$ {A'}_1 = \langle [0.00036{,}0.000769], 0.9961 \rangle, \langle [0.0012{,}0.00028], 0.9973 \rangle, \langle [0.9946{,}0.9968], 0.0019 \rangle $$

$$ {A'}_2 = \langle [0.0616{,}0.0792], 0.1996 \rangle, \langle [0.0528{,}0.0704], 0.2538 \rangle, \langle [0.8665{,}0.7627], 0.4677 \rangle $$

$$ {A'}_3 = \langle [0.1866{,}0.2129], 0.1071 \rangle, \langle [0.2391{,}0.4173], 0.1701 \rangle, \langle [0.3824{,}0.5724], 0.7588 \rangle $$

$$ {A'}_4 = \langle [0.0178{,}0.5582], 0.2260 \rangle, \langle [0.0711{,}0.3783], 0.0418 \rangle, \langle [0.1292{,}0.3728], 0.5989 \rangle $$

Now we compute the A_j score values (j=1,2,3,4).

$$ sc({A'}_1) = 0.3337, sc({A'}_2) = 0.3878, sc({A'}_3) = 0.4381, sc({A'}_4) = 0.5255 $$

Since $sc({A'}_4) > sc({A'}_3) > sc({A'}_2) > sc({A'}_1)$

$$ {A'}_4 > {A'}_3 > {A'}_1 > {A'}_2 $$

hence $A_{\delta(1)} = {A'}_4 = \langle [0.0178{,}0.5582], 0.2260 \rangle, \langle [0.0711{,}0.3783], 0.0418 \rangle, \langle [0.1292{,}0.3728], 0.5989 \rangle$

$$ A_{\delta(2)} = {A'}_3 = \langle [0.1866{,}0.2129], 0.1071 \rangle, \langle [0.2391{,}0.4173], 0.1701 \rangle, \langle [0.3824{,}0.5724], 0.7588 \rangle $$

$$ A_{\delta(3)} = {A'}_2 = \langle [0.0616{,}0.0792], 0.1996 \rangle, \langle [0.0528{,}0.0704], 0.2538 \rangle, \langle [0.8665{,}0.7627], 0.4677 \rangle $$

$$ A_{\delta(4)} = {A'}_1 = \langle [0.00036{,}0.000769], 0.9961 \rangle, \langle [0.0012{,}0.00028], 0.9973 \rangle, \langle [0.9946{,}0.9968], 0.0019 \rangle $$

For $A_j (j = 1{,}2,3{,}4)$ we have

$CPFEHWA(A_{\delta(1)}, A_{\delta(2)}, A_{\delta(3)}, A_{\delta(4)}) = \langle[0.1264{,}0.1623], > 0.0315\rangle, \langle[0.1301{,}0.1837], 0.0176\rangle, \langle[0.6720, 0.6849], 0.7516\rangle.$

In column-3 of Table 9, we have

$$A'_1 = \langle[0.00004{,}0.0002], 0.9937\rangle, \langle[0.0012{,}0.00016], 0.9817\rangle, \langle[0.9993{,}0.9997], 0.0042\rangle$$

$$A'_2 = \langle[0.1057{,}0.1234], 0.2431\rangle, \langle[0.0616{,}0.1145], 0.2106\rangle, \langle[0.1435{,}0.3586], 0.3109\rangle$$

$$A'_3 = \langle[0.0937{,}0.1203], 0.0853\rangle, \langle[0.0803{,}0.1070], 0.1223\rangle, \langle[0.56364{,}0.6493], 0.6483\rangle$$

$$A'_4 = \langle[0.0178{,}0.5582], 0.2260\rangle, \langle[0.0711{,}0.3783], 0.0418\rangle, \langle[0.1292{,}0.3728], 0.5989\rangle$$

Now we compute the A_j score values (j=1,2,3,4).

$$sc(A'_1) = 0.3360, sc(A'_2) = 0.4352, sc(A'_3) = 0.4539, sc(A'_4) = 0.5255$$

Since $sc(A'_4) > sc(A'_2) > sc(A'_3) > sc(A'_1)$

$$A'_4 > A'_3 > A'_2 > A'_1$$

hence

$$A_{\delta(1)} = A'_4 = \langle[0.0178{,}0.5582], 0.2260\rangle, \langle[0.0711{,}0.3783], 0.0418\rangle, \langle[0.1292{,}0.3728], 0.5989\rangle$$
$$A_{\delta(2)} = A'_3 = \langle[0.0937{,}0.1203], 0.0853\rangle, \langle[0.0803{,}0.1070], 0.1223\rangle, \langle[0.56364{,}0.6493], 0.6483\rangle$$
$$A_{\delta(3)} = A'_2 = \langle[0.1057{,}0.1234], 0.2431\rangle, \langle[0.0616{,}0.1145], 0.2106\rangle, \langle[0.1435{,}0.3586], 0.3109\rangle$$
$$A_{\delta(4)} = A'_1 = \langle[0.00004{,}0.0002], 0.9937\rangle, \langle[0.0012{,}0.00016], 0.9817\rangle, \langle[0.9993{,}0.9997], 0.0042\rangle$$

For $A_j(j = 1{,}2, 3{,}4)$ we have

$CPFEHWA(A_{\delta(1)}, A_{\delta(2)}, A_{\delta(3)}, A_{\delta(4)}) = \langle[0.1136{,}0.1678], 0.00311\rangle, \langle[0.0794{,}0.0854], 0.0125\rangle, \langle[0.5680, 0.5801], 0.5291\rangle$

In column-4 of Table 9, we have

$$A'_1 = \langle[0.000563{,}0.00064], 0.9935\rangle, \langle[0.00013{,}0.0072], 0.9949\rangle, \langle[0.9979{,}0.9988], 0.0030\rangle$$

$$A'_2 = \langle[0.0792{,}0.1677], 0.0320\rangle, \langle[0.0616{,}0.2628], 0.3996\rangle, \langle[0.2962{,}0.4605], 0.4024\rangle$$

$$A'_3 = \langle[0.0134{,}0.4418], 0.3458\rangle, \langle[0.0536{,}0.2910], 0.0925\rangle, \langle[0.2208{,}0.5281], 0.4781\rangle$$

$$A'_4 = \langle[0.1242{,}0.1593], 0.0376\rangle, \langle[0.1065{,}0.1417], 0.0603\rangle, \langle[0.4013{,}0.4695], 0.7723\rangle$$

Now we compute the A_j score values (j=1,2,3,4).

$$sc(A'_1) = 0.3327, sc(A'_2) = 0.3536, sc(A'_3) = 0.5248, sc(A'_4) = 0.4755$$

Since $sc(A'_3) > sc(A'_4) > sc(A'_2) > sc(A'_1)$

$$A'_3 > A'_4 > A'_2 > A'_1$$

hence $A_{\delta(1)} = A'_3 = \langle[0.0134{,}0.4418], 0.3458\rangle, \langle[0.0536{,}0.2910], 0.0925\rangle, \langle[0.2208{,}0.5281], 0.4781\rangle$

$$A_{\delta(2)} = A'_4 = \langle[0.1242{,}0.1593], 0.0376\rangle, \langle[0.1065{,}0.1417], 0.0603\rangle, \langle[0.4013{,}0.4695], 0.7723\rangle$$

$$A_{\delta(3)} = A'_2 s = \langle[0.0792{,}0.1677], 0.0320\rangle, \langle[0.0616{,}0.2628], 0.3996\rangle, \langle[0.2962{,}0.4605], 0.4024\rangle$$

$$A_{\delta(4)} = A'_1 = \langle[0.000563{,}0.00064], 0.9935\rangle, \langle[0.00013{,}0.0072], 0.9949\rangle, \langle[0.9979{,}0.9988], 0.0030\rangle$$

For $A_j(j = 1{,}2, 3{,}4)$ we have

$CPFEHWA(A_{\delta(1)}, A_{\delta(2)}, A_{\delta(3)}, A_{\delta(4)}) = \langle[0.0982{,}0.0996], 0.0075\rangle, \langle[0.0686{,}0.0730], 0.0205\rangle, \langle[0.6293, 0.8530], 0.7149\rangle$

In column-1 of Table 10, we have

$$A'_1 = \langle[0.00004{,}0.00014]\ , 0.9975\rangle, \langle[0.000160.00089], 0.9930\rangle, \langle[0.9958{,}0.9996], 0.0016\rangle$$

$$A'_2 = \langle[0.0088{,}0.3019], 0.5209\rangle, \langle[0.1934{,}0.0352], 0.2106\rangle, \langle[0.3792{,}0.7220], 0.3291\rangle$$

$$A'_3 = \langle[0.1203{,}0.2522], 0.0053\rangle, \langle[0.0937{,}0.3925], 0.2382\rangle, \langle[0.1538{,}0.2914], 0.5713\rangle$$

$$A'_4 = \langle[0.0178{,}0.0888], 0.555\rangle, \langle[0.0534{,}0.0711], 0.000275\rangle, \langle[0.4826{,}0.5552], 0.9527\rangle$$

Now we compute the A_j score values (j=1,2,3,4).

$$sc(A_1') = 0.3348, sc(A_2') = 0.51071, sc(A_3') = 0.3937, sc(A_4') = 0.5936$$

since $sc({A'}_4) > sc({A'}_2) > sc({A'}_3) > sc({A'}_1)$

$$A'_4 > A'_2 > A'_3 > A'_1$$

Hence

$$A_{\delta(1)} = A_4' = \langle [0.0178, 0.0888]\,, 0.555\rangle, \langle [0.0534, 0.0711]\,, 0.000275\rangle, \langle [0.4826, 0.5552]\,, 0.9527\rangle$$
$$A_{\delta(2)} = {A'}_2 = \langle [0.0088,0.3019], 0.5209\rangle, \langle [0.1934,0.0352], 0.2106\rangle, \langle [0.3792,0.7220], 0.3291\rangle$$
$$A_{\delta(3)} = {A'}_3 = \langle [0.1203,0.2522], 0.0053\rangle, \langle [0.0937,0.3925], 0.2382\rangle, \langle [0.1538,0.2914], 0.5713\rangle$$
$$A_{\delta(4)} = {A'}_1 = \langle [0.00004,0.00014,0.9975\rangle, \langle [0.000160.00089], 0.9930\rangle, \langle [0.9958,0.9996], 0.0016\rangle$$

For $A_j (j = 1,2,3,4)$ we have

$$CPFEHWA(A_{\delta(1)}, A_{\delta(2)}, A_{\delta(3)}, A_{\delta(4)}) = \langle [0.0512,0.0760], 0.0171\rangle, \langle [0.0388,0.0483], 0.0010\rangle, \langle [0.6373, 0.6669]\,, 0.8554\rangle$$

In column-2 of Table 10, we have

$${A'}_1 = \langle [0.00036,0.000769], 0.9961\rangle, \langle [0.00028,0.0012], 0.9973\rangle, \langle [0.9946,0.9968], 0.0019\rangle$$
$${A'}_2 = \langle [0.1057,0.1234], 0.2431\rangle, \langle [0.0616,0.1145], 0.2106\rangle, \langle [0.1435,0.3586], 0.3109\rangle$$
$$A_3' = \langle [0.1469, 02781]\,, 0.0647\rangle\,, \langle [0.2652, 0.4049]\,, 0.0582\rangle\,, \langle [0.2825, 0.3824]\,, 0.6268\rangle$$
$${A'}_4 = \langle [0.0178,0.5582], 0.2260\rangle, \langle [0.0711,0.3783], 0.0418\rangle, \langle [0.1292,0.3728], 0.5989\rangle$$

Now we compute the A_j score values (j=1,2,3,4).

$$sc(A_1') = 0.3337, sc(A_2') = 0.4352, sc(A_3') = 0.4146, sc(A_4') = 0.5255$$

Since $sc({A'}_4) > sc({A'}_2) > sc({A'}_3) > sc({A'}_1)$

$$A'_4 > A'_2 > A'_3 > A'_1$$

Hence

$$A_{\delta(1)} = {A'}_4 = \langle [0.0178,0.5582], 0.2260\rangle, \langle [0.0711,0.3783], 0.0418\rangle, \langle [0.1292,0.3728], 0.5989\rangle$$
$$A_{\delta(2)} = {A'}_2 = \langle [0.1057,0.1234], 0.2431\rangle, \langle [0.0616,0.1145], 0.2106\rangle, \langle [0.1435,0.3586], 0.3109\rangle$$
$$A_{\delta(3)} = {A'}_3 = \langle [0.1469,0.2781]\,, 0.0647\rangle, \langle [0.2652,0.4049], 0.0582\rangle, \langle [0.2825,0.3824], 0.6268\rangle$$
$$A_{\delta(4)} = {A'}_1 = \langle [0.00036,0.000769], 0.9961\rangle, \langle [0.00028,0.0012], 0.9973\rangle, \langle [0.9946,0.9968], 0.0019\rangle$$

For $A_j (j = 1,2,3,4)$ we have

$$CPFEHWA(A_{\delta(1)}, A_{\delta(2)}, A_{\delta(3)}, A_{\delta(4)}) = \langle [0.1138,0.1378], 0.0265\rangle, \langle [0.0796,0.0854], 0.0081\rangle, \langle [0.5023, 0.5694], 0.6792\rangle.$$

In column-3 of Table 10, we have

$${A'}_1 = \langle [0.00004,0.0002], 0.9937\rangle, \langle [0.0012,0.00016], 0.9817\rangle, \langle [0.9993,0.9997], 0.0042\rangle$$
$${A'}_2 = \langle [0.1057,0.1234], 0.2431\rangle, \langle [0.0616,0.1145], 0.2106\rangle, \langle [0.1435,0.3586], 0.3109\rangle$$
$${A'}_3 = \langle [0.1469,0.2781]\,, 0.0647\rangle, \langle [0.2652,0.4049], 0.0582\rangle, \langle [0.2825,0.3824], 0.6268\rangle$$
$${A'}_4 = \langle [0.0178,0.5582], 0.2260\rangle, \langle [0.0711,0.3783], 0.0418\rangle, \langle [0.1292,0.3728], 0.5989\rangle$$

Now we compute the A_j score values (j=1,2,3,4).

$$sc(A_1') = 0.3360, sc(A_2') = 0.4352, sc(A_3') = 0.4146, sc(A_4') = 0.5255$$

Since $sc({A'}_4) > sc({A'}_2) > sc({A'}_3) > sc({A'}_1)$

$$A'_4 > A'_2 > A'_3 > A'_1$$

Hence

$$A_{\delta(1)} = A'_4 = \langle [0.0178, 0.5582], 0.2260 \rangle, \langle [0.0711, 0.3783], 0.0418 \rangle, \langle [0.1292, 0.3728], 0.5989 \rangle$$
$$A_{\delta(2)} = A'_2 = \langle [0.1057,0.1234], 0.2431 \rangle, \langle [0.0616,0.1145], 0.2106 \rangle, \langle [0.1435,0.3586], 0.3109 \rangle$$
$$A_{\delta(3)} = A'_3 = \langle [0.1469,0.2781], 0.0647 \rangle, \langle [0.2652,0.4049], 0.0582 \rangle, \langle [0.2825,0.3824], 0.6268 \rangle$$
$$A_{\delta(4)} = A'_1 = \langle [0.00004,0.0002], 0.9937 \rangle, \langle [0.0012,0.00016], 0.9817 \rangle, \langle [0.9993,0.9997], 0.0042 \rangle$$

For $A_j (j = 1,2,3,4)$ we have

$$CPFEHWA(A_{\delta(1)}, A_{\delta(2)}, A_{\delta(3)}, A_{\delta(4)}) = \langle [0.1138,0.1378], 0.0219 \rangle, \langle [0.0794,0.0854], 0.0080 \rangle, \langle [0.5029, 0.5699], 0.5291 \rangle$$

In column-4 of Table 10 we have

$$A'_1 = \langle [0.00004,0.00014], 0.9975 \rangle, \langle [0.000160.00089], 0.9930 \rangle, \langle [0.9958,0.9996], 0.0016 \rangle$$
$$A'_2 = \langle [0.0969, 0.1854], 0.0647 \rangle, \langle [0.1765, 0.2748], 0.0582 \rangle, \langle [0.2825, 0.3824], 0.6268 \rangle$$
$$A'_3 = \langle [0.0134,0.4418], 0.3458 \rangle, \langle [0.0536,0.2910], 0.0925 \rangle, \langle [0.2208,0.5281], 0.4781 \rangle$$
$$A'_4 = \langle [0.0178,0.0888], 0.555 \rangle, \langle [0.0534,0.0711], 0.000275 \rangle, \langle [0.4826,0.5552], 0.9527 \rangle$$

Now we compute the A_j score values (j=1,2,3,4).

$$sc(A'_1) = 0.3348, sc(A'_2) = 0.4273, sc(A'_3) = 0.5248, sc(A'_4) = 0.5936$$
$$\text{Since } sc(A'_4) > sc(A'_3) > sc(A'_2) > sc(A'_1)$$
$$A'_4 > A'_3 > A'_2 > A'_1$$

Hence $A_{\delta(1)} = A'_4 = \langle [0.0178, 0.0888], 0.555 \rangle, \langle [0.0534, 0.0711], 0.000275 \rangle, \langle [0.4826, 0.5552], 0.9527 \rangle$

$$A_{\delta(2)} = A'_3 = \langle [0.0134,0.4418], 0.3458 \rangle, \langle [0.0536,0.2910], 0.0925 \rangle, \langle [0.2208,0.5281], 0.4781 \rangle$$
$$A_{\delta(3)} = A'_2 = \langle [0.0969,0.1854], 0.0647 \rangle, \langle [0.1765,0.2748], 0.0582 \rangle, \langle [0.2825,0.3824], 0.6268 \rangle$$
$$A_{\delta(4)} = A'_1 = \langle [0.00004,0.00014], 0.9975 \rangle, \langle [0.000160.00089], 0.9930 \rangle, \langle [0.9958,0.9996], 0.0016 \rangle$$

For $A_j (j = 1,2,3,4)$ we have

$$CPFEHWA(A_{\delta(1)}, A_{\delta(2)}, A_{\delta(3)}, A_{\delta(4)}) = \langle [0.0764,0.0873], 0.00143 \rangle, \langle [0.0548,0.0632], 0.00026 \rangle, \langle [0.6404, 0.6533], 0.8808 \rangle$$

Data availability

The datasets generated and/or analysed during the current study are not publicly available but are available from the corresponding author on reasonable request.

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Author contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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Declarations

Competing interests

The authors declare no conflict of interest.

Additional information

Correspondence and requests for materials should be addressed to Z.M.