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Study of phase separation process in multi-component mixtures using analytical methods and decomposition variational iteration method for the fourth-order Cahn–Hilliard equation

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In this paper, the fourth-order Cahn–Hilliard equation is studied, which plays an important role in the development of the spinodal decomposition, phase separation, and phase ordering dynamics. The $\tan(\phi/2)$-expansion method (TEM), Jacobi elliptic function expansion scheme (JEFES), rational multi wave functions (RMWFs), and decomposition variational iteration method (DVIM) are considered to investigate the exact traveling wave solutions to nonlinear evolution equations (NLEEs) in the domain of applied physics and engineering. This equation can be utilized to explain the contact between the modes in describing the process of phase separation of a binary alloy under the critical temperature, phase separation, phase-ordering dynamics, and spinodal decomposition, fluid mechanics, and fluid flow. The dynamics of the assessed solutions in terms of understanding the real phenomena for such nonlinear model is demonstrated by plotting their 3D, 2D, contour, and density profiles using proper parametric values. Consequently, we obtain distinct types of solutions, containing dark, bright, kink, singular, combo kink singular, and combo dark singular soliton solutions. These results are essential to the explanation of several mesmerizing and intricate physical phenomena. Also, the decomposition variational iteration method of the proposed model is analyzed and conditions are developed accordingly. The soliton solutions demonstrate the competency of the proposed technique in identifying traveling wave solutions, offering a useful tool for tackling a variety of NLEEs. We believe that our results would pave a way for future research generating optical memories based on the nonlinear solitons.

Keywords Fourth-order Cahn–Hilliard equation, Spinodal decomposition, Phase separation, $\tan(\phi/2)$-expansion method, Jacobi elliptic function expansion scheme, Rational multi wave functions, Decomposition variational iteration method, Periodic wave solution

In the past decades, investigating nonlinear models and solving them has been a challenge for scientists and experts in various fields, including: plasma physics[1], optics[3], electromagnetic-circuital-termal-mechanical multi physics[4], granular termo dynamic migration model[5] and multi-scale spatial-temporal interaction fusion network[6]. The study of wave propagation for nonlinear evolution equations (NLEEs) has drawn interest from the scientific community in recent years. NLEEs are frequently found in a variety of domains due to

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the extensive presence of nonlinearity in both natural and artificial frameworks, such as medical imaging, population dynamics, and material science. Advanced computational approaches, numerical simulations, and mathematical methodologies have all been developed in an effort to understand and solve NLEEs. Nonlinear partial differential equation is a type of equation with important research significance, which is applied in many subjects. At present, in the academic field, an important academic wave is to solve the exact solution of nonlinear partial differential equations[2]. Nonlinearity plays a major role to describe the behavior of optical solitons and wave packets in nonlinear media. With the continuous exploration of researchers, a series of methods have been explored, such as the $\exp(-\Omega(\eta))$-expansion method[3], the homotopy analysis method[7], the homotopy perturbation method[8], the Exp-function method[9], the Hirota bilinear method[10], the first integral method[11], the inverse scattering method[12], Lie group method[13], the Jacobi elliptic function expansion method[14], the variational iteration method[15], Laplace-residual power series method[16] and Legendre approximation method[17].

Numerous academic domains, such as signal and image processing, biology, chemistry, economics, physics, electricity, and aerodynamics, and many more, have been greatly impacted by nonlinear models[18–22]. Additionally, this understanding greatly enhances the real-world consequences and practical applicability of ordinary differential equations (ODEs) and partial differential equations (PDEs)[23–27]. Hirota bilinear scheme is powerful technique to obtain the analytical solutions. Therefore, to find the lump and their interactions solutions of such nonlinear models, physicists and mathematicians have worked very hard. Many researchers have been worked on this method for some nonlinear equations, including the generalized Bogoyavlensky-Konopelchenko equation[28], the generalized Hietarinta equation[29], the dispersive wave systems[30], the variable-coefficient generalized nonlinear wave equation[31], symmetric nonlinear dispersive wave model[32], the (3+1) dimensional KPB like equation[33], the spectral functional method[34], application of cubic B-splines method[35] and fractional order Sturm-Liouville problems[36].

The true structure of nonlinear behaviors can be revealed by solitons to nonlinear phenomena. A traveling wave is one that maintains its shape while moving in a specific direction. In addition to this, a traveling wave is associated with maintaining a steady velocity during its propagation[37]. In 1834, the early observations by John Scott Russell on the solitary wave were received with skepticism. However, his research was eventually confirmed by the theoretical studies of the Dutch physicists Diederik Korteweg and Gustav de Vries. In 1895, Korteweg and de Vries developed a mathematical model to describe the transmission of a solitary wave. Since the effective theoretical modeling of solitary waves, there has been an increased concern in nonlinear waves. The idea of solitons was discovered in the field of integrable models. In 1965, Kruskal and Zabusky proved the steady solitary wave solution of the KdV equation through a numerical experiment and termed it "soliton" because of its particle-like character. Later on, Gardner discovered the Inverse Scattering Transform to solve nonlinear evolution equations analytically. Among the mentioned analytical techniques that generate soliton solutions for NPDEs the $\tan(\phi/2)$-expansion approach stands as the fundamental easy and effective algebraic procedure. The transformation of NPDEs to NODEs through wave transportation enables the strategic application of the $\tan(\phi/2)$-expansion that results in a set of nonlinear algebraic equations when a solution form series assumption exists. The use of Maple computational software generates multiple soliton solutions containing rational exponential trigonometric and hyperbolic functions when implementing solutions to the resulting problem. A solitary self-reinforcing wave packet referred to as a soliton travels through a medium without shape or velocity modification. Academic researchers value soliton solutions for NPDEs because their higher detail level exceeds standard solution approaches. The extended existence and reliability of soliton solutions provide numerous technical applications in scientific domains. Nonlinear systems benefit from solitons through efficient information transfer processes which also lead to extensive concordance retention[38–40].

Cahn and Hilliard[41] established the Cahn–Hilliard equation in 1958 to depict the process of phase separation of a binary alloy under the critical temperature. This model is significant in a variety of noteworthy scientific processes, including phase separation, phase-ordering dynamics and spinodal decomposition[42,43]. One of the important and essential nonlinear wave phenomena in mathematics and physics is the Cahn–Hilliard (CH) equation has many applications. Exact solutions for the Cahn–Hilliard equation in terms of Weierstrass-elliptic and Jacobi-elliptic functions were obtained[44]. The Cahn–Hilliard model assumes a fundamental role in elucidating phase separation phenomena within physical systems, notably in the context of alloys[45,46]. Exact solutions of convective diffusive Cahn–Hilliard equation using extended direct algebraic method were created[47]. Authors of[48] used the natural decomposition approach with non-singular kernel derivatives to find the solution to nonlinear fractional Gardner and Cahn–Hilliard equations. The approximate solutions of nonlinear temporal fractional models of Gardner and Cahn–Hilliard equations were investigated in[49]. Novel approximations to the fourth-order fractional Cahn–Hillard equations using the Tantawy Technique and variational iteration transform method and the homotopy perturbation transform method were made[50]. Asymptotics of large deviations of finite difference method for stochastic Cahn Hilliard equation were made in[51]. A periodic boundary value problem for the generalized Cahn–Hilliard equation was considered in[52]. The viscous Cahn–Hilliard equation with a degenerate, phase-dependent mobility was considered to investigate weak solution and establish the existence of a solution[53].

The general form of the Cahn–Hilliard equation is given as follows[49]

$$\frac{\partial Y(x,t)}{\partial t} = \lambda \frac{\partial Y(x,t)}{\partial x} + \frac{\partial^2}{\partial x^2}\left(Y(x,t)^3 - Y(x,t) - \frac{\partial^2 Y(x,t)}{\partial x^2} \right), \tag{1.1}$$

in which λ is a free constant. Also, the nonlinear terms denote the chemical potential of the model, while $\frac{\partial^4 Y(x,t)}{\partial x^4}$ denotes the dispersive wave effect of the fourth order system. That equation is often used to simulate phase segregation of a binary alloy system, but many other applications, such as, image processing, planet formation and cancer growth are encountered in the literature. The Cahn–Hilliard equation is a basic partial differential equation in the context of so-called phase field models, which are also called diffuse interface models. It is used to describe the mixture of two conserved components, e.g. two different kinds of atoms in a binary alloy or two different fluids.

This paper employs the novel strategies based on the $\tan(\phi/2)$-expansion method, Jacobi elliptic function expansion scheme, rational multi wave functions and decomposition variational iteration method for the fourth order Cahn–Hilliard equation. This technique allows scholars to compute accurate optical solutions for differential equations of both integer and fractional order. Firstly, the fourth order CH equation is utilized to generate various novel soliton solutions. In addition, we explain how the selection of parameters affects the profile of wave dynamics. This analysis highlights the effectiveness and reliability of the method in constructing soliton using the $\tan(\phi/2)$-expansion method and Jacobi elliptic function expansion scheme to the fourth-order Cahn–Hilliard equation. The soliton solutions offer novel or enlarged types of wave dynamics and stability when compared to previous solutions in the literature. Future technological and scientific research may be able to take new paths thanks to the possible uses of these solutions in water waves, plasmas, and other fields. In this study, we aimed to use the RMWFs and one and two waves to the CH equation to investigate its capability of modeling wave propagation in multidimensional media. As demonstrated by applying the RMWFs, generating exact, implicit solutions for this nonlinear partial differential equation turned out to be highly effective, providing clear evidence for the utility of this method in solving NPDEs.

The sixth-order highly nonlinear generalized Korteweg–de Vries–Kadomtsev–Petviashvili dynamical model was studied with the unified method and the auxiliary method and the stability analysis[54]. The infinitesimal generators of symmetries were found by Lie symmetry invariance analysis using the modified Sardar sub equation method[55]. The Radhakrishnan–Kundu–Lakshmanan equation was investigated using the extended direct algebraic method and the modified Sardar sub-equation method[56]. The (2+1)-dimensional complex modified Korteweg–de Vries system was utilized by the auxiliary equation method and the Hirota bilinear method with novel solitons[57]. A comprehensive nonclassical symmetry analysis of the Tzitzéica-type nonlinear evolution model arising in quantum field theory and nonlinear optics was provided using chaotic behavior and sensitive analysis[58]. An integrable spin reduced Hirota–Maxwel–Bloch equation incorporating Atangana's conformable derivative was investigated modeling magnetization reversal and femtosecond pulse propagation[59]. A (2+1)-dimensional nonlinear electrical transmission line model with the Atangana Baleanu derivative was employed the generalized exponential rational function method including exponential, hyperbolic, periodic, dark, kink, singular, and combo forms[60].

The dynamics of solitons in electrical microtubule model, which describes the propagation of waves in nonlinear dynamical system were analyzed using the solitary wave ansatz to extract these solitons[61]. The optical soliton solutions for the (1+2)-dimensional chiral nonlinear Schrödinger's equation were extracted using the first integral method and the sine-Gordon expansion method[62]. The extreme efficient decision making units in stochastic data envelopment analysis was studied to investigate deterministic equivalent of stochastic mode[63]. The generalized Davey–Stewartson equation that was used to investigate the dynamics of wave propagation in water of finite depth under the effects of gravity force and surface tension using $exp(-\Phi(\xi))$-expansion method, the first integral method and the Sine-Gordon expansion method[64]. The extraction of optical solitons with Radhakrishnan–Kundu–Lakshmanan model in the presence of Kerr law media was investigated with the aid of the modified simple equation and $exp(-\varphi(q))$ method[65]. Generalized trial equation and enhance modified extended tanh function methods were used to the cubic–quintic nonlinear Helmholtz equation[66]. The Chupin Liu's theorem for obtaining the grey and black optical solitons was evaluated to study the optical metamaterials model[67].

Our motivation for using this model with the proposed methods including $\tan(\phi/2)$-expansion method, Jacobi elliptic function expansion scheme, rational multi wave functions is to obtain analytical solutions with a greater variety of solution categories. We also used the semi-analytical decomposition variational iteration method and could find closed form solution for this mentioned model.

This paper is organized into nine sections. "Introduction" section introduces the fourth order CH equation and also its significance in describing the phase ordering dynamics. The "The $\tan(\phi/2)$-expansion method" section discusses $\tan(\phi/2)$-expansion method and application of this method. In the "Main fundamentals concepts of Jacobi elliptic function expansion scheme" section, using Jacobi elliptic function expansion scheme the traveling wave solutions are extracted. "Application of RMWFs for 4th order CH equation" section offers rational multi wave functions and its application. "Decomposition variational iteration method" section presents decomposition variational iteration method. The graphical results and discussion are found in "Graphical results" section. The "Conclusion" section provides the conclusion.

The $\tan(\phi/2)$-expansion method

The approach of the $\tan(\phi/2)$-expansion method (TEM) is related to a first-order equation. We explain the auxiliary equation below before describing the main steps of the considered technique[68]. The equation is given below: Accepting that the arrangement of condition (1.1) can be communicated by the taking after ansatz:

$$Y(\theta) = \mu_0 + \sum_{i=1}^{\phi} \mu_i \tan(Z/2)^i + \sum_{i=1}^{\phi} \lambda_i \cot(Z/2)^i, \tag{2.1}$$

where $\mu_i(0 \leq i \leq \phi), \lambda_i(1 \leq i \leq \phi)$ are the parameters to be decided and $\mu_\phi \neq 0, \lambda_\phi \neq 0$ and $Z = Z(\theta)$ fulfills within the ODE as takes after:

$$Z'(\theta) = q_1 \sin(Z(\theta)) + q_2 \cos(Z(\theta)) + q_3. \tag{2.2}$$

The specific arrangements of condition (2.2) will be examined as:

Product 1: With $F = q_1^2 + q_2^2 - q_3^2 < 0$ and $q_2 - q_3 \neq 0$, then $\Phi(\xi) = 2\arctan\left[\frac{q_1}{q_2-q_3} - \frac{\sqrt{-F}}{q_2-q_3}\tan\left(\frac{\sqrt{-F}}{2}\theta\right)\right]$.

Product 2: With $F > 0$ and $q_2 - q_3 \neq 0$, then $\Phi(\xi) = 2\arctan\left[\frac{q_1}{q_2-q_3} + \frac{\sqrt{F}}{q_2-q_3}\tanh\left(\frac{\sqrt{F}}{2}\theta\right)\right]$.

Product 3: With $F > 0, F_1 = q_1^2 + q_2^2$, $q_2 \neq 0$ and $q_3 = 0$, then $\Phi(\xi) = 2\arctan\left[\frac{q_1}{q_2} + \frac{\sqrt{F_1}}{q_2}\tanh\left(\frac{\sqrt{F_1}}{2}\theta\right)\right]$.

Product 4: With $F < 0, F_2 = q_3^2 - q_1^2$, $q_3 \neq 0$ and $q_2 = 0$, then $\Phi(\xi) = 2\arctan\left[-\frac{q_1}{q_3} + \frac{\sqrt{F_2}}{q_3}\tan\left(\frac{\sqrt{F_2}}{2}\theta\right)\right]$.

Product 5: With $F > 0, F_3 = q_2^2 - q_3^2$, $q_2 - q_3 \neq 0$ and $q_1 = 0$, then $\Phi(\xi) = 2\arctan\left[\sqrt{\frac{q_2+q_3}{q_2-q_3}}\tanh\left(\frac{\sqrt{F_3}}{2}\theta\right)\right]$.

Product 6: With $q_1 = 0$ and $q_3 = 0$, then $\Phi(\xi) = \arctan\left[\frac{e^{2q_2\theta}-1}{e^{2q_2\theta}+1}, \frac{2e^{q_2\theta}}{e^{2q_2\theta}+1}\right]$.

Product 7: With $q_2 = 0$ and $q_3 = 0$, then $\Phi(\xi) = \arctan\left[\frac{2e^{q_1\theta}}{e^{2q_1\theta}+1}, \frac{e^{2q_1\theta}-1}{e^{2q_1\theta}+1}\right]$.

Product 8: With $F = 0$, then $\Phi(\xi) = -2\arctan\left[\frac{(q_2+q_3)(q_1\theta+2)}{q_1^2\theta}\right]$.

Product 9: With $q_1 = 0, q_2 = -q_3$, then $\Phi(\xi) = -2\arctan\left[\frac{1}{q_3\,\theta}\right]$.

Product 10: With $q_1 = 0, q_2 = q_3$, then $\Phi(\xi) = 2\arctan\left[C_1 + q_3\,\theta\right]$.

Product 11: With $q_3 = 0, q_2 = i\,q_1$, then $\Phi(\xi) = -2i\,arctanh\left[\frac{q_1\theta+2}{q_1\theta}\right]$.

For see the rest cases refer to Ref.[69–71]. Also, $\mu_i, \lambda_i (i = 1, 2, \ldots, \phi), q_1, q_2$ and q_3 are also the values to be explored later.

Application of TEM for 4th order CH equation

By using the transformation wave $\theta = m\,x + n\,t$, Eq. (1.1) changes to

$$(n - m\,\lambda)\,\frac{dY}{d\xi} - m^2\,\frac{d^2}{d\xi^2}\left(Y^3 - Y - m^2\,\frac{d^2Y}{d\xi^2}\right) = 0. \tag{2.3}$$

After integrating once Eq. (2.3) respect to θ, we get

$$(n - m\,\lambda)\,Y - m^2\,\frac{d}{d\xi}\left(Y^3 - Y - m^2\,\frac{d^2Y}{d\xi^2}\right) + g_0 = 0. \tag{2.4}$$

Employing the principle of homogeneous balance to the terms $(Y)^2\frac{dY}{d\xi}$ and $\frac{d^3Y}{d\xi^3}$ in Eq. (2.4) yields $\phi = 1$. So, Eq. (2.1) can be re-written as follows:

$$Y(\theta) = \mu_0 + \mu_1\,\tan(\phi/2) + \lambda_1\,\cot(\phi/2). \tag{2.5}$$

Insert the Eq. (2.5) and its derivatives into equation (2.4) get to the related nonlinear arithmetical condition. The sets of solutions are as follow:

Merge I:

$$n = \lambda m,\quad g_0 = 0,\quad \lambda_1 = 0,\quad \mu_0 = -\frac{\sqrt{m^2\left(-q_2^2+q_3^2\right)+2}}{\sqrt{2}},\quad \mu_1 = \frac{(q_2-q_3)m}{\sqrt{2}},\quad q_1 = \frac{\sqrt{m^2\left(-q_2^2+q_3^2\right)+2}}{m}. \tag{2.6}$$

Through product 2, $F = \frac{2}{m^2}$ the exact solution will be as:

$$Y_1(x,t) = -\frac{\sqrt{m^2\left(-q_2^2+q_3^2\right)+2}}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left[\sqrt{m^2\left(-q_2^2+q_3^2\right)+2} + \sqrt{2}\tanh\left(\frac{\sqrt{2}}{2}\,(x+\lambda\,t)\right)\right]. \tag{2.7}$$

Merge II:

$$n = \lambda m,\quad g_0 = 0,\quad \mu_1 = 0,\quad \mu_0 = \frac{\sqrt{m^2\left(-q_2^2+q_3^2\right)+2}}{\sqrt{2}},\quad \lambda_1 = \frac{(q_2+q_3)m}{\sqrt{2}},\quad q_1 = \frac{\sqrt{m^2\left(-q_2^2+q_3^2\right)+2}}{m}. \tag{2.8}$$

Through product 2, $F = \frac{2}{m^2}$ the exact solution will be as:

$$Y_2(x,t) = \frac{\sqrt{m^2\left(-q_2^2+q_3^2\right)+2}}{\sqrt{2}} + \frac{(q_2+q_3)}{\sqrt{2}}\left[\frac{\sqrt{m^2\left(-q_2^2+q_3^2\right)+2}}{q_2-q_3} + \frac{\sqrt{2}}{q_2-q_3}\tanh\left(\frac{\sqrt{2}}{2}\,(x+\lambda\,t)\right)\right]^{-1}. \tag{2.9}$$

Merge III:

$$m = \frac{1}{\sqrt{-q_2^2+q_3^2}}, \quad n = \frac{\lambda}{\sqrt{-q_2^2+q_3^2}}, \quad g_0 = 0, \quad \mu_0 = 0, \quad \lambda_1 = \sqrt{-\frac{q_2+q_3}{2q_2-2q_3}}, \quad \mu_1 = \sqrt{-\frac{q_2-q_3}{2q_2+2q_3}}, \ q_1 = 0. \tag{2.10}$$

Through product 5, $F_3 = q_2^2 - q_3^2$ the soliton-singular solution will be as:

$$Y_3(x,t) = \sqrt{-\frac{q_2-q_3}{2q_2+2q_3}}\left[\sqrt{\frac{q_2+q_3}{q_2-q_3}}\tanh\left(\frac{\sqrt{F_3}}{2}\theta\right)\right] + \sqrt{-\frac{q_2+q_3}{2q_2-2q_3}}\left[\sqrt{\frac{q_2-q_3}{q_2+q_3}}\coth\left(\frac{\sqrt{F_3}}{2}\theta\right)\right], \quad \theta = \frac{1}{\sqrt{-q_2^2+q_3^2}}(x+\lambda t). \tag{2.11}$$

Through product 6, the kink solution will be as:

$$Y_4(x,t) = \frac{i}{\sqrt{2}}\tan\left(\frac{1}{2}\arctan\left[\frac{e^{2q_2\theta}-1}{e^{2q_2\theta}+1}, \ \frac{2e^{q_2\theta}}{e^{2q_2\theta}+1}\right]\right) + \frac{i}{\sqrt{2}}\cot\left(\frac{1}{2}\arctan\left[\frac{e^{2q_2\theta}-1}{e^{2q_2\theta}+1}, \ \frac{2e^{q_2\theta}}{e^{2q_2\theta}+1}\right]\right), \quad \theta = \frac{1}{\sqrt{-q_2^2}}(x+\lambda t). \tag{2.12}$$

Merge IV:

$$m = \frac{1}{\sqrt{2q_2^2-2q_3^2}}, \quad n = \frac{\lambda}{\sqrt{2q_2^2-2q_3^2}}, \quad g_0 = 0, \quad \mu_0 = 0, \quad \lambda_1 = \sqrt{-\frac{q_2+q_3}{4q_2-4q_3}}, \quad \mu_1 = \sqrt{-\frac{q_2-q_3}{4q_2+4q_3}}, \ q_1 = 0. \tag{2.13}$$

Through product 5, $F_3 = q_2^2 - q_3^2$ the soliton-singular solution will be as:

$$Y_5(x,t) = \sqrt{-\frac{q_2-q_3}{4q_2+4q_3}}\left[\sqrt{\frac{q_2+q_3}{q_2-q_3}}\tanh\left(\frac{\sqrt{F_3}}{2}\theta\right)\right] + \sqrt{-\frac{q_2+q_3}{4q_2-4q_3}}\left[\sqrt{\frac{q_2-q_3}{q_2+q_3}}\coth\left(\frac{\sqrt{F_3}}{2}\theta\right)\right], \quad \theta = \frac{1}{\sqrt{2q_2^2-2q_3^2}}(x+\lambda t). \tag{2.14}$$

Through product 6, the kink solution will be as:

$$Y_6(x,t) = \frac{i}{2}\tan\left(\frac{1}{2}\arctan\left[\frac{e^{2q_2\theta}-1}{e^{2q_2\theta}+1}, \ \frac{2e^{q_2\theta}}{e^{2q_2\theta}+1}\right]\right) + \frac{i}{2}\cot\left(\frac{1}{2}\arctan\left[\frac{e^{2q_2\theta}-1}{e^{2q_2\theta}+1}, \ \frac{2e^{q_2\theta}}{e^{2q_2\theta}+1}\right]\right), \quad \theta = \frac{1}{\sqrt{2q_2^2}}(x+\lambda t). \tag{2.15}$$

Merge V:

$$m = \frac{i}{12\sqrt{2}q_3}, \quad n = \frac{i\lambda}{12\sqrt{2}q_3}, \quad g_0 = 0, \quad \mu_0 = 0, \quad \lambda_1 = \frac{3}{4}i, \quad \mu_1 = -\frac{2}{3}i, \ q_1 = 0, \quad q_2 = 17q_3. \tag{2.16}$$

Through product 5, $F_3 = 288q_3^2$ the soliton-singular solution will be as:

$$Y_7(x,t) = -\frac{i}{\sqrt{2}}\ \tanh\left(\frac{i\sqrt{288}}{24\sqrt{2}}(x+\lambda t)\right) + \frac{i}{\sqrt{2}}\ \coth\left(\frac{i\sqrt{288}}{24\sqrt{2}}(x+\lambda t)\right) = \frac{1}{\sqrt{2}}\ \tan\left(\frac{\sqrt{288}}{24\sqrt{2}}(x+\lambda t)\right) + \frac{1}{\sqrt{2}}\ \cot\left(\frac{\sqrt{288}}{24\sqrt{2}}(x+\lambda t)\right). \tag{2.17}$$

Merge VI:

$$m = \pm\frac{1}{24q_3}, \quad n = \pm\frac{\lambda}{24q_3}, \quad g_0 = 0, \quad \mu_0 = 0, \quad \lambda_1 = \frac{3}{4\sqrt{2}}, \quad \mu_1 = \frac{2}{3\sqrt{2}}, \ q_1 = 0, \quad q_2 = 17q_3. \tag{2.18}$$

Through product 5, $F_3 = 288q_3^2$ the soliton-singular solution will be as:

$$Y_8(x,t) = \frac{1}{2}\ \tanh\left(\frac{\sqrt{288}}{48}(\pm x \pm \lambda t)\right) + \frac{1}{2}\ \coth\left(\frac{\sqrt{288}}{48}(\pm x \pm \lambda t)\right). \tag{2.19}$$

Merge VII:

$$m = \frac{i}{4\sqrt{3}q_3}, \quad n = \frac{i\lambda}{4\sqrt{3}q_3}, \quad g_0 = 0, \quad \mu_0 = 0, \quad \lambda_1 = \frac{\sqrt{6}}{3}i, \quad \mu_1 = -\frac{\sqrt{6}}{4}i, \ q_1 = 0, \quad q_2 = 7q_3. \tag{2.20}$$

Through product 5, $F_3 = 48q_3^2$ the periodic-singular solution will be as:

$$Y_9(x,t) = -\frac{i}{2}\ \tanh\left(\frac{i}{2}(x+\lambda t)\right) + \frac{i}{2}\ \coth\left(\frac{i}{2}(x+\lambda t)\right) = \frac{1}{2}\ \tan\left(\frac{1}{2}(x+\lambda t)\right) + \frac{1}{2}\ \cot\left(\frac{1}{2}(x+\lambda t)\right). \tag{2.21}$$

Merge VIII:

$$m=\frac{1}{4\sqrt{6q_3}},\quad n=\frac{\lambda}{4\sqrt{6q_3}},\quad g_0=0,\quad \mu_0=0,\quad \lambda_1=\frac{1}{\sqrt{3}},\quad \mu_1=-\frac{3}{4\sqrt{3}},\ q_1=0,\quad q_2=7q_3. \tag{2.22}$$

Through product 5, $F_3=48q_3^2$ the periodic-singular solution will be as:

$$Y_{10}(x,t)=\frac{1}{2}\,\tanh\left(\frac{1}{2\sqrt{2}}(x+\lambda\,t)\right)+\frac{1}{2}\,\coth\left(\frac{1}{2\sqrt{2}}(x+\lambda\,t)\right). \tag{2.23}$$

Merge IX:

$$m=\frac{i}{\sqrt{3q_3}},\quad n=\frac{i\lambda}{\sqrt{3q_3}},\quad g_0=0,\quad \mu_0=0,\quad \lambda_1=\frac{\sqrt{6}}{2}i,\quad \mu_1=-\frac{\sqrt{6}}{2}i,\ q_1=0,\quad q_2=2q_3. \tag{2.24}$$

Through product 5, $F_3=3q_3^2$ the periodic-singular solution will be as:

$$\begin{gathered}Y_{11}(x,t)=-\frac{3\sqrt{2}i}{2}\,\tanh\left(\frac{i}{2}(x+\lambda\,t)\right)+\frac{3\sqrt{2}i}{2}\,\coth\left(\frac{i}{2}(x+\lambda\,t)\right)=\\ \frac{3\sqrt{2}}{2}\,\tan\left(\frac{1}{2}(x+\lambda\,t)\right)+\frac{3\sqrt{2}}{2}\,\cot\left(\frac{1}{2}(x+\lambda\,t)\right).\end{gathered} \tag{2.25}$$

Merge X:

$$m=\frac{1}{\sqrt{6q_3}},\quad n=\frac{\lambda}{\sqrt{6q_3}},\quad g_0=0,\quad \mu_0=0,\quad \lambda_1=\frac{\sqrt{3}}{2},\quad \mu_1=\frac{\sqrt{3}}{6},\ q_1=0,\quad q_2=2q_3. \tag{2.26}$$

Through product 5, $F_3=3q_3^2$ the periodic-singular solution will be as:

$$Y_{12}(x,t)=\frac{1}{2}\,\tanh\left(\frac{1}{2\sqrt{2}}(x+\lambda\,t)\right)+\frac{1}{2}\,\coth\left(\frac{1}{2\sqrt{2}}(x+\lambda\,t)\right). \tag{2.27}$$

Merge XI:

$$n=\lambda\,m,\quad g_0=0,\quad \lambda_1=-\frac{(q_2+q_3)m}{\sqrt{2}},\quad \mu_0=\mu_0=-\frac{\sqrt{2m^2\left(q_2^2-q_3^2\right)+2}}{\sqrt{2}},\quad \mu_1=\frac{(q_2-q_3)m}{\sqrt{2}},\ q_1=\frac{\sqrt{2m^2\left(q_2^2-q_3^2\right)+2}}{m}. \tag{2.28}$$

Through product 1, $F=\frac{3m^2q_2^2-3m^2q_3^2+2}{m^2}<0$ the soliton solution will be as:

$$\begin{aligned}Y_{13}(x,t)=&-\frac{\sqrt{2m^2\left(q_2^2-q_3^2\right)+2}}{\sqrt{2}}+\frac{(q_2-q_3)m}{\sqrt{2}}\left[\frac{\sqrt{2m^2\left(q_2^2-q_3^2\right)+2}}{m(q_2-q_3)}-\frac{\sqrt{3m^2(q_3^2-q_2^2)-2}}{m(q_2-q_3)}\tan\left(\frac{\sqrt{3m^2(q_3^2-q_2^2)-2}}{2}(x+\lambda\,t)\right)\right]\\ &-\frac{(q_2+q_3)m}{\sqrt{2}}\left[\frac{\sqrt{2m^2\left(q_2^2-q_3^2\right)+2}}{m(q_2-q_3)}-\frac{\sqrt{3m^2(q_3^2-q_2^2)-2}}{m(q_2-q_3)}\tan\left(\frac{\sqrt{3m^2(q_3^2-q_2^2)-2}}{2}(x+\lambda\,t)\right)\right]^{-1}.\end{aligned} \tag{2.29}$$

Through product 2, $F=\frac{3m^2q_2^2-3m^2q_3^2+2}{m^2}>0$ the soliton solution will be as:

$$\begin{aligned}Y_{14}(x,t)=&-\frac{\sqrt{2m^2\left(q_2^2-q_3^2\right)+2}}{\sqrt{2}}+\frac{(q_2-q_3)m}{\sqrt{2}}\left[\frac{\sqrt{2m^2\left(q_2^2-q_3^2\right)+2}}{m(q_2-q_3)}+\frac{\sqrt{-3m^2(q_3^2-q_2^2)+2}}{m(q_2-q_3)}\tanh\left(\frac{\sqrt{-3m^2(q_3^2-q_2^2)+2}}{2}(x+\lambda\,t)\right)\right]\\ &-\frac{(q_2+q_3)m}{\sqrt{2}}\left[\frac{\sqrt{2m^2\left(q_2^2-q_3^2\right)+2}}{m(q_2-q_3)}+\frac{\sqrt{-3m^2(q_3^2-q_2^2)+2}}{m(q_2-q_3)}\tanh\left(\frac{\sqrt{-3m^2(q_3^2-q_2^2)+2}}{2}(x+\lambda\,t)\right)\right]^{-1}.\end{aligned} \tag{2.30}$$

Through product 6, the kink soliton solution will be as:

$$\begin{aligned}Y_{15}(x,t)=&-\frac{\sqrt{2m^2\left(q_2^2\right)+2}}{\sqrt{2}}+\frac{q_2m}{\sqrt{2}}\tan\left(\frac{1}{2}\arctan\left[\frac{e^{2q_2\theta}-1}{e^{2q_2\theta}+1},\ \frac{2e^{q_2\theta}}{e^{2q_2\theta}+1}\right]\right)\\ &-\frac{(q_2)m}{\sqrt{2}}\cot\left(\frac{1}{2}\arctan\left[\frac{e^{2q_2\theta}-1}{e^{2q_2\theta}+1},\ \frac{2e^{q_2\theta}}{e^{2q_2\theta}+1}\right]\right),\quad \theta=m(x+\lambda t).\end{aligned} \tag{2.31}$$

Main fundamentals concepts of Jacobi elliptic function expansion scheme

The following is the expression for the NLPDE:

$$N(u,u_x,u_t,u_{xx},u_{xxx},\ldots)=0, \tag{3.33}$$

where N is a polynomial that contains $u(x,t)$ and its partial derivative. Consider the following transformation

$$u(x,t)=Y(\theta),\quad \theta=m\,x+n\,t. \tag{3.34}$$

In this way, Eq. (3.33) turns into a NODE, which is described as

$$M(Y,Y',Y'',\ldots)=0. \tag{3.35}$$

No.	r_0	r_2	r_4	$F(\theta)$
j = 1	1	$-1-\chi^2$	χ^2	$sn(\theta)$
j = 2	$1-\chi^2$	$2\chi^2-1$	$-\chi^2$	$cn(\theta)$
j = 3	χ^2	$-1-\chi^2$	1	$ns(\theta)$
j = 4	$-\chi^2$	$-1+2\chi^2$	$1-\chi^2$	$nc(\theta)$
j = 5	$\frac{1}{4}$	$\frac{1-2\chi^2}{2}$	$\frac{1}{4}$	$ns(\theta)\mp cs(\theta)$
j = 6	$\frac{1-\chi^2}{4}$	$\frac{1+\chi^2}{2}$	$\frac{1-\chi^2}{4}$	$nc(\theta)\mp sc(\theta)$ or $\frac{cn(\theta)}{1\mp sn(\theta)}$
j = 7	$\frac{1}{4}$	$\frac{\chi^2-2}{2}$	$\frac{\chi^2}{4}$	$\frac{sn(\theta)}{1\mp dn(\theta)}$
j = 8	1	$2-\chi^2$	$1-\chi^2$	$sc(\theta)$
j = 9	$1-\chi^2$	$2-\chi^2$	1	$cs(\theta)$
j = 10	χ^2-1	$2-\chi^2$	-1	$dn(\theta)$
j = 11	$\frac{\chi^4}{4}$	$\frac{\chi^2-2}{2}$	$\frac{1}{4}$	$ns(\theta)\mp ds(\theta)$
j = 12	$\frac{1}{4}$	$\frac{1+\chi^2}{2}$	$\frac{(1-\chi^2)^2}{4}$	$\frac{sn(\theta)}{dn(\theta)\mp cn(\theta)}$

Table 1. The Jacobi elliptic functions.

No.		$\chi\rightarrow 0$	$\chi\rightarrow 1$
j = 1	$sn(\theta)$	$\sin(\theta)$	$\tanh(\theta)$
j = 2	$cn(\theta)$	$\cos(\theta)$	$sech(\theta)$
j = 3	$dn(\theta)$	1	$sech(\theta)$
j = 4	$cd(\theta)$	$\cos(\theta)$	1
j = 5	$sd(\theta)$	$\sin(\theta)$	$\sinh(\theta)$
j = 6	$nd(\theta)$	1	$\cosh(\theta)$
j = 7	$dc(\theta)$	$sec(\theta)$	1
j = 8	$nc(\theta)$	$sec(\theta)$	$\cosh(\theta)$
j = 9	$sc(\theta)$	$\tan(\theta)$	$\sinh(\theta)$
j = 10	$ns(\theta)$	$csc(\theta)$	$\coth(\theta)$
j = 11	$ds(\theta)$	$csc(\theta)$	$csch(\theta)$
j = 12	$cs(\theta)$	$\cot(\theta)$	$csch(\theta)$

Table 2. The Jacobi functions for $\chi\rightarrow 0$ and $\chi\rightarrow 1$.

To get to the exact solutions, we have:

$$Y(\theta)=\mu_0+\sum_{i=0}^{\phi}\left(\frac{F(\theta)}{1+F^2(\theta)}\right)^{i-1}\left(\mu_i\frac{F(\theta)}{1+F^2(\theta)}+\lambda_i\frac{1-F^2(\theta)}{1+F^2(\theta)}\right). \tag{3.36}$$

where μ_i and λ_i are free constants ($\mu_i\neq 0$ or $\lambda_i\neq 0$). The function $F(\theta)$ is expressed as follows:

$$F'(\theta)=\sqrt{r_0+r_2F^2(\theta)+r_4F^4(\theta)}. \tag{3.37}$$

where r_0, r_2, and r_4 are constants, then the solution of Eq. (3.37) is given below Tables 1 and 2, with the conditions of r_0, r_2, and r_4. Table 1 contains the Jacobi elliptic functions, while Table 2 includes functions of trigonometric and hyperbolic functions.

Application of JEFES for 4th order CH equation

According to Eq. (2.4) and obtained balance number, using (3.36) solution can be re-written as follows:

$$Y(\theta)=\mu_0+\mu_1\frac{F(\theta)}{1+F^2(\theta)}+\lambda_1\frac{1-F^2(\theta)}{1+F^2(\theta)}. \tag{3.38}$$

With their derivatives

$$
\begin{aligned}
\frac{dY(\theta)}{d\theta} &= \frac{\left(\frac{d}{d\theta}F(\theta)\right)\mu_1}{1+F(\theta)^2} - \frac{2F(\theta)^2\mu_1\left(\frac{d}{d\theta}F(\theta)\right)}{\left(1+F(\theta)^2\right)^2} - \frac{2F(\theta)\left(\frac{d}{d\theta}F(\theta)\right)\lambda_1}{1+F(\theta)^2} - \frac{2\left(1-F(\theta)^2\right)\lambda_1 F(\theta)\left(\frac{d}{d\theta}F(\theta)\right)}{\left(1+F(\theta)^2\right)^2},\\
\frac{d^2Y(\theta)}{d\theta^2} &= \frac{\left(\frac{d^2}{d\theta^2}F(\theta)\right)\mu_1}{1+F(\theta)^2} - \frac{6\left(\frac{d}{d\theta}F(\theta)\right)^2\mu_1 F(\theta)}{\left(1+F(\theta)^2\right)^2} + \frac{8F(\theta)^3\mu_1\left(\frac{d}{d\theta}F(\theta)\right)^2}{\left(1+F(\theta)^2\right)^3} - \frac{2F(\theta)^2\mu_1\left(\frac{d^2}{d\theta^2}F(\theta)\right)}{\left(1+F(\theta)^2\right)^2} - \frac{2\left(\frac{d}{d\theta}F(\theta)\right)^2\lambda_1}{1+F(\theta)^2}\\
&\quad - \frac{2F(\theta)\left(\frac{d^2}{d\theta^2}F(\theta)\right)\lambda_1}{1+F(\theta)^2} + \frac{8F(\theta)^2\left(\frac{d}{d\theta}F(\theta)\right)^2\lambda_1}{\left(1+F(\theta)^2\right)^2} + \frac{8\left(1-F(\theta)^2\right)\lambda_1 F(\theta)^2\left(\frac{d}{d\theta}F(\theta)\right)^2}{\left(1+F(\theta)^2\right)^3} - \frac{2\left(1-F(\theta)^2\right)\lambda_1\left(\frac{d}{d\theta}F(\theta)\right)^2}{\left(1+F(\theta)^2\right)^2}\\
&\quad - \frac{2\left(1-F(\theta)^2\right)\lambda_1 F(\theta)\left(\frac{d^2}{d\theta^2}F(\theta)\right)}{\left(1+F(\theta)^2\right)^2},\\
\frac{d^3Y(\theta)}{d\theta^3} &= \frac{24F(\theta)^3\mu_1\left(\frac{d}{d\theta}F(\theta)\right)\left(\frac{d^2}{d\theta^2}F(\theta)\right)}{\left(1+F(\theta)^2\right)^3} + \frac{24F(\theta)^2\left(\frac{d^2}{d\theta^2}F(\theta)\right)\lambda_1\left(\frac{d}{d\theta}F(\theta)\right)}{\left(1+F(\theta)^2\right)^2} - \frac{48\left(1-F(\theta)^2\right)\lambda_1 F(\theta)^3\left(\frac{d}{d\theta}F(\theta)\right)^3}{\left(1+F(\theta)^2\right)^4}\\
&\quad - \frac{6\left(1-F(\theta)^2\right)\lambda_1\left(\frac{d}{d\theta}F(\theta)\right)\left(\frac{d^2}{d\theta^2}F(\theta)\right)}{\left(1+F(\theta)^2\right)^2} - \frac{2\left(1-F(\theta)^2\right)\lambda_1 F(\theta)\left(\frac{d^3}{d\theta^3}F(\theta)\right)}{\left(1+F(\theta)^2\right)^2} + \frac{\left(\frac{d^3}{d\theta^3}F(\theta)\right)\mu_1}{1+F(\theta)^2} - \frac{6\left(\frac{d}{d\theta}F(\theta)\right)^3\mu_1}{\left(1+F(\theta)^2\right)^2}\\
&\quad + \frac{48\left(\frac{d}{d\theta}F(\theta)\right)^3\mu_1 F(\theta)^2}{\left(1+F(\theta)^2\right)^3} - \frac{48F(\theta)^4\mu_1\left(\frac{d}{d\theta}F(\theta)\right)^3}{\left(1+F(\theta)^2\right)^4} - \frac{2F(\theta)^2\mu_1\left(\frac{d^3}{d\theta^3}F(\theta)\right)}{\left(1+F(\theta)^2\right)^2} - \frac{6\left(\frac{d}{d\theta}F(\theta)\right)\lambda_1\left(\frac{d^2}{d\theta^2}F(\theta)\right)}{1+F(\theta)^2} + \frac{24\left(\frac{d}{d\theta}F(\theta)\right)^3\lambda_1 F(\theta)}{\left(1+F(\theta)^2\right)^2}\\
&\quad - \frac{2F(\theta)\left(\frac{d^3}{d\theta^3}F(\theta)\right)\lambda_1}{1+F(\theta)^2} - \frac{48F(\theta)^3\left(\frac{d}{d\theta}F(\theta)\right)^3\lambda_1}{\left(1+F(\theta)^2\right)^3} + \frac{24\left(1-F(\theta)^2\right)\lambda_1 F(\theta)\left(\frac{d}{d\theta}F(\theta)\right)^3}{\left(1+F(\theta)^2\right)^3} + \frac{24\left(1-F(\theta)^2\right)\lambda_1 F(\theta)^2\left(\frac{d}{d\theta}F(\theta)\right)\left(\frac{d^2}{d\theta^2}F(\theta)\right)}{\left(1+F(\theta)^2\right)^3}\\
&\quad - \frac{18\left(\frac{d^2}{d\theta^2}F(\theta)\right)\mu_1 F(\theta)\left(\frac{d}{d\theta}F(\theta)\right)}{\left(1+F(\theta)^2\right)^2}.
\end{aligned}
\tag{3.39}
$$

Insert the Eq. (3.38) and its derivatives (3.39) into Eq. (2.4) get to the related nonlinear arithmetical condition. The sets of solutions are as follow:

Merge I.

$$
\begin{aligned}
&m = \frac{\sqrt{-2\left(3r_0^2-4r_0r_2-6r_0r_4+4r_2^2-4r_2r_4+3r_4^2\right)(r_0-r_2+r_4)}}{3r_0^2-4r_0r_2-6r_0r_4+4r_2^2-4r_2r_4+3r_4^2}, \quad n = \frac{\lambda\sqrt{-2\left(3r_0^2-4r_0r_2-6r_0r_4+4r_2^2-4r_2r_4+3r_4^2\right)(r_0-r_2+r_4)}}{3r_0^2-4r_0r_2-6r_0r_4+4r_2^2-4r_2r_4+3r_4^2},\\
&g_0 = 0, \quad \lambda_1 = \frac{2r_0-2r_2+2r_4}{\sqrt{3r_0^2-4r_0r_2-6r_0r_4+4r_2^2-4r_2r_4+3r_4^2}}, \quad \mu_0 = \frac{r_0-r_4}{\sqrt{3r_0^2-4r_0r_2-6r_0r_4+4r_2^2-4r_2r_4+3r_4^2}},
\end{aligned}
\tag{3.40}
$$

$$
\begin{aligned}
Y(\theta) &= -\frac{(r_0-2r_2+3r_4)F(\theta)^2-3r_0+2r_2-r_4}{\sqrt{3r_0^2-4r_0r_2-6r_0r_4+4r_2^2-4r_2r_4+3r_4^2}\left(1+F(\theta)^2\right)},\\
\theta &= \frac{\sqrt{-2\left(3r_0^2-4r_0r_2-6r_0r_4+4r_2^2-4r_2r_4+3r_4^2\right)(r_0-r_2+r_4)}}{3r_0^2-4r_0r_2-6r_0r_4+4r_2^2-4r_2r_4+3r_4^2}(x+\lambda t).
\end{aligned}
\tag{3.41}
$$

Case 1 Take the case $r_4 = \chi^2, r_2 = -1-\chi^2, r_0 = 1$ and $F(\theta) = sn(\theta,\chi)$. Then we can write the solution according (3.41) as

$$
Y(\theta) = -\frac{\left(5\chi^2+3\right)sn^2(\theta,\chi)-5-3\chi^2}{\sqrt{7-2\chi^2+4\left(-\chi^2-1\right)^2-4\left(-\chi^2-1\right)\chi^2+3\chi^4}\left(1+sn^2(\theta,\chi)\right)}, \quad \theta = \frac{2\sqrt{-\left(11\chi^4+10\chi^2+11\right)\left(\chi^2+1\right)}(\lambda t+x)}{11\chi^4+10\chi^2+11}.
\tag{3.42}
$$

When $\chi = 0$, then (3.42) gives the soliton solution as

$$
Y_1(x,t) = -\frac{3\sin^2\left(\frac{2\sqrt{11}i}{11}(x+\lambda t)\right)-5}{\sqrt{11}\left(1+\sin^2\left(\frac{2\sqrt{11}i}{11}(x+\lambda t)\right)\right)} = -\frac{-3\sinh^2\left(\frac{2\sqrt{11}}{11}(x+\lambda t)\right)-5}{\sqrt{11}\left(1-\sinh^2\left(\frac{2\sqrt{11}}{11}(x+\lambda t)\right)\right)}.
\tag{3.43}
$$

When $\chi = 1$, then (3.42) gives the periodic solution as

$$
Y_2(x,t) = \sqrt{2}\,\frac{1-\tanh^2\left(\frac{i}{2}(x+\lambda t)\right)}{1+\tanh^2\left(\frac{i}{2}(x+\lambda t)\right)} = \sqrt{2}\,\frac{1+\tan^2\left(\frac{1}{2}(x+\lambda t)\right)}{1-\tan^2\left(\frac{1}{2}(x+\lambda t)\right)}.
\tag{3.44}
$$

Case 2 Take the case $r_4 = -\chi^2, r_2 = 2\chi^2-1, r_0 = 1-\chi^2$ and $F(\theta) = cn(\theta,\chi)$. Then we can write the solution according (3.41) as

$$
Y(\theta) = \frac{\left(8\chi^2-3\right)cn^2(\theta,\chi)-8\chi^2+5}{\sqrt{32\chi^4-32\chi^2+11}\left(cn^2(\theta,\chi)+1\right)}, \qquad \theta = \frac{2\sqrt{\left(32\chi^4-32\chi^2+11\right)\left(2\chi^2-1\right)}(\lambda t+x)}{32\chi^4-32\chi^2+11}.
\tag{3.45}
$$

When $\chi = 0$, then (3.42) gives the soliton solution as

$$
Y_3(x,t) = \frac{-3\cos^2\left(\frac{2\sqrt{11}i}{11}(x+\lambda t)\right)+5}{\sqrt{11}\left(\cos^2\left(\frac{2\sqrt{11}i}{11}(x+\lambda t)\right)+1\right)} = \frac{-3\cosh^2\left(\frac{2\sqrt{11}}{11}(x+\lambda t)\right)+5}{\sqrt{11}\left(\cosh^2\left(\frac{2\sqrt{11}}{11}(x+\lambda t)\right)+1\right)}.
\tag{3.46}
$$

When $\chi = 1$, then (3.42) gives the soliton solution as

$$Y_4(x,t)=\frac{-3\,sech^2\left(\frac{2\sqrt{11}}{11}\,(x+\lambda\,t)\right)+5}{\sqrt{11}\left(sech^2\left(\frac{2\sqrt{11}}{11}\,(x+\lambda\,t)\right)+1\right)}. \tag{3.47}$$

Case 3 Take the case $r_4=1, r_2=-\chi^2-1, r_0=\chi^2$ and $F(\theta)=ns(\theta,\chi)$. Then we can write the solution according (3.41) as

$$Y(\theta)=-\frac{\left(3\chi^2+5\right)ns^2(\theta,\chi)-5\chi^2-3}{\sqrt{11\chi^4+10\chi^2+11}\left(ns^2(\theta,\chi)+1\right)},\qquad \theta=\frac{2\sqrt{-\left(11\chi^4+10\chi^2+11\right)\left(\chi^2+1\right)}\,(\lambda t+x)}{11\chi^4+10\chi^2+11}. \tag{3.48}$$

When $\chi=0$, then (3.42) gives the singular soliton solution as

$$Y_5(x,t)=-\frac{5\csc^2\left(\frac{2\sqrt{11}i}{11}\,(x+\lambda\,t)\right)-3}{\sqrt{11}\left(\csc^2\left(\frac{2\sqrt{11}i}{11}\,(x+\lambda\,t)\right)+1\right)}=-\frac{-5\,csch^2\left(\frac{2\sqrt{11}}{11}\,(x+\lambda\,t)\right)-3}{\sqrt{11}\left(-csch^2\left(\frac{2\sqrt{11}}{11}\,(x+\lambda\,t)\right)+1\right)}. \tag{3.49}$$

When $\chi=1$, then (3.42) gives the singular periodic solution as

$$Y_6(x,t)=-\frac{8\,\coth^2\left(\frac{i}{2}\,(x+\lambda\,t)\right)-8}{\sqrt{32}\left(\coth^2\left(\frac{i}{2}\,(x+\lambda\,t)\right)+1\right)}=-\frac{-8\,\cot^2\left(\frac{1}{2}\,(x+\lambda\,t)\right)-8}{\sqrt{32}\left(-\cot^2\left(\frac{1}{2}\,(x+\lambda\,t)\right)+1\right)}. \tag{3.50}$$

Case 4 Take the case $r_4=1-\chi^2, r_2=2\chi^2-1, r_0=-\chi^2$ and $F(\theta)=nc(\theta,\chi)$. Then we can write the solution according (3.41) as

$$Y(\theta)=\frac{\left(8\chi^2-5\right)nc^2(\theta,\chi)-8\chi^2+3}{\sqrt{32\chi^4-32\chi^2+11}\left(nc^2(\theta,\chi)+1\right)},\qquad \theta=\frac{2\sqrt{\left(32\chi^4-32\chi^2+11\right)\left(2\chi^2-1\right)}\,(\lambda t+x)}{32\chi^4-32\chi^2+11}. \tag{3.51}$$

When $\chi=0$, then (3.42) gives the bright soliton solution as

$$Y_7(x,t)=\frac{-5\sec^2\left(\frac{2\sqrt{11}i}{11}\,(x+\lambda\,t)\right)3}{\sqrt{11}\left(\sec^2\left(\frac{2\sqrt{11}i}{11}\,(x+\lambda\,t)\right)+1\right)}=\frac{-5sech^2\left(\frac{2\sqrt{11}}{11}\,(x+\lambda\,t)\right)3}{\sqrt{11}\left(sech^2\left(\frac{2\sqrt{11}}{11}\,(x+\lambda\,t)\right)+1\right)}. \tag{3.52}$$

When $\chi=1$, then (3.42) gives the soliton solution as

$$Y_8(x,t)=\frac{3\cosh^2\left(\frac{2\sqrt{11}}{11}\,(x+\lambda\,t)\right)-5}{\sqrt{11}\left(\cosh^2\left(\frac{2\sqrt{11}}{11}\,(x+\lambda\,t)\right)+1\right)}. \tag{3.53}$$

Case 5 Take the case $r_4=\frac{1}{4}, r_2=\frac{1-2\chi^2}{2}, r_0=\frac{1}{4}$ and $F(\theta)=ns(\theta,\chi)\mp cs(\theta,\chi)$. Then we can write the solution according (3.41) as

$$Y(\theta)=-\frac{\sqrt{2}\chi^2\left((ns(\theta,\chi)\mp cs(\theta,\chi))^2-1\right)}{\sqrt{2\chi^4-\chi^2}\left((ns(\theta,\chi)\mp cs(\theta,\chi))^2+1\right)},\qquad \theta=\frac{\sqrt{-\left(2\chi^2-1\right)\chi^4}\,(\lambda t+x)}{\left(2\chi^2-1\right)\chi^2}. \tag{3.54}$$

When $\chi=1$, then (3.42) gives the soliton solution as

$$Y_9(x,t)=-\frac{\sqrt{2}\left((\coth(i\,(x+\lambda\,t))\mp csch(i\,(x+\lambda\,t)))^2-1\right)}{\left((\coth(i\,(x+\lambda\,t))\mp csch(i\,(x+\lambda\,t)))^2+1\right)}=-\frac{\sqrt{2}\left((-\cot((x+\lambda\,t))\pm\csc((x+\lambda\,t)))^2-1\right)}{\left((-\cot((x+\lambda\,t))\pm\csc((x+\lambda\,t)))^2+1\right)}. \tag{3.55}$$

Case 6 Take the case $r_4=\frac{1-\chi^2}{4}, r_2=\frac{1+\chi^2}{2}, r_0=\frac{1-\chi^2}{4}$ and $F(\theta)=nc(\theta,\chi)\mp sc(\theta,\chi)$ or $\frac{cn(\theta,\chi)}{1\mp sn(\theta,\chi)}$. Then we can write the solution according (3.41) as

$$Y(\theta)=\frac{\sqrt{2}\chi^2\left((nc(\theta,\chi)\mp sc(\theta,\chi))^2-1\right)}{\sqrt{\chi^4+\chi^2}\left((nc(\theta,\chi)\mp sc(\theta,\chi))^2+1\right)},\quad or\quad Y(\theta)=\frac{\sqrt{2}\chi^2\left(\left(\frac{cn(\theta,\chi)}{1\mp sn(\theta,\chi)}\right)^2-1\right)}{\sqrt{\chi^4+\chi^2}\left(\left(\frac{cn(\theta,\chi)}{1\mp sn(\theta,\chi)}\right)^2+1\right)},$$
$$\theta=\frac{\sqrt{\chi^4\left(\chi^2+1\right)}\,(\lambda t+x)}{\chi^2\left(\chi^2+1\right)}. \tag{3.56}$$

When $\chi=1$, then (3.42) gives the soliton solution as

$$Y_{10}(x,t)=\frac{\left(\cosh\left(\frac{\sqrt{2}}{2}\,(x+\lambda\,t)\right)\mp\sinh\left(\frac{\sqrt{2}}{2}\,(x+\lambda\,t)\right)\right)^2-1}{\left(\cosh\left(\frac{\sqrt{2}}{2}\,(x+\lambda\,t)\right)\mp\sinh\left(\frac{\sqrt{2}}{2}\,(x+\lambda\,t)\right)\right)^2+1},$$
$$Y_{11}(x,t)=\frac{\left(\frac{sech\left(\frac{\sqrt{2}}{2}\,(x+\lambda\,t)\right)}{1\mp\tanh\left(\frac{\sqrt{2}}{2}\,(x+\lambda\,t)\right)}\right)^2-1}{\left(\frac{sech\left(\frac{\sqrt{2}}{2}\,(x+\lambda\,t)\right)}{1\mp\tanh\left(\frac{\sqrt{2}}{2}\,(x+\lambda\,t)\right)}\right)^2+1}. \tag{3.57}$$

Case 7 Take the case $r_4=\frac{\chi^2}{4}, r_2=\frac{\chi^2-2}{2}, r_0=\frac{1}{4}$ and $F(\theta)=\frac{sn(\theta,\chi)}{1\mp dn(\theta,\chi)}$. Then we can write the solution according (3.41) as

$$Y(\theta)=\frac{(3\chi^2-11)\left(\frac{sn(\theta,\chi)}{1\mp dn(\theta,\chi)}\right)^2-\chi^2+9}{\sqrt{11\chi^4-62\chi^2+83}\left(\left(\frac{sn(\theta,\chi)}{1\mp dn(\theta,\chi)}\right)^2+1\right)}, \quad \theta=\frac{2\sqrt{2}\sqrt{\left(11\chi^4-62\chi^2+83\right)\left(\chi^2-5\right)}\,(\lambda t+x)}{11\chi^4-62\chi^2+83}. \tag{3.58}$$

When $\chi=0$, then (3.42) gives the soliton solution as

$$Y_{12}(x,t)=\frac{-11\left(\frac{\sin\left(\frac{2\sqrt{830}i}{83}\,(x+\lambda\,t)\right)}{2}\right)^2+9}{\sqrt{83}\left(\left(\frac{\sin\left(\frac{2\sqrt{830}i}{83}\,(x+\lambda\,t)\right)}{2}\right)^2+1\right)}=\frac{-11\left(\frac{-\sinh\left(\frac{2\sqrt{830}}{83}\,(x+\lambda\,t)\right)}{2}\right)^2+9}{\sqrt{83}\left(\left(\frac{-\sinh\left(\frac{2\sqrt{830}}{83}\,(x+\lambda\,t)\right)}{2}\right)^2+1\right)}. \tag{3.59}$$

When $\chi=1$, then (3.42) gives the bright-dark soliton solution as

$$Y_{13}(x,t)=\frac{-8\left(\frac{\tanh\left(\frac{2\sqrt{830}}{83}\,(x+\lambda\,t)\right)}{1\mp sech\left(\frac{2\sqrt{830}}{83}\,(x+\lambda\,t)\right)}\right)^2+8}{\sqrt{32}\left(\left(\frac{\tanh\left(\frac{2\sqrt{830}}{83}\,(x+\lambda\,t)\right)}{1\mp sech\left(\frac{2\sqrt{830}}{83}\,(x+\lambda\,t)\right)}\right)^2+1\right)}. \tag{3.60}$$

Case 11 Take the case $r_4=\frac{1}{4}, r_2=\frac{\chi^2-2}{2}, r_0=\frac{\chi^4}{4}$ and $F(\theta)=ns(\theta,\chi)\mp ds(\theta,\chi)$. Then, we can write the solution according (3.41) as

$$\begin{aligned}Y(\theta)&=-\frac{\left(\chi^4-4\chi^2+11\right)(ns(\theta,m)\mp ds(\theta,m))^2-3\chi^4+4\chi^2-9}{\sqrt{3\chi^8-8\chi^6+26\chi^4-72\chi^2+83}\left((ns(\theta,m)\mp ds(\theta,m))^2+1\right)},\\ \theta&=\frac{2\sqrt{-2\left(3\chi^8-8\chi^6+26\chi^4-72\chi^2+83\right)\left(\chi^4-2\chi^2+5\right)}\,(\lambda t+x)}{3\chi^8-8\chi^6+26\chi^4-72\chi^2+83}.\end{aligned} \tag{3.61}$$

When $\chi=0$, then (3.42) gives the bright soliton solution as

$$Y_{14}(x,t)=-\frac{44\csc^2\left(\frac{2\sqrt{830}i}{83}\,(x+\lambda\,t)\right)-9}{\sqrt{83}\left(4\csc\left(\frac{2\sqrt{830}i}{83}\,(x+\lambda\,t)\right)^2+1\right)}=-\frac{-4csch^2\left(\frac{2\sqrt{830}}{83}\,(x+\lambda\,t)\right)-9}{\sqrt{83}\left(-4csch\left(\frac{2\sqrt{830}}{83}\,(x+\lambda\,t)\right)^2+1\right)}. \tag{3.62}$$

When $\chi=1$, then (3.42) gives the combined singular solution as

$$Y_{15}(x,t)=-\frac{8(\coth(i(x+\lambda\,t))\mp csch(i(x+\lambda\,t)))^2-8}{\sqrt{32}\left((\coth(i(x+\lambda\,t))\mp csch(i(x+\lambda\,t)))^2+1\right)}=-\frac{8(-\cot(x+\lambda\,t)\mp\csc(x+\lambda\,t))^2-8}{\sqrt{32}\left((-\cot(x+\lambda\,t)\mp\csc(x+\lambda\,t))^2+1\right)}. \tag{3.63}$$

Case 12 Take the case $r_4=\frac{(1+\chi^2)^2}{4}, r_2=\frac{1+\chi^2}{2}, r_0=\frac{1}{4}$ and $F(\theta)=\frac{sn(\theta,\chi)}{dn(\theta,\chi)\mp cn(\theta,\chi)}$. Then, we can write the solution according (3.41) as

$$\begin{aligned}Y(\theta)&=-\frac{\left(3\chi^4-10\chi^2\right)\left(\frac{sn(\theta,\chi)}{dn(\theta,\chi)\mp cn(\theta,\chi)}\right)^2-\chi^4+6\chi^2}{\sqrt{3\chi^8-20\chi^6+36\chi^4+32\chi^2}\left(\left(\frac{sn(\theta,\chi)}{dn(\theta,\chi)\mp cn(\theta,\chi)}\right)^2+1\right)},\\ \theta&=\frac{2\sqrt{-2\chi^4\left(3\chi^6-20\chi^4+36\chi^2+32\right)(\chi-2)(\chi+2)}\,(\lambda t+x)}{\chi^2\left(3\chi^6-20\chi^4+36\chi^2+32\right)}.\end{aligned} \tag{3.64}$$

When $\chi=1$, then (3.42) gives the dark-bright soliton solution as

$$Y_{16}(x,t)=-\frac{-7\left(\frac{\tanh\left(\frac{2\sqrt{34}}{17}\,(x+\lambda\,t)\right)}{4\,sech\left(\frac{2\sqrt{34}}{17}\,(x+\lambda\,t)\right)}\right)^2+5}{\sqrt{51}\left(\left(\frac{\tanh\left(\frac{2\sqrt{34}}{17}\,(x+\lambda\,t)\right)}{4\,sech\left(\frac{2\sqrt{34}}{17}\,(x+\lambda\,t)\right)}\right)^2+1\right)}. \tag{3.65}$$

<u>Merge II</u>.

$$\begin{aligned}&m=\frac{\sqrt{-2\left(2r_0r_2-12r_0r_4+r_2^2+2r_2r_4\right)(r_0-r_2+r_4)}}{2r_0r_2-12r_0r_4+r_2^2+2r_2r_4}, \quad n=\frac{\lambda\sqrt{-2\left(2r_0r_2-12r_0r_4+r_2^2+2r_2r_4\right)(r_0-r_2+r_4)}}{2r_0r_2-12r_0r_4+r_2^2+2r_2r_4},\\ &g_0=0, \quad \lambda_1=\frac{r_0-r_2+r_4}{\sqrt{2r_0r_2-12r_0r_4+r_2^2+2r_2r_4}}, \quad \mu_0=\frac{r_0-r_4}{\sqrt{2r_0r_2-12r_0r_4+r_2^2+2r_2r_4}},\\ &\mu_1=\frac{2\,\mathrm{I}(r_0-r_2+r_4)}{\sqrt{2r_0r_2-12r_0r_4+r_2^2+2r_2r_4}},\end{aligned} \tag{3.66}$$

$$
\begin{aligned}
Y(\theta) &= \frac{r_0-r_4}{\sqrt{2r_0r_2-12r_0r_4+r_2^2+2r_2r_4}} + \frac{2\,\mathrm{I}(r_0-r_2+r_4)}{\sqrt{2r_0r_2-12r_0r_4+r_2^2+2r_2r_4}}\,\frac{F(\theta)}{1+F(\theta)^2} + \frac{r_0-r_2+r_4}{\sqrt{2r_0r_2-12r_0r_4+r_2^2+2r_2r_4}}\,\frac{\left(1-F(\theta)^2\right)}{1+F(\theta)^2} \\
&= \frac{1}{\sqrt{2r_0r_2-12r_0r_4+r_2^2+2r_2r_4}}\left(\frac{F(\theta)r_2-2F(\theta)r_4-\mathrm{I}r_2+2\,\mathrm{I}r_0}{F(\theta)+\mathrm{I}}\right), \\
\theta &= \frac{\sqrt{-2\left(2r_0r_2-12r_0r_4+r_2^2+2r_2r_4\right)(r_0-r_2+r_4)}}{2r_0r_2-12r_0r_4+r_2^2+2r_2r_4}\,(x+\lambda\,t).
\end{aligned} \tag{3.67}
$$

Case 1 Take the case $r_4=\chi^2, r_2=-1-\chi^2, r_0=1$ and $F(\theta)=sn(\theta,\chi)$. Then we can write the solution according to (3.41) as

$$
Y(\theta) = -\frac{\left(3\chi^2+1\right)sn(\theta,\chi)-i(\chi^2+3)}{\sqrt{-\chi^4-14\chi^2-1}\,(i+sn(\theta,\chi))}, \quad \theta = -\frac{2\sqrt{\left(\chi^4+14\chi^2+1\right)\left(\chi^2+1\right)}\,(\lambda t+x)}{\chi^4+14\chi^2+1}. \tag{3.68}
$$

When $\chi=0$, then (3.68) gives the periodic solution as

$$
Y_1(x,t) = \frac{-\sin(2\,(x+\lambda\,t))-3i}{(1+i\,\sin(2\,(x+\lambda\,t)))}. \tag{3.69}
$$

When $\chi=1$, then (3.42) gives the dark soliton solution as

$$
Y_2(x,t) = \frac{-4\tanh\left(\frac{\sqrt{2}}{2}\,(x+\lambda\,t)\right)-4i}{\left(4+4i\,\tanh\left(\frac{\sqrt{2}}{2}\,(x+\lambda\,t)\right)\right)}. \tag{3.70}
$$

Case 2 Take the case $r_4=-\chi^2, r_2=2\chi^2-1, r_0=1-\chi^2$ and $F(\theta)=cn(\theta,\chi)$. Then we can write the solution according to (3.41) as

$$
Y(\theta) = \frac{\left(4\chi^2-1\right)cn(\theta,\chi)-i(4\chi^2-3i)}{\sqrt{-16\chi^4+16\chi^2-1}\,(i+cn(\theta,\chi))}, \quad \theta = -\frac{2\sqrt{-\left(16\chi^4-16\chi^2+1\right)\left(2\chi^2-1\right)}\,(\lambda t+x)}{16\chi^4-16\chi^2+1}. \tag{3.71}
$$

When $\chi=0$, then (3.42) gives the periodic solution as

$$
Y_3(x,t) = \frac{\cos(2\,(x+\lambda\,t))+3}{1-i\,\cos(2\,(x+\lambda\,t))}. \tag{3.72}
$$

When $\chi=1$, then (3.42) gives the singular-periodic solution as

$$
Y_4(x,t) = \frac{3sech(2i\,(x+\lambda\,t))-i(4-3i)}{-1+i\,sech(2i\,(x+\lambda\,t))} = \frac{3\sec(2\,(x+\lambda\,t))-i(4-3i)}{-1+i\,\sec(2\,(x+\lambda\,t))}. \tag{3.73}
$$

Case 3 Take the case $r_4=1, r_2=-\chi^2-1, r_0=\chi^2$ and $F(\theta)=ns(\theta,\chi)$. Then we can write the solution according to (3.41) as

$$
Y(\theta) = \frac{-\left(\chi^2+3\right)cn(\theta,\chi)+i(3\chi^2+1)}{\sqrt{-\chi^4-14\chi^2-1}\,(i+cn(\theta,\chi))}, \quad \theta = -\frac{2\sqrt{\left(\chi^4+14\chi^2+1\right)\left(\chi^2+1\right)}\,(\lambda t+x)}{\chi^4+14\chi^2+1}. \tag{3.74}
$$

When $\chi=0$, then (3.74) gives the singular periodic solution as

$$
Y_5(x,t) = \frac{3\csc(2\,(x+\lambda\,t))+i}{-1-i\,\csc(2\,(x+\lambda\,t))}. \tag{3.75}
$$

When $\chi=1$, then (3.74) gives the soliton solution as

$$
Y_6(x,t) = \frac{4\coth\left(\frac{\sqrt{2}}{2}\,(x+\lambda\,t)\right)+4i}{4\left(-1-i\,\coth\left(\frac{\sqrt{2}}{2}\,(x+\lambda\,t)\right)\right)}. \tag{3.76}
$$

Case 4 Take the case $r_4=1-\chi^2, r_2=2\chi^2-1, r_0=-\chi^2$ and $F(\theta)=nc(\theta,\chi)$. Then we can write the solution according to (3.67) as

$$
Y(\theta) = \frac{\left(4\chi^2-3\right)nc^2(\theta,\chi)-i(4\chi^2-1)}{\sqrt{-16\chi^4+16\chi^2-1}\,\left(nc^2(\theta,\chi)+1\right)}, \quad \theta = -\frac{2\sqrt{-\left(16\chi^4-16\chi^2+1\right)\left(2\chi^2-1\right)}\,(\lambda t+x)}{16\chi^4-16\chi^2+1}. \tag{3.77}
$$

When $\chi=0$, then (3.77) gives the periodic solution as

$$
Y_7(x,t) = \frac{-3\,\sec(2\,(x+\lambda\,t))+i}{-1+i\,\sec(2\,(x+\lambda\,t))}. \tag{3.78}
$$

When $\chi=1$, then (3.77) gives the periodic solution as

$$
Y_8(x,t) = \frac{\cosh(2i\,(x+\lambda\,t))-3i}{-1+i\,\cosh(2i\,(x+\lambda\,t))} = \frac{\cos(2\,(x+\lambda\,t))-3i}{-1+i\,\cos(2\,(x+\lambda\,t))}. \tag{3.79}
$$

Case 5 Take the case $r_4 = \frac{1}{4}, r_2 = \frac{1-2\chi^2}{2}, r_0 = \frac{1}{4}$ and $F(\theta) = ns(\theta,\chi) \mp cs(\theta,\chi)$. Then we can write the solution according (3.41) as

$$Y(\theta) = -\frac{\chi^2((ns(\theta,\chi)\mp cs(\theta,\chi))-i)}{\sqrt{\chi^4-2\chi^2}\,((ns(\theta,\chi)\mp cs(\theta,\chi))+i)}, \quad \theta = \frac{\sqrt{-2\chi^4(\chi^2-2)}\,(\lambda t+x)}{\chi^2(\chi^2-2)}. \tag{3.80}$$

When $\chi = 1$, then (3.80) gives the complex singular soliton solution as

$$Y_9(x,t) = -\frac{(-\coth(\sqrt{2}\,(x+\lambda\,t))\pm csch(\sqrt{2}\,(x+\lambda\,t)))-i}{i\,((-\coth(\sqrt{2}\,(x+\lambda\,t))\pm csch(\sqrt{2}\,(x+\lambda\,t)))+i)}. \tag{3.81}$$

Case 6 Take the case $r_4 = \frac{1-\chi^2}{4}, r_2 = \frac{1+\chi^2}{2}, r_0 = \frac{1-\chi^2}{4}$ and $F(\theta) = nc(\theta,\chi) \mp sc(\theta,\chi)$ or $\frac{cn(\theta,\chi)}{1\mp sn(\theta,\chi)}$. Then we can write the solution according (3.41) as

$$Y(\theta) = \frac{\chi^2((nc(\theta,\chi)\mp sc(\theta,\chi))-i)}{\sqrt{-\chi^4+2\chi^2}\,((nc(\theta,\chi)\mp sc(\theta,\chi))+i)}, \quad or \quad Y(\theta) = \frac{\chi^2\left(\frac{cn(\theta,\chi)}{1\mp sn(\theta,\chi)}-i\right)}{\sqrt{-\chi^4+2\chi^2}\left(\frac{cn(\theta,\chi)}{1\mp sn(\theta,\chi)}+i\right)},$$
$$\theta = -\frac{\sqrt{2}\sqrt{-\chi^4(\chi^2-2)}\,(\lambda t+x)}{\chi^2(\chi^2-2)}. \tag{3.82}$$

When $\chi = 1$, then (3.82) gives the complex combined soliton solution as

$$Y_{10}(x,t) = \frac{((\cosh(\sqrt{2}\,(x+\lambda\,t))\mp\sinh(\sqrt{2}\,(x+\lambda\,t)))-i)}{((\cosh(\sqrt{2}\,(x+\lambda\,t))\mp\sinh(\sqrt{2}\,(x+\lambda\,t)))+i)},$$
$$Y_{11}(x,t) = \frac{\frac{sech(\sqrt{2}\,(x+\lambda\,t))}{1\mp\tanh(\sqrt{2}\,(x+\lambda\,t))}-i}{\frac{sech(\sqrt{2}\,(x+\lambda\,t))}{1\mp\tanh(\sqrt{2}\,(x+\lambda\,t))}+i}. \tag{3.83}$$

Case 7 Take the case $r_4 = \frac{\chi^2}{4}, r_2 = \frac{\chi^2-2}{2}, r_0 = \frac{1}{4}$ and $F(\theta) = \frac{sn(\theta,\chi)}{1\mp dn(\theta,\chi)}$. Then we can write the solution according (3.41) as

$$Y(\theta) = \frac{2\frac{sn(\theta,\chi)}{1\mp dn(\theta,\chi)}+i(\chi^2-3)}{\sqrt{2\chi^4-8\chi^2+2}\left(\left(\frac{sn(\theta,\chi)}{1\mp dn(\theta,\chi)}\right)+i\right)}, \quad \theta = \frac{\sqrt{(\chi^4-4\chi^2+1)(\chi^2-5)}\,(\lambda t+x)}{\chi^4-4\chi^2+1}. \tag{3.84}$$

When $\chi = 0$, then (3.84) gives the soliton solution as

$$Y_{12}(x,t) = \frac{\sin(\sqrt{5}i\,(x+\lambda\,t))-3i}{\sqrt{2}\left(\left(\frac{\sin(\sqrt{5}i\,(x+\lambda\,t))}{2}\right)+i\right)} = \frac{\sinh(\sqrt{5}\,(x+\lambda\,t))-3}{\sqrt{2}\left(\frac{\sinh(\sqrt{5}\,(x+\lambda\,t))}{2}+1\right)}. \tag{3.85}$$

When $\chi = 1$, then (3.84) gives the complex bright-dark soliton solution as

$$Y_{13}(x,t) = \frac{2\frac{-\tanh(\sqrt{2}\,(x+\lambda\,t))}{1\mp sech(\sqrt{2}\,(x+\lambda\,t))}-2i}{2i\left(\frac{-\tanh(\sqrt{2}\,(x+\lambda\,t))}{1\mp sech(\sqrt{2}\,(x+\lambda\,t))}+i\right)}. \tag{3.86}$$

Case 10 Take the case $r_4 = -1, r_2 = 2-\chi^2, r_0 = \chi^2-1$ and $F(\theta) = dn(\theta,\chi)$. Then, we can write the solution according (3.41) as

$$Y(\theta) = \frac{(4-\chi^2)dn(\theta,\chi)+i(3\chi^2-4)}{\sqrt{-\chi^4+16\chi^2-16}\,(dn(\theta,\chi)+i)}, \quad \theta = -\frac{2\sqrt{(\chi^4-16\chi^2+16)(\chi^2-2)}\,(\lambda t+x)}{\chi^4-16\chi^2+16}. \tag{3.87}$$

When $\chi = 1$, then (3.87) gives the complex periodic solution as

$$Y_{14}(x,t) = \frac{3\,sech\left(\frac{\sqrt{2}}{2}i\,(x+\lambda\,t)\right)-i}{i\left(sech\left(\frac{\sqrt{2}}{2}i\,(x+\lambda\,t)\right)+i\right)} = \frac{3\,\sec\left(\frac{\sqrt{2}}{2}\,(x+\lambda\,t)\right)-i}{i\left(\sec\left(\frac{\sqrt{2}}{2}\,(x+\lambda\,t)\right)+i\right)}. \tag{3.88}$$

Case 11 Take the case $r_4 = \frac{1}{4}, r_2 = \frac{\chi^2-2}{2}, r_0 = \frac{\chi^4}{4}$ and $F(\theta) = ns(\theta,\chi) \mp ds(\theta,\chi)$. Then, we can write the solution according (3.67) as

$$Y(\theta) = \frac{(\chi^2-3)\,(ns(\theta,\chi)\mp ds(\theta,\chi))+i(\chi^2+2)}{\sqrt{(\chi^2+1)(\chi^4-5\chi^2+2)}\,(ns(\theta,\chi)\mp ds(\theta,\chi)+i)},$$
$$\theta = \frac{\sqrt{-2(\chi^2+1)(\chi^4-5\chi^2+2)(\chi^4-2\chi^2+5)}\,(\lambda t+x)}{(\chi^2+1)(\chi^4-5\chi^2+2)}. \tag{3.89}$$

When $\chi = 0$, then (3.89) gives the complex singular soliton solution as

$$Y_{15}(x,t) = \frac{-6\ \csc\left(\sqrt{5}i\,(x+\lambda\,t)\right)+2i}{\sqrt{2}\left(2\ \csc\left(\sqrt{5}i\,(x+\lambda\,t)\right)+i\right)} = \frac{-6i\,csch\left(\sqrt{5}\,(x+\lambda\,t)\right)+2i}{\sqrt{2}\left(-2i\,csch\left(\sqrt{5}\,(x+\lambda\,t)\right)+i\right)}. \tag{3.90}$$

When $\chi = 1$, then (3.89) gives the complex singular solution as

$$Y_{16}(x,t) = \frac{-2\,(-\coth\left(\sqrt{2}\,(x+\lambda\,t)\right)\pm csch\left(\sqrt{2}\,(x+\lambda\,t)\right))+3i}{2\,i\left(-\coth\left(\sqrt{2}\,(x+\lambda\,t)\right)\pm csch\left(\sqrt{2}\,(x+\lambda\,t)\right)+i\right)}. \tag{3.91}$$

Case 12 Take the case $r_4 = \frac{(1+\chi^2)^2}{4}, r_2 = \frac{1+\chi^2}{2}, r_0 = \frac{1}{4}$ and $F(\theta) = \frac{sn(\theta,\chi)}{dn(\theta,\chi)\mp cn(\theta,\chi)}$. Then, we can write the solution according (3.67) as

$$\begin{aligned} Y(\theta) &= -\frac{\chi^2\left(\chi^2-3\right)\frac{sn(\theta,\chi)}{dn(\theta,\chi)\mp cn(\theta,\chi)}+i\chi^2}{\sqrt{\chi^6-3\chi^4+8\chi^2}\left(\frac{sn(\theta,\chi)}{dn(\theta,\chi)\mp cn(\theta,\chi)}+i\right)}, \\ \theta &= \frac{\sqrt{-2\chi^2(\chi^6-3\chi^4+8\chi^2)\left(\chi^2-4\right)(\lambda t+x)}}{\chi^6-3\chi^4+8\chi^2}. \end{aligned} \tag{3.92}$$

When $\chi = 1$, then (3.92) gives the complex dark-bright soliton solution as

$$Y_{17}(x,t) = \frac{2\frac{\tanh(x+\lambda\,t)}{sech(x+\lambda\,t)}-i}{\sqrt{6}\left(\frac{\tanh(x+\lambda\,t)}{sech(x+\lambda\,t)}+i\right)}. \tag{3.93}$$

Application of RMWFs for 4th order CH equation

We consider two subsections:

One wave solution for Eq. (3.38)

Consider the one wave solution as

$$\begin{aligned} &F_1(x,t) = \lambda_0 + \lambda_1\,exp(f(x,t)), \\ &F_2(x,t) = 1 + \lambda_2\,exp(f(x,t)) + \lambda_3\,exp(2\,f(x,t)), \\ &u(x,t) = \frac{F_1(x,t)}{F_2(x,t)}. \end{aligned} \tag{4.94}$$

Using (4.94) and inserting (4.94) into (3.38), we can get the nonlinear algebra equations

$$\begin{aligned} &11\lambda_2^2\lambda_0a_1^4 - 9\lambda_2^2\lambda_0^3a_1^2 - 16\lambda_3\lambda_0a_1^4 - 11\lambda_2a_1^4\lambda_1 + 12\lambda_3\lambda_0^3a_1^2 + 21\lambda_2\lambda_1\lambda_0^2a_1^2 + 3\lambda_2^2\lambda_0\lambda a_1 - 12\lambda_1^2\lambda_0a_1^2 \\ &\quad -\lambda_2^2\lambda_0a_1^2 + 2\lambda_3\lambda_0\lambda a_1 - 3\lambda_2\lambda_1a_1\lambda - 4\lambda_3\lambda_0a_1^2 + \lambda_2a_1^2\lambda_1 - 3\lambda_2^2\lambda_0b_1 - 2\lambda_3\lambda_0b_1 + 3\lambda_2b_1\lambda_1 = 0, \\ &\quad -11\lambda_2^3\lambda_0a_1^4 + 77\lambda_3\lambda_2\lambda_0a_1^4 + 11\lambda_2^2a_1^4\lambda_1 - 33\lambda_3\lambda_2\lambda_0^3a_1^2 - 12\lambda_2^2\lambda_1\lambda_0^2a_1^2 + 3\lambda_2^3\lambda_0\lambda a_1 - 76\lambda_3a_1^4\lambda_1 \\ &\quad +66\lambda_3\lambda_1\lambda_0^2a_1^2 + 21\lambda_2\lambda_1^2\lambda_0a_1^2 + \lambda_2^3\lambda_0a_1^2 + 9\lambda_3\lambda_2\lambda_0a_1\lambda - 3\lambda_2^2\lambda_1a_1\lambda - 7\lambda_3\lambda_2\lambda_0a_1^2 - 9\lambda_1^3a_1^2 - \lambda_2^2a_1^2\lambda_1 \\ &\quad -3\lambda_2^3\lambda_0b_1 - 2\lambda_3\lambda_1a_1\lambda - 4\lambda_3a_1^2\lambda_1 - 9\lambda_3\lambda_2\lambda_0b_1 + 3\lambda_2^2b_1\lambda_1 + 2\lambda_3b_1\lambda_1 = 0, \\ &\lambda_2^4\lambda_0a_1^4 - 58\lambda_3\lambda_2^2\lambda_0a_1^4 - \lambda_2^3a_1^4\lambda_1 + \lambda_2^4\lambda_0\lambda a_1 + 176\lambda_3^2\lambda_0a_1^4 + 47\lambda_3\lambda_2a_1^4\lambda_1 - 36\lambda_3^2\lambda_0^3a_1^2 - 51\lambda_3\lambda_2\lambda_1\lambda_0^2a_1^2 \\ &\quad -3\lambda_2^2\lambda_1^2\lambda_0a_1^2 + \lambda_2^4\lambda_0a_1^2 + 12\lambda_3\lambda_2^2\lambda_0\lambda a_1 - \lambda_2^3\lambda_1a_1\lambda + 84\lambda_3\lambda_1^2\lambda_0a_1^2 + 2\lambda_3\lambda_2^2\lambda_0a_1^2 + 3\lambda_2\lambda_1^3a_1^2 \\ &\quad -\lambda_2^3a_1^2\lambda_1 - \lambda_2^4\lambda_0b_1 + 6\lambda_3^2\lambda_0\lambda a_1 - 3\lambda_3\lambda_2\lambda_1a_1\lambda - 4\lambda_3^2\lambda_0a_1^2 - 13\lambda_3\lambda_2a_1^2\lambda_1 - 12\lambda_3\lambda_2^2\lambda_0b_1 \\ &\quad +\lambda_2^3b_1\lambda_1 - 6\lambda_3^2\lambda_0b_1 + 3\lambda_3\lambda_2b_1\lambda_1 = 0, \\ &5a_1^4\lambda_0\lambda_2^3\lambda_3 - 115a_1^4\lambda_0\lambda_2\lambda_3^2 - 10a_1^4\lambda_1\lambda_2^2\lambda_3 + 5\lambda a_1\lambda_0\lambda_2^3\lambda_3 + 230a_1^4\lambda_1\lambda_3^2 - 75a_1^2\lambda_0^2\lambda_1\lambda_3^2 - 15a_1^2\lambda_0\lambda_1^2\lambda_2\lambda_3 \\ &\quad +5a_1^2\lambda_0\lambda_2^3\lambda_3 + 15\lambda a_1\lambda_0\lambda_2\lambda_3^2 + 5a_1^2\lambda_0\lambda_2\lambda_3^2 + 30a_1^2\lambda_1^3\lambda_3 - 10a_1^2\lambda_1\lambda_2^2\lambda_3 \\ &\quad -5b_1\lambda_0\lambda_2^3\lambda_3 - 10a_1^2\lambda_1\lambda_3^2 - 15b_1\lambda_0\lambda_2\lambda_3^2 = 0, \\ &11a_1^4\lambda_0\lambda_2^2\lambda_3^2 - a_1^4\lambda_1\lambda_2^3\lambda_3 - 176a_1^4\lambda_0\lambda_3^3 + 47a_1^4\lambda_1\lambda_2\lambda_3^2 + 9\lambda a_1\lambda_0\lambda_2^2\lambda_3^2 + \lambda a_1\lambda_1\lambda_2^3\lambda_3 \\ &\quad -48a_1^2\lambda_0\lambda_1^2\lambda_3^2 + 11a_1^2\lambda_0\lambda_2^2\lambda_3^2 + 3a_1^2\lambda_1^3\lambda_2\lambda_3 - a_1^2\lambda_1\lambda_2^3\lambda_3 + 6\lambda a_1\lambda_0\lambda_3^3 + 3\lambda a_1\lambda_1\lambda_2\lambda_3^2 \\ &\quad +4a_1^2\lambda_0\lambda_3^3 - 13a_1^2\lambda_1\lambda_2\lambda_3^2 - 9b_1\lambda_0\lambda_2^2\lambda_3^2 - b_1\lambda_1\lambda_2^3\lambda_3 - 6b_1\lambda_0\lambda_3^3 - 3b_1\lambda_1\lambda_2\lambda_3^2 = 0, \\ &\quad -a_1^4\lambda_0\lambda_2\lambda_3^3 + 11a_1^4\lambda_1\lambda_2^2\lambda_3^2 - 76a_1^4\lambda_1\lambda_3^3 + 7\lambda a_1\lambda_0\lambda_2\lambda_3^3 + 3\lambda a_1\lambda_1\lambda_2^2\lambda_3^2 + 11a_1^2\lambda_0\lambda_2\lambda_3^3 \\ &\quad -9a_1^2\lambda_1^3\lambda_3^2 - a_1^2\lambda_1\lambda_2^2\lambda_3^2 + 2\lambda a_1\lambda_1\lambda_3^3 - 4a_1^2\lambda_1\lambda_3^3 - 7b_1\lambda_0\lambda_2\lambda_3^3 - 3b_1\lambda_1\lambda_2^2\lambda_3^2 - 2b_1\lambda_1\lambda_3^3 = 0, \\ &16a_1^4\lambda_0\lambda_3^4 - 11a_1^4\lambda_1\lambda_2\lambda_3^3 + 2\lambda a_1\lambda_0\lambda_3^4 + 3\lambda a_1\lambda_1\lambda_2\lambda_3^3 + 4a_1^2\lambda_0\lambda_3^4 + a_1^2\lambda_1\lambda_2\lambda_3^3 - 2b_1\lambda_0\lambda_3^4 - 3b_1\lambda_1\lambda_2\lambda_3^3 = 0, \\ &a_1^4\lambda_1\lambda_3^4 + \lambda a_1\lambda_1\lambda_3^4 + a_1^2\lambda_1\lambda_3^4 - b_1\lambda_1\lambda_3^4 = 0, \\ &-\lambda_2\lambda_0a_1^4 + 3\lambda_2\lambda_0^3a_1^2 + a_1^4\lambda_1 - 3\lambda_1\lambda_0^2a_1^2 + \lambda_2\lambda_0a_1\lambda - \lambda_2\lambda_0a_1^2 - \lambda_1a_1\lambda + a_1^2\lambda_1 - \lambda_2\lambda_0b_1 + b_1\lambda_1 = 0. \end{aligned} \tag{4.95}$$

After solving the above system, we can receive the following results:

$$a_1 = \sqrt{2}, \quad b_1 = \lambda\sqrt{2}, \quad \lambda_0 = \pm 1, \quad \lambda_1 = \mp\lambda_2, \quad \lambda_3 = 0, \quad Y_{1,2}(x,t) = \frac{\mp\lambda_2 e^{(\lambda t+x)\sqrt{2}+c_1}\pm 1}{1+\lambda_2 e^{(\lambda t+x)\sqrt{2}+c_1}}. \tag{4.96}$$

$$a_1 = \sqrt{-1}, \quad b_1 = \lambda\sqrt{-1}, \quad \lambda_0 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = \frac{\lambda_1^2}{8}, \quad Y_3(x,t) = \frac{8\lambda_1 e^{I(\lambda t+x)+c_1}}{\lambda_1^2 e^{I(2\lambda t+2x)+2c_1}+8}. \tag{4.97}$$

Two wave solution for Eq. (3.38)

Consider the one wave solution as

$$
\begin{aligned}
&F_1(x,t) = \lambda_0 + \lambda_1\, exp(f(x,t)) + \lambda_2\, exp(g(x,t)) + \lambda_3\, exp(f(x,t)+g(x,t)),\\
&F_2(x,t) = 1 + \mu_1\, exp(f(x,t)) + \mu_2\, exp(g(x,t)) + \mu_3\, exp(2\, f(x,t)) + \mu_4\, exp(2\, g(x,t)),\\
&u(x,t) = \tfrac{F_1(x,t)}{F_2(x,t)}.
\end{aligned} \tag{4.98}
$$

Using (4.98) and inserting (4.98) into (3.38), we can get the nonlinear algebra equations. After solving the obtained system, we can receive the following results:

$$
\begin{aligned}
&a_1 = -a_2 + \sqrt{2}, \quad b_1 = \lambda\sqrt{2} - b_2, \quad \lambda_0 = -1, \quad \lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 0,\\
&Y_1(x,t) = \frac{-1+\mu_3 \mathrm{e}^{(\lambda t+x)\left(-a_2+\sqrt{2}\right)+(\lambda t+x)a_2+c_1+c_2}}{1+\mu_3 \mathrm{e}^{(\lambda t+x)\left(-a_2+\sqrt{2}\right)+(\lambda t+x)a_2+c_1+c_2}}.
\end{aligned} \tag{4.99}
$$

$$
\begin{aligned}
&a_1 = \sqrt{2}, \quad a_2 = b_2 = 0, \quad b_1 = \lambda\sqrt{2}, \quad \lambda_0 = -1, \quad \lambda_1 = \mu_1, \quad \lambda_3 = \mu_3, \quad \lambda_2 = \mu_2 = 0,\\
&Y_2(x,t) = \frac{-1+\mu_1 \mathrm{e}^{(\lambda t+x)\sqrt{2}+c_1}+\mu_3 \mathrm{e}^{(\lambda t+x)\sqrt{2}+c_1+c_2}}{1+\mu_1 \mathrm{e}^{(\lambda t+x)\sqrt{2}+c_1}+\mu_3 \mathrm{e}^{(\lambda t+x)\sqrt{2}+c_1+c_2}}.
\end{aligned} \tag{4.100}
$$

$$
\begin{aligned}
&a_1 = \sqrt{2}, \quad a_2 = b_2 = 0, \quad b_1 = \lambda\sqrt{2}, \quad \lambda_0 = 1, \quad \lambda_1 = -\mu_1, \quad \lambda_3 = -\mu_3, \quad \lambda_2 = \mu_2 = 0,\\
&Y_3(x,t) = \frac{-\mu_1 \mathrm{e}^{(\lambda t+x)\sqrt{2}+c_1}-\mu_3 \mathrm{e}^{(\lambda t+x)\sqrt{2}+c_1+c_2}+1}{1+\mu_1 \mathrm{e}^{(\lambda t+x)\sqrt{2}+c_1}+\mu_3 \mathrm{e}^{(\lambda t+x)\sqrt{2}+c_1+c_2}}.
\end{aligned} \tag{4.101}
$$

$$
\begin{aligned}
&a_2 = \sqrt{2}, \quad b_2 = \lambda\sqrt{2}, \quad \lambda_0 = 1, \quad \lambda_1 = \mu_1 = \tfrac{\mu_3}{\mu_2}, \quad \lambda_3 = -\mu_3, \quad \lambda_2 = -\mu_2,\\
&Y_4(x,t) = \frac{-\mu_2^2 \mathrm{e}^{(\lambda t+x)\sqrt{2}+c_2}-\mu_3 \mathrm{e}^{(\lambda t+x)\sqrt{2}+b_1 t+a_1 x+c_1+c_2}\mu_2+\mu_3 \mathrm{e}^{b_1 t+a_1 x+c_1}+\mu_2}{\mu_2^2 \mathrm{e}^{(\lambda t+x)\sqrt{2}+c_2}+\mu_3 \mathrm{e}^{(\lambda t+x)\sqrt{2}+b_1 t+a_1 x+c_1+c_2}\mu_2+\mu_3 \mathrm{e}^{b_1 t+a_1 x+c_1}+\mu_2}.
\end{aligned} \tag{4.102}
$$

$$
\begin{aligned}
&a_2 = \sqrt{2}, \quad b_2 = \lambda\sqrt{2}, \quad \lambda_0 = -1, \quad \lambda_1 = -\mu_1 = -\tfrac{\mu_3}{\mu_2}, \quad \lambda_3 = \mu_3, \quad \lambda_2 = \mu_2,\\
&Y_5(x,t) = \frac{\mu_2^2 \mathrm{e}^{(\lambda t+x)\sqrt{2}+c_2}+\mu_3 \mathrm{e}^{(\lambda t+x)\sqrt{2}+b_1 t+a_1 x+c_1+c_2}\mu_2-\mu_3 \mathrm{e}^{b_1 t+a_1 x+c_1}-\mu_2}{\mu_2^2 \mathrm{e}^{(\lambda t+x)\sqrt{2}+c_2}+\mu_3 \mathrm{e}^{(\lambda t+x)\sqrt{2}+b_1 t+a_1 x+c_1+c_2}\mu_2+\mu_3 \mathrm{e}^{b_1 t+a_1 x+c_1}+\mu_2}.
\end{aligned} \tag{4.103}
$$

$$
\begin{aligned}
&a_1 = b_1 = 0, \quad b_2 = a_2\left(a_2^3+\lambda+a_2\right), \quad \lambda_0 = \tfrac{\lambda_2}{\mu_2}, \quad \lambda_1 = \lambda_3 = 0, \quad \mu_3 = \mu_1\mu_2,\\
&Y_6(x,t) = \frac{\lambda_2\left(\mu_2 \mathrm{e}^{a_2\left(a_2^3+\lambda+a_2\right)t+a_2 x+c_2}+1\right)}{\mu_2\left(1+\mu_1 \mathrm{e}^{c_1}+\mu_2 \mathrm{e}^{a_2\left(a_2^3+\lambda+a_2\right)t+a_2 x+c_2}+\mu_1\mu_2 \mathrm{e}^{a_2\left(a_2^3+\lambda+a_2\right)t+a_2 x+c_1+c_2}\right)}.
\end{aligned} \tag{4.104}
$$

$$
\begin{aligned}
&a_1 = b_1 = 0, \quad b_2 = -a_2^4 + a_2\lambda - a_2^2, \quad \lambda_0 = 0, \quad \lambda_1 = \tfrac{\lambda_3}{\mu_2}, \quad \lambda_2 = 0, \quad \mu_3 = \mu_1\mu_2,\\
&Y_7(x,t) = \frac{\lambda_3\left(\mathrm{e}^{a_2\left(-a_2^3+\lambda-a_2\right)t+a_2 x+c_1+c_2}\mu_2+\mathrm{e}^{c_1}\right)}{\mu_2\left(1+\mu_1 \mathrm{e}^{c_1}+\mu_2 \mathrm{e}^{a_2\left(-a_2^3+\lambda-a_2\right)t+a_2 x+c_2}+\mu_1\mu_2 \mathrm{e}^{a_2\left(-a_2^3+\lambda-a_2\right)t+a_2 x+c_1+c_2}\right)}.
\end{aligned} \tag{4.105}
$$

$$
\begin{aligned}
&a_1 = b_1 = 0, \quad b_2 = -a_2^4 + 3a_2^2\lambda_0^2 + a_2\lambda - a_2^2, \quad \lambda_1 = \tfrac{\lambda_3}{\mu_2}, \quad \lambda_2 = \mu_2\lambda_0, \quad \mu_1 = \tfrac{\mu_3}{\mu_2},\\
&Y_8(x,t) = \frac{\mu_2^2\lambda_0 \mathrm{e}^{a_2\left(-a_2^3+\left(3\lambda_0^2-1\right)a_2+\lambda\right)t+a_2 x+c_2}+\lambda_3 \mathrm{e}^{a_2\left(-a_2^3+\left(3\lambda_0^2-1\right)a_2+\lambda\right)t+a_2 x+c_1+c_2}\mu_2+\lambda_3 \mathrm{e}^{c_1}+\mu_2\lambda_0}{\mu_2^2 \mathrm{e}^{a_2\left(-a_2^3+\left(3\lambda_0^2-1\right)a_2+\lambda\right)t+a_2 x+c_2}+\mu_3 \mathrm{e}^{a_2\left(-a_2^3+\left(3\lambda_0^2-1\right)a_2+\lambda\right)t+a_2 x+c_1+c_2}\mu_2+\mu_3 \mathrm{e}^{c_1}+\mu_2}.
\end{aligned} \tag{4.106}
$$

$$
\begin{aligned}
&a_1 = \sqrt{2},\ b_1 = \lambda\sqrt{2}, \quad \lambda_0 = -1, \quad \lambda_1 = \tfrac{\mu_3}{\mu_2}, \quad \lambda_2 = -\mu_2, \quad \lambda_3 = \mu_3, \quad \mu_1 = \tfrac{\mu_3}{\mu_2},\\
&Y_9(x,t) = \frac{-\mu_2^2 \mathrm{e}^{b_2 t+a_2 x+c_2}+\mu_3 \mathrm{e}^{(\lambda t+x)\sqrt{2}+b_2 t+a_2 x+c_1+c_2}\mu_2+\mu_3 \mathrm{e}^{(\lambda t+x)\sqrt{2}+c_1}-\mu_2}{\mu_2^2 \mathrm{e}^{b_2 t+a_2 x+c_2}+\mu_3 \mathrm{e}^{(\lambda t+x)\sqrt{2}+b_2 t+a_2 x+c_1+c_2}\mu_2+\mu_3 \mathrm{e}^{(\lambda t+x)\sqrt{2}+c_1}+\mu_2}.
\end{aligned} \tag{4.107}
$$

$$
\begin{aligned}
&a_1 = \sqrt{2},\ b_1 = \lambda\sqrt{2}, \quad \lambda_0 = 1, \quad \lambda_1 = -\tfrac{\mu_3}{\mu_2}, \quad \lambda_2 = \mu_2, \quad \lambda_3 = -\mu_3, \quad \mu_1 = \tfrac{\mu_3}{\mu_2},\\
&Y_{10}(x,t) = \frac{\mu_2^2 \mathrm{e}^{b_2 t+a_2 x+c_2}-\mu_3 \mathrm{e}^{(\lambda t+x)\sqrt{2}+b_2 t+a_2 x+c_1+c_2}\mu_2-\mu_3 \mathrm{e}^{(\lambda t+x)\sqrt{2}+c_1}+\mu_2}{\mu_2^2 \mathrm{e}^{b_2 t+a_2 x+c_2}+\mu_3 \mathrm{e}^{(\lambda t+x)\sqrt{2}+b_2 t+a_2 x+c_1+c_2}\mu_2+\mu_3 \mathrm{e}^{(\lambda t+x)\sqrt{2}+c_1}+\mu_2}.
\end{aligned} \tag{4.108}
$$

Decomposition variational iteration method

In this section, we will explain the decomposition variational iteration method which will be utilized to the mentioned nonlinear model. By taking the provided conditions in $Y(x,0) = \tanh\left(\frac{x}{\sqrt{2}}\right)$, the Eq. (3.38) is transformed to the following form,

$$
\begin{aligned}
&x,t)) + \mathfrak{R}(Y(x,t)) + \mathfrak{N}(Y(x,t)) = Y_t + \mathfrak{R}(Y(x,t)) + \mathfrak{N}(Y(x,t)) =\\
&Y_t - \lambda Y_x + Y_{xxxx} + Y_{xx} - 6YY_x^2 - 3Y^2 Y_{xx} = 0.
\end{aligned} \tag{5.109}
$$

For investigating the procedure, we consider the following iteration formula as:

$$
\begin{aligned}
&Y_{n+1}(x,t) = Y_n(x,t) + \int_0^t \phi(\omega)\left[\mathfrak{R}(Y_n(x,\omega) - Y_{n-1}(x,\omega)) + (Q_n(x,\omega) - Q_{n-1}(x,\omega))\right] d\omega,\\
&\mathfrak{N}(Y_n(x,t)) = Q_n(x,t) + O(t^{n+1}), \quad n \geq 1.
\end{aligned} \tag{5.110}
$$

where $\mathfrak{L} = \partial_t$, $\mathfrak{R}(Y(x,t))$ is a linear operator and $\mathfrak{N}(Y(x,t))$ is a nonlinear term. Also, $\phi(\omega)$ is the general Lagrange multiplier[72]. To compute the variation with respect to Y_n, we obtain the following issues:

$$\phi'(\omega) = 0, \quad 1 + \phi(\omega)|_{\omega=t} = 0, \tag{5.111}$$

thus we obtain $\phi(\omega) = -1$. So, the equality is changed to

$$\begin{array}{l} Y_{n+1}(x,t) = Y_n(x,t) - \int_0^t \left[\Re(Y_n(x,\omega) - Y_{n-1}(x,\omega)) + (Q_n(x,\omega) - Q_{n-1}(x,\omega))\right] d\omega, \\ where \\ Y_{-1}(x,t) = 0,\ Y_0(x,t) = l(x), \\ Y_1(x,t) = Y_0(x,t) - \int_0^t \left[\Re(Y_0(x,\omega) - Y_{-1}(x,\omega)) + (Q_0(x,\omega) - Q_{-1}(x,\omega))\right] d\omega. \end{array} \tag{5.112}$$

Then, we have

$$\begin{array}{l} Y_{n+1}(x,t) = Y_n(x,t) - \int_0^t [(-\lambda(Y_n)_x(x,\omega) + (Y_n)_{xx}(x,\omega) + (Y_n)_{xxxx}(x,\omega) + \lambda(Y_{n-1})_x(x,\omega) \\ \quad -(Y_{n-1})_{xx}(x,\omega) - (Y_{n-1})_{xxxx}(x,\omega)) + (A_n(x,\omega) - A_{n-1}(x,\omega))]\, d\omega, \\ where \\ A_n(x,t) = \sum_{j=0}^{n} \sum_{k=0}^{j} Y_k(x,t) Y_{j-k}(x,t) Y_{n-j}(x,t), \\ (A_0(x,t))_{xx} = 3Y_0(x)\left(\left(\frac{d^2}{dx^2}Y_0(x)\right)Y_0(x) + 2\left(\frac{d}{dx}Y_0(x)\right)^2\right), \\ (A_1(x,t))_{xx} = 6Y_1(x)\left(\frac{d^2}{dx^2}Y_0(x)\right)Y_0(x) + 3\left(\frac{d^2}{dx^2}Y_1(x)\right)Y_0(x)^2 + 6Y_1(x)\left(\frac{d}{dx}Y_0(x)\right)^2 \\ \quad +12Y_0(x)\left(\frac{d}{dx}Y_0(x)\right)\left(\frac{d}{dx}Y_1(x)\right), \\ (A_2(x,t))_{xx} = 6Y_2(x)\left(\frac{d^2}{dx^2}Y_0(x)\right)Y_0(x) + 3\left(\frac{d^2}{dx^2}Y_0(x)\right)Y_1(x)^2 \\ \quad +6Y_1(x)\left(\frac{d^2}{dx^2}Y_1(x)\right)Y_0(x) + 3\left(\frac{d^2}{dx^2}Y_2(x)\right)Y_0(x)^2 + 6Y_2(x)\left(\frac{d}{dx}Y_0(x)\right)^2 \\ \quad +12Y_1(x)\left(\frac{d}{dx}Y_0(x)\right)\left(\frac{d}{dx}Y_1(x)\right) + 12Y_0(x)\left(\frac{d}{dx}Y_0(x)\right)\left(\frac{d}{dx}Y_2(x)\right) + 6Y_0(x)\left(\frac{d}{dx}Y_1(x)\right)^2, \end{array} \tag{5.113}$$

$$\begin{array}{l} (A_3(x,t))_{xx} = 6Y_2(x)\left(\frac{d^2}{dx^2}Y_0(x)\right)Y_1(x) + 6Y_3(x)\left(\frac{d^2}{dx^2}Y_0(x)\right)Y_0(x) + 6Y_2(x)\left(\frac{d^2}{dx^2}Y_1(x)\right)Y_0(x) \\ \quad +6Y_1(x)\left(\frac{d^2}{dx^2}Y_2(x)\right)Y_0(x) + 3\left(\frac{d^2}{dx^2}Y_1(x)\right)Y_1(x)^2 + 3\left(\frac{d^2}{dx^2}Y_3(x)\right)Y_0(x)^2 \\ \quad +12Y_2(x)\left(\frac{d}{dx}Y_0(x)\right)\left(\frac{d}{dx}Y_1(x)\right) + 12Y_0(x)\left(\frac{d}{dx}Y_0(x)\right)\left(\frac{d}{dx}Y_3(x)\right) + 12Y_0(x)\left(\frac{d}{dx}Y_1(x)\right)\left(\frac{d}{dx}Y_2(x)\right) \\ \quad +6Y_1(x)\left(\frac{d}{dx}Y_1(x)\right)^2 + 6Y_3(x)\left(\frac{d}{dx}Y_0(x)\right)^2 + 12Y_1(x)\left(\frac{d}{dx}Y_0(x)\right)\left(\frac{d}{dx}Y_2(x)\right), \\ (A_4(x,t))_{xx} = 12Y_2(x)\left(\frac{d}{dx}Y_0(x)\right)\left(\frac{d}{dx}Y_2(x)\right) + 12Y_3(x)\left(\frac{d}{dx}Y_0(x)\right)\left(\frac{d}{dx}Y_1(x)\right) + 12Y_1(x)\left(\frac{d}{dx}Y_0(x)\right)\left(\frac{d}{dx}Y_3(x)\right) \\ \quad +12Y_1(x)\left(\frac{d}{dx}Y_1(x)\right)\left(\frac{d}{dx}Y_2(x)\right) + 12Y_0(x)\left(\frac{d}{dx}Y_0(x)\right)\left(\frac{d}{dx}Y_4(x)\right) \\ \quad +12Y_0(x)\left(\frac{d}{dx}Y_1(x)\right)\left(\frac{d}{dx}Y_3(x)\right) + 6Y_4(x)\left(\frac{d}{dx}Y_0(x)\right)^2 + 6Y_2(x)\left(\frac{d}{dx}Y_1(x)\right)^2 + 6Y_0(x)\left(\frac{d}{dx}Y_2(x)\right)^2 \\ \quad +3\left(\frac{d^2}{dx^2}Y_0(x)\right)Y_2(x)^2 + 3\left(\frac{d^2}{dx^2}Y_2(x)\right)Y_1(x)^2 + 3\left(\frac{d^2}{dx^2}Y_4(x)\right)Y_0(x)^2 + 6Y_3(x)\left(\frac{d^2}{dx^2}Y_1(x)\right)Y_0(x) \\ \quad +6Y_2(x)\left(\frac{d^2}{dx^2}Y_1(x)\right)Y_1(x) + 6Y_2(x)\left(\frac{d^2}{dx^2}Y_2(x)\right)Y_0(x) + 6Y_1(x)\left(\frac{d^2}{dx^2}Y_3(x)\right)Y_0(x) \\ \quad +6Y_3(x)\left(\frac{d^2}{dx^2}Y_0(x)\right)Y_1(x) + 6Y_4(x)\left(\frac{d^2}{dx^2}Y_0(x)\right)Y_0(x), \end{array} \tag{5.114}$$

$$\begin{array}{l} Y_{-1}(x,t) = 0,\ Y_0(x,t) = \tanh\left(\frac{x}{\sqrt{2}}\right) = z, \\ Y_1(x,t) = Y_0(x,t) - \int_0^t \left[R(Y_0(x,\omega) - Y_{-1}(x,\omega)) + (Q_0(x,\omega) - Q_{-1}(x,\omega))\right] d\omega = z - \frac{t\sqrt{2}\,z^2\lambda}{2} + \frac{t\lambda\sqrt{2}}{2}, \\ Y_2(x,t) = -\frac{t\sqrt{2}\,z^2\lambda}{2} + z + \frac{t\lambda\sqrt{2}}{2} + \frac{z^3\lambda^2t^2}{2} - \frac{z\,\lambda^2t^2}{2} + 18z^5t - 27z^3t + 9zt, \\ Y_3(x,t) = -\frac{t\sqrt{2}\,z^2\lambda}{2} + z + \frac{t\lambda\sqrt{2}}{2} + \frac{z^3\lambda^2t^2}{2} - \frac{z\,\lambda^2t^2}{2} + 54z^5t - 81z^3t + 27zt - \frac{\sqrt{2}\,z^4\lambda^3t^3}{4} + \frac{z^2\lambda^3t^3\sqrt{2}}{3} - 3024z^9t^2 \\ \quad + \frac{18279z^7t^2}{2} - 45\sqrt{2}\,z^6\lambda\,t^2 + \frac{171\sqrt{2}\,z^4t^2\lambda}{2} - \frac{\sqrt{2}\,\lambda^3t^3}{12} - \frac{19629z^5t^2}{2} - 45t^2\sqrt{2}\,z^2\lambda + \frac{8613z^3t^2}{2} + \frac{9t^2\lambda\sqrt{2}}{2} - \frac{1215z\,t^2}{2}, \\ Y_4(x,t) = z - \frac{1215t^3\lambda\sqrt{2}}{4} + \frac{z^5\lambda^4t^4}{4} + 135z^7\lambda^2t^3 + \frac{z\,\lambda^4t^4}{6} + 216z^3\lambda^2t^3 - 306z^5\lambda^2t^3 - 45z\,\lambda^2t^3 - \frac{5z^3\lambda^4t^4}{12} \\ \quad + \frac{z^3\lambda^2t^2}{2} - \frac{z\,\lambda^2t^2}{2} - \frac{\sqrt{2}\,\lambda^3t^3}{12} + \frac{27t^2\lambda\sqrt{2}}{2} + \frac{t\lambda\sqrt{2}}{2} - \frac{t\sqrt{2}\,z^2\lambda}{2} - \frac{\sqrt{2}\,z^4\lambda^3t^3}{4} + \frac{z^2\lambda^3t^3\sqrt{2}}{3} - 135\sqrt{2}\,z^6\lambda\,t^2 \\ \quad + \frac{513\sqrt{2}\,z^4t^2\lambda}{2} - 135t^2\sqrt{2}\,z^2\lambda - \frac{182385z^8\sqrt{2}\,\lambda\,t^3}{4} - 30996\sqrt{2}\,z^4t^3\lambda + \frac{13527t^3\sqrt{2}\,z^2\lambda}{2} + \frac{113049\sqrt{2}\,z^6\lambda\,t^3}{2} \\ \quad +13608\sqrt{2}\,z^{10}\lambda\,t^3 + 114z^5t - 171z^3t + 57zt - 6804z^9t^2 + 21627z^7t^2 - 24462z^5t^2 + 11340z^3t^2 \\ \quad -1701z\,t^2 + 2794176z^{13}t^3 - \frac{24357537z^{11}t^3}{2} + \frac{42478533z^9t^3}{2} - 18713997z^7t^3 + 8590077z^5t^3 - \frac{3732885z^3t^3}{2} + \frac{271377z\,t^3}{2}, \\ Y_5(x,t) = z - 7376624640z^{17}t^4 + \frac{335725854597z^{15}t^4}{8} - \frac{809066581515z^{13}t^4}{8} + \frac{13916397699z^3t^4}{8} - \frac{614564253z\,t^4}{8} \\ \quad + \frac{1070527025673z^{11}t^4}{8} - \frac{841318711479z^9t^4}{8} + \frac{395458650927z^7t^4}{8} - \frac{105615074529z^5t^4}{8} + 5709528z^{13}t^3 - \frac{8902791z^3t^3}{2} \\ \quad + \frac{676107z\,t^3}{2} - \frac{51104601z^{11}t^3}{2} + \frac{91660977z^9t^3}{2} - 41617476z^7t^3 + 19743102z^5t^3 - 7812z^9t^2 + 28926z^7t^2 - 37251z^5t^2 \\ \quad +19377z^3t^2 - 3240z\,t^2 + 204z^5t - 306z^3t + 102zt - 68040z^{11}\lambda^2t^4 + \frac{13527z\,\lambda^2t^4}{2} + \frac{463131z^5\lambda^2t^4}{2} - \frac{703917z^7\lambda^2t^4}{2} \\ \quad - \frac{137511z^3\lambda^2t^4}{2} + 250425z^9\lambda^2t^4 - 918z^5\lambda^2t^3 - \frac{5z^3\lambda^4t^4}{12} + \frac{z\,\lambda^4t^4}{6} + 405z^7\lambda^2t^3 - 135z\,\lambda^2t^3 + 648z^3\lambda^2t^3 + \frac{z^5\lambda^4t^4}{4} \\ \quad + \frac{z^3\lambda^2t^2}{2} - \frac{z\,\lambda^2t^2}{2} - \frac{315z^8\sqrt{2}\,\lambda^3t^4}{2} - \frac{z^6\sqrt{2}\,\lambda^5t^5}{8} - 18162144z^{14}\sqrt{2}\,\lambda\,t^4 - \frac{17z^2\lambda^5t^5\sqrt{2}}{120} + \frac{z^4\lambda^5t^5\sqrt{2}}{4} + \frac{340581483z^{12}\sqrt{2}\,\lambda\,t^4}{4} \\ \quad -162559926z^{10}\sqrt{2}\,\lambda\,t^4 + \frac{644302755z^8\sqrt{2}\,\lambda\,t^4}{4} + \frac{97099425\sqrt{2}\,z^4t^4\lambda}{4} - 2867508t^4z^2\sqrt{2}\,\lambda - 86974182z^6\sqrt{2}\,\lambda\,t^4 \\ \quad + \frac{825z^6\sqrt{2}\,\lambda^3t^4}{2} + \frac{231z^2\lambda^3t^4\sqrt{2}}{2} - 363z^4\sqrt{2}\,\lambda^3t^4 - \frac{t\,z^2\sqrt{2}\,\lambda}{2} - \frac{z^4\sqrt{2}\,\lambda^3t^3}{4} + \frac{z^2\lambda^3t^3\sqrt{2}}{3} - 285z^6\sqrt{2}\,\lambda\,t^2 \\ \quad + \frac{1083\sqrt{2}\,z^4t^2\lambda}{2} - 285t^2z^2\sqrt{2}\,\lambda + 30618z^{10}\sqrt{2}\,\lambda\,t^3 - \frac{212625z^8\sqrt{2}\,\lambda\,t^3}{2} - 78165\sqrt{2}\,z^4t^3\lambda + \frac{35721t^3z^2\sqrt{2}\,\lambda}{2} \\ \quad + \frac{273699z^6\sqrt{2}\,\lambda\,t^3}{2} - \frac{1701t^3\lambda\sqrt{2}}{2} + \frac{t\lambda\sqrt{2}}{2} - \frac{\sqrt{2}\,\lambda^3t^3}{12} + \frac{57t^2\lambda\sqrt{2}}{2} + \frac{271377t^4\lambda\sqrt{2}}{4} + \frac{\sqrt{2}\,\lambda^5t^5}{60} - \frac{15\sqrt{2}\,\lambda^3t^4}{2}, \\ \vdots \end{array} \tag{5.115}$$

Subsequently, we define the series form solutions as follows

$$Y(x,t) = \sum_{k=0}^{\infty} Y_k(x,t) = Y_0(x,t) + Y_1(x,t) + Y_2(x,t) + Y_3(x,t) + \cdots, \tag{5.116}$$

where the exact solution is given as $Y(x,t) = \tanh\left(\frac{x+t}{\sqrt{2}}\right)$.

Convergence analysis of DVIM

We discuss the convergence by the following theorems:

Theorem 5.1 Suppose that $\mathfrak{L}$ and $\mathfrak{N}$ are Lipschitz functions as in the last theorems; then, the relation of (5.109) is convergent.

Proof Assume $H = (C[J], \| \, . \, \|)$ be the Banach space defined before, and let $Y_k = \sum_{i=0}^{k} Y_i(x,t)$. To offer that Y_k is a Cauchy sequence in *H*. Let

$$
\begin{aligned}
&\| Y_{k_1} - Y_{k_2} \| = \max_{t \in J} \mid \sum_{i=k_2+1}^{k_1} \mid, \quad k_2 = 1,2,3,\dots \\
&\quad \leq \mid \mathfrak{N}^{-1} \left(\mathfrak{N} \sum_{i=k_2+1}^{k_1} (\mathfrak{L}(Y_{i-1}) + \mathfrak{R}(Y_{i-1}) + \mathfrak{N}(Y_{i-1})) \right) \mid \\
&\quad = \mid \mathfrak{N}^{-1} \left(\mathfrak{N} \sum_{i=k_2+1}^{k_1-1} (\mathfrak{L}(Y_{i}) + \mathfrak{R}(Y_{i}) + \mathfrak{N}(Y_{i})) \right) \mid \\
&\quad \leq \mid \mathfrak{N}^{-1} \left\{ \mathfrak{N}(\mathfrak{L}(Y_{k_1-1}) - \mathfrak{L}(Y_{k_2-1}) + \mathfrak{R}(Y_{k_1-1}) - \mathfrak{R}(Y_{k_2-1}) + \mathfrak{N}(Y_{k_1-1}) - \mathfrak{N}(Y_{k_2-1})) \right\} \mid \\
&\quad \leq g_1 \mid \mathfrak{N}^{-1} \left\{ \mathfrak{N}(\mathfrak{L}(Y_{k_1-1}) - \mathfrak{L}(Y_{k_2-1})) \right\} \mid \\
&\qquad + g_2 \mid \mathfrak{N}^{-1} \left\{ \mathfrak{N}(\mathfrak{R}(Y_{k_1-1}) - \mathfrak{R}(Y_{k_2-1})) \right\} \mid \\
&\qquad + g_3 \mid \mathfrak{N}^{-1} \left\{ \mathfrak{N}(\mathfrak{N}(Y_{k_1-1}) - \mathfrak{N}(Y_{k_2-1})) \right\} \mid \\
&\quad = (g_1 + g_2 + g_3) \, \| Y_{k_1-1} - Y_{k_2-1} \| \, .
\end{aligned}
\tag{5.117}
$$

Let $k_1 = k_2 + 1$, then we have

$$
\| Y_{k_2+1} - Y_{k_2} \| \leq g \, \| Y_{k_2} - Y_{k_2-1} \| \leq g^2 \, \| Y_{k_2-1} - Y_{k_2-2} \| \leq \cdots \leq g^{k_2} \, \| Y_1 - Y_0 \|,
\tag{5.118}
$$

in which $g = g_1 + g_2 + g_3$, we get

$$
\begin{aligned}
&\| Y_{k_1} - Y_{k_2} \| \leq \| Y_{k_2+1} - Y_{k_2} \| + \| Y_{k_2+2} - Y_{k_2+1} \| + \cdots + \| Y_{k_1} - Y_{k_1-1} \| \\
&\quad \leq \left(g^{k_2} + g^{k_2+1} + \cdots + g^{k_1-1} \right) \| Y_1 - Y_0 \| \\
&\quad \leq g^{k_1} \left(\frac{1 - g^{k_1-k_2}}{1-g} \right) \| Y_1 \|,
\end{aligned}
\tag{5.119}
$$

if $0 < g < 1$, then $1 - g^{k_1-k_2} < 1$, hence we have

$$
\begin{aligned}
&\| Y_{k_1} - Y_{k_2} \| \leq \left(\frac{g^{k_1}}{1-g} \right) \max_{t \in J} \| Y_1 \|, \\
&when \quad \| Y_1 \| < \infty, \quad and \ n \to \infty \ then \quad \| Y_{k_1} - Y_{k_2} \| \to 0.
\end{aligned}
\tag{5.120}
$$

One can see that Y_{k_1} is a Cauchy sequence in *H*. So, the series Y_{k_1} is convergent.

Graphical results

In this particular section, a detailed analysis of the generated solutions with those from earlier research demonstrates the unique originality and innovation of the current study. Using the Maple software, a number of graphical representations are produced for the problems, ranging from Figs. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 and 13, at different selected values. This section aims to generate graphical illustrations of obtained solutions of fourth-order Cahn Hilliard Equation, which are obtained by the $\tan(\phi/2)$-expansion technique, Jacobi elliptic function expansion Scheme and one and two waves solutions. The results showcase a variety of visual depictions, including 3D, 2D and density plots. Here, we elaborate and investigate the dynamics of the some of the obtained solutions. These results generally shows dark, periodic, kink, bright, M-shaped and W-shaped soliton solutions. Figure 1 visualize the graphs associated with the breather wave solution for the given Eq. (3.43) using $\lambda = 0.2$. Figure 2 illustrates the characteristics of periodic wave solutions for Eq. (3.44) using the given parameters $\lambda = 0.2$. Figure 3 illustrates the characteristics of soliton form solutions for Eq. (3.45) by applying the given parameters $\lambda = 0.2$. Figure 4 illustrates the characteristics of bright soliton solutions for Eq. (3.46) using the given parameters $\lambda = 0.2$. Figure 5 displays the characteristics of the periodic wave solution graph for the solution of Eq. (3.55) for $\lambda = 0.2$. Figure 6 depicts the graph of the soliton solution for the solution of Eq. (3.62) for $\lambda = 0.2$. Figure 7 visualizes the graph associated with the periodic wave solution for the solution of Eq. (3.63) for $\lambda = 0.2$. Figure 8 provides the graph associated with the periodic wave solution for the solution of Eq. (3.69) for $\lambda = 2$. Figure 9 provides the graph associated with the soliton solution for the solution of Eq. (3.70) for $\lambda = 0.2$.

Figure 10 provides the graph associated with the one wave solution for the solution of Eq. (3.72) for $\lambda = 2$. Figure 11 visualizes the graph associated with the periodic wave solution for the solution of Eq. (3.73) for $\lambda_2 = 0.2$. Figure 12 provides the graph associated with the periodic form wave solution for the solution of Eq. (3.75) for $\lambda = 2$.

Dark solitons are defined as solutions to the nonlinear Schrödinger equation that exhibit a dip in intensity within a uniform background, characterized by an abrupt phase change at the dip center. Bright and dark solitons are usually considered from the perspective of modulation instability of the carrier wave train: bright solitons are observed when the carrier wave is unstable with respect to long-wave modulations, while dark solitons are observed when the carrier wave is modulationally stable. Solitons can be classified in numerous ways, such as considering their topology or their profiles. The solitons that will be focused on are called the Kink soliton. The kink soliton is topological, meaning that the boundary conditions at infinity for the wave are topologically different to the vacuum. These kink solitons are also characterized by their permanent profiles, which mean that they do not change with time. We also report a soliton-induced change in the topology of the density profiles of the two-species condensates at phase separation in Figs. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12.

The first, second, third, fourth, fifth and sixth order approximations are compared with the exact solution for the described case with known condition in Fig. 13 and show that the decomposition VIM is a working method in obtaining the fourth order Cahn–Hilliard equation as a model of fluid flow.

We furnish the comparison of the above reported results with previously published works on fourth-order Cahn–Hilliard equation. Hassana et al. utilized optimal Homotopy asymptotic method to derive approximate solutions of the fractional Cahn–Hilliard equation[73]. Al-Smadi et al. studied the approximate solutions of nonlinear temporal fractional model of Cahn–Hilliard equation[49]. El-Tantawy et al. utilized homotopy perturbation transform method to derive semi-analytical solutions of the fourth-order time-fractional Cahn–Hilliard models[50]. Only few works are published in the literature on the analysis of fourth-order Cahn–Hilliard model. Some researchers derive trigonometric, hyperbolic and rational trigonometric solutions with aid ansatz methods. We obtained abundant solutions by help of proposed schemes. We implemented the $\tan(\phi/2)$ -expansion approach, Jacobi elliptic function expansion scheme, rational multi wave functions method and decomposition variational iteration method to derive hyperbolic, periodic, dark soliton, bright soliton, kink soliton, and singular soliton solutions of the fourth-order Cahn–Hilliard model. In this work, we derive more soliton solutions with efficient approach. The solutions are expressed by trigonometric, hyperbolic and Jacobi-elliptic functions. These solutions are new for the fourth-order Cahn–Hilliard equation, which not been reported elsewhere, confirming the originality of the reported solutions. The main advantages of three analytical methods are the good transparency between calculated results and input data, less effort in modeling and programming, and very short computation time. Depending on the degree of simplification, the results are sometimes not very accurate.

Conclusion

In this article, the TEM, JEFES, RMWFs and DVIM were acknowledged as the strong techniques for finding solutions of NLEEs. TEM was presented for the nonlinear CH equation. By utilizing JEFES, RMWFs and DVIM techniques, we found several solutions of distinct types. Maple is a widely validated program used in many scientific and mathematical domains. We used the latest version of Maple to influence changes or improvements made to its algorithms. The acquired results demonstrate that the mentioned schemes were a strong mathematical technique from which novel, exact, precise solutions are frequently obtained. From a theoretical and practical standpoint, these approaches were quite valid, modest, practical, and efficient. In the vast field of applied nonlinear science, it is crucial to note that these approaches may be more commonly appropriate for the broad category of physical problems. The behavior of such nonlinear models can be better determined, and decisions made to further resolve corresponding difficulties in the future. The results of this research present new insights into soliton behavior, and scholars can use the method and models suggested in this article. The result will significantly enhance nonlinear science and serve as a valuable resource for further research. Numerical simulation has been carried out to highlight the ability of the DVIM method. We obtained a good approximation for the nonlinear model with minimal numerical calculations. In the future, this article will

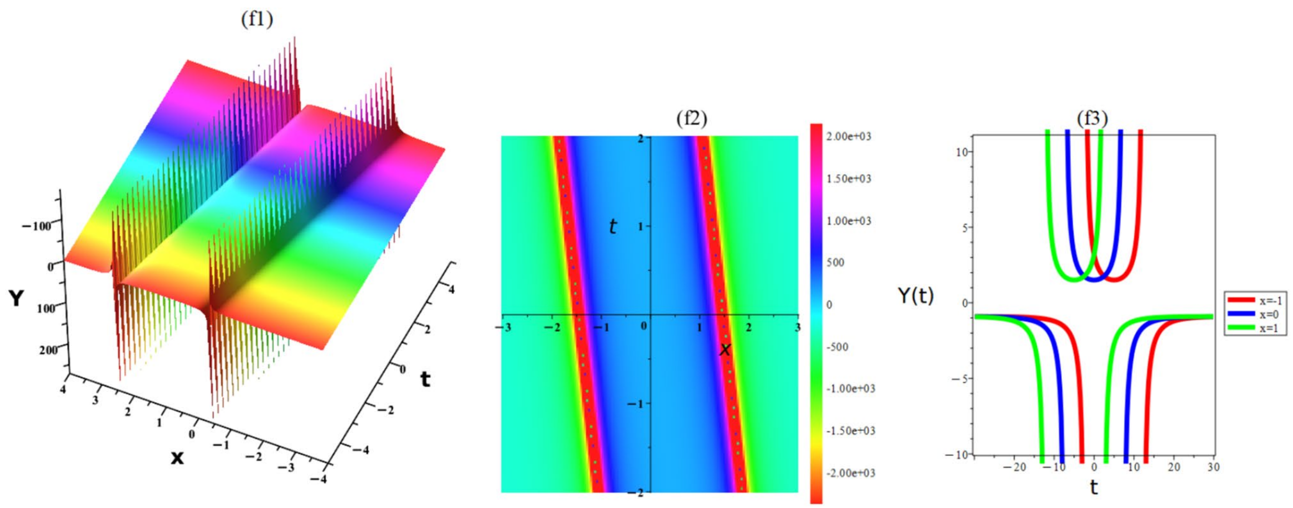

Figure 1. Dynamic Graphical visualization of the obtained result of Eq. (3.43) of JEFES method, gives breather wave solution such as (f1) 3D surface, (f2) density plot, (f3) 2D surface for $Y_1(x,t)$: $\lambda = 0.2, \theta = m\,(x + \lambda\,t)$, with $-4 \leq x \leq 4$ and $-4 \leq t \leq 4$ for 3D plot, $-3 \leq x \leq 3$ and $-2 \leq t \leq 2$ for density plot, and $-30 \leq t \leq 30$ for 2D plot.

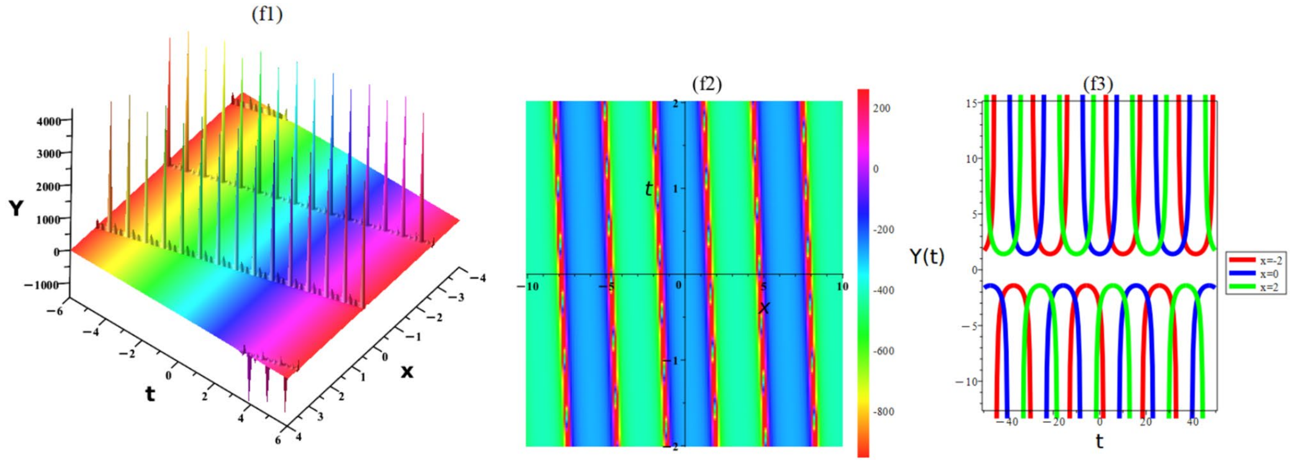

Figure 2. Dynamic Graphical visualization of the obtained result of Eq. (3.44) of JEFES method, gives periodic wave solution such as (f1) 3D surface, (f2) density plot, (f3) 2D surface for $Y_2(x,t)$: $\lambda = 0.2, \theta = m\,(x + \lambda\,t)$, with $-4 \le x \le 4$ and $-6 \le t \le 6$ for 3D plot, $-10 \le x \le 10$ and $-2 \le t \le 2$ for density plot, and $-40 \le t \le 40$ for 2D plot.

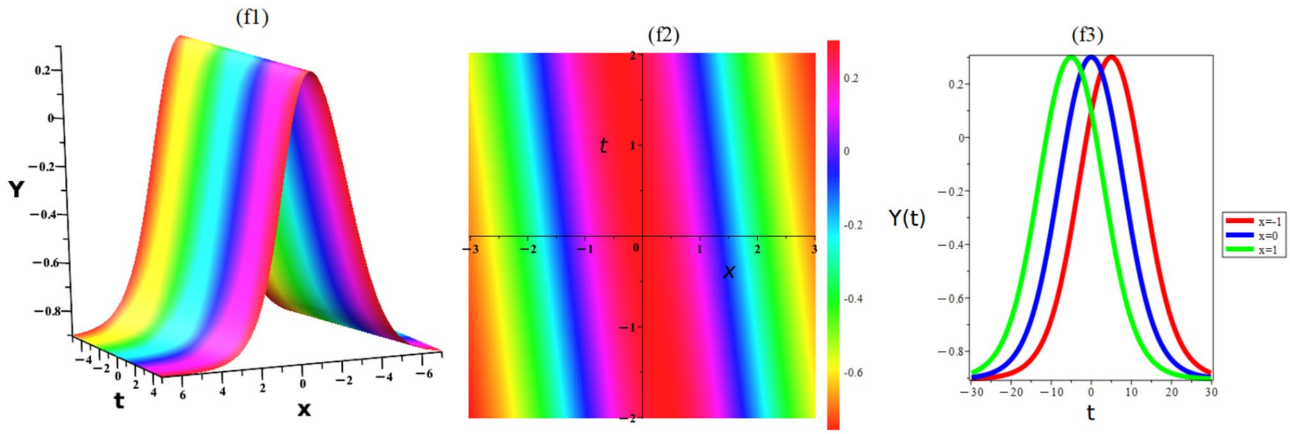

Figure 3. Dynamic Graphical visualization of the obtained result of Eq. (3.46) of JEFES method, gives soliton solution such as (f1) 3D surface, (f2) density plot, (f3) 2D surface for $Y_3(x,t)$: $\lambda = 0.2, \theta = m\,(x + \lambda\,t)$, with $-6 \le x \le 6$ and $-4 \le t \le 4$ for 3D plot, $-3 \le x \le 3$ and $-2 \le t \le 2$ for density plot, and $-30 \le t \le 30$ for 2D plot.

also help the researchers to use these techniques on higher-dimensional nonlinear models and to the system of coupled equations. In addition, by selecting appropriate parameters, we plot three-dimensional graphs, density plots (Figs. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13). The visual depictions act as a significant means of verification, supporting the main findings and offering a thorough and detailed analysis of the solutions. The obtained solutions can be used in different physical systems such interfacial fluid flow, polymer science and in industrial applications. These images vividly illustrate the physical characteristics of rogue wave solutions. Two and one waves for two cases of the mentioned equations were analyzed and investigated with plenty of solutions. In future, the applied approaches can be utilized to more advanced problems in integrable systems. Compared to many other methods, these solutions demonstrate and highlight the proposed method's simplicity, reliability, and effectiveness.

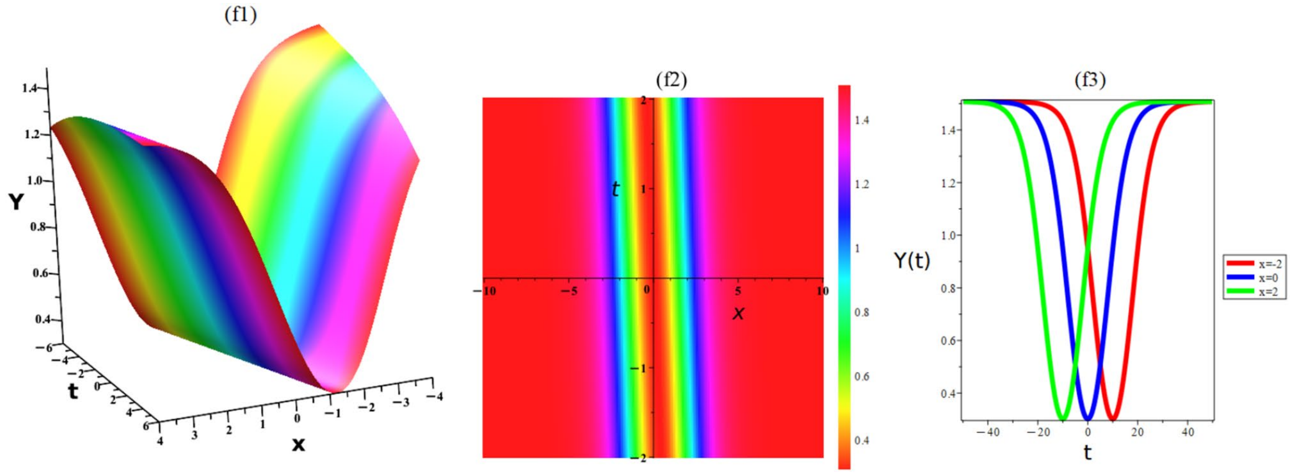

Figure 4. Dynamic Graphical visualization of the obtained result of Eq. (3.47) of JEFES method, gives bright soliton solution such as (f1) 3D surface, (f2) density plot, (f3) 2D surface for $Y_4(x,t)$: $\lambda = 0.2, \theta = m\,(x + \lambda\,t)$, with $-4 \le x \le 4$ and $-6 \le t \le 6$ for 3D plot, $-10 \le x \le 10$ and $-2 \le t \le 2$ for density plot, and $-40 \le t \le 40$ for 2D plot.

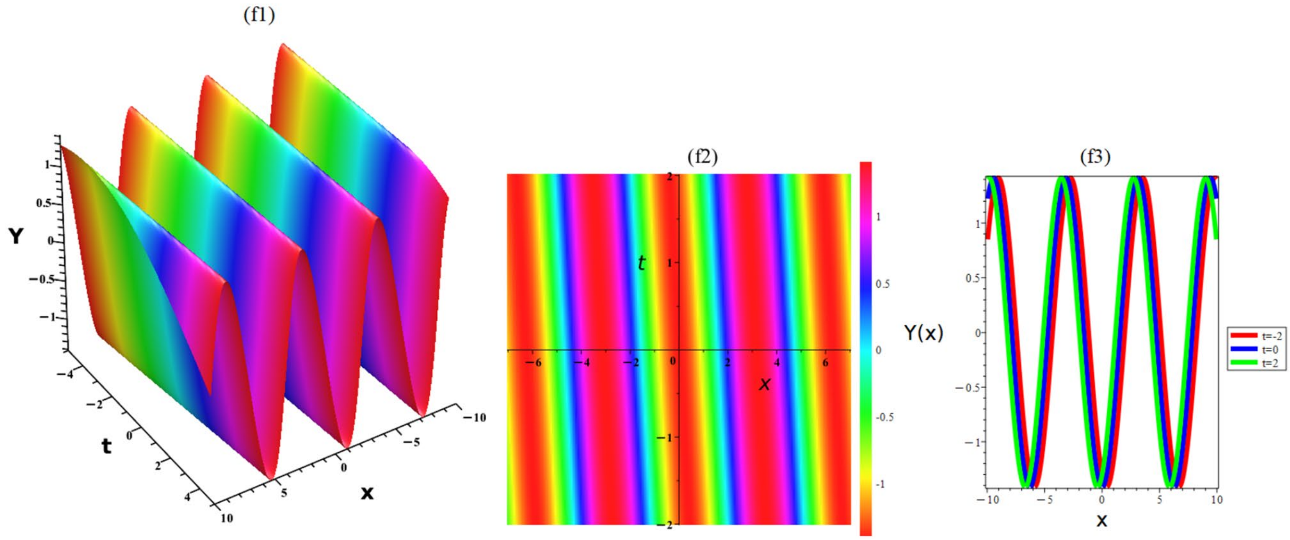

Figure 5. Dynamic Graphical visualization of the obtained result of Eq. (3.55) of JEFES method, gives periodic wave solution such as (f1) 3D surface, (f2) density plot, (f3) 2D surface for $Y_9(x,t)$: $\lambda = 0.2, \theta = m\,(x + \lambda\,t)$, with $-10 \le x \le 10$ and $-4 \le t \le 4$ for 3D plot, $-6 \le x \le 6$ and $-2 \le t \le 2$ for density plot, and $-10 \le x \le 10$ for 2D plot.

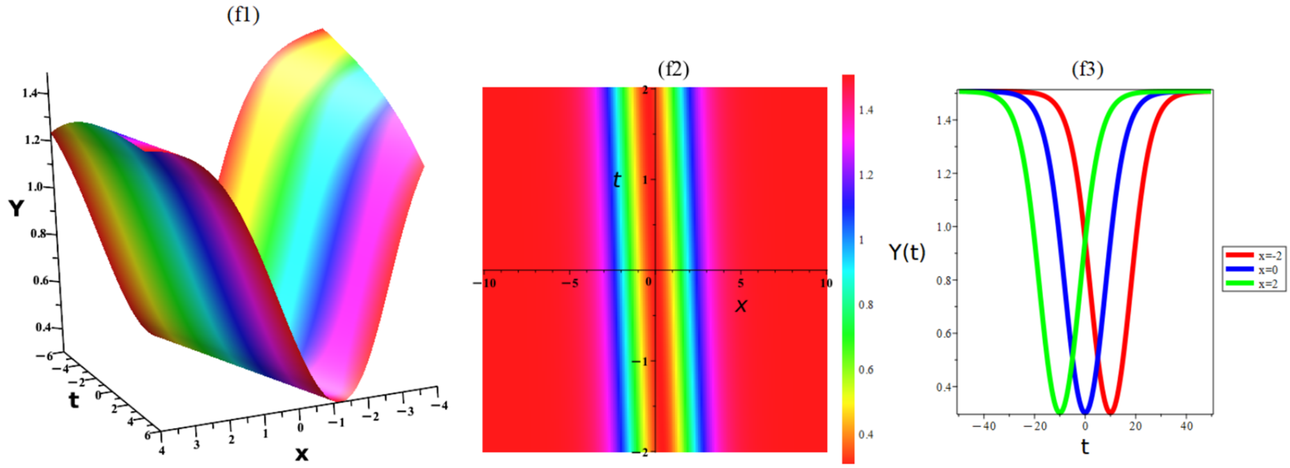

Figure 6. Dynamic Graphical visualization of the obtained result of Eq. (3.62) of JEFES method, gives soliton solution such as (f1) 3D surface, (f2) density plot, (f3) 2D surface for $Y_{14}(x,t)$: $\lambda = 0.2, \theta = m\,(x + \lambda\, t)$, with $-4 \leq x \leq 4$ and $-6 \leq t \leq 6$ for 3D plot, $-10 \leq x \leq 10$ and $-2 \leq t \leq 2$ for density plot, and $-40 \leq t \leq 40$ for 2D plot.

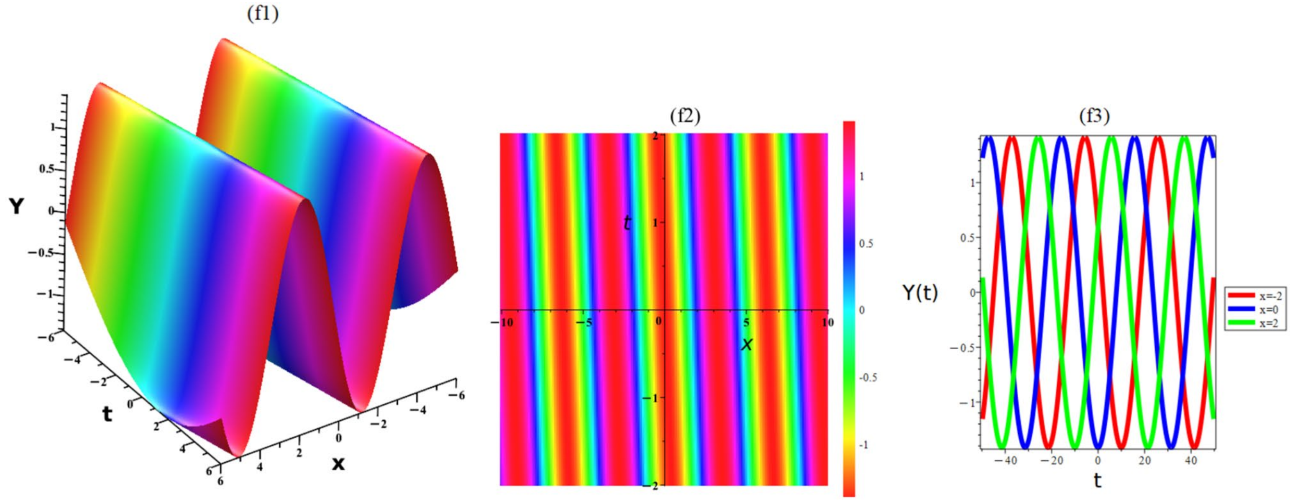

Figure 7. Dynamic Graphical visualization of the obtained result of Eq. (3.63) of JEFES method, gives periodic wave solution such as (f1) 3D surface, (f2) density plot, (f3) 2D surface for $Y_{15}(x,t)$: $\lambda = 0.2, \theta = m\,(x + \lambda\, t)$, with $-6 \leq x \leq 6$ and $-6 \leq t \leq 6$ for 3D plot, $-10 \leq x \leq 10$ and $-2 \leq t \leq 2$ for density plot, and $-40 \leq t \leq 40$ for 2D plot.

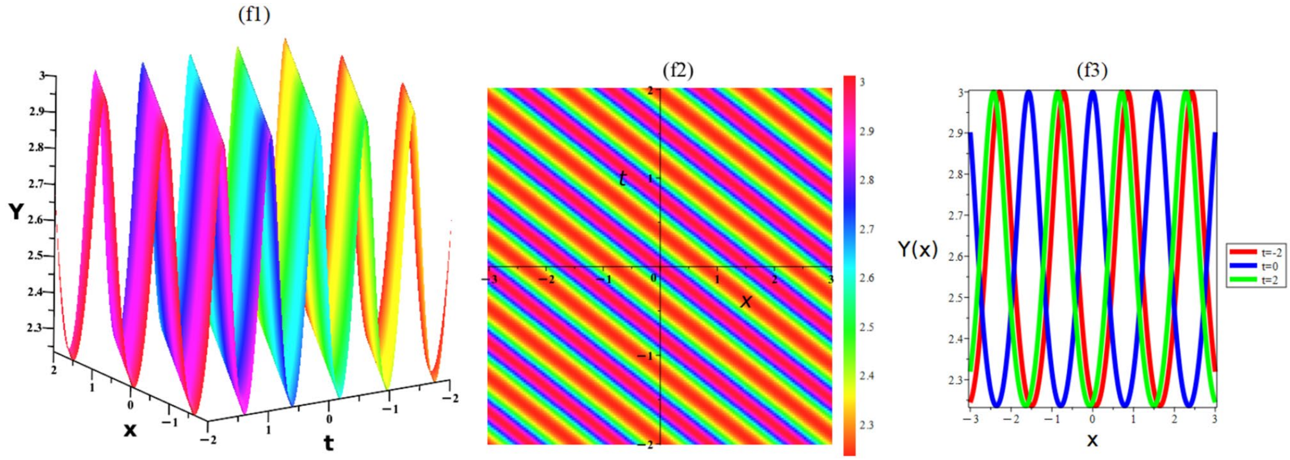

Figure 8. Dynamic Graphical visualization of the obtained result of Eq. (3.69) of JEFES method, gives periodic wave solution such as (f1) 3D surface, (f2) density plot, (f3) 2D surface for $Y_1(x,t)$: $\lambda = 2, \theta = m\,(x + \lambda\,t)$, with $-2 \le x \le 2$ and $-2 \le t \le 2$ for 3D plot, $-3 \le x \le 3$ and $-2 \le t \le 2$ for density plot, and $-3 \le x \le 3$ for 2D plot.

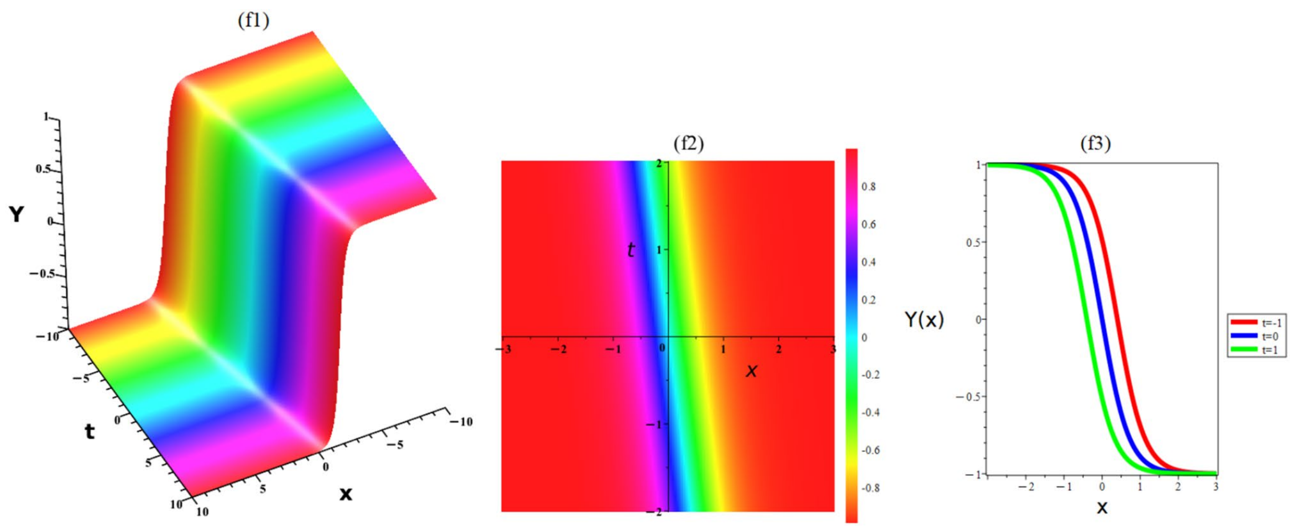

Figure 9. Dynamic Graphical visualization of the obtained result of Eq. (3.70) of JEFES method, gives soliton solution such as (f1) 3D surface, (f2) density plot, (f3) 2D surface for $Y_2(x,t)$: $\lambda = 0.2, \theta = m\,(x + \lambda\,t)$, with $-10 \le x \le 10$ and $-10 \le t \le 10$ for 3D plot, $-3 \le x \le 3$ and $-2 \le t \le 2$ for density plot, and $-3 \le x \le 3$ for 2D plot.

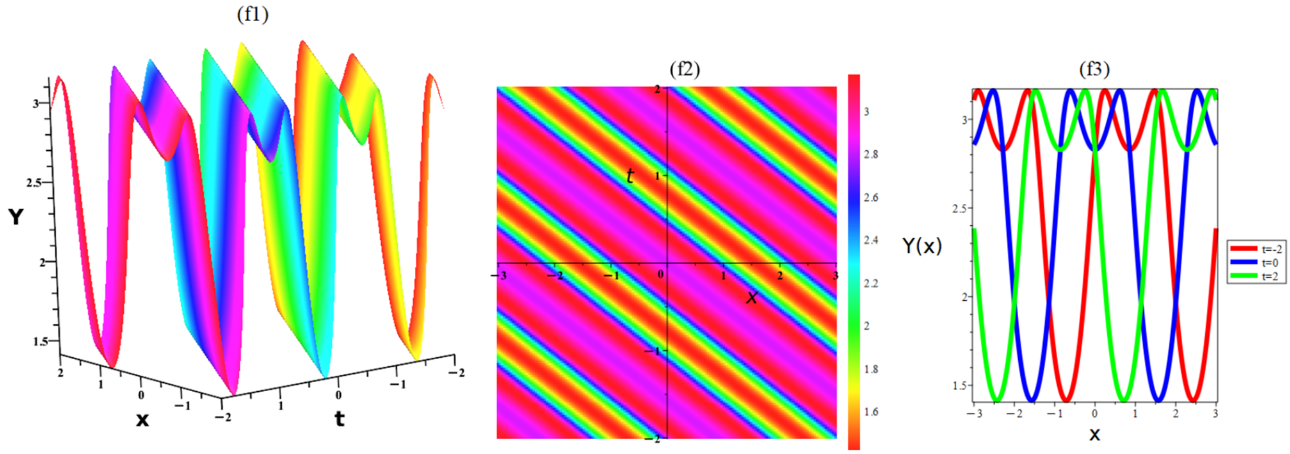

Figure 10. Dynamic Graphical visualization of the obtained result of Eq. (3.72) of JEFES method, gives M-shaped soliton solution such as (f1) 3D surface, (f2) density plot, (f3) 2D surface for $Y_3(x,t)$: $\lambda = 2, \theta = m\,(x + \lambda\,t)$, with $-2 \le x \le 2$ and $-2 \le t \le 2$ for 3D plot, $-3 \le x \le 3$ and $-2 \le t \le 2$ for density plot, and $-3 \le x \le 3$ for 2D plot.

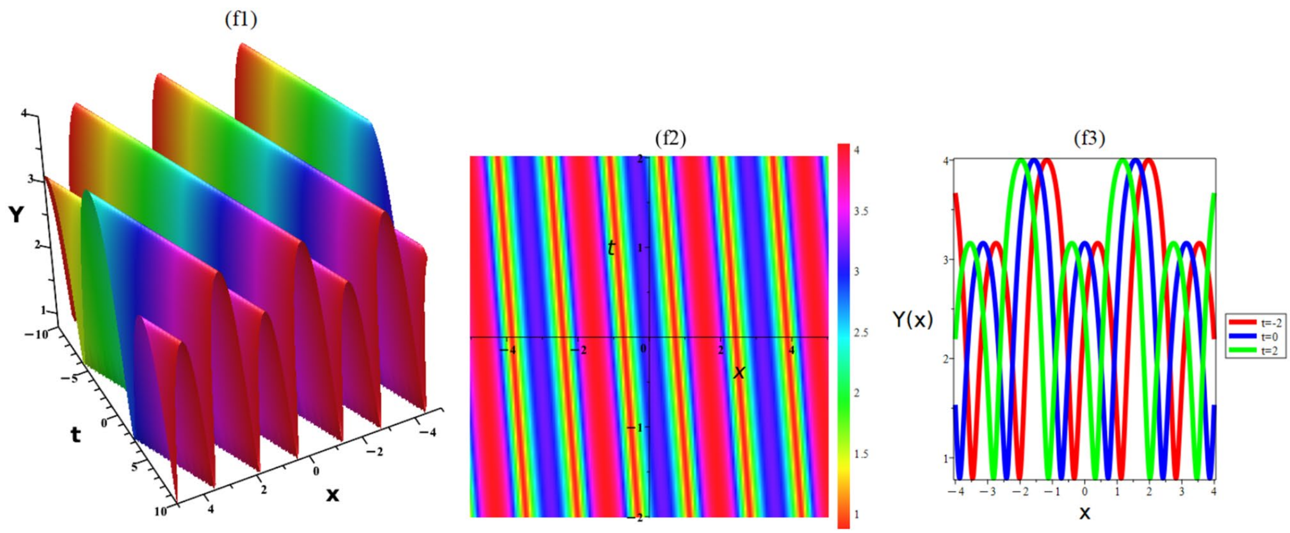

Figure 11. Dynamic Graphical visualization of the obtained result of Eq. (3.73) of JEFES method, gives W-shaped soliton solution such as (f1) 3D surface, (f2) density plot, (f3) 2D surface for $Y_4(x,t)$: $\lambda = 0.2, \theta = m\,(x + \lambda\,t)$, with $-4 \le x \le 4$ and $-10 \le t \le 10$ for 3D plot, $-4 \le x \le 4$ and $-2 \le t \le 2$ for density plot, and $-4 \le x \le 4$ for 2D plot.

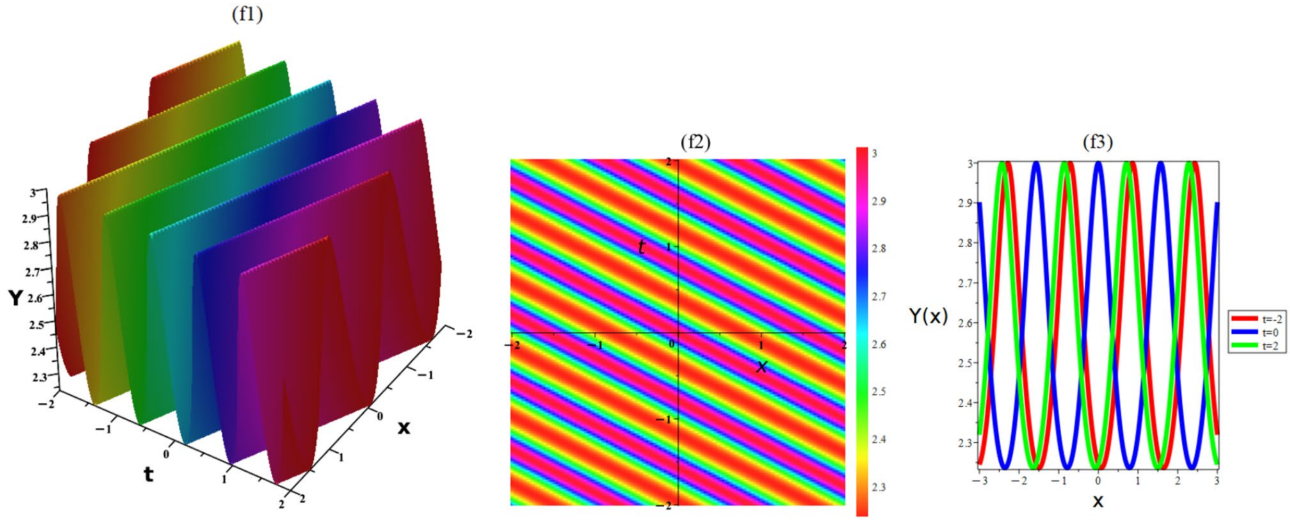

Figure 12. Dynamic Graphical visualization of the obtained result of Eq. (3.75) of JEFES method, gives periodic wave solution such as (f1) 3D surface, (f2) density plot, (f3) 2D surface for $Y_5(x,t)$: $\lambda = 0.2, \theta = m\,(x + \lambda\,t)$, with $-2 \leq x \leq 2$ and $-2 \leq t \leq 2$ for 3D plot, $-2 \leq x \leq 2$ and $-2 \leq t \leq 2$ for density plot, and $-3 \leq x \leq 3$ for 2D plot.

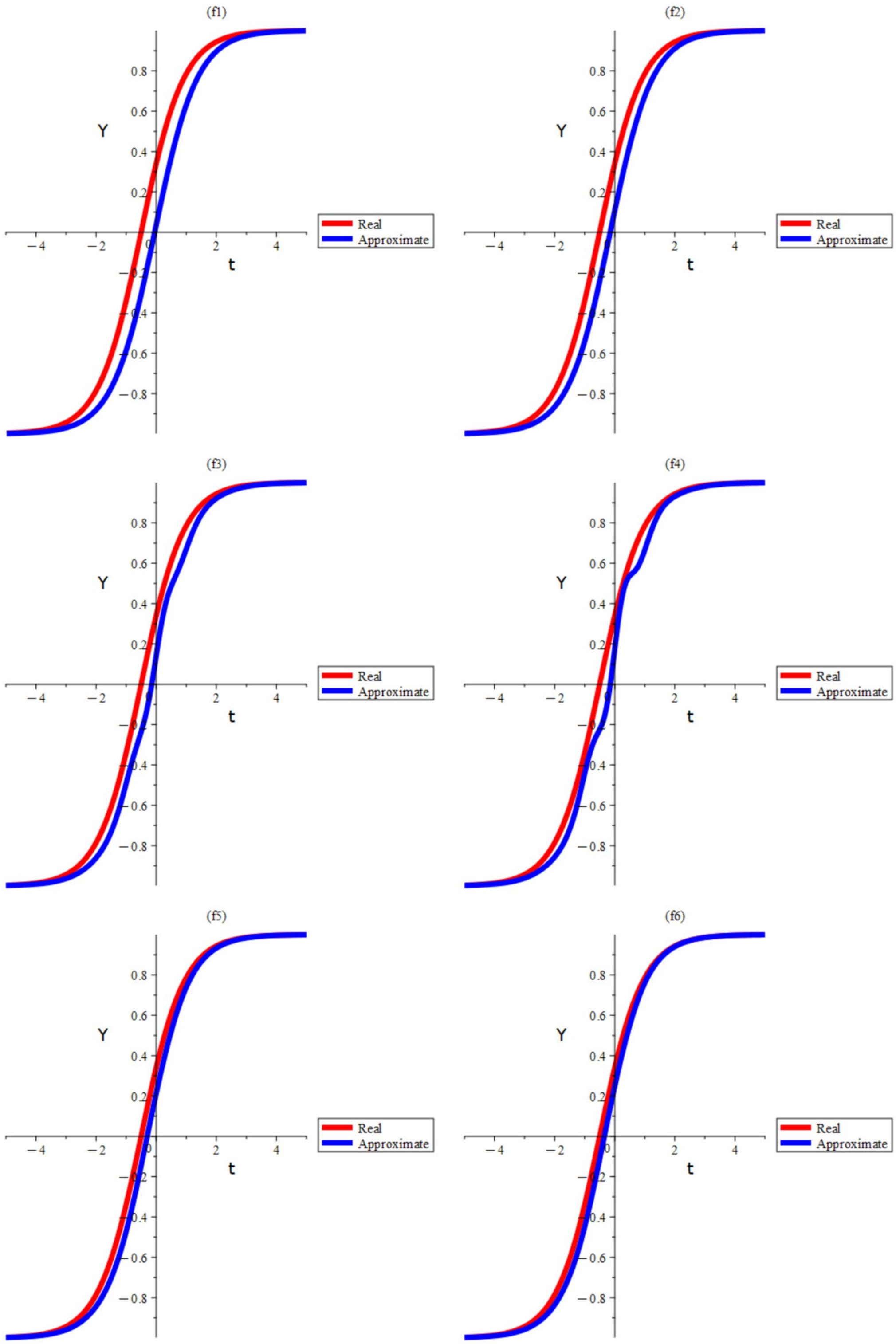

Figure 13. Comparing the exact and approximate solution in a two-dimensional graph (5.113) (f1) the first-order, (f2) the second-order, (f3) the third-order, (f4) the fourth-order, (f5) the fifth-order and (f6) the sixth-order of 4OCHEq by dVIM.

Data availability

The datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

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Author contributions

The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript. Conceptualization, J.L., J.M.; Methodology, J.L., J.M., A.A. and O.A.I.; Software, J.M., B.E., A.A.F., K.H.M.; Investigation, A.A.F., A.A. M.F.A., and O.A.I.; Resources, O.A.I., A.A., and M.F.A; Writing-original draft, J.L., J.M., A.A., O.A.I., B.E., A.A.F., K.H.M., M.F.A.; Supervision, J.M. All authors have read and agreed to the published version of the manuscript.

Declarations

Competing interests
The authors declare no competing interests.

Additional information
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