



OPEN Periodic structures of solitons and shock wave solutions in the fractional nonlinear Shynaray-IIA equation via a generalized analytical method

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In this research, we utilized the Jacobi elliptic function expansion method to study the truncated M-fractional nonlinear Shynaray-IIA (S-IIA) equation with computational simulations. With the use of a truncated M-fractional derivative, the nonlinear (1 + 1)-dimensional Shynaray-IIA equation can be effectively solved using the Jacobi elliptic function expansion approach. This method yields new exact optical soliton wave solutions that display a range of intriguing features. In particular, we derive the solutions in terms of Jacobi elliptic functions, which are very useful for understanding complex physical events. Additionally, solitary wave and shock wave solutions emerge in the limiting instances for $m \rightarrow 1$ and $m \rightarrow 0$ respectively, offering information on periodic oscillations and localized wave behavior. To aid in understanding and enable a more thorough examination of their features, a number of these solutions are graphically depicted in several dimensions 2D, 3D and contour plots. There are numerous fields in which the Jacobi elliptic function expansion method is used, including fluid dynamics, plasma physics and quantum mechanics. This approach enhances our understanding and makes it possible to predict real-world phenomena more accurately.

Keywords Fractional nonlinear Shynaray-IIA equation, Jacobi elliptic function expansion method, Shock wave solutions, Fractional impacts, Analytical solutions, Solitons

The complexity and importance of real-world phenomena in a range of scientific domains resulted in an ongoing search for dependable and effective analytical methods in nonlinear partial differential equations (NLPDEs). The introduction of fractional calculus, in particular, has broadened the possibilities for mathematical modeling, allowing for the representation of systems with complicated behaviors that are beyond the realm of classical structures, such as anomalous diffusion and viscoelasticity. Nonlinear fractional partial differential equations (NLPDEs) are important in physics because they allow us to simulate a wide range of phenomena across multiple disciplines. For examples the applications include fractional resonant nonlinear Schrödinger equations¹. Fractional order longitudinal wave equation in a magneto-elastic circular rod². Properties and applications of fractional partial differential equations with gamma, beta and hypergeometric functions^{3–5}. Soliton solutions of fractional extended nonlinear Schrödinger equations which are arising in plasma physics and nonlinear optical fiber^{6–12}. Soliton solutions to the nonlinear fractional Kairat-II and Kairat-X equations^{13–15}. Fractional soliton solutions of dynamical system arising in plasma physics¹⁶. Optical solutions of conformable fractional perturbed Gerdjikov-Ivanov equation in mathematical nonlinear optics¹⁷. Soliton solutions and dynamical investigation for

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fractional planer Hamiltonian system of Fokas model¹⁸. Optical solitons with conformable fractional evolution for the (3 + 1)-dimensional Sasa–Satsuma equation¹⁹. Novel soliton solutions of the fractional Riemann wave equation via a mathematical method²⁰. The different kinds of nonlinear partial differential equations and its soliton solutions having significant role in the area of science and engineering such as fiber optics, plasma physics, nonlinear optics, communication system, fluid dynamics, mechanics, and many others^{21–25}. Some examples are: novel optical soliton solutions for the nonlinear complex Ginzburg–Landau equation²⁶. The two-dimensional nonlinear Kadomtsev–Petviashvili-modified equal width (KP-MEW) equation²⁷. Using the extended modified rational expansion method and symbolic computation, multiple solitary wave solutions are constructed for the nonlinear two-dimensional Jaulent–Miodek Hierarchy (JMh) equation²⁸. Various soliton solutions for the complex nonlinear Kuralay–IIA equation were obtained using utilizing an enhanced F-expansion method²⁹. The nonlinear longitudinal wave equation (LWE) is analyzed utilizing the auxiliary equation mapping method and the extended direct algebraic method³⁰ and other kinds of nonlinear partial differential equations^{31–34}. Currently, the significance of soliton theory has grown exponentially, as it has become a special topic within nonlinear science. Solitons have garnered profound importance due to their extraordinary properties. One of their most important features is their ability to maintain both shape and velocity after interaction and stability. Solitons have various forms including, dark, periodic, singular, dark bright, kink, anti-kink, and many more. There are different methods to derive these forms of soliton solutions, such as the auxiliary equation method^{35–38}, sardar sub equation method and riccati equation mapping method³⁹, extended simple equation method^{40–42}, Darbous and Hirota technique⁴³, Improved F-expansion method^{44–46}, extended modified rational expansion method⁴⁷, modified auxiliary equation method⁴⁸, and other analytical techniques^{49–55}. Some other useful techniques to examine the nonlinear models^{56–59}. Consider the nonlinear (1 + 1)-dimensional Shynaray-IIA equation in^{60,61}.

$$\begin{aligned}ig_t + g_{x,t} - i(hg)_x &= 0 \\if_t - f_{x,t} - i(hf)_x &= 0 \\h_x - 2q(fg)_t &= 0\end{aligned}\quad (1)$$

Gives the S-IIA Eq. (1). When $f = \epsilon \bar{g}$ ($\epsilon = \pm 1$), the S-IIA equation takes on the following form, as shown in⁶¹:

$$\begin{aligned}ig_t + g_{x,t} - i(hg)_x &= 0 \\h_x - 2\epsilon q(|g|^2)_t &= 0\end{aligned}\quad (2)$$

The truncated M-fractional Shynaray-IIA (S-IIA) equation is presented as:

$$\begin{aligned}iD_{M,t}^{\alpha,\Upsilon}g + D_{M,x,t}^{2\alpha,\Upsilon}g - iD_{M,x}^{\alpha,\Upsilon}(hg) &= 0 \\iD_{M,t}^{\alpha,\Upsilon}f - D_{M,x,t}^{2\alpha,\Upsilon}f - iD_{M,x}^{\alpha,\Upsilon}(hf) &= 0 \\D_{M,x}^{\alpha,\Upsilon}h - 2qD_{M,t}^{\alpha,\Upsilon}(|g|^2) &= 0\end{aligned}\quad (3)$$

Presents the S-IIA equation as Eq. (3). Under the condition $f = \epsilon \bar{g}$ ($\epsilon = \pm 1$), the S-IIA equation is simplified to:

$$\begin{aligned}iD_{M,t}^{\alpha,\Upsilon}g + D_{M,x,t}^{2\alpha,\Upsilon}g - iD_{M,x}^{\alpha,\Upsilon}(hg) &= 0 \\D_{M,x}^{\alpha,\Upsilon}h - 2\epsilon qD_{M,t}^{\alpha,\Upsilon}(|g|^2) &= 0\end{aligned}\quad (4)$$

where

$$D_{M,x}^{\alpha,\Upsilon}g(x) = \lim_{\tau \rightarrow 0} \frac{g(xE_{\Upsilon}(\tau x^{1-\alpha})) - g(x)}{\tau}, 0 < \alpha \leq 1, \Upsilon \in (0, \infty) \quad (5)$$

Here, $E_{\Upsilon}(\cdot)$ represents a truncated Mittag–Leffler (TML) function cited in⁶³.

In general, the system consists of a set of interconnected nonlinear fractional partial differential equations characterized by real constants p, q and ϵ , where $p^2 = 2q^2$. The function $g = g(x, t)$ is a complex-valued, whereas $h = h(x, t)$ is a real-valued function. The model under consideration, the Shynaray-IIA equation, is a coupled system of NLFPEs that is integrable and possesses soliton solutions. The above equation represents the integrable motion of space curves and finds applications in nonlinear optics, water waves, plasma physics, and various other modern scientific domains.

Previous studies have tackled this model using a limited array of techniques; some optical wave solutions have been obtained through an improved modified Sardar sub-equation method⁶⁰. While exact optical wave solutions have been achieved by employing the ϕ^6 -model expansion method⁶¹. In this paper, we investigate new exact optical soliton wave solutions for the truncated M-fractional nonlinear (1 + 1)-dimensional Shynaray-IIA equation by applying the Jacobi elliptic function expansion method. Specifically, we obtain the solutions expressed in terms of Jacobi elliptic functions, which are especially helpful in the comprehension of intricate physical phenomena. Furthermore, in the limiting sense for $m \rightarrow 1$ and $m \rightarrow 0$, solitary wave and shock wave solutions arise to provide information on localized wave behavior and insight into periodic oscillation. The flow chart of our proposed work is illustrated in Fig. 1. This flow chart represented to our investigation step by step.

The paper consists of various sections, in section 1, we discussed introduction and mathematical model truncated M-fractional Shynaray-IIA equation, in section 2, we discussed the main descriptions of the Jacobi elliptic function expansion methodology, in section 3, we utilized the Jacobi Elliptic Function Expansion Method to find the new exact optical soliton wave solutions for truncated M-fractional nonlinear $(1 + 1)$ -dimensional Shynaray-IIA equation, in section 4, we explained the results and discussion of the obtained solutions and the suggested approach with other traditional methods, in section 5, we plotted the graphs of the solutions in various dimensions, in section 6, we addressed the conclusion of the study.

The details of the JEFEM

In this section, we simply discussed the main description of the Jacobi elliptic function expansion method.

Generally, the nonlinear partial differential equations have the following mathematical format.

$$F(u, u^2 u_x u_t, u_{xx}, u_{tt}, u_{xt}, \dots) = 0 \quad (6)$$

Transforming Eq. (6) into a nonlinear ordinary differential equation (NODE).

$$S(u, u', u'', u''', \dots) = 0 \quad (7)$$

Employing the given wave transformations.

$$u(x, t) = u(\xi), \xi = k(x - ct) \quad (8)$$

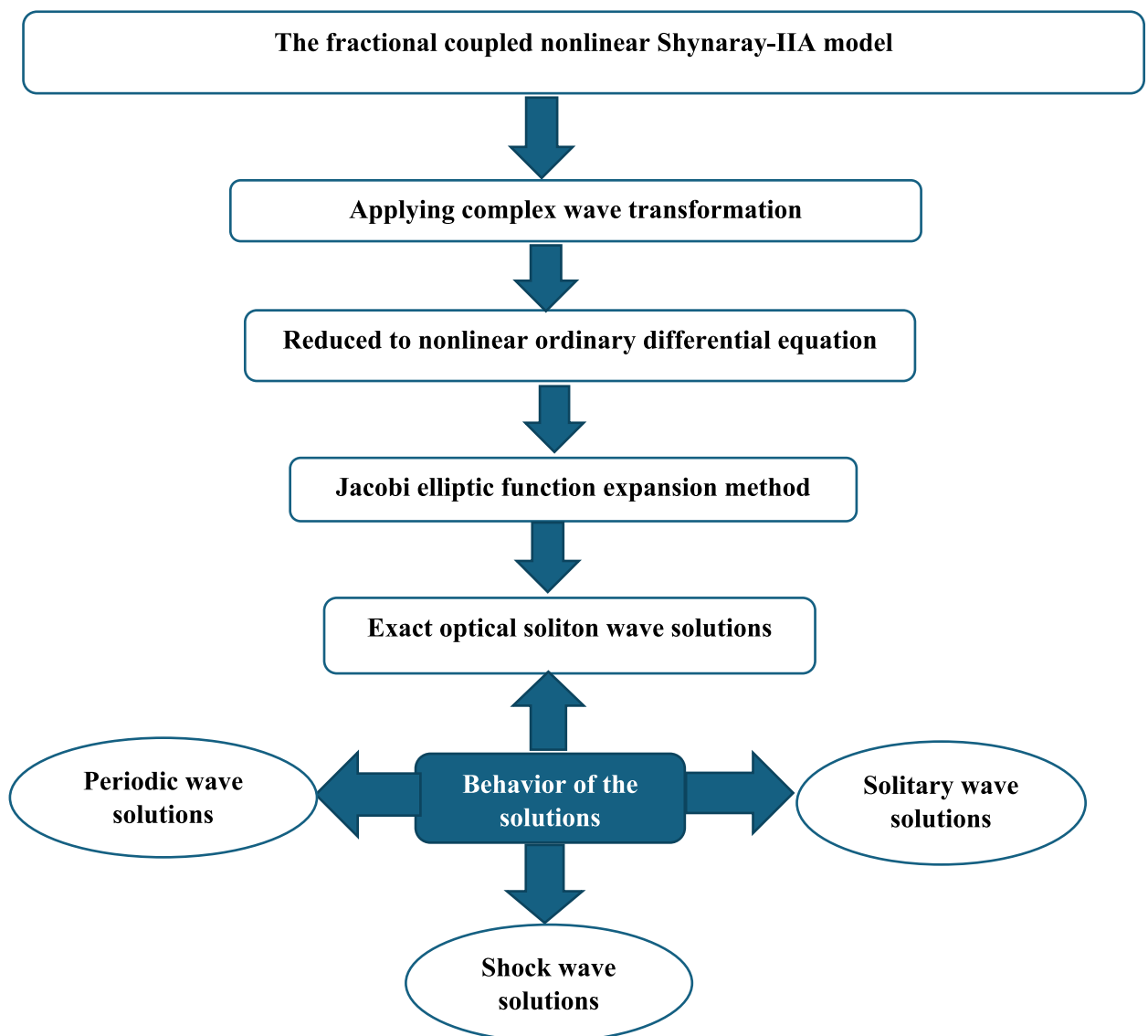


Fig. 1. Work flow chart.

Here, “ k ” represents wave number and “ c ” represents wave speed.

Moreover, the main goal of using this extended indirect method is to increase the probability of finding solutions to an additional ordinary differential equation (the first kind in the parameter Jacobian equation). Creating a large number of Jacobian elliptic function solutions for the given equation is the objective. This effort aims to produce a significant number of Jacobian elliptic function solutions. Additionally, it is also possible to visualize the auxiliary equation.

$$(F')^2(\xi) = m_2 F^4(\xi) + m_1 F^2(\xi) + m_0 \quad (9)$$

In this context, where $F' = \frac{dF}{d\xi}$, $\xi = \xi(x, t)$, Table 1 contains the solutions of Eq. (9) With $i^2 = -1$. The Jacobi Elliptic Functions $sn\xi = sn(\xi, r)$, $cn\xi = cn(\xi, r)$ and $dn\xi = dn(\xi, r)$ are involved, where r represents the modulus within the range $0 < r < 1$. The Eq. (9) could yield numerous solutions for the function based on the selected values for m_2 , m_1 , and m_0 .

As r approaches 1 and 0, respectively, a limiting procedure reduces the Jacobi elliptic functions, as shown in Table 2, to hyperbolic and trigonometric functions.

This leads to a variety of solutions for the given problem, including Jacobian elliptic function, hyperbolic, and trigonometric solutions. It is possible to express $u(\xi)$ as a finite series of Jacobi elliptic functions by using the Jacobi elliptic function expansion method.

$$u(\xi) = \sum_{i=0}^n a_i F^i(\xi) \quad (10)$$

Here, $F(\xi)$ represents the solution of the nonlinear ordinary differential Eq. (9) and n, a_i (where $i = 0, 1, 2, 3, \dots$) are constants that we determined later,

	m_2	m_1	m_0	F
1	r^2	$-(1+r^2)$	1	sn, cd
2	r^2	$2r^2 - 1$	$1 - r^2$	cn
3	-1	$2 - r^2$	$r^2 - 1$	dn
4	1	$-(1+r^2)$	r^2	ns, dc
5	$1 - r^2$	$2r^2 - 1$	-1	nd
6	$r^2 - 1$	$2 - r^2$	-1	nd
7	$1 - r^2$	$2 - r^2$	1	sc
8	$-r^2(1 - r^2)$	$2r^2 - 1$	1	sd
9	1	$2 - r^2$	$1 - r^2$	cs
10	1	$2r^2 - 1$	$r^2(1 - r^2)$	ds
11	$\frac{-1}{4}$	$\frac{r^2+1}{2}$	$\frac{-(1-r^2)^2}{4}$	$rcn \mp dn$
12	$\frac{1}{4}$	$\frac{-2r^2+1}{2}$	$\frac{1}{4}$	$ns \mp cs$
13	$\frac{1-r^2}{4}$	$\frac{r^2+1}{2}$	$\frac{1-r^2}{4}$	$nc \mp sc$
14	$\frac{1}{4}$	$\frac{r^2-2}{2}$	$\frac{r^4}{4}$	$ns \mp ds$
15	$\frac{r^2}{4}$	$\frac{r^2-2}{2}$	$\frac{r^2}{4}$	$sn \mp icn, \frac{sn}{\sqrt{1-r^2sn \mp cn}}$
16	$\frac{1}{4}$	$\frac{1-2r^2}{2}$	$\frac{1}{4}$	$rcn \mp idn, \frac{sn}{1 \mp cn}$
17	$\frac{r^2}{4}$	$\frac{r^2-2}{2}$	$\frac{1}{4}$	$\frac{sn}{1 \mp dn}$
18	$\frac{r^2-1}{4}$	$\frac{r^2+1}{2}$	$\frac{r^2-1}{4}$	$\frac{dn}{1 \mp rsn}$
19	$\frac{1-r^2}{4}$	$\frac{r^2+1}{2}$	$\frac{-r^2+1}{4}$	$\frac{cn}{1 \mp sn}$
20	$\frac{(1-r^2)^2}{4}$	$\frac{r^2+1}{2}$	$\frac{1}{4}$	$\frac{sn}{dn \mp cn}$
21	$\frac{r^4}{4}$	$\frac{r^2-2}{2}$	$\frac{1}{4}$	$\frac{cn}{\sqrt{1-r^2 \mp dn}}$

Table 1. The Eq. (9) could yield numerous solutions for the function F based on the selected values for m_2 , m_1 and m_0 ^{62,63}.

	$r \rightarrow 1$					$r \rightarrow 1$	$r \rightarrow 0$
1	snu	$\tanh u$	$\sin u$	7	dcu	1	$\sec u$
2	cnu	$\operatorname{sech} u$	$\cos u$	8	ncu	$\cosh u$	$\sec u$
3	dnu	$\operatorname{sech} u$	1	9	scu	$\sinh u$	$\tan u$
4	cdu	1	$\cos u$	10	nsu	$\coth u$	$\csc u$
5	sdu	$\sinh u$	$\sin u$	11	dsu	$\operatorname{csch} u$	$\csc u$
6	ndu	$\cosh u$	1	12	csu	$\operatorname{csch} u$	$\cot u$

Table 2. In terms of constraining aspects for both $r \rightarrow 1$ and $r \rightarrow 0$, the Jacobi elliptic functions simplify to^{63,64}.

$$O\left(\frac{d^p u}{d\xi^p}\right) = n + p, p = 0, 1, 2, 3, \dots \quad (11)$$

And the highest-order nonlinear term is,

$$O\left(u^q \frac{d^p u}{d\xi^p}\right) = (q + 1)n + p, p = 0, 1, 2, 3, \dots, q = 1, 2, 3, \dots \quad (12)$$

We obtain a set of algebraic equations for the coefficients, a_i ($i = 0, 1, 2, 3, \dots$) by substituting Eq. (10) into Eq. (7) and setting the power of F to zero. We can find the desired solution by solving this set using the given value of m_2, m_1 and m_0 from Table 1. By utilizing the previously mentioned technique of merging data with the auxiliary equation, Exact solutions for Eq. (6) can be obtained.

Utilizing mathematical methods to solve the governing equation

Considering the transformation below:

$$g(x, t) = G(\xi) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(-\theta x^\alpha + \tau t^\alpha) + \vartheta\right)}, h(x, t) = H(\xi), \xi = \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \lambda t^\alpha) \quad (13)$$

where $G(\xi)$, θ , τ , ϑ and λ represents pulse shape, frequency of the soliton, wave number of the soliton, phase constant and velocity of the soliton respectively.

$$D_{M,t}^{\alpha,\Upsilon} g = (i\tau G - \lambda G') \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(-\theta x^\alpha + \tau t^\alpha) + \vartheta\right)} \quad (14)$$

$$D_{M,x}^{2\alpha,\Upsilon} g = (-\lambda G'' + \theta i \lambda G' + i\tau G' + \theta \tau G) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(-\theta x^\alpha + \tau t^\alpha) + \vartheta\right)} \quad (15)$$

$$D_{M,x}^{\alpha,\Upsilon} (hg) = (HG' + GH' - \theta i GH) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(-\theta x^\alpha + \tau t^\alpha) + \vartheta\right)} \quad (16)$$

$$D_{M,x}^{\alpha,\Upsilon} h = H' \quad (17)$$

$$D_{M,t}^{\alpha,\Upsilon} (|g|^2) = -2\lambda GG' \quad (18)$$

By substituting Eqs. (14), (15) and (16) into the first equation and Eqs. (17) and (18) into the second equation of system Eq. (4), we find:

$$\begin{aligned} \lambda G'' + \tau(1 - \theta)G + \theta HG + i(\tau - \lambda(1 - \theta))G' - HG' - H'G &= 0, \\ H' + \frac{2\lambda\epsilon p^2}{q}GG' &= 0 \end{aligned} \quad (19)$$

Integrating the second equation of the system in Eq. (19), we find:

$$H(\xi) = -\frac{\lambda\epsilon p^2}{q}G^2(\xi) \quad (20)$$

By substituting Eq. (20) into the first equation of Eq. (19), we get the real part given as:

$$\lambda G'' + \tau(1 - \theta)G - \frac{\theta\lambda\epsilon p^2}{q}G^3 = 0 \quad (21)$$

The imaginary part is given as:

$$(\tau - \lambda(1 - \theta))G' + \frac{3\lambda\epsilon p^2}{q}G^2G' = 0 \quad (22)$$

The constrain condition appears as follows:

$$\tau = \lambda(1 - \theta) \quad (23)$$

By using the Homogeneous balancing procedure, balancing the terms G^3 and G'' in Eq. (21), we determine the value of $n = 1$.

Now, we will solve Eq. (21) using the previously mentioned approach. For $n = 1$ the solution of Eq. (21) can be expressed as:

$$G(\xi) = a_0 + a_1F(\xi) \quad (24)$$

Substituting Eq. (24) along with Eq. (9) into Eq. (21), we get:

$$a_0 = 0, a_1 = \frac{\sqrt{\frac{2qr^2}{\theta\epsilon}}}{p}, \lambda = -\frac{\tau(1 - \theta)}{m_1} \quad (25)$$

Jacobian elliptic function solutions

By utilizing the data presented in Tables 1 and 2, and combining the corresponding values as per Eq. (24), we can obtain the Jacobi elliptic function solutions which are in periodic nature for Eq. (21) as outlined below.

$$g_{1,1}(x, t) = \pm \frac{\sqrt{\frac{2qr^2}{\theta\epsilon}}}{p} \operatorname{sn} \left(\frac{\Gamma(1 + \Upsilon)}{\alpha} \left(x^\alpha - \frac{\tau(1 - \theta)}{(1 + r^2)} t^\alpha \right) \right) \times e^{i \left(\frac{\Gamma(1 + \Upsilon)}{\alpha} (-\theta x^\alpha + \tau t^\alpha) + \vartheta \right)} \quad (26)$$

$$h_{1,1}(x, t) = -\frac{\lambda\epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{2qr^2}{\theta\epsilon}}}{p} \operatorname{sn} \left(\frac{\Gamma(1 + \Upsilon)}{\alpha} \left(x^\alpha - \frac{\tau(1 - \theta)}{(1 + r^2)} t^\alpha \right) \right) \right)^2 \quad (27)$$

$$g_{1,2}(x, t) = \pm \frac{\sqrt{\frac{2qr^2}{\theta\epsilon}}}{p} \operatorname{cd} \left(\frac{\Gamma(1 + \Upsilon)}{\alpha} \left(x^\alpha - \frac{\tau(1 - \theta)}{(1 + r^2)} t^\alpha \right) \right) \times e^{i \left(\frac{\Gamma(1 + \Upsilon)}{\alpha} (-\theta x^\alpha + \tau t^\alpha) + \vartheta \right)} \quad (28)$$

$$h_{1,2}(x, t) = -\frac{\lambda\epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{2qr^2}{\theta\epsilon}}}{p} \operatorname{cd} \left(\frac{\Gamma(1 + \Upsilon)}{\alpha} \left(x^\alpha - \frac{\tau(1 - \theta)}{(1 + r^2)} t^\alpha \right) \right) \right)^2 \quad (29)$$

$$g_{1,3}(x, t) = \pm \frac{\sqrt{\frac{-2qr^2}{\theta\epsilon}}}{p} \operatorname{cn} \left(\frac{\Gamma(1 + \Upsilon)}{\alpha} \left(x^\alpha + \frac{\tau(1 - \theta)}{(2r^2 - 1)} t^\alpha \right) \right) \times e^{i \left(\frac{\Gamma(1 + \Upsilon)}{\alpha} (-\theta x^\alpha + \tau t^\alpha) + \vartheta \right)} \quad (30)$$

$$h_{1,3}(x, t) = -\frac{\lambda\epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{-2qr^2}{\theta\epsilon}}}{p} \operatorname{cn} \left(\frac{\Gamma(1 + \Upsilon)}{\alpha} \left(x^\alpha + \frac{\tau(1 - \theta)}{(2r^2 - 1)} t^\alpha \right) \right) \right)^2 \quad (31)$$

$$g_{1,4}(x, t) = \pm \frac{\sqrt{\frac{-2q}{\theta\epsilon}}}{p} \operatorname{dn} \left(\frac{\Gamma(1 + \Upsilon)}{\alpha} \left(x^\alpha + \frac{\tau(1 - \theta)}{(2 - r^2)} t^\alpha \right) \right) \times e^{i \left(\frac{\Gamma(1 + \Upsilon)}{\alpha} (-\theta x^\alpha + \tau t^\alpha) + \vartheta \right)} \quad (32)$$

$$h_{1,4}(x, t) = -\frac{\lambda\epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{-2q}{\theta\epsilon}}}{p} \operatorname{dn} \left(\frac{\Gamma(1 + \Upsilon)}{\alpha} \left(x^\alpha + \frac{\tau(1 - \theta)}{(2 - r^2)} t^\alpha \right) \right) \right)^2 \quad (33)$$

$$g_{1,5}(x, t) = \pm \frac{\sqrt{\frac{2q}{\theta\epsilon}}}{p} \operatorname{ns} \left(\frac{\Gamma(1 + \Upsilon)}{\alpha} \left(x^\alpha - \frac{\tau(1 - \theta)}{(1 + r^2)} t^\alpha \right) \right) \times e^{i \left(\frac{\Gamma(1 + \Upsilon)}{\alpha} (-\theta x^\alpha + \tau t^\alpha) + \vartheta \right)} \quad (34)$$

$$h_{1,5}(x, t) = -\frac{\lambda\epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{2q}{\theta\epsilon}}}{p} \operatorname{ns} \left(\frac{\Gamma(1 + \Upsilon)}{\alpha} \left(x^\alpha - \frac{\tau(1 - \theta)}{(1 + r^2)} t^\alpha \right) \right) \right)^2 \quad (35)$$

$$g_{1,6}(x, t) = \pm \frac{\sqrt{\frac{2q}{\theta\epsilon}}}{p} \operatorname{dc} \left(\frac{\Gamma(1 + \Upsilon)}{\alpha} \left(x^\alpha - \frac{\tau(1 - \theta)}{(1 + r^2)} t^\alpha \right) \right) \times e^{i \left(\frac{\Gamma(1 + \Upsilon)}{\alpha} (-\theta x^\alpha + \tau t^\alpha) + \vartheta \right)} \quad (36)$$

$$h_{1,6}(x, t) = -\frac{\lambda \epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{2q}{\theta \epsilon}}}{p} dc \left(\frac{\Gamma(1 + \Upsilon)}{\alpha} \left(x^\alpha - \frac{\tau(1 - \theta)}{(1 + r^2)} t^\alpha \right) \right) \right)^2 \quad (37)$$

$$g_{1,7}(x, t) = \pm \frac{\sqrt{\frac{2q(1-r^2)}{\theta\epsilon}}}{p} nc \left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{\tau(1-\theta)}{(2r^2-1)} t^\alpha \right) \right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(-\theta x^\alpha + \tau t^\alpha) + \vartheta\right)} \quad (38)$$

$$h_{1,7}(x, t) = -\frac{\lambda \epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{2q(1-r^2)}{\theta \epsilon}}}{p} n c \left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{\tau(1-\theta)}{(2r^2-1)} t^\alpha \right) \right) \right)^2 \quad (39)$$

$$g_{1,8}(x, t) = \pm \frac{\sqrt{\frac{2q(1-r^2)}{\theta\epsilon}}}{p} \text{sc} \left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{\tau(1-\theta)}{(2-r^2)} t^\alpha \right) \right) \times e^{i \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (-\theta x^\alpha + \tau t^\alpha) + \vartheta \right)} \quad (40)$$

$$h_{1,8}(x, t) = -\frac{\lambda \epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{2q(1-r^2)}{\theta \epsilon}}}{p} sc \left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{\tau(1-\theta)}{(2-r^2)} t^\alpha \right) \right) \right)^2 \quad (41)$$

$$g_{1,9}(x, t) = \pm \frac{\sqrt{\frac{2q}{\theta\epsilon}}}{p} c_S \left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{\tau(1-\theta)}{(2-r^2)} t^\alpha \right) \right) \times e^{i \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (-\theta x^\alpha + \tau t^\alpha) + \vartheta \right)} \quad (42)$$

$$h_{1,9}(x, t) = -\frac{\lambda \epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{2q}{\theta \epsilon}}}{p} c_s \left(\frac{\Gamma(1 + \Upsilon)}{\alpha} \left(x^\alpha + \frac{\tau(1 - \theta)}{(2 - r^2)} t^\alpha \right) \right) \right)^2 \quad (43)$$

$$g_{1,10}(x,t) = \pm \frac{\sqrt{\frac{2q}{\theta\epsilon}}}{p} ds \left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{\tau(1-\theta)}{(2r^2-1)} t^\alpha \right) \right) \times e^{i \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (-\theta x^\alpha + \tau t^\alpha) + \vartheta \right)} \quad (44)$$

$$h_{1,10}(x, t) = -\frac{\lambda \epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{2q}{\theta \epsilon}}}{p} ds \left(\frac{\Gamma(1 + \Upsilon)}{\alpha} \left(x^\alpha + \frac{\tau(1 - \theta)}{(2r^2 - 1)} t^\alpha \right) \right) \right)^2 \quad (45)$$

$$g_{1,11}(x, t) = \pm \frac{\sqrt{2\theta\epsilon}}{p} \left(r c n \left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(r^2+1)} t^\alpha \right) \right) \pm d n \left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(r^2+1)} t^\alpha \right) \right) \right) \times e^{i \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (-\theta x^\alpha + \tau t^\alpha) + \vartheta \right)} \quad (46)$$

$$h_{1,11}(x, t) = -\frac{\lambda \epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{q}{2\theta\epsilon}} rcn}{p} \left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(r^2+1)} t^\alpha \right) \right) \pm dn \left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(r^2+1)} t^\alpha \right) \right) \right)^2 \quad (47)$$

$$g_{1,12}(x,t) = \pm \frac{\sqrt{\frac{9}{26\epsilon}}}{2b\epsilon} \left(ns \left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(-2r^2+1)} t^\alpha \right) \right) \pm cs \left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(-2r^2+1)} t^\alpha \right) \right) \right) \times e^{i \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (-\theta x^\alpha + \tau t^\alpha) + \vartheta \right)} \quad (48)$$

$$h_{1,12}(x, t) = -\frac{\lambda \epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{q}{2\theta\epsilon}}}{p} n s \left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(-2r^2+1)} t^\alpha \right) \right) \pm c s \left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(-2r^2+1)} t^\alpha \right) \right) \right)^2 \quad (49)$$

$$g_{1,13}(x, t) = \pm \frac{\sqrt{\frac{q(1-r^2)}{2\theta c}}}{p} \left(nc \left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(r^2+1)} t^\alpha \right) \right) \pm sc \left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(r^2+1)} t^\alpha \right) \right) \right) \times e^{i \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (-\theta x^\alpha + \tau t^\alpha) + \theta \right)} \quad (50)$$

$$h_{1,13}(x, t) = -\frac{\lambda \epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{q(1-r^2)}{2\theta\epsilon}}}{p} n c \left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(r^2+1)} t^\alpha \right) \right) \pm s c \left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(r^2+1)} t^\alpha \right) \right) \right)^2 \quad (51)$$

$$g_{1,14}(x,t) = \pm \sqrt{\frac{q}{2\theta\epsilon}} \left(ns \left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(r^2-2)} t^\alpha \right) \right) \pm ds \left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(r^2-2)} t^\alpha \right) \right) \right) \times e^{i \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (-\theta x^\alpha + \tau t^\alpha) + \vartheta \right)} \quad (52)$$

$$h_{1,14}(x, t) = -\frac{\lambda \epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{q}{2\theta \epsilon}}}{p} n_s \left(\frac{\Gamma(1 + \Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1 - \theta)}{(r^2 - 2)} t^\alpha \right) \right) \pm ds \left(\frac{\Gamma(1 + \Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1 - \theta)}{(r^2 - 2)} t^\alpha \right) \right) \right)^2 \quad (53)$$

$$g_{1,15}(x, t) = \pm \frac{\sqrt{\frac{q\tau^2}{2\theta c}}}{p} \left(\operatorname{sn} \left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(\tau^2-2)} t^\alpha \right) \right) \mp i \operatorname{cn} \left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(\tau^2-2)} t^\alpha \right) \right) \right) \times e^{i \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (-\theta x^\alpha + \tau t^\alpha) + \vartheta \right)} \quad (54)$$

$$h_{1,15}(x, t) = -\frac{\lambda \epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{gr^2}{2\theta\epsilon}}}{p} sn \left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(r^2-2)} t^\alpha \right) \right) \mp icn \left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(r^2-2)} t^\alpha \right) \right) \right)^2 \quad (55)$$

$$g_{1,16}(x, t) = \pm \frac{\sqrt{\frac{qr^2}{2\theta\epsilon}}}{p} \frac{sn\left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(r^2-2)} t^\alpha\right)\right)}{\sqrt{1-r^2 sn\left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(r^2-2)} t^\alpha\right)\right) \mp cn\left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(r^2-2)} t^\alpha\right)\right)}} \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(-\theta x^\alpha + \tau t^\alpha) + \vartheta\right)} \quad (56)$$

$$h_{1,16}(x, t) = -\frac{\lambda\epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{qr^2}{2\theta\epsilon}}}{p} \frac{sn\left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(r^2-2)} t^\alpha\right)\right)}{\sqrt{1-m^2 sn\left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(r^2-2)} t^\alpha\right)\right) \mp cn\left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(r^2-2)} t^\alpha\right)\right)}} \right)^2, \quad (57)$$

$$g_{1,17}(x, t) = \pm \frac{\sqrt{\frac{q}{2\theta\epsilon}}}{p} \left(r sn\left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(1-2r^2)} t^\alpha\right)\right) \mp idn\left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(1-2r^2)} t^\alpha\right)\right) \right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(-\theta x^\alpha + \tau t^\alpha) + \vartheta\right)} \quad (58)$$

$$h_{1,17}(x, t) = -\frac{\lambda\epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{q}{2\theta\epsilon}}}{p} r sn\left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(1-2r^2)} t^\alpha\right)\right) \mp idn\left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(1-2r^2)} t^\alpha\right)\right) \right)^2 \quad (59)$$

$$g_{1,18}(x, t) = \pm \frac{\sqrt{\frac{q}{2\theta\epsilon}}}{p} \frac{sn\left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(1-2r^2)} t^\alpha\right)\right)}{1 \mp dn\left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(1-2r^2)} t^\alpha\right)\right)} \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(-\theta x^\alpha + \tau t^\alpha) + \vartheta\right)} \quad (60)$$

$$h_{1,18}(x, t) = -\frac{\lambda\epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{q}{2\theta\epsilon}}}{p} \frac{sn\left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(1-2r^2)} t^\alpha\right)\right)}{1 \mp dn\left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(1-2r^2)} t^\alpha\right)\right)} \right)^2 \quad (61)$$

$$g_{1,19}(x, t) = \pm \frac{\sqrt{\frac{qr^2}{2\theta\epsilon}}}{p} \frac{sn\left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(r^2-2)} t^\alpha\right)\right)}{1 \mp dn\left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(r^2-2)} t^\alpha\right)\right)} \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(-\theta x^\alpha + \tau t^\alpha) + \vartheta\right)} \quad (62)$$

$$h_{1,19}(x, t) = -\frac{\lambda\epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{qr^2}{2\theta\epsilon}}}{p} \frac{sn\left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(r^2-2)} t^\alpha\right)\right)}{1 \mp dn\left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(r^2-2)} t^\alpha\right)\right)} \right)^2 \quad (63)$$

$$g_{1,20}(x, t) = \pm \frac{\sqrt{\frac{q(1-r^2)}{2\theta\epsilon}}}{p} \frac{cn\left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(r^2+1)} t^\alpha\right)\right)}{1 \mp sn\left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(r^2+1)} t^\alpha\right)\right)} \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(-\theta x^\alpha + \tau t^\alpha) + \vartheta\right)} \quad (64)$$

$$h_{1,20}(x, t) = -\frac{\lambda\epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{q(1-r^2)}{2\theta\epsilon}}}{p} \frac{cn\left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(r^2+1)} t^\alpha\right)\right)}{1 \mp sn\left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(r^2+1)} t^\alpha\right)\right)} \right)^2 \quad (65)$$

$$g_{1,21}(x, t) = \pm \frac{\sqrt{\frac{q(1-r^2)^2}{2\theta\epsilon}}}{p} \frac{sn\left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{\tau(1-\theta)}{(r^2+1)} t^\alpha\right)\right)}{dn\left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{\tau(1-\theta)}{(r^2+1)} t^\alpha\right)\right) \mp cn\left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{\tau(1-\theta)}{(r^2+1)} t^\alpha\right)\right)} \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(-\theta x^\alpha + \tau t^\alpha) + \vartheta\right)} \quad (66)$$

$$h_{1,21}(x, t) = -\frac{\lambda\epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{q(1-r^2)^2}{2\theta\epsilon}}}{p} \frac{sn\left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{\tau(1-\theta)}{(r^2+1)} t^\alpha\right)\right)}{dn\left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{\tau(1-\theta)}{(r^2+1)} t^\alpha\right)\right) \mp cn\left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{\tau(1-\theta)}{(r^2+1)} t^\alpha\right)\right)} \right)^2 \quad (67)$$

$$g_{1,22}(x, t) = \pm \frac{\sqrt{\frac{qr^4}{2\theta\epsilon}}}{p} \frac{cn\left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(r^2-2)} t^\alpha\right)\right)}{\sqrt{1-r^2 \mp dn\left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(r^2-2)} t^\alpha\right)\right)}} \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(-\theta x^\alpha + \tau t^\alpha) + \vartheta\right)} \quad (68)$$

$$h_{1,22}(x, t) = -\frac{\lambda\epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{qr^4}{2\theta\epsilon}}}{p} \frac{cn\left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(r^2-2)} t^\alpha\right)\right)}{\sqrt{1-r^2 \mp dn\left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{(r^2-2)} t^\alpha\right)\right)}} \right)^2 \quad (69)$$

Solitary wave solutions

When $m \rightarrow 1$, in this case some of the Jacobi elliptic function solutions degenerate to the solitary wave solutions and $g_{1,7}$, $g_{1,8}$, $g_{1,13}$, $g_{1,21}$ become zero. The solitary wave solutions for Eq. (21) are outlined below

$$g_{1,23}(x, t) = \pm \frac{\sqrt{\frac{2q}{\theta\epsilon}}}{p} \tanh\left(\frac{\Gamma(1+\Upsilon)}{\alpha}\left(x^\alpha - \frac{\tau(1-\theta)}{2}t^\alpha\right)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(-\theta x^\alpha + \tau t^\alpha) + \vartheta\right)} \quad (70)$$

$$h_{1,23}(x, t) = -\frac{\lambda \epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{2q}{\theta \epsilon}}}{p} \tanh \left(\frac{\Gamma(1 + \Upsilon)}{\alpha} \left(x^\alpha - \frac{\tau(1 - \theta)}{2} t^\alpha \right) \right) \right)^2 \quad (71)$$

$$g_{1,24}(x, t) = \pm \frac{\sqrt{\frac{2q}{\theta\epsilon}}}{p} \cosh\left(\frac{\Gamma(1+\Upsilon)}{\alpha}\left(x^\alpha - \frac{\tau(1-\theta)}{2}t^\alpha\right)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(-\theta x^\alpha + \tau t^\alpha) + \vartheta\right)} \quad (72)$$

$$h_{1,24}(x, t) = -\frac{\lambda \epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{2q}{\theta \epsilon}}}{p} \cosh \left(\frac{\Gamma(1 + \Upsilon)}{\alpha} \left(x^\alpha - \frac{\tau(1 - \theta)}{2} t^\alpha \right) \right) \right)^2 \quad (73)$$

$$g_{1,25}(x, t) = \pm \frac{\sqrt{\frac{-2q}{\theta\epsilon}}}{p} \operatorname{sech}\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha + \tau(1-\theta)t^\alpha)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(-\theta x^\alpha + \tau t^\alpha) + \vartheta\right)} \quad (74)$$

$$h_{1,25}(x, t) = -\frac{\lambda \epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{-2q}{\theta \epsilon}}}{p} \operatorname{sech} \left(\frac{\Gamma(1 + \Upsilon)}{\alpha} (x^\alpha + \tau(1 - \theta)t^\alpha) \right) \right)^2 \quad (75)$$

$$g_{1,26}(x, t) = \pm \pm \frac{\sqrt{\frac{2q}{\theta \epsilon}}}{p} \coth \left(\frac{\Gamma(1 + \Upsilon)}{\alpha} \left(x^\alpha - \frac{\tau(1 - \theta)}{2} t^\alpha \right) \right) \times e^{i \left(\frac{\Gamma(1 + \Upsilon)}{\alpha} (-\theta x^\alpha + \tau t^\alpha) + \vartheta \right)} \quad (76)$$

$$h_{1,26}(x, t) = -\frac{\lambda \epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{2q}{\theta \epsilon}}}{p} \coth \left(\frac{\Gamma(1 + \Upsilon)}{\alpha} \left(x^\alpha - \frac{\tau(1 - \theta)}{2} t^\alpha \right) \right) \right)^2 \quad (77)$$

$$g_{1,27}(x, t) = \pm \frac{\sqrt{\frac{2q}{\theta\epsilon}}}{p} \operatorname{csch}\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha + \tau(1-\theta)t^\alpha)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(-\theta x^\alpha + \tau t^\alpha) + \vartheta\right)} \quad (78)$$

$$h_{1,27}(x, t) = -\frac{\lambda \epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{2q}{\theta \epsilon}}}{p} \operatorname{csch} \left(\frac{\Gamma(1 + \Upsilon)}{\alpha} (x^\alpha + \tau(1 - \theta)t^\alpha) \right) \right)^2 \quad (79)$$

$$g_{1,28}(x, t) = \pm \frac{\sqrt{\frac{q}{2\theta\epsilon}}}{p} \text{sech} \left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1-\theta)}{2} t^\alpha \right) \right) \times e^{i \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (-\theta x^\alpha + \tau t^\alpha) + \vartheta \right)} \quad (80)$$

$$h_{1,28}(x, t) = -\frac{\lambda \epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{q}{2\theta\epsilon}}}{p} 2 \operatorname{sech} \left(\frac{\Gamma(1 + \Upsilon)}{\alpha} \left(x^\alpha + \frac{2\tau(1 - \theta)}{2} t^\alpha \right) \right) \right)^2 \quad (81)$$

$$g_{1,29}(x,t) = \pm \frac{\sqrt{\frac{q}{2\theta\epsilon}}}{p} \left(\coth \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (x^\alpha - 2\tau(1-\theta)t^\alpha) \right) \pm \operatorname{csch} \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (x^\alpha - 2\tau(1-\theta)t^\alpha) \right) \right) \times e^{i \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (-\theta x^\alpha + \tau t^\alpha) + \vartheta \right)} \quad (82)$$

$$h_{1,29}(x, t) = -\frac{\lambda \epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{q}{2\theta \epsilon}}}{p} \coth \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (x^\alpha - 2\tau(1-\theta)t^\alpha) \right) \pm \operatorname{csch} \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (x^\alpha - 2\tau(1-\theta)t^\alpha) \right) \right)^2 \quad (83)$$

$$g_{1,30}(x,t) = \pm \frac{\sqrt{\frac{9}{2\epsilon}}}{p\theta} \left(\coth \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (x^\alpha - 2\tau(1-\theta)t^\alpha) \right) \pm \operatorname{csch} \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (x^\alpha - 2\tau(1-\theta)t^\alpha) \right) \right) \times e^{i \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (-\theta x^\alpha + \tau t^\alpha) + \vartheta \right)} \quad (84)$$

$$h_{1,30}(x, t) = -\frac{\lambda \epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{q}{2\theta \epsilon}}}{p} \coth \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (x^\alpha - 2\tau(1-\theta)t^\alpha) \right) \pm \operatorname{csch} \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (x^\alpha - 2\tau(1-\theta)t^\alpha) \right) \right)^2 \quad (85)$$

$$g_{1,31}(x,t) = \pm \frac{\sqrt{\frac{9}{29\epsilon}}}{2b} \left(\tanh \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (x^\alpha - 2\tau(1-\theta)t^\alpha) \right) \mp \operatorname{sech} \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (x^\alpha - 2\tau(1-\theta)t^\alpha) \right) \right) \times e^{i \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (-\theta x^\alpha + \tau t^\alpha) + \vartheta \right)} \quad (86)$$

$$h_{1,31}(x, t) = -\frac{\lambda \epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{q}{2\theta c}}}{p} \tanh \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (x^\alpha - 2\tau(1-\theta)t^\alpha) \right) \mp \operatorname{sech} \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (x^\alpha - 2\tau(1-\theta)t^\alpha) \right) \right)^2 \quad (87)$$

$$g_{1,32}(x, t) = \pm \frac{\sqrt{\frac{q}{2\theta\epsilon}}}{p} \frac{\tanh\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - 2\tau(1-\theta)t^\alpha)\right)}{\sqrt{1 - \tanh\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - 2\tau(1-\theta)t^\alpha)\right) \mp \operatorname{sech}\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - 2\tau(1-\theta)t^\alpha)\right)}} \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(-\theta x^\alpha + \tau t^\alpha) + \vartheta\right)} \quad (88)$$

$$h_{1,32}(x, t) = -\frac{\lambda p^2}{q} \left(\pm \frac{\sqrt{\frac{q}{2\theta\epsilon}}}{p} \frac{\tanh\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - 2\tau(1-\theta)t^\alpha)\right)}{\sqrt{1 - \tanh\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - 2\tau(1-\theta)t^\alpha)\right) \pm \operatorname{sech}\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - 2\tau(1-\theta)t^\alpha)\right)}} \right)^2 \quad (89)$$

$$g_{1,33}(x, t) = \pm \frac{\sqrt{\frac{q}{2\theta\epsilon}}}{p} \frac{\tanh\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - 2\tau(1-\theta)t^\alpha)\right)}{1 \mp \operatorname{sech}\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - 2\tau(1-\theta)t^\alpha)\right)} \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(-\theta x^\alpha + \tau t^\alpha) + \vartheta\right)} \quad (90)$$

$$h_{1,33}(x, t) = -\frac{\lambda\epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{q}{2\theta\epsilon}}}{p} \frac{\tanh\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - 2\tau(1-\theta)t^\alpha)\right)}{1 \mp \operatorname{sech}\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - 2\tau(1-\theta)t^\alpha)\right)} \right)^2 \quad (91)$$

$$g_{1,34}(x, t) = \pm \frac{\sqrt{\frac{q}{2\theta\epsilon}}}{p} \frac{\tanh\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - 2\tau(1-\theta)t^\alpha)\right)}{1 \mp \operatorname{sech}\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - 2\tau(1-\theta)t^\alpha)\right)} \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(-\theta x^\alpha + \tau t^\alpha) + \vartheta\right)} \quad (92)$$

$$h_{1,34}(x, t) = -\frac{\lambda\epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{q}{2\theta\epsilon}}}{p} \frac{\tanh\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - 2\tau(1-\theta)t^\alpha)\right)}{1 \mp \operatorname{sech}\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - 2\tau(1-\theta)t^\alpha)\right)} \right)^2 \quad (93)$$

$$g_{1,35}(x, t) = \pm \frac{\sqrt{\frac{q}{2\theta\epsilon}}}{p} \frac{\operatorname{sech}\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - 2\tau(1-\theta)t^\alpha)\right)}{\sqrt{1 \mp \operatorname{sech}\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - 2\tau(1-\theta)t^\alpha)\right)}} \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(-\theta x^\alpha + \tau t^\alpha) + \vartheta\right)} \quad (94)$$

$$h_{1,35}(x, t) = -\frac{\lambda\epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{q}{2\theta\epsilon}}}{p} \frac{\operatorname{sech}\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - 2\tau(1-\theta)t^\alpha)\right)}{\sqrt{1 \mp \operatorname{sech}\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - 2\tau(1-\theta)t^\alpha)\right)}} \right)^2 \quad (95)$$

Shock wave solutions

When $m \rightarrow 0$, in this case some of the Jacobi elliptic functions solutions degenerate to the trigonometric solutions and $g_{1,1}, g_{1,2}, g_{1,3}, g_{1,11}, g_{1,15}, g_{1,16}, g_{1,19}, g_{1,22}$ become zero. The Shock wave solutions for Eq. (21) are outlined below.

$$g_{1,36}(x, t) = \pm \frac{\sqrt{\frac{2q}{\theta\epsilon}}}{p} \csc\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \tau(1-\theta)t^\alpha)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(-\theta x^\alpha + \tau t^\alpha) + \vartheta\right)} \quad (96)$$

$$h_{1,36}(x, t) = -\frac{\lambda\epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{2q}{\theta\epsilon}}}{p} \csc\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \tau(1-\theta)t^\alpha)\right) \right)^2 \quad (97)$$

$$g_{1,37}(x, t) = \pm \frac{\sqrt{\frac{2q}{\theta\epsilon}}}{p} \sec\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \tau(1-\theta)t^\alpha)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(-\theta x^\alpha + \tau t^\alpha) + \vartheta\right)} \quad (98)$$

$$h_{1,37}(x, t) = -\frac{\lambda\epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{2q}{\theta\epsilon}}}{p} \sec\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \tau(1-\theta)t^\alpha)\right) \right)^2 \quad (99)$$

$$g_{1,38}(x, t) = \pm \frac{\sqrt{\frac{2q}{\theta\epsilon}}}{p} \sec\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \tau(1-\theta)t^\alpha)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(-\theta x^\alpha + \tau t^\alpha) + \vartheta\right)} \quad (100)$$

$$h_{1,38}(x, t) = -\frac{\lambda\epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{2q}{\theta\epsilon}}}{p} \sec\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha - \tau(1-\theta)t^\alpha)\right) \right)^2 \quad (101)$$

$$g_{1,39}(x, t) = \pm \frac{\sqrt{\frac{2q}{\theta\epsilon}}}{p} \tan\left(\frac{\Gamma(1+\Upsilon)}{\alpha}\left(x^\alpha + \frac{\tau(1-\theta)}{2}t^\alpha\right)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(-\theta x^\alpha + \tau t^\alpha) + \vartheta\right)} \quad (102)$$

$$h_{1,39}(x, t) = -\frac{\lambda\epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{2q}{\theta\epsilon}}}{p} \tan\left(\frac{\Gamma(1+\Upsilon)}{\alpha}\left(x^\alpha + \frac{\tau(1-\theta)}{2}t^\alpha\right)\right) \right)^2 \quad (103)$$

$$g_{1,40}(x, t) = \pm \frac{\sqrt{\frac{2q}{\theta\epsilon}}}{p} \cot\left(\frac{\Gamma(1+\Upsilon)}{\alpha}\left(x^\alpha + \frac{\tau(1-\theta)}{2}t^\alpha\right)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(-\theta x^\alpha + \tau t^\alpha) + \vartheta\right)} \quad (104)$$

$$h_{1,40}(x, t) = -\frac{\lambda\epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{2q}{\theta\epsilon}}}{p} \cot\left(\frac{\Gamma(1+\Upsilon)}{\alpha}\left(x^\alpha + \frac{\tau(1-\theta)}{2}t^\alpha\right)\right) \right)^2 \quad (105)$$

$$g_{1,41}(x, t) = \pm \frac{\sqrt{\frac{q}{2\theta\epsilon}}}{p} \csc\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha + 2\tau(1-\theta)t^\alpha)\right) \pm \cot\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha + 2\tau(1-\theta)t^\alpha)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(-\theta x^\alpha + \tau t^\alpha) + \vartheta\right)} \quad (106)$$

$$h_{1,41}(x, t) = -\frac{\lambda \epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{q}{2\theta\epsilon}}}{p} \csc \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (x^\alpha + 2\tau(1-\theta)t^\alpha) \right) \pm \cot \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (x^\alpha + 2\tau(1-\theta)t^\alpha) \right) \right)^2 \quad (107)$$

$$g_{1,42}(x, t) = \pm \frac{\sqrt{\frac{q}{2\theta\epsilon}}}{p} \left(\sec \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (x^\alpha + 2\tau(1-\theta)t^\alpha) \right) \pm \tan \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (x^\alpha + 2\tau(1-\theta)t^\alpha) \right) \right) \times e^{i \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (-\theta x^\alpha + \tau t^\alpha) + \vartheta \right)} \quad (108)$$

$$h_{1,42}(x, t) = -\frac{\lambda \epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{q}{2\theta\epsilon}}}{p} \sec \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (x^\alpha + 2\tau(1-\theta)t^\alpha) \right) \pm \tan \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (x^\alpha + 2\tau(1-\theta)t^\alpha) \right) \right)^2 \quad (109)$$

$$g_{1,43}(x, t) = \pm \frac{\sqrt{\frac{q}{2\theta\epsilon}}}{p} \left(\csc \left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha - \frac{2\tau(1-\theta)}{2} t^\alpha \right) \right) \pm \cot \left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha - \frac{2\tau(1-\theta)}{2} t^\alpha \right) \right) \right) \times e^{i \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (-\theta x^\alpha + \tau t^\alpha) + \vartheta \right)} \quad (110)$$

$$h_{1,43}(x, t) = -\frac{\lambda \epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{q}{2\theta\epsilon}}}{p} \csc \left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha - \frac{2\tau(1-\theta)}{2} t^\alpha \right) \right) \pm \cot \left(\frac{\Gamma(1+\Upsilon)}{\alpha} \left(x^\alpha - \frac{2\tau(1-\theta)}{2} t^\alpha \right) \right) \right)^2 \quad (111)$$

$$g_{1,44}(x, t) = \pm \frac{\sqrt{\frac{q}{2\theta\epsilon}}}{p} \sin \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (x^\alpha + 2\tau(1-\theta)t^\alpha) \right) \times e^{i \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (-\theta x^\alpha + \tau t^\alpha) + \vartheta \right)} \quad (112)$$

$$h_{1,44}(x, t) = -\frac{\lambda \epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{q}{2\theta\epsilon}}}{p} \sin \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (x^\alpha + 2\tau(1-\theta)t^\alpha) \right) \right)^2 \quad (113)$$

$$g_{1,45}(x, t) = \pm \frac{\sqrt{\frac{q}{2\theta\epsilon}}}{p} \frac{\cos \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (x^\alpha + 2\tau(1-\theta)t^\alpha) \right)}{1 \mp \sin \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (x^\alpha + 2\tau(1-\theta)t^\alpha) \right)} \times e^{i \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (-\theta x^\alpha + \tau t^\alpha) + \vartheta \right)} \quad (114)$$

$$h_{1,45}(x, t) = -\frac{\lambda \epsilon p^2}{q} \left(\pm \frac{\sqrt{\frac{q}{2\theta\epsilon}}}{p} \frac{\cos \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (x^\alpha + 2\tau(1-\theta)t^\alpha) \right)}{1 \mp \sin \left(\frac{\Gamma(1+\Upsilon)}{\alpha} (x^\alpha + 2\tau(1-\theta)t^\alpha) \right)} \right)^2 \quad (115)$$

Results and discussion

In this section, we compare and discuss the solutions obtained from our investigation with those from previous studies. By employing the Jacobi elliptic function expansion method, we derived a variety of interesting optical soliton wave solutions for the truncated M-fractional nonlinear (1+1)-dimensional Shynaray-IIA equations. It is important to note that, among these solutions are exact periodic solutions. This method also enabled us to produce some shock wave and solitary wave solutions. To highlight the similarities and differences between solutions obtained through different methods, we conducted a comparative analysis. Previous studies have addressed similar nonlinear equations using various methods, such as the Kudryashov method, the exp-function method and the modified simple equation method^{59,60}. These approaches are instrument in understanding the dynamics of phenomena like tsunamis and tidal waves.

Our finding exhibits notable novelties, diversity in physical structures, and general applicability. Comparing our new, more general soliton wave solutions with those from earlier research underscores the robustness, reliability, simplicity, and efficiency of the Jacobi elliptic function expansion method. This comparison demonstrates the superiority of the Jacobi elliptic function expansion method over previously used techniques. Our results not only enhance the understanding of the Shynaray-IIA equation but also illustrate the method potential in solving a broader class of nonlinear problems.

Graphical representation

This section shows the graphical depiction of the soliton wave solutions for the truncated M-fractional nonlinear (1+1)-dimensional Shynaray-IIA equation that were found using the previously mentioned Jacobi elliptic function approach. Using the proper values for the free variables, we present these graphs in 3D, 2D, and contour formats. Figures 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19 represented to the physical structure of some examined solutions with different soliton and solitary wave profiles by using appropriate values for the arbitrary parameters. These graphs shows the physical structure in periodic wave solitons, bright and dark solitons, mixed bright and dark periodic wave solitons, peakon periodic wave solitons and solitary wave structure. Figures 2, 3, 4, 5, 6, 7 represented to peakon periodic wave soliton structure for $|g_{1,1}(x, t)|$ and $|g_{1,4}(x, t)|$. Figures 8, 9, 10, 11, 12, 13 represented to bell type periodic wave soliton structure for $|g_{1,5}(x, t)|$ and $|g_{1,7}(x, t)|$. Figure 14 represented to bell type bright and dark soliton structure for $|g_{1,9}(x, t)|$. Figures 15, 16 represented to bell type periodic wave soliton structure for $|g_{1,9}(x, t)|$, and Fig. 17 represented to peakon bright soliton structure for $|g_{1,11}(x, t)|$. Figures 18, 19 represented to peakon periodic wave soliton structure for $|g_{1,11}(x, t)|$, by using appropriate values for the arbitrary parameters.

These visual aids are crucial because they allow us to confirm the accuracy of the theoretical conclusions we previously reached. Remember that *Mathematica* was used to create these figures and graphs. However, the influence of waves is also examined, and it is observed that by regulating the soliton propagation with wave number, physicists and researchers can obtain the necessary results. A thorough explanation of the figure, together with the relevant variables or constants utilized in each one, is provided below:

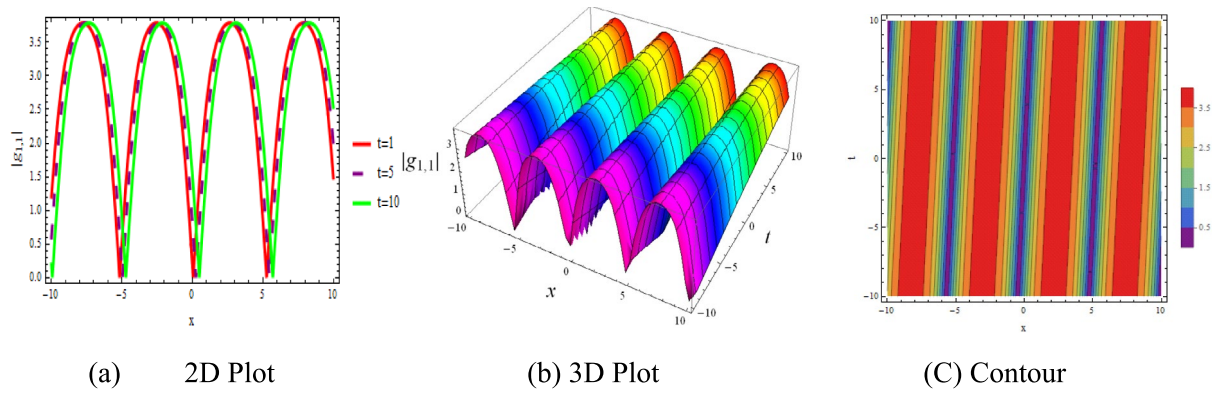


Fig. 2. 2-D plot for the wave propagation of $|g_{1,1}(x, t)|$ for varying t , contour and 3-D for the parameters $m = 0.8, \gamma = 0.5, \epsilon = 0.5, \alpha = 1, p = 2; q = 3, \theta = 0.2, \vartheta = 0.2$ and $\tau = 0.1$.

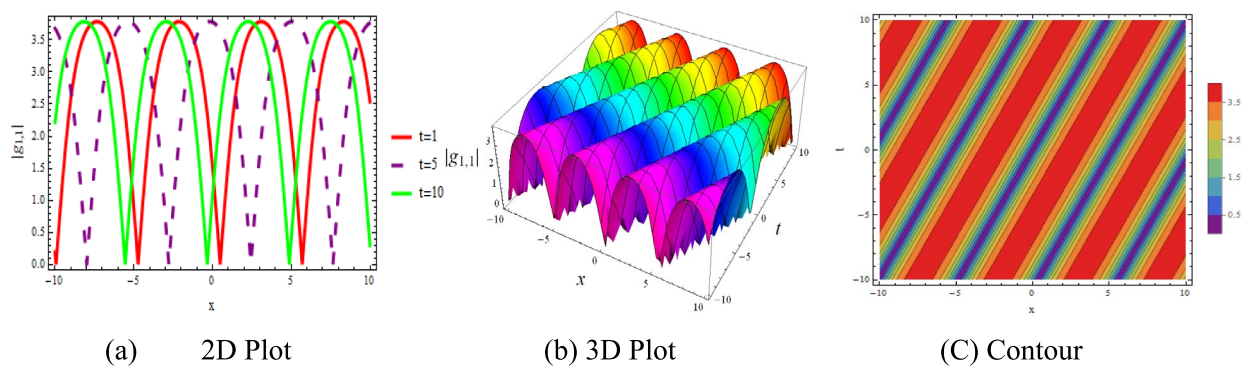


Fig. 3. 2-D plot for the wave propagation of $|g_{1,1}(x, t)|$ for varying t , contour and 3-D for the parameters $m = 0.8, \gamma = 0.5, \epsilon = 0.5, \alpha = 1, p = 2; q = 3, \theta = 0.2, \vartheta = 0.2$ and $\tau = 1$.

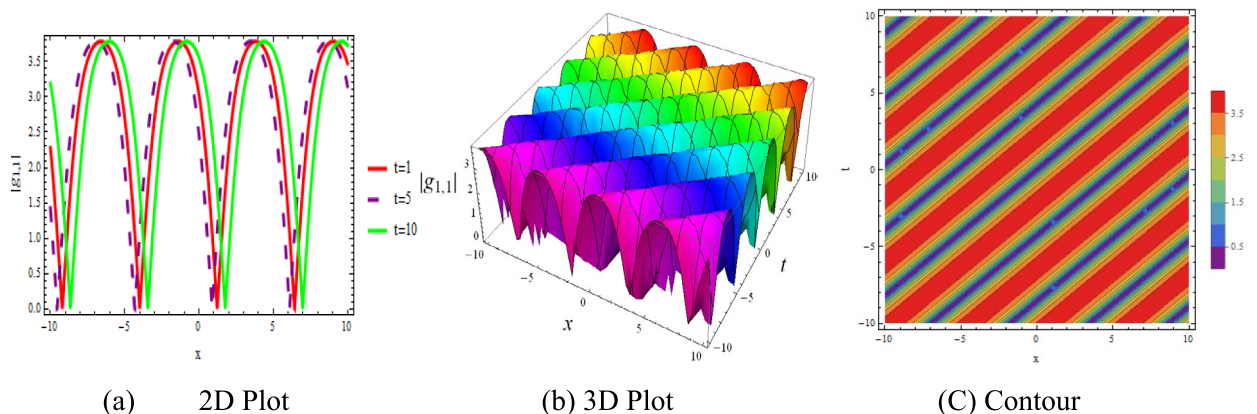


Fig. 4. 2-D plot for the wave propagation of $|g_{1,1}(x, t)|$ for varying t , contour and 3-D for the parameters $m = 0.8, \gamma = 0.5, \epsilon = 0.5, \alpha = 1, p = 2; q = 3, \theta = 0.2, \vartheta = 0.2$ and $\tau = 2.5$.

Conclusion

This study demonstrates that the Jacobi elliptic function expansion method is an efficient approach for solving the nonlinear $(1 + 1)$ -dimensional Shynaray-IIA equation based on the truncated M-fractional derivative. By Utilizing JEFEM, exact solutions are obtained that exhibit a variety of interesting behaviors, especially when expressed in terms of Jacobi elliptic functions. These solutions provide valuable insights into complex physical phenomena. Moreover, the appearance of both solitary wave and shock wave solutions offers important information regarding localized wave behavior and periodic fluctuations. Graphical representations of these

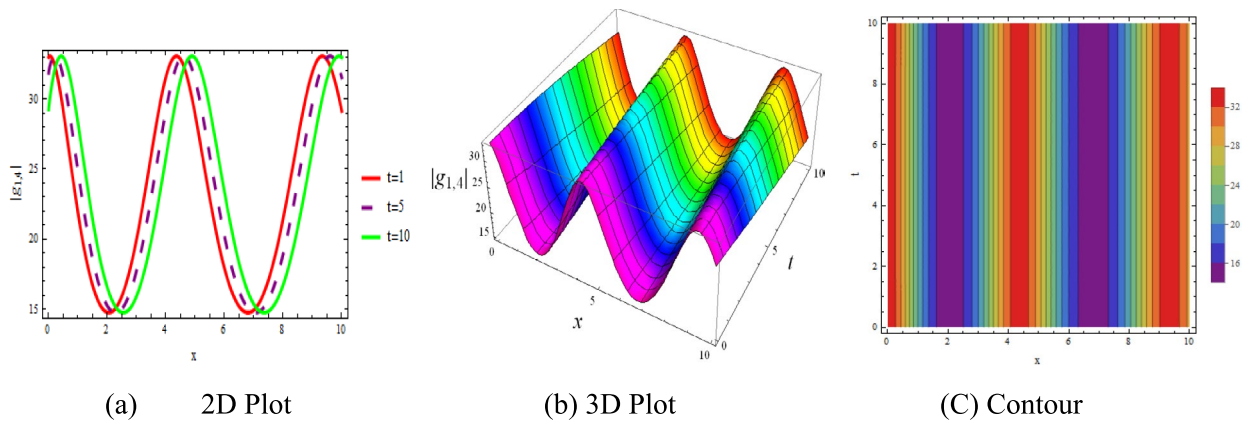


Fig. 5. 2-D plot for the wave propagation of $|g_{1,4}(x,t)|$ for varying t , contour and 3-D for the parameters $m = 0.8, \gamma = 0.9, \epsilon = 0.5, \alpha = 0.9, p = -1, q = 1, \theta = 0.2, \vartheta = 2$ and $\tau = 0.1$.

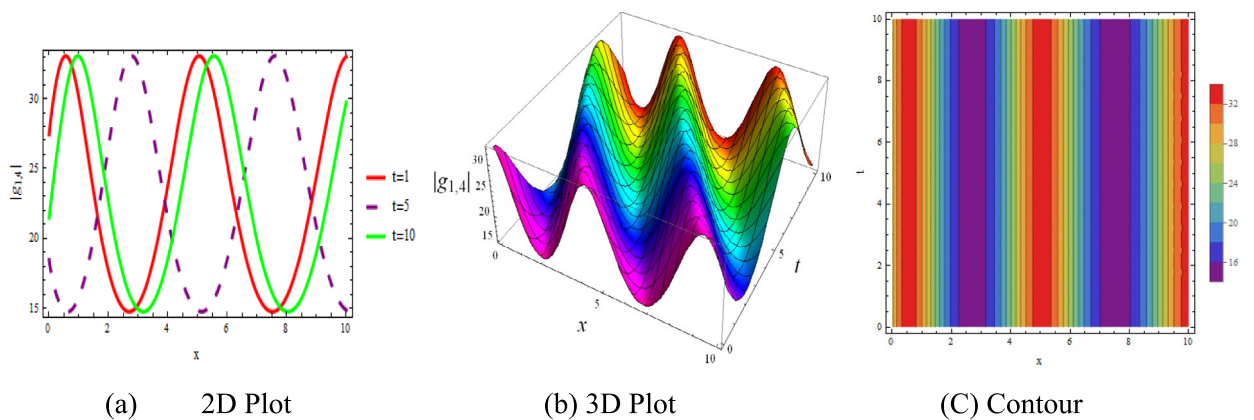


Fig. 6. 2-D plot for the wave propagation of $|g_{1,4}(x,t)|$ for varying t , contour and 3-D for the parameters $m = 0.8, \gamma = 0.9, \epsilon = 0.5, \alpha = 0.9, p = -1, q = 1, \theta = 0.2, \vartheta = 2$ and $\tau = 1$.

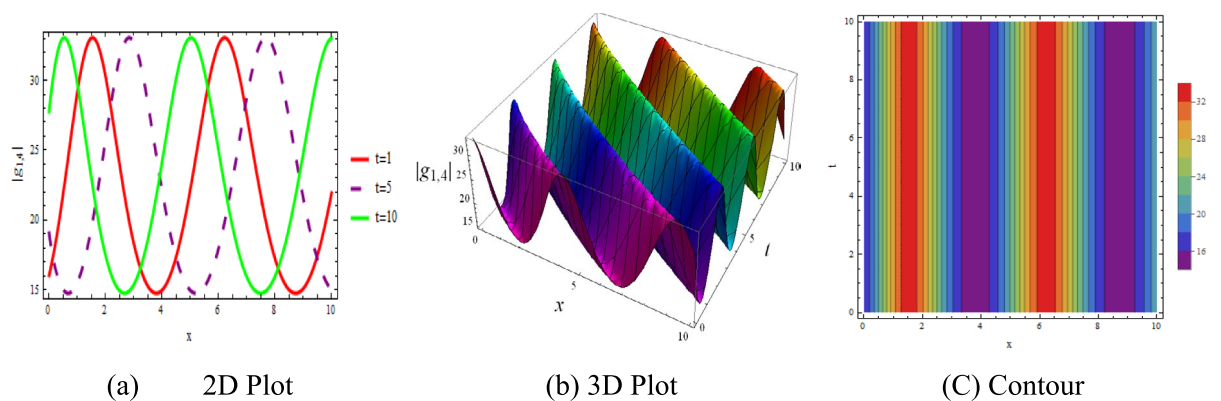


Fig. 7. 2-D plot for the wave propagation of $|g_{1,4}(x,t)|$ for varying t , contour and 3-D for the parameters $m = 0.8, \gamma = 0.9, \epsilon = 0.5, \alpha = 0.9, p = -1, q = 1, \theta = 0.2, \vartheta = 2$ and $\tau = 2.5$.

solutions in various dimensions enhanced our comprehension and facilitate a more thorough exploration of their properties. The application of the Jacobi elliptic function expansion method promises to improve our understanding and prediction of real-world phenomena in a variety of fields, such as fluid dynamics and quantum mechanics.

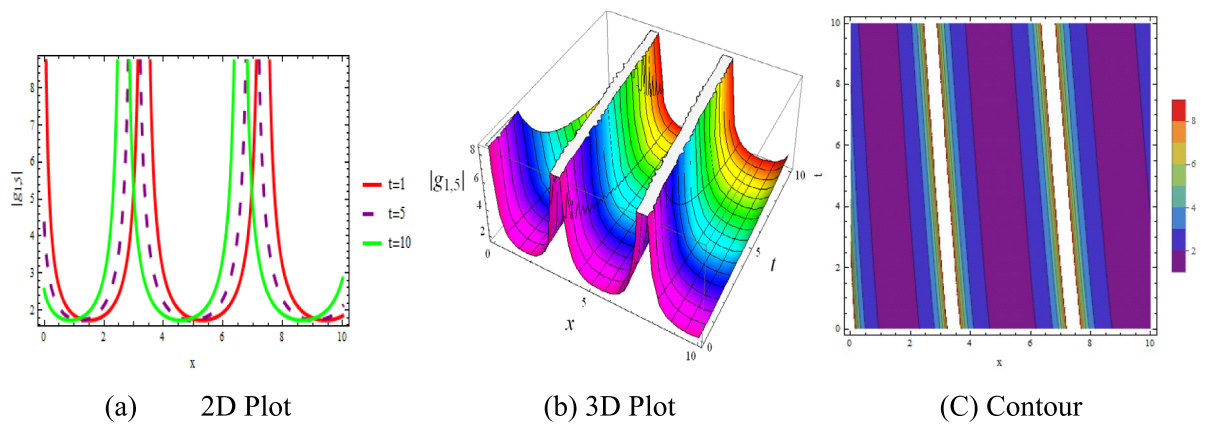


Fig. 8. 2-D plot for the wave propagation of $|g_{1,5}(x, t)|$ for varying t , contour and 3-D for the parameters $m = 0.5, \gamma = 0.9, \epsilon = 0.5, \alpha = 0.9, p = 1, q = 1, \theta = 2, \vartheta = 0.2, \tau = 0.1$.

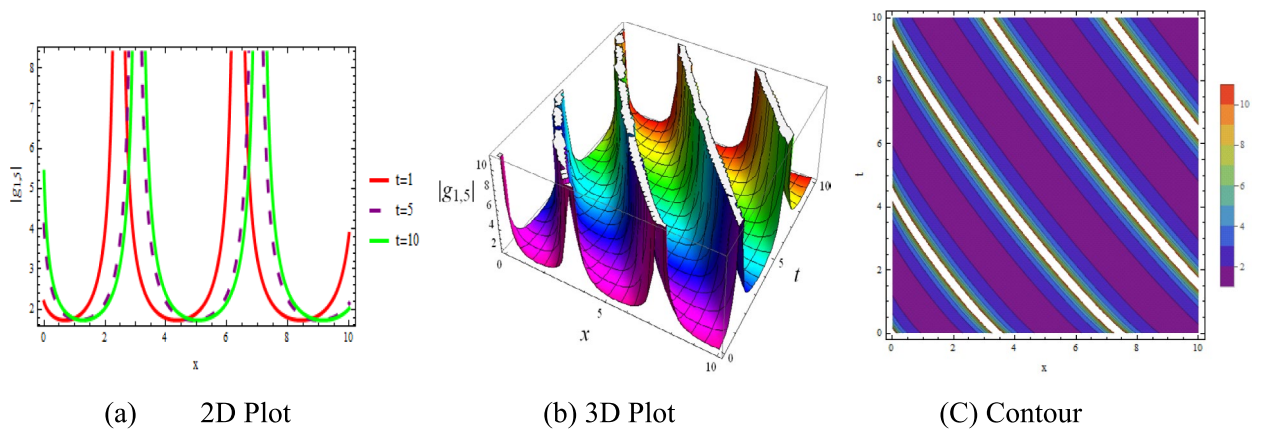


Fig. 9. 2-D plot for the wave propagation of $|g_{1,5}(x, t)|$ for varying t , contour and 3-D for the parameters $m = 0.5, \gamma = 0.9, \epsilon = 0.5, \alpha = 0.9, p = 1, q = 1, \theta = 2, \vartheta = 0.2, \tau = 1$.

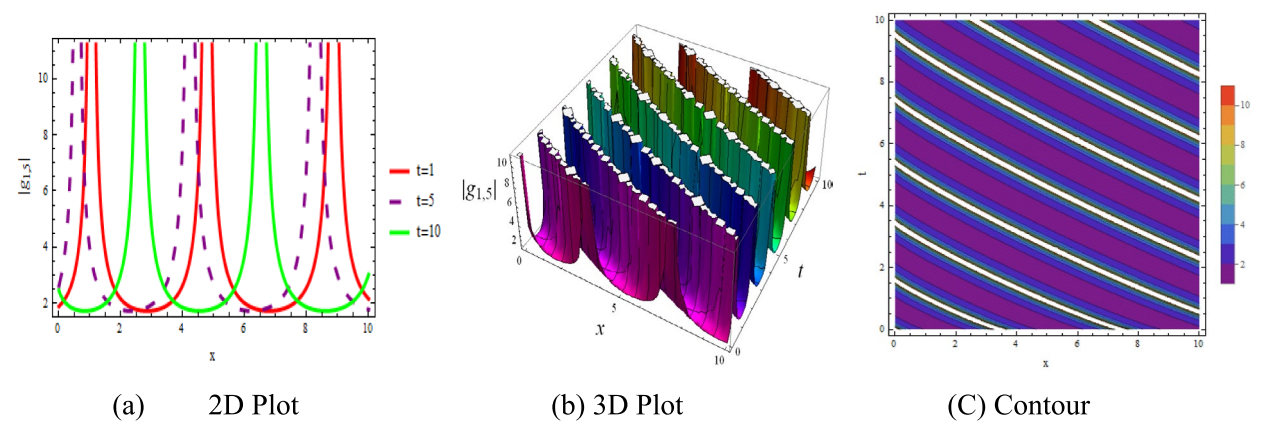


Fig. 10. 2-D plot for the wave propagation of $|g_{1,5}(x, t)|$ for varying t , contour and 3-D for the parameters $m = 0.5, \gamma = 0.9, \epsilon = 0.5, \alpha = 0.9, p = 1, q = 1, \theta = 2, \vartheta = 0.2, \tau = 2.5$.

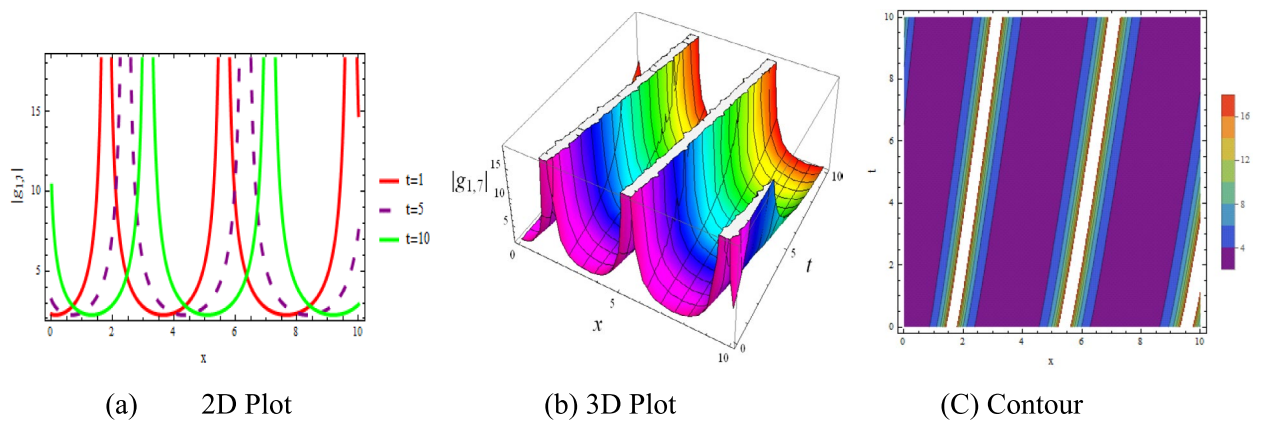


Fig. 11. 2-D plot for the wave propagation of $|g_{1,7}(x, t)|$ for varying t , contour and 3-D for the parameters $m = 0.5, \gamma = 0.9, \epsilon = 1.5, \alpha = 0.9, p = 1, q = 1, \theta = 0.2, \vartheta = 4$, and $\tau = 0.1$.

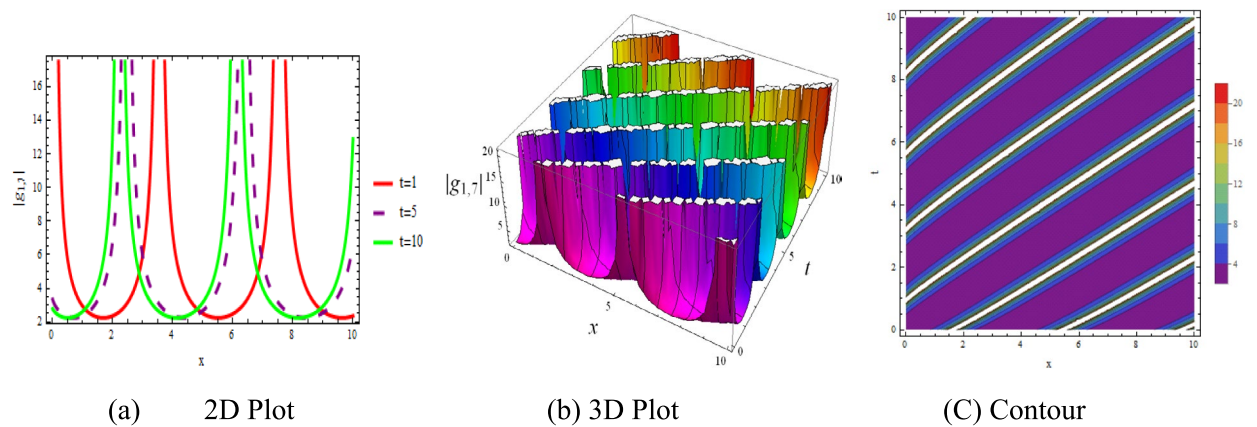


Fig. 12. 2-D plot for the wave propagation of $|g_{1,7}(x, t)|$ for varying t , contour and 3-D for the parameters $m = 0.5, \gamma = 0.9, \epsilon = 1.5, \alpha = 0.9, p = 1, q = 1, \theta = 0.2, \vartheta = 4$, and $\tau = 1$.

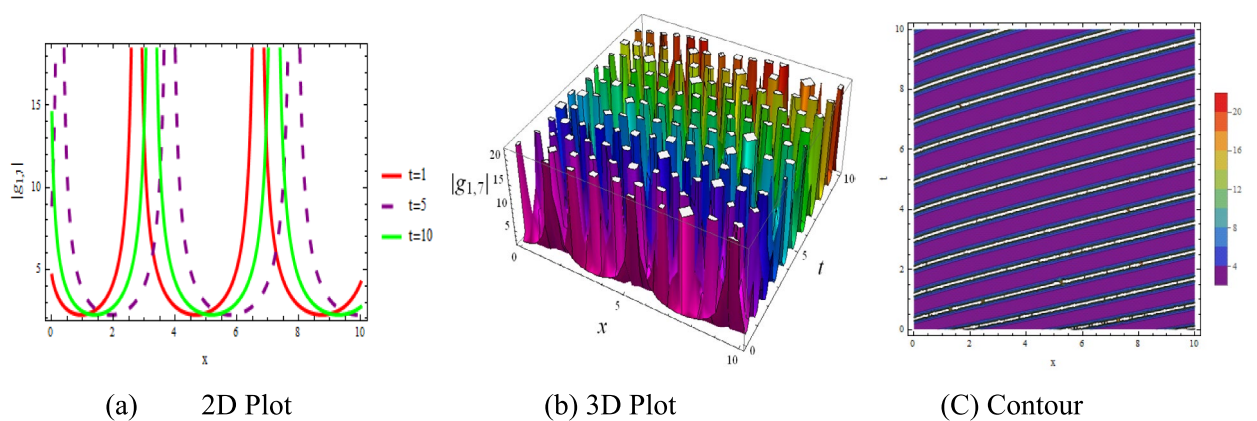


Fig. 13. 2-D plot for the wave propagation of $|g_{1,7}(x, t)|$ for varying t , contour and 3-D for the parameters $m = 0.5, \gamma = 0.9, \epsilon = 1.5, \alpha = 0.9, p = 1, q = 1, \theta = 0.2, \vartheta = 4$, and $\tau = 2.5$.

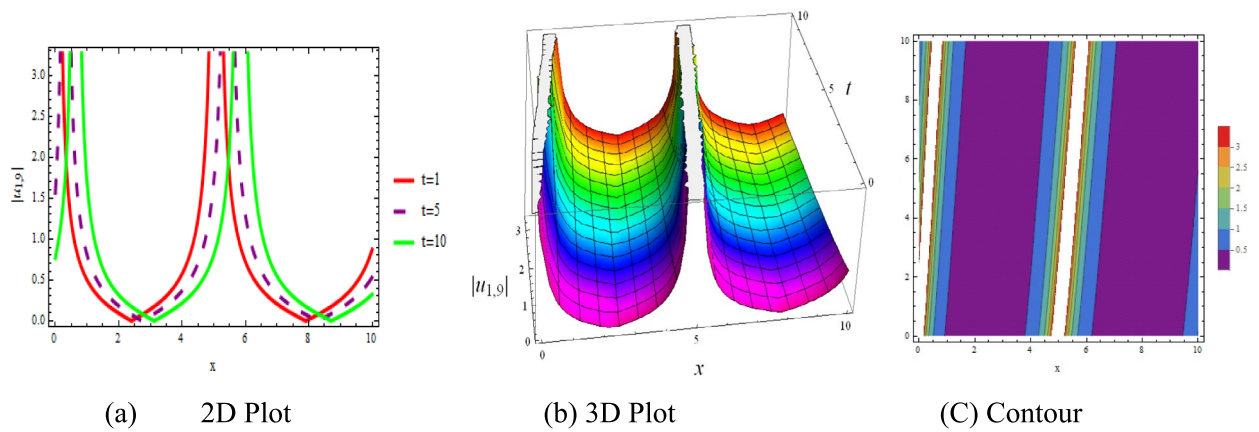


Fig. 14. 2-D plot for the wave propagation of $|g_{1,9}(x, t)|$ for varying t , contour and 3-D for the parameters $m = 0.9, \gamma = 0.9, \epsilon = 0.5\epsilon = 0.9, p = 4, q = 4, \theta = 2, \vartheta = 2$ and $\tau = 0.1$.

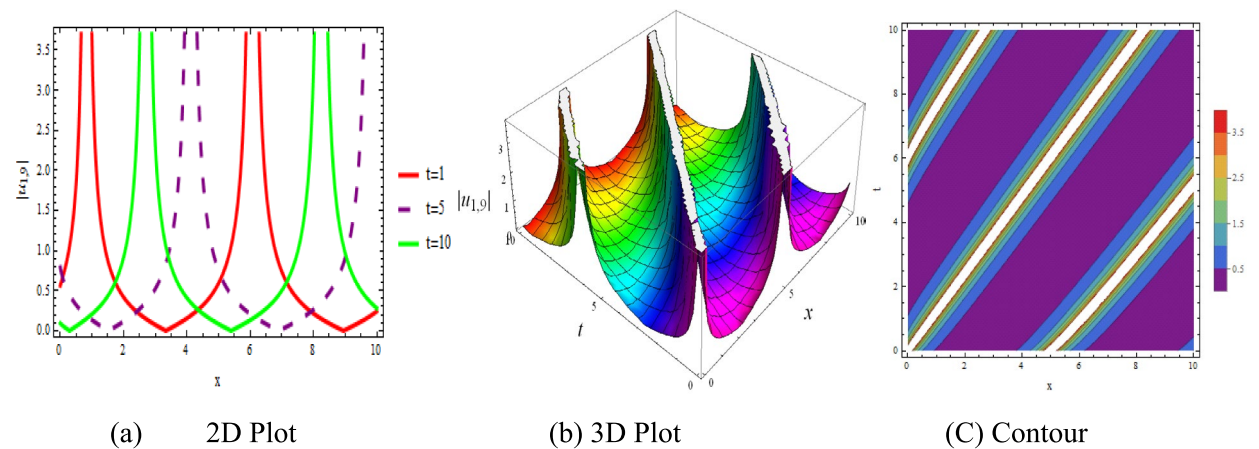


Fig. 15. 2-D plot for the wave propagation of $|g_{1,9}(x, t)|$ for varying t , contour and 3-D for the parameters $m = 0.9, \gamma = 0.9, \epsilon = 0.5\epsilon = 0.9, p = 4, q = 4, \theta = 2, \vartheta = 2$ and $\tau = 1$.

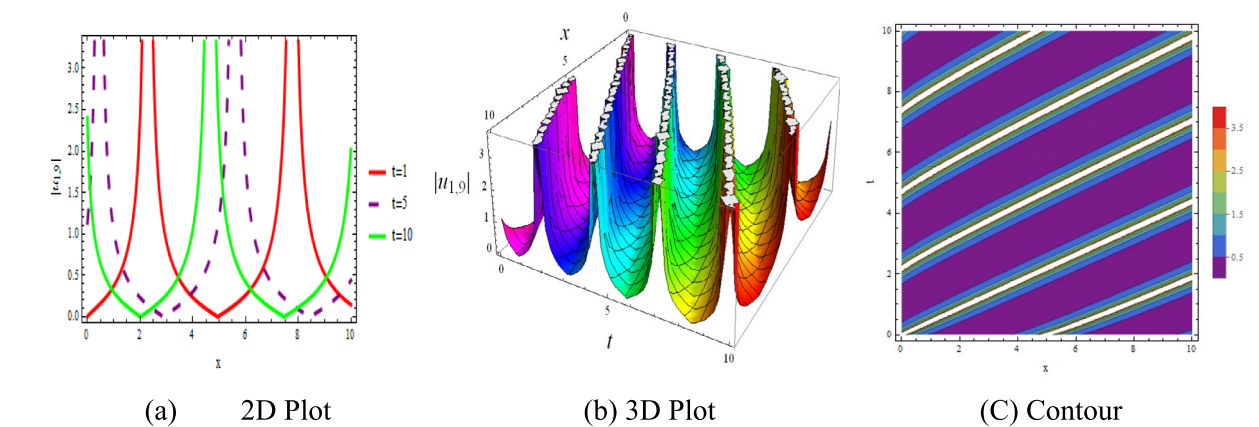


Fig. 16. 2-D plot for the wave propagation of $|g_{1,9}(x, t)|$ for varying t , contour and 3-D for the parameters $m = 0.9, \gamma = 0.9, \epsilon = 0.5\epsilon = 0.9, p = 4, q = 4, \theta = 2, \vartheta = 2$ and $\tau = 2.5$.

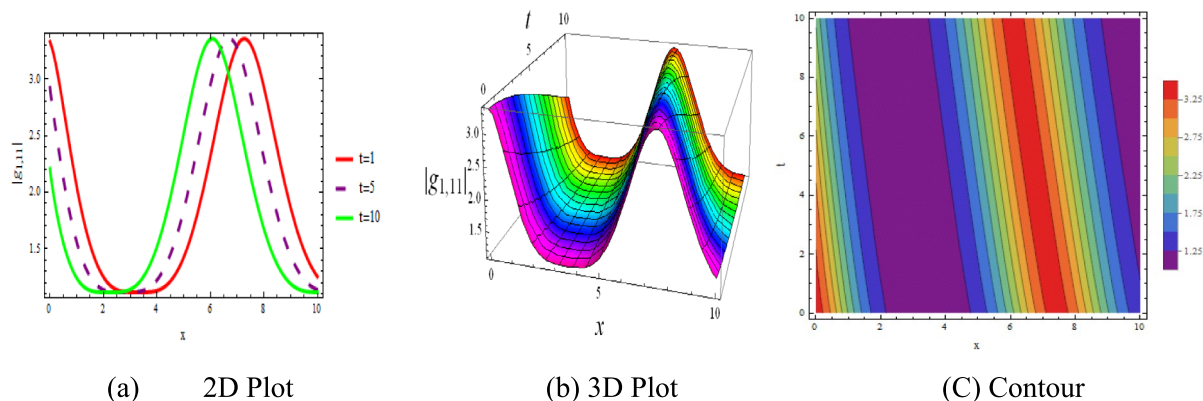


Fig. 17. 2-D plot for the wave propagation of $|g_{1,11}(x,t)|$ for varying t , contour and 3-D for the parameters $m = 0.5, \gamma = 0.9; \epsilon = 0.5; \alpha = 0.9; p = 1; q = 1, \theta = 0.2, \vartheta = 4$ and $\tau = 0.1$.

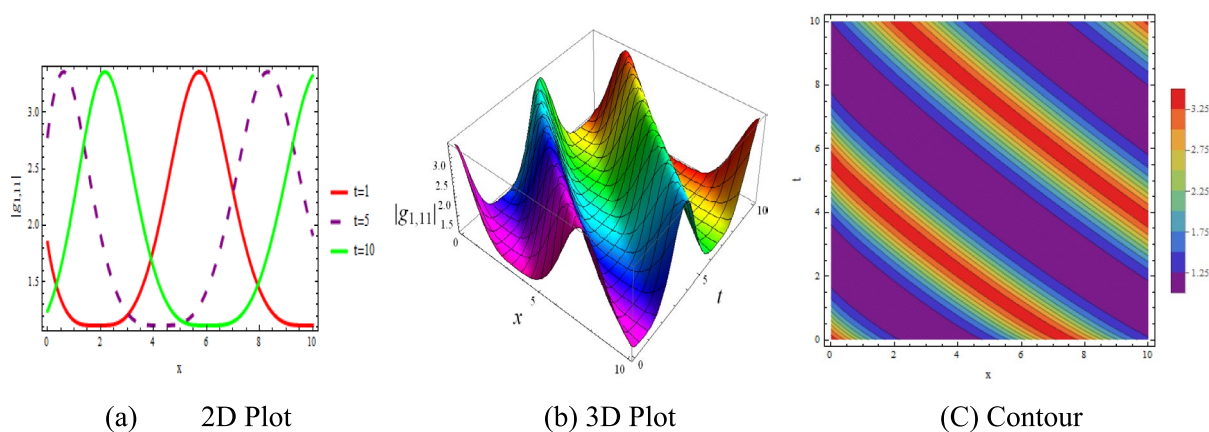


Fig. 18. 2-D plot for the wave propagation of $|g_{1,11}(x,t)|$ for varying t , contour and 3-D for the parameters $m = 0.5, \gamma = 0.9; \epsilon = 0.5; \alpha = 0.9; p = 1; q = 1, \theta = 0.2, \vartheta = 4$ and $\tau = 1$.

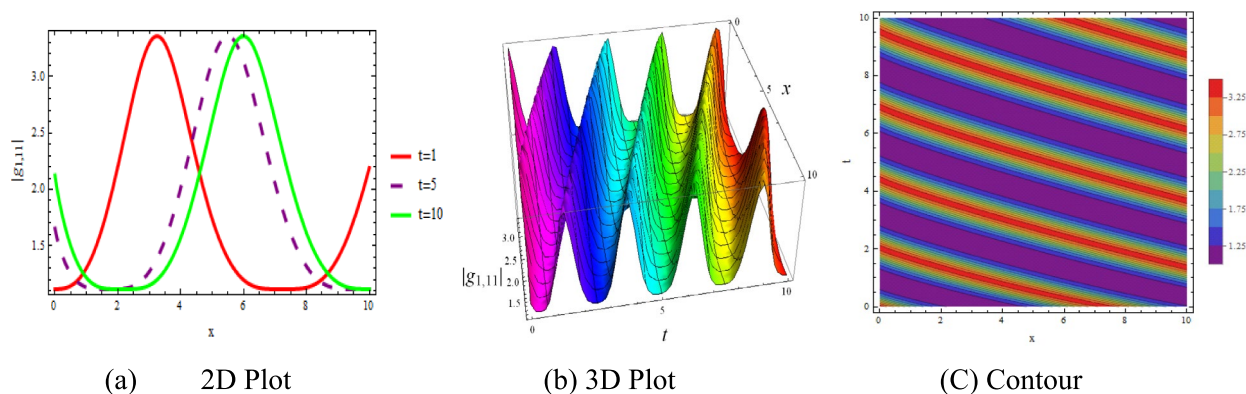


Fig. 19. 2-D plot for the wave propagation of $|g_{1,11}(x,t)|$ for varying t , contour and 3-D for the parameters $m = 0.5, \gamma = 0.9; \epsilon = 0.5; \alpha = 0.9; p = 1; q = 1, \theta = 0.2, \vartheta = 4$ and $\tau = 2.5$.

Data availability

All data that support the findings of this study are included in the article.

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Author contributions

MI: Writing-Original draft preparation, Methodology. MIK: Supervision, Conceptualization. HDA: Data curation, Investigation. AA: Software, Writing-reviewing & editing. RA: Visualization, Writing-reviewing & editing. FAHA: Validation, Data curation. NM: Investigation, Formal analysis. SA: Visualization, Formal analysis.

Declarations

Competing interests

The authors declare no competing interests.

Additional information

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