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# Existence and uniqueness of solutions for fuzzy fractional integro-differential equations with boundary conditions

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**The analytical behavior of fractional-order differential equations under uncertainty is often difficult to investigate. To address this challenge, this study considers Caputo-type fuzzy fractional Volterra integro-differential equations (FFVIDEs) with boundary conditions. An iterative numerical approach based on the Adomian decomposition method (ADM) is proposed to obtain approximate fuzzy solutions. The existence and uniqueness of the solution are established using Banach's fixed point theorem. Numerical simulations are carried out in MATLAB to illustrate the symmetry between the lower and upper $\rho$-cut representations of the fuzzy solutions. The graphical results demonstrate the accuracy and efficiency of the proposed method in handling uncertainty in fractional systems.**

**Keywords** Fractional differential equation, Adomian decomposition method, fixed point theorem, fuzzy fractional differential equations.

Aside from pure mathematics, there is a strong desire to learn more about the subject applications in fluid flow, biology, fractal theory, control theory, electrochemistry, and viscoelasticity stimulate the study of fractional calculus (see[1–3]). Fixed point theorems are commonly used to investigate the existence and uniqueness of solutions in the domain of fractional calculus (see[4–6]). The solution properties were discovered in[7,8], while several numerical approaches, including quadrature, product integration, variational iteration, and fuzzy transforms, were discovered in (see[5,9]). The use of fuzzy sets is well-suited for models with uncertainty, leading to a focus on fuzzy fractional calculus for addressing these types of problems. The study of fuzzy fractional differential equations (FFDEs) is a new area of fuzzy mathematics that is expanding steadily. To investigate a novel sort of dynamical system, the notion of fractional order fuzzy differential equations (FDEs) was recently developed[10]. The authors Agarwal, Lakshmikantham, and Nieto[11] originated the fractional case generalization of the H-differentiability. In[12] authors discovered the existence and uniqueness of results for FFDEs. Several studies and solutions of order $0 < \eta \leq 1$, (see[13,14]) FFDEs have been published in recent years. However, uncertainty can take a heavy toll on real-world issues at any time. Incomplete data, measurement errors, or identifying initial conditions can all lead to uncertainty.

The area of fuzzy fractional integro-differential equations (FFIDEs) has past few years piqued the interest of many researchers since it is recognized as a powerful tool for presenting imprecise parameters and handling their dynamical systems in real fuzzy conditions. Analyzing the majority of FFIDE's problems is difficult. Some real-world scenarios deal with this, including the golden mean, quantum optics, gravity, engineering phenomena, practical systems, and medical science. The existence of solutions for various fuzzy fractional integral equations is important, particularly in the context of developing efficient numerical methods for approximating these solutions[15]. By employing the generalized Hukuhara derivative[16], the notion of fractional Caputo H-differentiability was utilized to study fuzzy fractional differential[17] and integro-differential[18] equations. In[19], weakly contractive mappings were employed to establish the existence of a unique solution for FFVIDEs without Liptchitz conditions. In[20] the existence and uniqueness of the solution for FFVIDEs in Caputo's sense were proven, as well as the usage of Adomian decomposition to approximate the solution. In[21–23], stability properties for FFVIDEs were explored, including the Ulam-Hyers stability. The variational iteration method was used in[24]

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to get numerical solutions for FFDEs. The fuzzy Caputo derivative was implemented in[25] using the modified fractional Euler (MFE) technique.

The investigation for interval-valued functions begun in[26] continued in[27], which describes the MFE and Adams generalization methods. Since the initial value problem (IVP) for the FFDEs and the corresponding FFVIDEs are not equivalent in general, a desirable condition is presented in[28] to make this equivalence valid. The fractional Euler method is used[29] to prove the solution of FFDEs under Caputo differentiability using generalized Taylor's expansion as fuzzy-valued functions.

Very recently, Alexandru Mihai Bica et al.,[30] studied the numerical method for fractional fuzzy integral equations, and Ehsan Ul Haq et al.,[31] investigated a reliable method for a system of fuzzy solutions. The method of Adomian decomposition has been used in this seller to provide approximate results for the system of fuzzy fractional initial value problems. A new fractional-order method based on Jacobi polynomials was introduced by Bidari et al.[32] for finding the solution to FFIDEs. They developed a set of equations to represent the suggested problem using operational matrices. They proved that under certain conditions, the proposed method converges. The fuzzy product integration approach was utilized by Bica et al.[33] to develop an iterative method for solving fuzzy fractional Volterra integral equations. The fuzzy Volterra integral equation with weak singularity was investigated in[34] using a piece-wise spline collocation approach. Additionally, they have shown that the solution exists and converges to content validity. FFIDE of the Volterra and Fredholm types can be solved using the Chebyshev spectral method according to a recent proposal by Kumar et al.,[35].

Caputo-type fuzzy fractional Volterra integro-differential equations (FFVIDEs) emerge naturally in real-life situations where memory, heredity, and uncertainty play a significant role in the system's behavior. These equations combine three key concepts-fractional calculus, fuzzy logic, and integro-differential operators-making them suitable for modeling complex real-world processes that cannot be accurately described using classical differential equations.

In real life, many physical, biological, and engineering systems exhibit memory effects (dependence on past states) and imprecise information due to environmental fluctuations or measurement errors. The fractional derivative in the Caputo sense captures memory and hereditary properties, while fuzziness handles uncertainty or vagueness in system parameters and initial conditions. For instance: In engineering, they model viscoelastic materials where stress depends not only on the current strain but also on the entire deformation history. In biology, fuzzy fractional models describe population dynamics or enzyme kinetics under uncertain environmental influences. In electrical circuits, they represent RLC circuits with fractional elements and uncertain parameters. In control theory, they help design controllers for systems with uncertain or partially known dynamics.

The usage of fuzzy fractional calculus, also known as fuzzy fractional integro-differential equations, has grown significantly in recent years due to their extensive use in simulating a variety of physical industrial processes, including heat and mass transport, bio-mechanics, electromagnetic fields, etc. For the approximate solutions of fractional fuzzy Volterra integro-differential equations, several researchers have developed some numerical approaches using different types of fractional derivatives[36–46].

Several authors have contributed significantly to recent developments in fuzzy fractional calculus. For example,[47] investigated a novel method for solving fractional Abel k-integral equations and linear FDEs in a fuzzy environment. This study shows the effectiveness of applying fractional calculus with fuzzy systems to improve solution accuracy. Similarly,[48] examined fuzzy Langevin fractional delay differential equations using the granular derivative, demonstrating the possibility for combining both mathematical frameworks. Furthermore,[49] investigated fuzzy fractional delay integro-differential equations using the extended Atangana-Baleanu fractional derivative, giving a unique approach to solving these complicated equations. Additionally,[50] investigated the existence and stability of solutions for fuzzy neutral fractional integro-differential equations involving the Caputo fractional generalized Hukuhara derivative.

The existence of the FFVIDEs has been further investigated using Schauder's fixed point theorem and the uniqueness has been studied using the Banach contraction principle in[51]. In addition, recently the Adomian decomposition method (ADM) with initial condition[52] and ADM with boundary conditions[38,39], Shehu Adomian decomposition method[53], successive approximation method[54], monotone iterative technique[55] two-dimensional legendre wavelet method[56] have all been used to examine the solutions of FFVIDEs.

| Reference | Method Used | Type of Problem Studied | Remarks / Contributions |
|---|---|---|---|
| 1. Savla, S. et al. (2024) | Shehu Adomian decomposition method | Solving linear and nonlinear fuzzy fractional Volterra -Fredholm integro-differential equations (IVP) | Provided approximate analytical solutions; Via Adomian Polynimoals. |
| 2. Vu, H. et al. (2019) | Successive approximation method | Stability for initial value problems of fuzzy Volterra integro-differential equation with fractional order derivative | Discussed existence and uniqueness; limited to simple approximation method. |
| 3. Shabestari, M.R.M. et al. (2018) | Two-dimensional Legendre wavelet method | Numerical solution of fuzzy fractional integro-differential equation via (IVP) | Focused on deterministic systems and investigated existence and uniqueness under Hukuhara differentiability |
| **Proposed Work** | Adomian Decomposition Method | Fuzzy fractional Volterra integro -differential equations with boundary conditions (BVP) | ADM improves convergence and handles uncertainty, efficiently, and requires no discretization or linearization. |

In this work, we try to solve FFVIDEs under Caputo derivative using the ADM. In order to solve FFVIDEs with our proposed approach. To the best of our knowledge, this is the first time in the literature that numerical solution of the FFVIDEs are examined under Caputo derivative employing ADM with the fractional order $0 < \eta \leq 1$ and fuzzy initial conditions are investigated. The goal of this work is to look into it.

### Motivation

Numerous authors have focused on exploring numerical solutions for FFVIDEs, as many of these equations are difficult to solve analytically. Therefore, researchers are starting to use several numerical approaches. Some examples of these numerical methods include the Laplace transform method, variational iteration method, semi-analytical method, and Adomian decomposition method (ADM) to obtain better approximation solutions for FFIDEs. All of these studies encouraged us to develop a numerical approach to solving FFVIDE. Additionally, our analysis of the literature revealed that numerous investigations of numerical solutions of FFIDEs were solved only with initial conditions. Hamoud[57] proposed the ADM to investigate the fractional Volterra integro-differential equation's solution without a fuzzy concept. Also in[20] N. Ahmad et al. only addressed the FFVIDEs' initial value problem. As a result, this study inspires us to adapt research for FFVIDE with boundary conditions. Therefore, FFVIDEs must be developed with boundary conditions.

### Novelty

The following is a summary of the work's primary contribution:

1. The authors provide a new framework for the Adomian decomposition approach to solve FFVIDEs with generalized boundary conditions.
2. Comparing to the existing studies in (see[20,57]) we present boundary conditions for the FFVIDEs investigated.
3. To demonstrate the existence of the unique solution using the Banach fixed point theory to the considered equation. To provide lower/uppercut solutions for FFVIDEs with the help of the Adomian decomposition technique.
4. To establish the effectiveness and reliability of the ADM, we provided a sufficient number of examples to verify the theory and proposed an algorithm for solving these types of equations.

### Overview of the paper's sections

The paper is organized as follows: The introduction is included in the first part. Additionally, the motivation, novelty, and structure of the article are described. Section 2 presents some basic ideas about fuzzy numbers and the basics of fractional calculus, whereas Section 3 is focused on establishing the existence and uniqueness of the FFBVPs under consideration. Section 4 presents the methodology of Adomian decomposition. In Section 5, we analyze the suggested method's accuracy using a numerical example and Section 6 concludes with a brief overview of this investigation.

## Preliminaries

The definitions of fuzzy calculus and its key notions are provided in this section.

**Definition 1** [20] A fuzzy set $A$ is said to be a fuzzy number, if the following conditions are satisfied.

1. $A$ is normal; there exist, $y_0 \in \mathcal{R}$ such that $A(y_0) = 1$,
2. $A(\lambda y_1 + (1-\lambda)y_2) \geq \min(A(y_1), A(y_2))$ holds, for all, $y_1, y_2 \in \mathcal{R}$ and $0 \leq \lambda \leq 1$ i.e, $A$ is convex,
3. $A$ is upper semi-continuous,
4. $A$ is compactly supported that is $cl\{y \in \mathcal{R} \mid A(y) > 0\} \subseteq \mathcal{R}$ is compact.

**Definition 2** [57] A fuzzy number $A$ is $\xi$-level set characterised as

$$[A]_\xi = \{y \in R : A(y) \geq \xi\},$$

where $\xi \in (0,1]$ and $y \in R$ For a fuzzy number $A$, its $\xi$-cuts are closed intervals in $R$ and we denoted by $[A]_\xi = [\underline{\xi}, \overline{\xi}]$.

**Definition 3** [29] The RL fractional integral of order $\eta$ of fuzzy function $\tilde{f}(y,\xi)$, based on its $\xi$-level illustrations:

$\left[I^\eta \tilde{f}(y,\xi)\right] = \left[I^\eta \underline{f}(y,\xi), I^\eta \overline{f}(y,\xi)\right]$ where,

$$I^\eta \underline{f}(y,\xi) = \frac{1}{\Gamma(\eta)} \int_0^y (y-t)^{\eta-1} \underline{f}(t,\xi)dt,$$
$$I^\eta \overline{f}(y,\xi) = \frac{1}{\Gamma(\eta)} \int_0^y (y-t)^{\eta-1} \overline{f}(t,\xi)dt.$$

**Note:** In this paper, we consider ($i$) differentiable type to find the solution for FFVIDEs (from the theorem 2.1 in[38]). However, a type ($ii$) differentiable solution is the same as type ($i$) differentiable.

**Definition 4** [38] The Caputo fractional derivative of order $\eta$ of fuzzy function $\tilde{f}(y,\xi)$, it can be stated as follows, based on its $\xi$-level illustrations:

$\left[D^{\eta}\tilde{f}(y,\xi)\right] = \left[D^{\eta}\underline{f}(y,\xi), D^{\eta}\overline{f}(y,\xi)\right]$ where,

$$D^{\eta}[\underline{f}(y,\xi)] = \begin{cases} \frac{1}{\Gamma(r-\eta)}\int\limits_0^y (y-t)^{r-\eta-1}\underline{f}^r(y,\xi)dt, & r-1<\eta<r \\ \frac{d^r}{dt^r}\underline{f}(y,\xi), \quad \eta = r, r\in N, \end{cases}$$

$$D^{\eta}[\overline{f}(y,\xi)] = \begin{cases} \frac{1}{\Gamma(r-\eta)}\int\limits_0^y (y-t)^{r-\eta-1}\overline{f}^r(y,\xi)dt, & r-1<\eta<r \\ \frac{d^r}{dt^r}\overline{f}(y,\xi), \quad \eta = r, r\in N. \end{cases}$$

## Problem formulation

In this study, we use the fuzzy idea to extend the concept to FVIDEs across the Caputo-type of order $\eta \in (0,1]$.

$$D^{\eta}[\tilde{\vartheta}(y,\xi)] = \left(\tilde{f}(y,\xi) + \int_0^y \Upsilon_1(y,\varrho)\Lambda_1(\tilde{\vartheta}(\varrho,\xi))d\varrho + \int_0^T \Upsilon_2(y,\varrho)\Lambda_2(\tilde{\vartheta}(\varrho,\xi))d\varrho\right) \tag{1}$$

with the boundary condition

$$p_1\tilde{\vartheta}(0,\xi) + p_2\tilde{\vartheta}(T,\xi) = \tilde{p_3}, \quad p_1,p_2\in\mathcal{R},\ p_1+p_2\neq 0,\ \tilde{p_3}\in\mathbb{R}_f. \tag{2}$$

Where $\tilde{\vartheta}(y,\xi)$ is a fuzzy function, $D^{\eta}(.), \eta\in(0,1]$, is the Caputo fractional derivative, $\tilde{f}: X\to\mathcal{R}_f, \Upsilon_i : X\times X\to\mathcal{R}$ for $i=1,2$ are continuous functions, $\Lambda_i:\mathcal{R}\to\mathcal{R}$ are Lipschitz continuous function, where $X=[0,1]$.

## Main result

The focus of this section will be on the discussion of the existence and uniqueness of the solution to Eq. (1) with boundary condition (2), based on the hypothesis.

$\mathbb{H}1$ : For any $\tilde{\vartheta}_1,\tilde{\vartheta}_2\in C(X,\mathcal{R}_f)$, there exists two constants $c_1,c_2>0$ such that,

$$|\Lambda_1(\tilde{\vartheta}_1(y,\xi)) - \Lambda_1(\tilde{\vartheta}_2(y,\xi))| \le c_1|\tilde{\vartheta}_1-\tilde{\vartheta}_2|,$$
$$|\Lambda_2(\tilde{\vartheta}_1(y,\xi)) - \Lambda_2(\tilde{\vartheta}_2(y,\xi))| \le c_2|\tilde{\vartheta}_1-\tilde{\vartheta}_2|.$$

$\mathbb{H}2$ : For the set of all positive continuous function on $J=\{(y,\varrho)\in\mathcal{R}\times\mathcal{R}: 0\le\varrho\le y\le 1\}$, there exists the functions ${\Upsilon_1}^*, {\Upsilon_2}^*$ such that

$${\Upsilon_1}^* = \sup_{y\in[0,1]}\int_0^y |\Upsilon_1(y,\varrho)|d\varrho < \infty$$

and

$${\Upsilon_2}^* = \sup_{y\in[0,1]}\int_0^1 |\Upsilon_2(y,\varrho)|d\varrho < \infty.$$

$\mathbb{H}3$ : The function $\tilde{f}:[0,1]\to\mathcal{R}_f$ is continuous, there exists a constant $L>0$, such that $L=\sup\{|\tilde{f}(y,\xi)| : 0\le y\le T\}$.

## Theorems and Lemma

*Lemma 1 If a function $\tilde{\vartheta}(y,\xi)\in C[0,T]$ satisfies $(\mathbb{H}1)-(\mathbb{H}3)$ in [0, T], then the problems (1)-(2) are similar to obtaining a continuous solution of the FFVIDEs.*

$$\begin{aligned}\tilde{\vartheta}(y,\xi) = &\int_0^y \frac{(y-t)^{\eta-1}}{\Gamma(\eta)}\Big(\tilde{f}(t,\xi) + \int_0^t \Upsilon_1(t,\varrho)\Lambda_1(\tilde{\vartheta}(\varrho,\xi))d\varrho \\ &+ \int_0^T \Upsilon_2(t,\varrho)\Lambda_2(\tilde{\vartheta}(\varrho,\xi))d\varrho\Big)dt \\ &- \frac{1}{p_1+p_2}\Big[\int_0^T \frac{p_2(T-t)^{\eta-1}}{\Gamma(\eta)}\Big(\tilde{f}(t,\xi) + \int_0^t \Upsilon_1(t,\varrho)\Lambda_1(\tilde{\vartheta}(\varrho,\xi))d\varrho \\ &+ \int_0^T \Upsilon_2(t,\varrho)\Lambda_2(\tilde{\vartheta}(\varrho,\xi))d\varrho\Big)dt - \tilde{p_3}\Big].\end{aligned} \tag{3}$$

*The above equation is equivalent to*

$$\begin{aligned}\tilde{\vartheta}(y,\xi) =&\tilde{h}(y,\xi) - \frac{1}{p_1+p_2}\int_0^T \frac{p_2(T-t)^{\eta-1}}{\Gamma(\eta)}\Big(\int_0^t \Upsilon_1(t,\varrho)\Lambda_1(\tilde{\vartheta}(\varrho,\xi))d\varrho \\ &+\int_0^T \Upsilon_2(t,\varrho)\Lambda_2(\tilde{\vartheta}(\varrho,\xi))d\varrho\Big)dt \\ &+\int_0^y \frac{(y-t)^{\eta-1}}{\Gamma(\eta)}\left(\int_0^t \Upsilon_1(t,\varrho)\Lambda_1(\tilde{\vartheta}(\varrho,\xi))d\varrho + \int_0^T \Upsilon_2(t,\varrho)\Lambda_2(\tilde{\vartheta}(\varrho,\xi))d\varrho\right)dt\end{aligned} \tag{4}$$

*where,*

$$\tilde{h}(y,\xi) =\frac{\tilde{p_3}}{p_1+p_2} - \frac{1}{p_1+p_2}\int_0^T \frac{p_2(T-t)^{\eta-1}}{\Gamma(\eta)}(\tilde{f}(t,\xi))dt + \int_0^y \frac{(y-t)^{\eta-1}}{\Gamma(\eta)}\big(\tilde{f}(t,\xi)\big)\,dt. \tag{5}$$

**Theorem 1** (Existence theorem) *Let the hypotheses* ($\mathbb{H}1$) *and* ($\mathbb{H}2$) *be satisfied and* $\Lambda_1(\tilde{\vartheta}(y,\xi) \le \mu_1, \Lambda_2(\tilde{\vartheta}(y,\xi) \le \mu_2$ *for all* $y \in [0,T]$ *and for all* $\tilde{\vartheta}(y,\xi) \in \mathcal{R}_f$*, Under this hypothesis our considered equation has at least one solution in [0, T].*

*Proof* Let the operator $\Upsilon : C(X,\mathcal{R}_f) \to C(X,\mathcal{R}_f)$ characterized by

$$\begin{aligned}(\Upsilon\tilde{\vartheta})(y,\xi) =&\int_0^y \frac{(y-t)^{\eta-1}}{\Gamma(\eta)}\Big(\tilde{f}(t,\xi) + \int_0^t \Upsilon_1(t,\varrho)\Lambda_1(\tilde{\vartheta}(\varrho,\xi))d\varrho \\ &+\int_0^T \Upsilon_2(t,\varrho)\Lambda_2(\tilde{\vartheta}(\varrho,\xi))d\varrho\Big)dt \\ &-\frac{1}{p_1+p_2}\Big[\int_0^T \frac{p_2(T-t)^{\eta-1}}{\Gamma(\eta)}\Big(\tilde{f}(t,\xi) + \int_0^t \Upsilon_1(t,\varrho)\Lambda_1(\tilde{\vartheta}(\varrho,\xi))d\varrho \\ &+\int_0^T \Upsilon_2(t,\varrho)\Lambda_2(\tilde{\vartheta}(\varrho,\xi))d\varrho\Big)dt - \tilde{p_3}\Big],\ \ for\ y \in [0,1].\end{aligned} \tag{6}$$

We begin by showing that the operator $\Upsilon$ is completely continuous.

**(1) To prove $\Upsilon$ is continuous:** Let $\tilde{\vartheta}_n$ be a sequence converges to $\tilde{\vartheta}$ in $C(X,\mathcal{R}_f)$, for each $\tilde{\vartheta}_n, \tilde{\vartheta} \in C(X,\mathcal{R}_f)$ and for any $y \in X$, we have

$$\begin{aligned}|\Upsilon\tilde{\vartheta_n}(y,\xi) - \Upsilon\tilde{\vartheta}(y,\xi)| \le& \frac{1}{\Gamma(\eta)}\int_0^y (y-t)^{\eta-1}\Big(\int_0^t |\Upsilon_1(t,\varrho)||\Lambda_1(\tilde{\vartheta_n}(\varrho,\xi)) - \Lambda_1(\tilde{\vartheta}(\varrho,\xi))|d\varrho \\ &+\int_0^T |\Upsilon_2(t,\varrho)||\Lambda_2(\tilde{\vartheta_n}(\varrho,\xi)) - \Lambda_2(\tilde{\vartheta}(\varrho,\xi))|d\varrho\Big)dt \\ &+\frac{|p_2|}{|p_1+p_2|\Gamma(\eta)}\int_0^T (T-t)^{\eta-1}\Big(\int_0^t |\Upsilon_1(t,\varrho)||\Lambda_1(\tilde{\vartheta_n}(\varrho,\xi)) \\ &-\Lambda_1(\tilde{\vartheta}(\varrho,\xi))|d\varrho + \int_0^T |\Upsilon_2(t,\varrho)||\Lambda_2(\tilde{\vartheta_n}(\varrho,\xi)) - \Lambda_2(\tilde{\vartheta}(\varrho,\xi))|d\varrho\Big)dt.\end{aligned}$$

Taking supremum on both sides,

$$\begin{aligned}\|\Upsilon\tilde{\vartheta_n}(y,\xi) - \Upsilon\tilde{\vartheta}(y,\xi)\| \le& \frac{1}{\Gamma(\eta)}\Big({\Upsilon_1}^*\|\Lambda_1(\tilde{\vartheta_n}(\varrho,\xi)) - \Lambda_1(\tilde{\vartheta}(\varrho,\xi))\| \\ &+{\Upsilon_2}^*\|\Lambda_2(\tilde{\vartheta_n}(\varrho,\xi)) - \Lambda_2(\tilde{\vartheta}(\varrho,\xi))\|\Big)\int_0^y (y-t)^{\eta-1}dt \\ &+\frac{|p_2|}{|p_1+p_2|\Gamma(\eta)}\Big({\Upsilon_1}^*\|\Lambda_1(\tilde{\vartheta_n}(\varrho,\xi)) - \Lambda_1(\tilde{\vartheta}(\varrho,\xi))\| \\ &+{\Upsilon_2}^*\|\Lambda_2(\tilde{\vartheta_n}(\varrho,\xi)) - \Lambda_2(\tilde{\vartheta}(\varrho,\xi))\|\Big)\int_0^T (T-t)^{\eta-1}dt\end{aligned}$$

Since, $\Lambda_1$ and $\Lambda_2$ are continuous, $\int_0^y (y-t)^{\eta-1}dt$ is bounded. As a result, we can deduce that $\|\Upsilon\tilde{\vartheta}_n(y,\xi) - \Upsilon\tilde{\vartheta}(y,\xi)\|_\infty \to 0$ as $n \to \infty$. Thus $\Upsilon$ is continuous.

**(2) To prove $\Upsilon$ is bounded:** $\Upsilon$ maps bounded sets into bounded sets in $C\left(X,\mathcal{R}_f\right)$ . We demonstrate that a positive constant $\lambda$ exists, for every $\rho > 0$ such that for any $\tilde{\vartheta} \in \mathcal{B}_\rho = \{\tilde{\vartheta} \in C(X,\mathcal{R}_f) : \|\tilde{\vartheta}\|_\infty \leqslant \eta\}$. One has $\|\tilde{\vartheta}\|_\infty \leqslant \lambda$. Let $\mu_1 = \sup_{\tilde{\vartheta}\in X\times[0,\eta]} \Lambda_1(\tilde{\vartheta}(y,\xi)) + 1$ and $\mu_2 = \sup_{\tilde{\vartheta}\in X\times[0,\eta]} \Lambda_2(\tilde{\vartheta}(y,\xi)) + 1$. Also for each $y \in X$ and for any $\tilde{\vartheta} \in \mathcal{B}_{r'}$, we have

$$
\begin{aligned}
0 \leq |\Upsilon\tilde{\vartheta}(y,\xi)| =& \Big| \int_0^y \frac{(y-t)^{\eta-1}}{\Gamma(\eta)} \Big( \tilde{f}(t,\xi) + \int_0^t \Upsilon_1(t,\varrho)\Lambda_1(\tilde{\vartheta}(\varrho,\xi))d\varrho \\
&+ \int_0^T \Upsilon_2(t,\varrho)\Lambda_2(\tilde{\vartheta}(\varrho,\xi))d\varrho \Big) dt \\
&- \frac{1}{p_1+p_2} \Big[ \int_0^T \frac{p_2(T-t)^{\eta-1}}{\Gamma(\eta)} \Big( \tilde{f}(t,\xi) + \int_0^t \Upsilon_1(t,\varrho)\Lambda_1(\tilde{\vartheta}(\varrho,\xi))d\varrho \\
&+ \int_0^T \Upsilon_2(t,\varrho)\Lambda_2(\tilde{\vartheta}(\varrho,\xi))d\varrho \Big) dt - \tilde{p_3} \Big] \Big|, \ for\ y \in [0,T] \\
\leq& \int_0^y \frac{(y-t)^{\eta-1}}{\Gamma(\eta)} \Big( |\tilde{f}(t,\xi)| + \int_0^t |\Upsilon_1(t,\varrho)||\Lambda_1(\tilde{\vartheta}(\varrho,\xi))|d\varrho \\
&+ \int_0^T |\Upsilon_2(t,\varrho)||\Lambda_2(\tilde{\vartheta}(\varrho,\xi))|d\varrho \Big) dt \\
&+ \frac{1}{|p_1+p_2|} \int_0^T \frac{|p_2|(T-t)^{\eta-1}}{\Gamma(\eta)} \Big( |\tilde{f}(t,\xi)| + \int_0^t |\Upsilon_1(t,\varrho)||\Lambda_1(\tilde{\vartheta}(\varrho,\xi)|)d\varrho \\
&+ \int_0^T |\Upsilon_2(t,\varrho)||\Lambda_2(\tilde{\vartheta}(\varrho,\xi)|)d\varrho \Big) dt + \frac{|\tilde{p_3}|}{|p_1+p_2|}.
\end{aligned}
$$

Taking the supremum on both sides, we obtain

$$
\begin{aligned}
\|\Upsilon\tilde{\vartheta}(y,\xi)\|_\infty \leq& \int_0^y \frac{(y-t)^{\eta-1}}{\Gamma(\eta)} \Big( L + [\Upsilon_1^*(\mu_1) + \Upsilon_2^*(\mu_2)] \Big) dt \\
&+ \frac{1}{|p_1+p_2|} \int_0^T \frac{|p_2|(T-t)^{\eta-1}}{\Gamma(\eta)} \Big( L + [\Upsilon_1^*(\mu_1) + \Upsilon_2^*(\mu_2)] \Big) dt + \frac{|\tilde{p_3}|}{|p_1+p_2|} \\
\leq& \frac{T^\eta \Big( L + [\Upsilon_1^*(\mu_1) + \Upsilon_2^*(\mu_2)] \Big)}{\Gamma(\eta+1)} \left( 1 + \frac{|p_2|}{|p_1+p_2|} \right) + \frac{|\tilde{p_3}|}{|p_1+p_2|} \\
\leq& \frac{\Big( L + [\Upsilon_1^*(\mu_1) + \Upsilon_2^*(\mu_2)] \Big)}{\Gamma(\eta+1)} \left( 1 + \frac{|p_2|}{|p_1+p_2|} \right) + \frac{|\tilde{p_3}|}{|p_1+p_2|} = \lambda.
\end{aligned}
$$

Thus, for every $\|\Upsilon\tilde{\vartheta}\| \leq \lambda$ for every $\tilde{\vartheta} \subset \mathcal{B}_{r'}$, which that $\Upsilon\mathcal{B}_{r'} \subset \mathcal{B}_\lambda$.

**(3) To prove $\Upsilon$ is equicontinuous:** $\Upsilon$ maps bounded sets into equicontinuous sets of $C(X, \mathcal{R}_f)$. Let $y_1, y_2 \in [0,1]$ with $y_1 < y_2$. For $\tilde{\vartheta} \in \mathcal{B}_\rho$, (defined in last step $\mathcal{B}_\rho$) we have

$$
\begin{aligned}
|(\Upsilon\tilde{\vartheta})(y_2,\xi) - (\Upsilon\tilde{\vartheta})(y_1,\xi)| \leq& \Big| \int_0^{y_2} \frac{(y_2-t)^{\eta-1}}{\Gamma(\eta)} \Big( |\tilde{f}(t,\xi)| + \int_0^t |\Upsilon_1(t,\varrho)||\Lambda_1(\tilde{\vartheta}(\varrho,\xi))|d\varrho \\
&+ \int_0^T |\Upsilon_2(t,\varrho)||\Lambda_2(\tilde{\vartheta}(\varrho,\xi))|d\varrho \Big) dt - \int_0^{y_1} \frac{(y_1-t)^{\eta-1}}{\Gamma(\eta)} \Big( |\tilde{f}(t,\xi)| \\
&+ \int_0^t |\Upsilon_1(t,\varrho)||\Lambda_1(\tilde{\vartheta}(\varrho,\xi))|d\varrho + \int_0^T |\Upsilon_2(t,\varrho)||\Lambda_2(\tilde{\vartheta}(\varrho,\xi))|d\varrho \Big) dt \Big| \\
\leq& \Big| \int_0^{y_1} \frac{(y_2-t)^{\eta-1} - (y_1-t)^{\eta-1}}{\Gamma(\eta)} \Big( \tilde{f}(t,\xi) + \int_0^t \Upsilon_1(t,\varrho)\Lambda_1(\tilde{\vartheta}(\varrho,\xi))d\varrho \\
&+ \int_0^T \Upsilon_2(t,\varrho)\Lambda_2(\tilde{\vartheta}(\varrho,\xi))d\varrho \Big) dt \Big| + \Big| \int_{y_1}^{y_2} \frac{(y_2-t)^{\eta-1}}{\Gamma(\eta)} \Big( \tilde{f}(t,\xi) \\
&+ \int_0^t \Upsilon_1(t,\varrho)\Lambda_1(\tilde{\vartheta}(\varrho,\xi))d\varrho + \int_0^T \Upsilon_2(t,\varrho)\Lambda_2(\tilde{\vartheta}(\varrho,\xi))d\varrho \Big) dt \Big| \\
\leq& \frac{1}{\Gamma(\eta)} \int_0^{y_1} [(y_1-t)^{\eta-1} - (y_2-t)^{\eta-1}] \Big( |\tilde{f}(t,\xi)| \\
&+ \int_0^t |\Upsilon_1(t,\varrho)||\Lambda_1(\tilde{\vartheta}(\varrho,\xi))|d\varrho + \int_0^T |\Upsilon_2(t,\varrho)||\Lambda_2(\tilde{\vartheta}(\varrho,\xi))|d\varrho \Big) dt \\
&+ \frac{1}{\Gamma(\eta)} \int_{y_1}^{y_2} (y_2-t)^{\eta-1} \Big( |\tilde{f}(t,\xi)| + \int_0^t |\Upsilon_1(t,\varrho)||\Lambda_1(\tilde{\vartheta}(\varrho,\xi))|d\varrho \\
&+ \int_0^T |\Upsilon_2(t,\varrho)||\Lambda_2(\tilde{\vartheta}(\varrho,\xi))|d\varrho \Big) dt.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|\Upsilon\tilde{\vartheta}(y_2,\xi)-\Upsilon\tilde{\vartheta}(y_1,\xi)| &\leq \frac{1}{\Gamma(\eta)}\int_0^{y_1}\Big[(y_1-t)^{\eta-1}-(y_2-t)^{\eta-1}\Big]\Big[L+\Upsilon_1(\mu_1)+\Upsilon_2(\mu_2)\Big]dt\\
&\leq \frac{1}{\Gamma(\eta)}\int_{y_1}^{y_2}\Big[(y_2-t)^{\eta-1}\Big]\Big[L+\Upsilon_1(\mu_1)+\Upsilon_2(\mu_2)\Big]dt\\
&\leq \frac{[L+\Upsilon_1(\mu_1)+\Upsilon_2(\mu_2)]}{[\eta+1]}\Big[{y_1}^{\eta}-{y_2}^{\eta}+2(y_2-y_1)^{\eta}\Big]\\
|\Upsilon\tilde{\vartheta}(y_2,\xi)-\Upsilon\tilde{\vartheta}(y_1,\xi)| &\leq \frac{2[L+\Upsilon_1(\mu_1)+\Upsilon_2(\mu_2)]}{[\eta+1]}|y_2-y_1|^{\eta}.
\end{aligned}
$$

$\Upsilon\tilde{\vartheta}(y_2,\xi)-\Upsilon\tilde{\vartheta}(y_1,\xi)\to 0$ as $y_2-y_1\to 0$. This implies that,

$$
|(\Upsilon\tilde{\vartheta})(y_2,\xi)-(\Upsilon\tilde{\vartheta})(y_1,\xi)|\to 0.
$$

The set $\Upsilon\mathcal{B}_\rho$ is equicontinuous. Using the Arzelà–Ascoli theorem, we determine that $\Upsilon$ is completely continuous.

For the final step, let us consider the set defined by

$$
E=\{\tilde{\vartheta}\in C([0,T],\mathcal{R}_f):\tilde{\vartheta}=\delta\Upsilon(\tilde{\vartheta})\quad for\quad 0<\delta<1\}.
$$

Now, we show that the above set $E$ is bounded.

Let us take $\tilde{\vartheta}\in E$. For every $y\in[0,T]$, we have, from Eq. (6)

$$
\begin{aligned}
\tilde{\vartheta}(y,\xi)=&\delta\Bigg[\int_0^y\frac{(y-t)^{\eta-1}}{\Gamma(\eta)}\Big(\tilde{f}(t,\xi)+\int_0^t\Upsilon_1(t,\varrho)\Lambda_1(\tilde{\vartheta}(\varrho,\xi))d\varrho\\
&+\int_0^T\Upsilon_2(t,\varrho)\Lambda_2(\tilde{\vartheta}(\varrho,\xi))\Big)dt\\
&-\frac{1}{p_1+p_2}\Big[\int_0^T\frac{p_2(T-t)^{\eta-1}}{\Gamma(\eta)}\Big(\tilde{f}(t,\xi)+\int_0^t\Upsilon_1(t,\varrho)\Lambda_1(\tilde{\vartheta}(\varrho,\xi))d\varrho\\
&+\int_0^T\Upsilon_2(t,\varrho)\Lambda_2(\tilde{\vartheta}(\varrho,\xi))d\varrho\Big)dt-\tilde{p_3}\Big]\Bigg],\ for\ y\in[0,T]\\
|\tilde{\vartheta}(y,\xi)|=&\Bigg|\delta\Bigg[\int_0^y\frac{(y-t)^{\eta-1}}{\Gamma(\eta)}\Big(\tilde{f}(t,\xi)+\int_0^t\Upsilon_1(t,\varrho)\Lambda_1(\tilde{\vartheta}(\varrho,\xi))d\varrho\\
&+\int_0^T\Upsilon_2(t,\varrho)\Lambda_2(\tilde{\vartheta}(\varrho,\xi))s\Big)dt\\
&-\frac{1}{p_1+p_2}\Big[\int_0^T\frac{p_2(T-t)^{\eta-1}}{\Gamma(\eta)}\Big(\tilde{f}(t,\xi)+\int_0^t\Upsilon_1(t,\varrho)\Lambda_1(\tilde{\vartheta}(\varrho,\xi))d\varrho\\
&+\int_0^T\Upsilon_2(t,\varrho)\Lambda_2(\tilde{\vartheta}(\varrho,\xi))d\varrho\Big)dt-\tilde{p_3}\Big]\Bigg]\Bigg|,\ for\ y\in[0,T]\\
\leq&\Bigg|\delta\Bigg[\tilde{h}(y,\xi)+\int_0^y\frac{(y-t)^{\eta-1}}{\Gamma(\eta)}\Big[\int_0^t\Upsilon_1(t,\varrho)\Lambda_1(\tilde{\vartheta}(\varrho,\xi))d\varrho\\
&+\int_0^T\Upsilon_2(t,\varrho)\Lambda_2(\tilde{\vartheta}(\varrho,\xi))d\varrho\Big]dt\\
&-\frac{1}{p_1+p_2}\Big(\int_0^T\frac{p_2(T-t)^{\eta-1}}{\Gamma(\eta)}\Big[\int_0^t\Upsilon_1(t,\varrho)\Lambda_1(\tilde{\vartheta}(\varrho,\xi))d\varrho\\
&+\int_0^T\Upsilon_2(t,\varrho)\Lambda_2(\tilde{\vartheta}(\varrho,\xi))d\varrho\Big]dt\Big)\Bigg]\Bigg|
\end{aligned}
$$

where, $\tilde{h}(y,\xi)$ is defined in Lemma (1).

$$
\begin{aligned}
\tilde{h}(y,\xi)=&\frac{\tilde{p_3}}{p_1+p_2}-\frac{1}{p_1+p_2}\int_0^T\frac{p_2(T-t)^{\eta-1}}{\Gamma(\eta)}(\tilde{f}(t,\xi))dt+\int_0^y\frac{(y-t)^{\eta-1}}{\Gamma(\eta)}\big(\tilde{f}(t,\xi)\big)\,dt\\
|\tilde{h}(y,\xi)|\leq&A.
\end{aligned}
$$

We already obtain $|\Lambda_1(\tilde{\vartheta}(y,\xi)|\leq\mu_1, |\Lambda_2(\tilde{\vartheta}(y,\xi)|\leq\mu_2$, for every $y\in[0,T]$.

$$
\begin{aligned}
|\tilde{\vartheta}(y,\xi)| &= |\tilde{h}(y,\xi)| + \frac{1}{\Gamma(\eta)}\int_0^y (y-t)^{\eta-1}\Big[\int_0^t |\Upsilon_1(t,\varrho)||\Lambda_1(\tilde{\vartheta}(\varrho,\xi))|d\varrho \\
&\quad + \int_0^T |\Upsilon_2(t,\varrho)||\Lambda_2(\tilde{\vartheta}(\varrho,\xi))|d\varrho\Big]dt \\
&\quad - \frac{|p_2|}{|p_1+p_2|}\frac{1}{\Gamma(\eta)}\int_0^T (T-t)^{\eta-1}\Big[\int_0^t |\Upsilon_1(t,\varrho)||\Lambda_1(\tilde{\vartheta}(\varrho,\xi))|d\varrho \\
&\quad + \int_0^T |\Upsilon_2(t,\varrho)||\Lambda_2(\tilde{\vartheta}(\varrho,\xi))|d\varrho\Big]dt \\
&\le A + \left(\frac{[\Upsilon_1\mu_1+\Upsilon_2\mu_2]}{\Gamma(\eta)}\int_0^y (y-t)^{\eta-1}dt\right)\left(1+\frac{|p_2|}{|p_1+p_2|}\right) \\
&\le A + \left(\frac{[\Upsilon_1\mu_1+\Upsilon_2\mu_2]y^\eta}{\Gamma(\eta+1)}\right)\left(1+\frac{|p_2|}{|p_1+p_2|}\right) \\
&\le A + \left(\frac{[\Upsilon_1\mu_1+\Upsilon_2\mu_2]T^\eta}{\Gamma(\eta+1)}\right)\left(1+\frac{|p_2|}{|p_1+p_2|}\right) \\
&\le d.
\end{aligned}
$$

Define $A+\left(\frac{[\Upsilon_1\mu_1+\Upsilon_2\mu_2]T^\eta}{\Gamma(\eta+1)}\right)\left(1+\frac{|p_2|}{|p_1+p_2|}\right)=d$. Therefore $\|\tilde{\vartheta}\|\le d$. This shows that any $\tilde{\vartheta}\in E$ is bounded. Therefore, the set $E$ is bounded.

Furthermore, the operator $\Upsilon$ has a fixed point, according to Schaefer's fixed point theorem. This shows that at least one solution $\tilde{\vartheta}(y)$ for all $y\in[0,T]$ exists.

Now, we finish up the consequence of the hypothesis dependent on the contraction mapping rule. $\Upsilon$ has a unique fixed point. This shows that the problems (1)-(2) has unique solution. □

**Theorem 2** (Uniqueness theorem) *Assume that* $\mathbb{H}1$ *to* $\mathbb{H}3$ *holds. If*

$$
\frac{\left([\Upsilon_1^*(c_1)+\Upsilon_2^*(c_2)]\right)}{\Gamma(\eta+1)}\left(1+\frac{|p_2|}{|p_1+p_2|}\right)<1,
$$

*then there exists a unique solution of the Eq.(1).*

*Proof* $\tilde{\vartheta}(y,\xi)$ is the solution of the Eq. (1), with given boundary condition, if $\tilde{\vartheta}(y,\xi)$ satisfies

$$
\begin{aligned}
\tilde{\vartheta}(y,\xi) &= \int_0^y \frac{(y-t)^{\eta-1}}{\Gamma(\eta)}\Big(\tilde{f}(t,\xi)+\int_0^t \Upsilon_1(t,\varrho)\Lambda_1(\tilde{\vartheta}(\varrho,\xi))d\varrho \\
&\quad + \int_0^T \Upsilon_2(t,\varrho)\Lambda_2(\tilde{\vartheta}(\varrho,\xi))d\varrho\Big)dt \\
&\quad - \frac{1}{p_1+p_2}\Big[\int_0^T \frac{p_2(T-t)^{\eta-1}}{\Gamma(\eta)}\Big(\tilde{f}(t,\xi)+\int_0^t \Upsilon_1(t,\varrho)\Lambda_1(\tilde{\vartheta}(\varrho,\xi))d\varrho \\
&\quad + \int_0^T \Upsilon_2(t,\varrho)\Lambda_2(\tilde{\vartheta}(\varrho,\xi))d\varrho\Big)dt - \tilde{p_3}\Big],\ \ for\ y\in[0,1].
\end{aligned}
$$

Let's define the operator $\Upsilon$ according to the Theorem (1). If $\tilde{\vartheta}\in C(X,\mathcal{R}_f)$ is a fixed point of $\Upsilon$, then $\tilde{\vartheta}$ is the solution of the Eq. (1). Let $\tilde{\vartheta}_1,\tilde{\vartheta}_2\in C(X,\mathcal{R}_f)$, then

$$
\begin{aligned}
|\Upsilon\tilde{\vartheta}_1(y,\xi)-\Upsilon\tilde{\vartheta}_2(y,\xi)| &= \frac{1}{\Gamma(\eta)}\int_0^y (y-t)^{\eta-1}\Big(\int_0^t |\Upsilon_1(t,\varrho)||\Lambda_1(\tilde{\vartheta}_1(\varrho,\xi))-\Lambda_1(\tilde{\vartheta}_2(\varrho,\xi))|d\varrho \\
&\quad + \int_0^T |\Upsilon_2(t,\varrho)||\Lambda_2(\tilde{\vartheta}_1(\varrho,\xi))-\Lambda_2(\tilde{\vartheta}_2(\varrho,\xi))|d\varrho\Big)dt \\
&\quad + \frac{|p_2|}{|p_1+p_2|\Gamma(\eta)}\int_0^T (T-t)^{\eta-1}\Big(\int_0^t |\Upsilon_1(t,\varrho)||\Lambda_1(\tilde{\vartheta}_1(\varrho,\xi)) \\
&\quad - \Lambda_1(\tilde{\vartheta}_2(\varrho,\xi))|d\varrho + \int_0^T |\Upsilon_2(t,\varrho)||\Lambda_2(\tilde{\vartheta}_1(\varrho,\xi))-\Lambda_2(\tilde{\vartheta}_2(\varrho,\xi))|d\varrho\Big)dt
\end{aligned}
$$

Taking supremum on both sides, we can get,

$$\|\Upsilon\tilde{\vartheta}_1(y,\xi)-\Upsilon\tilde{\vartheta}_2(y,\xi)\|_\infty \le \frac{1}{\Gamma(\eta)}\Big(\Upsilon_1^*(c_1)+\Upsilon_2^*(c_2)\Big)\left\|\tilde{\vartheta}_1-\tilde{\vartheta}_2\right\|_\infty\int_0^y (y-t)^{\eta-1}dt$$
$$+\frac{|p_2|}{|p_1+p_2|\Gamma(\eta)}\Big(\Upsilon_1^*(c_1)+\Upsilon_2^*(c_2)\Big)\left\|\tilde{\vartheta}_1-\tilde{\vartheta}_2\right\|_\infty\int_0^T (T-t)^{\eta-1}dt$$
$$\|\Upsilon\tilde{\vartheta}_1(y,\xi)-\Upsilon\tilde{\vartheta}_2(y,\xi)\|_\infty \le \frac{\Big(\Upsilon_1^*c_1+\Upsilon_2^*c_2\Big)}{\Gamma(\eta+1)}\left(1+\frac{|p_2|}{|p_1+p_2|}\right)\left\|\tilde{\vartheta}_1-\tilde{\vartheta}_2\right\|_\infty<1$$
$$\|\Upsilon\tilde{\vartheta}_1(y,\xi)-\Upsilon\tilde{\vartheta}_2(y,\xi)\|_\infty < \left\|\tilde{\vartheta}_1-\tilde{\vartheta}_2\right\|_\infty.$$

Hence, $\Upsilon$ is a contraction. Thus, by the Banach contraction principle, using the theorem (2.12) in[20], $\Upsilon$ has a unique fixed point $\tilde{\vartheta}\in C([0,1],\mathcal{R}_f)$. □

## Description of ADM

The Adomian decomposition method, abbreviated ADM, is the subject of this essay. ADM is a kind of method that uses a decomposition approach to generate approximative solutions and even exact solutions for nonlinear systems when the proper initial data. Numerous benefits of this approach include. The diagrammatic representation of the proposed method is shown in Fig. 1.

### Advantages of the adomian decomposition method

- Numerous types of nonlinear systems, including algebraic equations, differential equations, integral equations, integro-differential equations, and so on, can be solved using this method, and it is relatively simple to use.
- It avoids the time-consuming Picard method integrations.
- It can solve certain nonlinear problems that conventional numerical approaches cannot address.
- The number of variables has no impact on applications since it need not directly affect the solutions series.

Before using ADM to construct approximations of solutions for nonlinear equations with boundary conditions.
Consider this equation: When the operator $I^\eta$ is applied to both sides of Eq. (1), we obtain

$$D^\eta\tilde{\vartheta}(y,\xi)=\tilde{h}(y,\xi)-\frac{1}{p_1+p_2}\int_0^T\frac{p_2(T-t)^{\eta-1}}{\Gamma(\eta)}\Big(\int_0^t\Upsilon_1(t,\varrho)\Lambda_1(\tilde{\vartheta}(\varrho,\xi))d\varrho$$
$$+\int_0^T\Upsilon_2(t,\varrho)\Lambda_2(\tilde{\vartheta}(\varrho,\xi))d\varrho\Big)dt$$
$$+\int_0^y\frac{(y-t)^{\eta-1}}{\Gamma(\eta)}\Big(\int_0^t\Upsilon_1(t,\varrho)\Lambda_1(\tilde{\vartheta}(\varrho,\xi))d\varrho$$
$$+\int_0^T\Upsilon_2(t,\varrho)\Lambda_2(\tilde{\vartheta}(\varrho,\xi))d\varrho\Big)dt.$$

In the form of a series, Adomian's method provides the solution $\tilde{\vartheta}(y,\xi)$,

$$\tilde{\vartheta}=\sum_{n=0}^{\infty}\tilde{\vartheta}_n$$

and the decomposition of the nonlinear $\Lambda_1$ and $\Lambda_2$ is

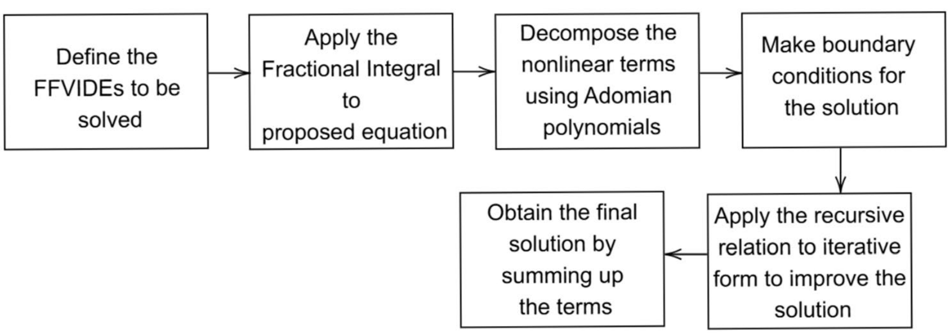



**Fig. 1**. Diagrammatic representation of proposed method.

$$\Lambda_1 = \sum_{n=0}^{\infty} M_n, \quad \Lambda_2 = \sum_{n=0}^{\infty} N_n.$$

Where $M_n$ and $N_n$ are Adomian polynomials and the expression for each of them is:

$$M_n = \frac{1}{n!}\frac{d^n}{dy^n}\left[\Lambda_1\left(\sum_{l=0}^{\infty} U^l \tilde{\vartheta}_l\right)\right]_{U=0}$$
$$N_n = \frac{1}{n!}\frac{d^n}{dy^n}\left[\Lambda_2\left(\sum_{l=0}^{\infty} U^l \tilde{\vartheta}_l\right)\right]_{U=0}.$$

Therefore,

$$\begin{aligned}
M_0 &= \Lambda_1\left(\tilde{\vartheta}_0\right),\\
M_1 &= \tilde{\vartheta}_1\Lambda_1'\left(\tilde{\vartheta}_0\right),\\
M_2 &= \tilde{\vartheta}_2\Lambda_1'\left(\tilde{\vartheta}_0\right) + \frac{1}{2}\tilde{\vartheta}_1^2\Lambda_1''\left(\tilde{\vartheta}_0\right),\\
M_3 &= \tilde{\vartheta}_3\Lambda_1'\left(\tilde{\vartheta}_0\right) + \tilde{\vartheta}_1\tilde{\vartheta}_2\Lambda_1''\left(\tilde{\vartheta}_0\right) + \frac{1}{3}\tilde{\vartheta}_1^3\Lambda_1'''\left(\tilde{\vartheta}_0\right),\\
&\vdots
\end{aligned}$$

and

$$\begin{aligned}
N_0 &= \Lambda_2\left(\tilde{\vartheta}_0\right),\\
N_1 &= \tilde{\vartheta}_1\Lambda_2'\left(\tilde{\vartheta}_0\right),\\
N_2 &= \tilde{\vartheta}_2\Lambda_2'\left(\tilde{\vartheta}_0\right) + \frac{1}{2}\tilde{\vartheta}_1^2\Lambda_2''\left(\tilde{\vartheta}_0\right),\\
N_3 &= \tilde{\vartheta}_3\Lambda_2'\left(\tilde{\vartheta}_0\right) + \tilde{\vartheta}_1\tilde{\vartheta}_2\Lambda_2''\left(\tilde{\vartheta}_0\right) + \frac{1}{3}\tilde{\vartheta}_1^3\Lambda_2'''\left(\tilde{\vartheta}_0\right),\\
&\vdots
\end{aligned}$$

The components $\tilde{\vartheta}_0, \tilde{\vartheta}_1, \tilde{\vartheta}_2, \ldots$ are determined in a recursive manner by

$$\begin{aligned}
\tilde{\vartheta}_0(y,\xi) =& \tilde{h}(y,\xi)\\
\tilde{\vartheta}_1(y,\xi) =& -\frac{1}{p_1+p_2}\int_0^T \frac{p_2(T-t)^{\eta-1}}{\Gamma(\eta)}\Big(\int_0^t \Upsilon_1(t,\varrho)M_0 d\varrho + \int_0^T \Upsilon_2(t,\varrho)N_0 d\varrho\Big)dt\\
&+\int_0^y \frac{(y-t)^{\eta-1}}{\Gamma(\eta)}\Big(\int_0^t \Upsilon_1(t,\varrho)M_0 d\varrho + \int_0^T \Upsilon_2(t,\varrho)N_0 d\varrho\Big)dt\\
\tilde{\vartheta}_{k+1}(y,\xi) =& -\frac{1}{p_1+p_2}\int_0^T \frac{p_2(T-t)^{\eta-1}}{\Gamma(\eta)}\Big(\int_0^t \Upsilon_1(t,\varrho)M_k d\varrho + \int_0^T \Upsilon_2(t,\varrho)N_k d\varrho\Big)dt\\
&+\int_0^y \frac{(y-t)^{\eta-1}}{\Gamma(\eta)}\Big(\int_0^t \Upsilon_1(t,\varrho)M_k d\varrho + \int_0^T \Upsilon_2(t,\varrho)N_k dr\varrho\Big)dt, \quad k \geq 1.
\end{aligned}$$

We approximate the BVP solution by solving the relation using boundary conditions. The recurrence relations are easily considered, in contrast to the boundary conditions for (3), which are important in the development of the solution.

*Remark 1* Note that the notation $\Lambda_1(\tilde{\vartheta}_0(y,\xi))$ always must be replaced by $\Lambda_1(\tilde{\vartheta}_0)$.

## Limitations of the adomian decomposition method

- The method's accuracy relies heavily on the choice of the initial fuzzy approximation, which can affect the convergence and stability of the solution.
- ADM does not provide a straightforward way to estimate approximation errors in fuzzy fractional contexts.
- For FFVIDEs with strong nonlinear kernels or long-memory fractional terms, ADM may require many iterations to achieve acceptable accuracy.

## Numerical experiments

The examples in this section demonstrate how precisely the methodology yields the FFVIDEs solution. The existence and uniqueness requirement in (1-2) is required for BVPs to solve the problem.

*Example 1* Consider the Caputo-type Volterra-Fredholm integro-differential equation with fuzzy boundary conditions.

$$\begin{cases} D^{\frac{1}{2}}\tilde{\vartheta}(y,\xi) = (2+\frac{y^2}{3})[\xi-1, 1-\xi] + \frac{1}{20}\int\limits_0^y (3y+4\varrho)\tilde{\vartheta}(\varrho,\xi)d\varrho + \frac{1}{8}\int\limits_0^1 (3y-1)\tilde{\vartheta}(\varrho,\xi)d\varrho \\ 2\tilde{\vartheta}(0,\xi) + \tilde{\vartheta}(1,\xi) = [\xi-1, 1-\xi], \quad y \in (0,1] \end{cases}$$

$$p_1, p_2 \in R,\ \tilde{p}_3 \in \mathcal{R}_f \quad p_1 = 2, p_2 = 1, \tilde{p}_3 = [\underline{p}_3, \overline{p}_3] = [\xi-1, 1-\xi].$$

The equivalent form of the above equation for type(i)-differentiability is shown below. ( using the definition 2.9 in[38])

$$\begin{cases} D^{\frac{1}{2}}\underline{\vartheta}(y,\xi) = (2+\frac{y^2}{3})(\xi-1) + \frac{1}{20}\int\limits_0^x (3y+4\varrho)\underline{\vartheta}(\varrho,\xi)d\varrho + \frac{1}{8}\int\limits_0^1 (3y-1)\underline{\vartheta}(\varrho,\xi)d\varrho, \\ D^{\frac{1}{2}}\overline{\vartheta}(y,\xi) = (2+\frac{y^2}{3})(1-\xi) + \frac{1}{20}\int\limits_0^x (3y+4\varrho)\overline{\vartheta}(\varrho,\xi)d\varrho + \frac{1}{8}\int\limits_0^1 (3y-1)\overline{\vartheta}(\varrho,\xi)d\varrho. \end{cases}$$

let's construct $\underline{\vartheta}(y,\xi)$ and apply the operator $I^{\frac{1}{2}}$ on both sides

$$\begin{cases} \underline{\vartheta}(y,\xi) = & \underline{h}(y,\xi) - \frac{p_2}{p_1+p_2}\frac{1}{\Gamma(1/2)}\int\limits_0^1 (1-t)^{-1/2}\left[\int\limits_0^y \frac{(3y+4\varrho)}{20}\underline{\vartheta}(\varrho,\xi)d\varrho + \int\limits_0^1 \frac{(3y-1)}{8}\underline{\vartheta}(\varrho,\xi)d\varrho\right]dt \\ & + \frac{1}{(1/2)}\int\limits_0^y (y-t)^{-1/2}\left[\int\limits_0^x \frac{(3y+4\varrho)}{20}\underline{\vartheta}(\varrho,\xi)d\varrho + \int\limits_0^1 \frac{(3y-1)}{8}\underline{\vartheta}(\varrho,\xi)d\varrho\right]dt, \\ \overline{\vartheta}(y,\xi) = & \overline{h}(y,\xi) - \frac{p_2}{p_1+p_2}\frac{1}{\Gamma(1/2)}\int\limits_0^1 (1-t)^{-1/2}\left[\int\limits_0^y \frac{(3y+4\varrho)}{20}\overline{\vartheta}(\varrho,\xi)d\varrho + \int\limits_0^1 \frac{(3y-1)}{8}\overline{\vartheta}(\varrho,\xi)d\varrho\right]dt \\ & + \frac{1}{(1/2)}\int\limits_0^y (y-t)^{-1/4}\left[\int\limits_0^y \frac{(3y+4\varrho)}{20}\overline{\vartheta}(\varrho,\xi)d\varrho + \int\limits_0^1 \frac{(3y-1)}{8}\overline{\vartheta}(\varrho,\xi)d\varrho\right]dt. \end{cases}$$

The next step is to determine the solution of $\underline{\vartheta}(y,\xi)$ by using the ADM.

$$\underline{\vartheta}_0(y,\xi) = \underline{h}(y,\xi)$$

$$\underline{\vartheta}_0(y,\xi) = \underline{h}(y,\xi) = \frac{\underline{p}_3}{p_1+p_2} - \frac{1}{p_1+p_2}\int_0^1 \frac{p_2(1-t)^{\eta-1}}{\Gamma(\eta)}[f(t,\xi)]dt + \int_0^y \frac{(y-t)^{\eta-1}}{\Gamma(\eta)}[f(t,\xi)]dt$$

$$\underline{\vartheta}_0(y,\xi) = \underline{h}(y,\xi) = \frac{(\xi-1)}{3} - \frac{(\xi-1)}{3\Gamma(1/2)}\int_0^1 (1-t)^{-1/2}[t^3+3t^2+1]dt + \frac{(\xi-1)}{\Gamma(1/2)}\int_0^y (y-t)^{-1/2}[t^3+3t^2+1]dt$$

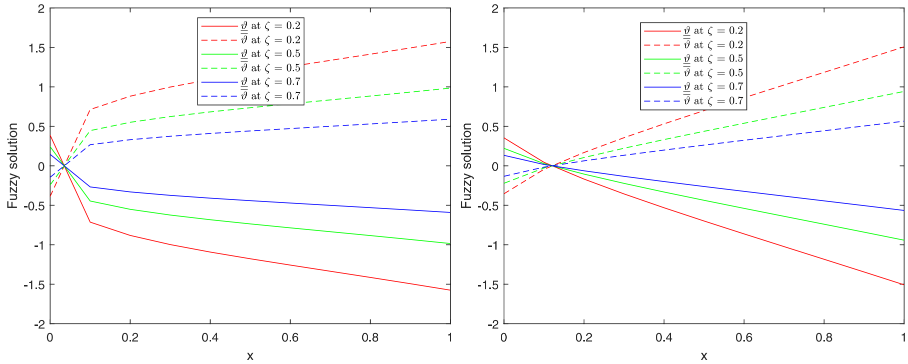
$$\underline{\vartheta}_1(y,\xi) = -\frac{p_2}{p_1+p_2}\frac{1}{\Gamma(1/2)}\int\limits_0^1 (1-t)^{-1/2}\left[\frac{1}{20}\int\limits_0^y (3y+4\varrho)M_0 d\varrho + \frac{1}{8}\int\limits_0^1 (3y-1)N_0 d\varrho\right]dt + \frac{1}{\Gamma(1/2)}\int\limits_0^y (y-t)^{-1/2}\left[\frac{1}{20}\int\limits_0^y (3y+4\varrho)\ M_0 d\varrho + \frac{1}{8}\int\limits_0^1 (3y-1)N_0 d\varrho\right]dt$$

In a similar manner, we can determine the following terms: $\underline{\vartheta}_2(y,\xi), \underline{\vartheta}_3(y,\xi), \ldots$ and obtain the result.

$$\underline{\vartheta}(y,\xi) = \sum_{n=0}^{\infty}\underline{\vartheta}_n(y,\xi) = \frac{(\xi-1)}{3} - \frac{2(\xi-1)}{3\Gamma(3/2)} - \frac{2(\xi-1)}{9\Gamma(7/2)} + \frac{2(\xi-1)y^{1/2}}{\Gamma(3/2)} + \frac{2(\xi-1)y^{5/2}}{3\Gamma(7/2)} + \ldots$$

In the same way, we can find type(ii) differentiable,

$$\overline{\vartheta}(y,\xi) = \sum_{n=0}^{\infty}\overline{\vartheta}_n(y,\xi) = \frac{(1-\xi)}{3} - \frac{2(1-\xi)}{3\Gamma(3/2)} - \frac{2(1-\xi)}{9\Gamma(7/2)} + \frac{2(1-\xi)y^{1/2}}{\Gamma(3/2)} + \frac{2(1-\xi)y^{5/2}}{3\Gamma(7/2)} + \ldots$$




(a) $\eta = 1/5$ (b) $\eta = 3/4$

**Fig. 2.** Visualizes the approximated fuzzy results of $\underline{\vartheta}(y,\xi), \overline{\vartheta}(y,\xi)$ at different values of uncertainty $\xi$= 0.2,0.5,0.7 and Caputo derivative $\eta = 1/5, 3/4$ for Example 1.

| $\eta = 1/5$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $\xi = 0.2$ | | $\xi = 0.4$ | | $\xi = 0.6$ | | $\xi = 0.8$ | |
| $y$ | $\underline{\vartheta}(y,\xi)$ | $\overline{\vartheta}(y,\xi)$ | $\underline{\vartheta}(y,\xi)$ | $\overline{\vartheta}(y,\xi)$ | $\underline{\vartheta}(y,\xi)$ | $\overline{\vartheta}(y,\xi)$ | $\underline{\vartheta}(y,\xi)$ | $\overline{\vartheta}(y,\xi)$ |
| 0.2 | -0.8818 | 0.8818 | -0.6614 | 0.6614 | -0.4409 | 0.4409 | -0.2205 | 0.2205 |
| 0.4 | -1.0926 | 1.0926 | -0.8194 | 0.8194 | -0.5463 | 0.5463 | -0.2731 | 0.2731 |
| 0.6 | -1.2573 | 1.2573 | -0.943 | 0.943 | -0.6287 | 0.6287 | -0.3143 | 0.3143 |
| 0.8 | -1.4137 | 1.4137 | -1.0603 | 1.0603 | -0.7068 | 0.7068 | -0.3534 | 0.3534 |

**Table 1.** Numerical results for Example 1, $\underline{\vartheta}(y,\xi)$ & $\overline{\vartheta}(y,\xi)$ at different values of uncertainty $\xi$ and fractional value $\eta = 1/5$..

Using MATLAB, we obtain 2D and 3D graphs of the simulation result of approximated fuzzy solutions displayed in the figure and table values for different levels of uncertainty $\xi = 0.2$(red), $\xi = 0.5$(green), $\xi = 0.7$(blue) shown in Fig. 2a, 2b and Table. 1, 2, 3 where us plots of $\xi$-cut representation of the approximated solution for Caputo derivative $\eta = 1/5$, $\eta = 3/4$ respectively. Different values of space variable $y = 0.2, 0.5, 0.7$ are shown in Fig. 3a,3b, and both uncertainty $\xi$ and space variable *y* with the fuzzy solutions are shown in Fig. 4a, 4b the surface plot of the fuzzy solutions are presented respectively (green and blue are the upper and lower solutions respectively). The fuzzy solution and the solution representation changed when we changed the $\eta$ order. While this modification is tiny, it may make a significant effect when applied to real-world issues.

**Note:** Here, The 1st non linear term $\underline{\vartheta}_0(y,\xi) = M_0$,
and 2nd non linear term $\underline{\vartheta}_0(y,\xi) = N_0$.

*Example 2* Consider the Caputo-type Volterra-Fredholm integro-differential equation with fuzzy boundary conditions.

$$\begin{cases} D^{\frac{3}{4}}\tilde{\vartheta}(y,\xi) = (y+1)[\xi-1, 1-\xi] + \frac{1}{10}\int\limits_0^y(\varrho^2 e^y)\tilde{\vartheta}(\varrho,\xi)d\varrho + \int\limits_0^1 y^2\tilde{\vartheta}(\varrho,\xi)d\varrho \\ 2\tilde{\vartheta}(0,\xi) + \tilde{\vartheta}(1,\xi) = [\xi-1, 1-\xi], \quad y \in (0,1] \end{cases}$$

$$p_1, p_2 \in R,\ \tilde{p}_3 \in \mathcal{R}_f \quad p_1 = 2, p_2 = 1, \tilde{p}_3 = [\underline{p}_3, \overline{p}_3] = [\xi-1, 1-\xi].$$

The equivalent form of the above equation for type(i)-differentiability is shown below.

$$\begin{cases} D^{\frac{3}{4}}\underline{\vartheta}(y,\xi) = (y+1)(\xi-1) + \frac{1}{10}\int\limits_0^x(\varrho^2 e^y)\underline{\vartheta}(\varrho,\xi)d\varrho + \int\limits_0^1 y^2\underline{\vartheta}(\varrho,\xi)d\varrho \\ D^{\frac{3}{4}}\overline{\vartheta}(y,\xi) = (y+1)(1-\xi) + \frac{1}{10}\int\limits_0^y(\varrho^2 e^y)\overline{\vartheta}(\varrho,\xi)d\varrho + \int\limits_0^1 y^2\overline{\vartheta}(\varrho,\xi)d\varrho \end{cases}$$

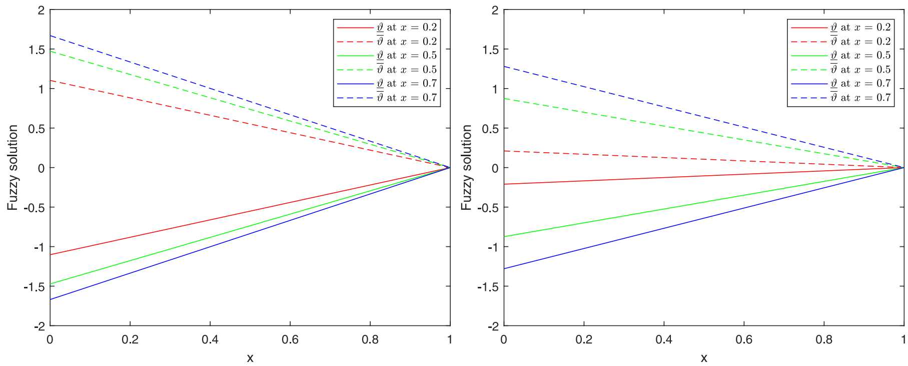
let's construct $\underline{\vartheta}(y,\xi)$ and apply the operator $I^{\frac{3}{4}}$ on both sides

| $\eta = 3/4$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $\xi = 0.2$ | | $\xi = 0.4$ | | $\xi = 0.6$ | | $\xi = 0.8$ | |
| $y$ | $\underline{\vartheta}(y,\xi)$ | $\overline{\vartheta}(y,\xi)$ | $\underline{\vartheta}(y,\xi)$ | $\overline{\vartheta}(y,\xi)$ | $\underline{\vartheta}(y,\xi)$ | $\overline{\vartheta}(y,\xi)$ | $\underline{\vartheta}(y,\xi)$ | $\overline{\vartheta}(y,\xi)$ |
| 0.2 | -0.1683 | 0.1683 | -0.1262 | 0.1262 | -0.0841 | 0.0841 | -0.0421 | 0.0421 |
| 0.4 | -0.5315 | 0.5315 | -0.3986 | 0.3986 | -0.2658 | 0.2658 | -0.1329 | 0.1329 |
| 0.6 | -0.8626 | 0.8626 | -0.6469 | 0.6469 | -0.4313 | 0.4313 | -0.2156 | 0.2156 |
| 0.8 | -1.1841 | 1.1841 | -0.8881 | 0.8881 | -0.592 | 0.592 | -0.296 | 0.296 |

**Table 2.** Numerical results for Example 1, $\underline{\vartheta}(y,\xi)$ & $\overline{\vartheta}(y,\xi)$ at different values of uncertainty $\xi$ and fractional value $\eta = 3/4$.

| | | $\eta = 1/5$ | | $\eta = 3/4$ | |
|---|---|---|---|---|---|
| $\xi$ | $\underline{p}_3$ | $\underline{\vartheta}_0(y,\xi)$ | $\underline{\vartheta}_1(y,\xi)$ | $\underline{\vartheta}_0(y,\xi)$ | $\underline{\vartheta}_1(y,\xi)$ |
| 0.2 | -0.8 | 0.387 | -1.575 | 0.353 | -1.507 |
| 0.5 | -0.5 | 0.242 | -0.985 | 0.221 | -0.942 |
| 0.7 | -0.3 | 0.145 | -0.590 | 0.132 | -0.565 |
| $\xi$ | $\overline{p}_3$ | $\overline{\vartheta}_0(y,\xi)$ | $\overline{\vartheta}_1(y,\xi)$ | $\overline{\vartheta}_0(y,\xi)$ | $\overline{\vartheta}_1(y,\xi)$ |
| 0.2 | 0.8 | -0.387 | 1.575 | -0.353 | 1.507 |
| 0.5 | 0.5 | -0.242 | 0.985 | -0.221 | 0.942 |
| 0.7 | 0.7 | -0.145 | 0.590 | -0.132 | 0.565 |

**Table 3.** Numerical results for Example 1, where $p_1 = 2, p_2 = 1$ and boundary points $\underline{\vartheta}_0(y,\xi), \underline{\vartheta}_1(y,\xi), \overline{\vartheta}_0(y,\xi), \overline{\vartheta}_1(y,\xi)$ at different values of uncertainty $\xi$ and fractional values $\eta = 1/5$, $\eta = 3/4$.

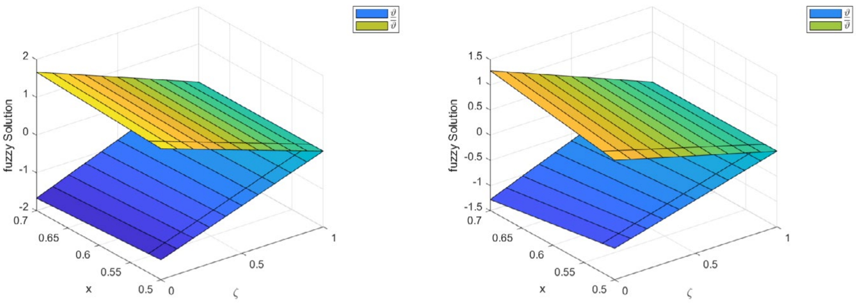



(a) $\eta = 1/5$ (b) $\eta = 3/4$

**Fig. 3.** Visualizes the fuzzy results of $\underline{\vartheta}(y,\xi), \overline{\vartheta}(y,\xi)$ at different values of $\eta = 1/5, 3/4$ and space variable $y = 0.2, 0.5, 0.7$ for Example 1.

$$\begin{cases} \underline{\vartheta}(y,\xi) = & \underline{h}(y,\xi) - \frac{p_2}{p_1+p_2}\frac{1}{\Gamma(\eta)}\int_0^1 (1-t)^{-1/4}\left[\frac{1}{10}\int_0^y (\varrho^2 e^y)\underline{\vartheta}(\varrho,\xi)d\varrho + \int_0^1 t^2\underline{\vartheta}(\varrho,\xi)d\varrho\right]dt \\ & +\frac{1}{(\eta)}\int_0^y (y-t)^{-1/4}\left[\frac{1}{10}\int_0^y (\varrho^2 e^y)\ \underline{\vartheta}(\varrho,\xi)d\varrho + \int_0^1 t^2\underline{\vartheta}(\varrho,\xi)d\varrho\right]dt \\ \overline{\vartheta}(y,\xi) = & \overline{h}(t,\xi) - \frac{p_2}{p_1+p_2}\frac{1}{\Gamma(\eta)}\int_0^1 (1-t)^{-1/4}\left[\frac{1}{10}\int_0^y (\varrho^2 e^y)\overline{\vartheta}(\varrho,\xi)d\varrho + \int_0^1 t^2\overline{\vartheta}(\varrho,\xi)d\varrho\right]dt \\ & +\frac{1}{(\eta)}\int_0^y (y-t)^{-1/4}\left[\frac{1}{10}\int_0^y (\varrho^2 e^y)\overline{\vartheta}(\varrho,\xi)d\varrho + \int_0^1 t^2\overline{\vartheta}(\varrho,\xi)d\varrho\right]dt. \end{cases}$$

The next step is to determine the solution of $\underline{\vartheta}(y,\xi)$ by using the ADM.




(a) $\eta = 1/5$ (b) $\eta = 3/4$

**Fig. 4**. Visualizes the fuzzy results of $\underline{\vartheta}(y,\xi), \overline{\vartheta}(y,\xi)$ at different values of fractional order $\eta$, uncertainty $\xi$ and space variable *y* for Example 1.

$$\begin{aligned}
\underline{\vartheta}_0(y,\xi) =& \underline{h}(y,\xi)\\
\underline{h}(y,\xi) =& \frac{\underline{p}_3}{p_1+p_2} - \frac{1}{p_1+p_2}\int_0^1 \frac{p_2(1-t)^{\eta-1}}{\Gamma(\eta)}[f(t,\xi)]dt + \int_0^y \frac{(y-t)^{\eta-1}}{\Gamma(\eta)}[f(t,\xi)]dt\\
\underline{\vartheta}_0(y,\xi) =& \underline{h}(y,\xi) = \frac{(\xi-1)}{2+1} - \frac{1}{2+1}\int_0^1 \frac{(1-t)^{-1/4}}{\Gamma(3/4)}[(t+1)(\xi-1)]dt\\
&+ \int_0^y \frac{(y-t)^{-1/4}}{\Gamma(3/4)}[(t+1)(\xi-1)]dt\\
\underline{\vartheta}_1(y,\xi) =& -\frac{p_2}{p_1+p_2}\frac{1}{\Gamma(\eta)}\int_0^1 (1-t)^{-1/4}\left[\frac{1}{10}\int_0^y (\varrho^2 e^y)M_0 d\varrho + \int_0^1 t^2 N_0 d\varrho\right] dt\\
&+ \frac{1}{\Gamma(\eta)}\int_0^y (y-t)^{-1/4}\left[\frac{1}{10}\int_0^y (\varrho^2 e^y)\ M_0 d\varrho + \int_0^1 t^2 N_0 d\varrho\right] dt
\end{aligned}$$

In a similar manner, we can determine the following terms: $\underline{\vartheta}_2(y,\xi), \underline{\vartheta}_3(y,\xi)$ and obtain the result.

$$\underline{\vartheta}(y,\xi) = \sum_{n=0}^{\infty} \underline{\vartheta}(y,\xi)_n = \frac{(\xi-1)}{3} - \frac{(\xi-1)}{3\Gamma(11/4)} - \frac{(\xi-1)}{3\Gamma(7/4)} + \frac{(\xi-1)y^{7/4}}{\Gamma(11/4)} + \frac{(\xi-1)y^{3/4}}{\Gamma(7/4)} + \ldots$$

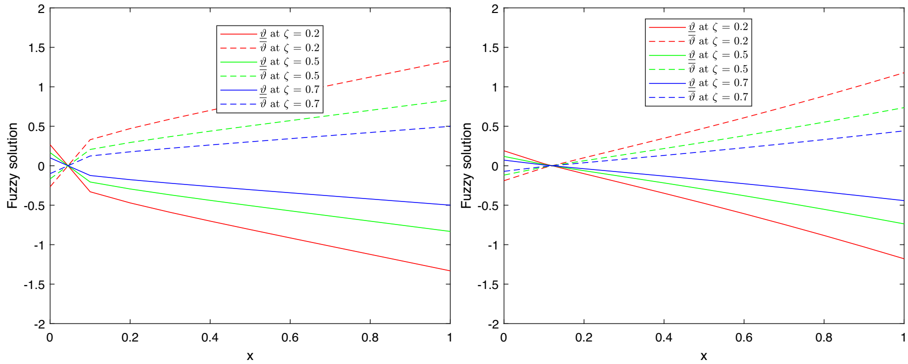
In the same way, we can find type(ii) differentiable,

$$\overline{\vartheta}(y,\xi) = \sum_{n=0}^{\infty} \overline{\vartheta}(y,\xi)_n = \frac{(1-\xi)}{3} - \frac{(1-\xi)}{3\Gamma(11/4)} - \frac{(1-\xi)}{3\Gamma(7/4)} + \frac{(1-\xi)y^{7/4}}{\Gamma(11/4)} + \frac{(1-\xi)y^{3/4}}{\Gamma(7/4)} + \ldots$$

Using MATLAB, we obtain 2D and 3D graphs of the simulation result of approximated fuzzy solutions displayed in the figure and table values for different levels of uncertainty $\xi = 0.2$(red), $\xi = 0.5$(green), $\xi = 0.7$(blue) shown in Fig. 5a, 5b and Table. 4, 5, 6 where us plots of $\xi$-cut representation of the approximated solution for Caputo derivative $\eta = 1/5, \eta = 3/4$ respectively. Different values of space variable $y = 0.2, 0.5, 0.7$ are shown in Fig. 6a, 6b, and both uncertainty $\xi$ and space variable *y* with the fuzzy solutions are shown in Fig. 7a, 7b the surface plot of the fuzzy solutions are presented respectively(green and blue are the upper and lower solutions respectively.)

## Conclusions

In this study, we examined a class of fuzzy fractional Volterra integro-differential equations (FFVIDEs) using both theoretical and numerical approaches based on the Adomian decomposition method. The existence and uniqueness of the solution were established through the contraction mapping principle under the given fuzzy boundary conditions. The proposed approach, derived from the decomposition rule, effectively generates approximate solutions whose behavior was analyzed both mathematically and graphically. The simulation results indicate that even a slight variation in the $\xi$-cut value can significantly affect the solution, confirming the




(a) $\eta = 1/5$ (b) $\eta = 3/4$

**Fig. 5**. Visualizes the fuzzy results of $\underline{\vartheta}(y,\xi), \overline{\vartheta}(y,\xi)$ at different values of uncertainty $\xi = 0.2, 0.5, 0.7$ and fractional order $\eta = 1/5, 3/4$ for Example 2.

| $\eta = 1/5$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $\xi = 0.2$ | | $\xi = 0.4$ | | $\xi = 0.6$ | | $\xi = 0.8$ | |
| $y$ | $\underline{\vartheta}(y,\xi)$ | $\overline{\vartheta}(y,\xi)$ | $\underline{\vartheta}(y,\xi)$ | $\overline{\vartheta}(y,\xi)$ | $\underline{\vartheta}(y,\xi)$ | $\overline{\vartheta}(y,\xi)$ | $\underline{\vartheta}(y,\xi)$ | $\overline{\vartheta}(y,\xi)$ |
| 0.2 | -0.471 | 0.471 | -0.3532 | 0.3532 | -0.2355 | 0.2355 | -0.1177 | 0.1177 |
| 0.4 | -0.7014 | 0.7014 | -0.5261 | 0.5261 | -0.3507 | 0.3507 | -0.1754 | 0.1754 |
| 0.6 | -0.9142 | 0.9142 | -0.6857 | 0.6857 | -0.4571 | 0.4571 | -0.2286 | 0.2286 |
| 0.8 | -1.123 | 1.123 | -0.8422 | 0.8422 | -0.5615 | 0.5615 | -0.2807 | 0.2807 |

**Table 4**. Numerical results for Example 2, $\underline{\vartheta}(y,\xi)$ & $\overline{\vartheta}(y,\xi)$ at different values of uncertainty $\xi$ and fractional value $\eta = 1/5$.

| $\eta = 3/4$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $\xi = 0.2$ | | $\xi = 0.4$ | | $\xi = 0.6$ | | $\xi = 0.8$ | |
| $y$ | $\underline{\vartheta}(y,\xi)$ | $\overline{\vartheta}(y,\xi)$ | $\underline{\vartheta}(y,\xi)$ | $\overline{\vartheta}(y,\xi)$ | $\underline{\vartheta}(y,\xi)$ | $\overline{\vartheta}(y,\xi)$ | $\underline{\vartheta}(y,\xi)$ | $\overline{\vartheta}(y,\xi)$ |
| 0.2 | -0.1008 | 0.1008 | -0.0756 | 0.0756 | -0.0504 | 0.0504 | -0.0252 | 0.0252 |
| 0.4 | -0.3486 | 0.3486 | -0.2615 | 0.2615 | -0.1743 | 0.1743 | -0.0872 | 0.0872 |
| 0.6 | -0.6076 | 0.6076 | -0.4557 | 0.4557 | -0.3038 | 0.3038 | -0.1519 | 0.1519 |
| 0.8 | -0.8836 | 0.8836 | -0.6627 | 0.6627 | -0.4418 | 0.4418 | -0.2209 | 0.2209 |

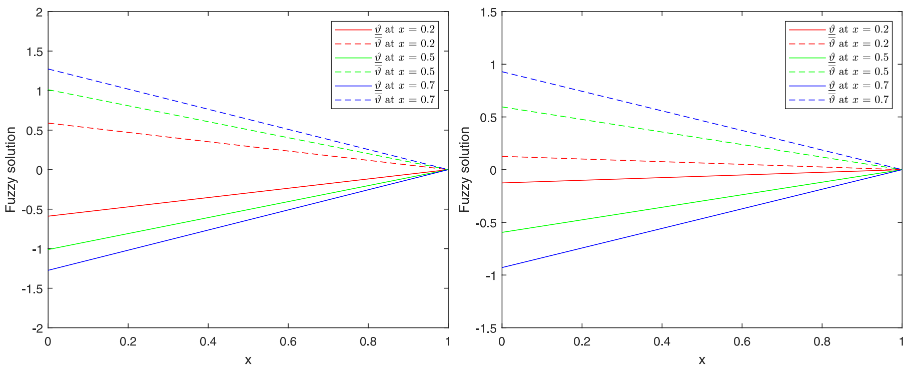
**Table 5**. Numerical results for Example 2, $\underline{\vartheta}(y,\xi)$ & $\overline{\vartheta}(y,\xi)$ at different values of uncertainty $\xi$ and fractional value $\eta = 3/4$.

sensitivity and accuracy of the numerical approach. Overall, the findings demonstrate that the proposed method provides reliable and precise fuzzy fractional solutions.

For future research, we intend to extend this work by incorporating the concept of time delay, as time-delay systems play a vital role in modeling real-world processes in engineering, biology, and control theory. The study will further explore variable-order fuzzy fractional Volterra integro-differential equations (VOFFVIDEs) using alternative and more advanced numerical techniques.

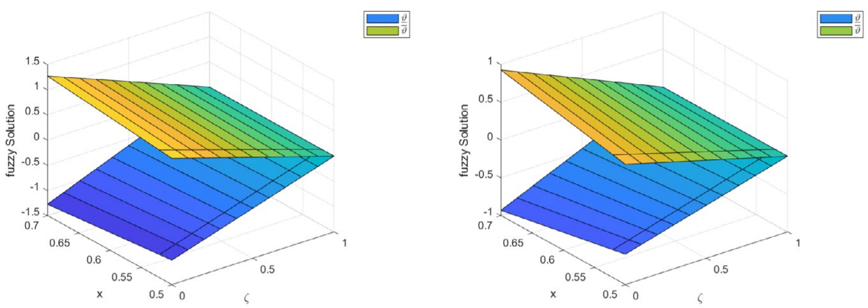
| | | $\eta = 1/5$ | | $\eta = 3/4$ | |
|---|---|---|---|---|---|
| $\xi$ | $\underline{p}_3$ | $\underline{\vartheta}_0(y,\xi)$ | $\underline{\vartheta}_1(y,\xi)$ | $\underline{\vartheta}_0(y,\xi)$ | $\underline{\vartheta}_1(y,\xi)$ |
| 0.2 | -0.8 | 0.265 | -1.331 | 0.189 | -1.178 |
| 0.5 | -0.5 | 0.166 | -0.832 | 0.118 | -0.736 |
| 0.7 | -0.3 | 0.099 | -0.499 | 0.070 | -0.441 |
| $\xi$ | $\overline{p}_3$ | $\overline{\vartheta}_0(y,\xi)$ | $\overline{\vartheta}_1(y,\xi)$ | $\overline{\vartheta}_0(y,\xi)$ | $\overline{\vartheta}_1(y,\xi)$ |
| 0.2 | 0.8 | -0.265 | 1.331 | -0.189 | 1.178 |
| 0.5 | 0.5 | -0.166 | 0.832 | -0.118 | 0.736 |
| 0.7 | 0.3 | -0.099 | 0.499 | -0.070 | 0.441 |

**Table 6**. Numerical results for Example 2, where $p_1 = 2, p_2 = 1$ and boundary points $\underline{\vartheta}_0(y,\xi), \underline{\vartheta}_1(y,\xi), \overline{\vartheta}_0(y,\xi), \overline{\vartheta}_1(y,\xi)$ at different values of uncertainty $\xi$ and fractional values $\eta = 1/5$, $\eta = 3/4$.




(a) $\eta = 1/5$ (b) $\eta = 3/4$

**Fig. 6**. Visualizes the fuzzy results of $\underline{\vartheta}(y,\xi), \overline{\vartheta}(y,\xi)$ at different values of $\eta = 1/5, 3/4$ and space variable $y = 0.2, 0.5, 0.7$ for Example 2.




(a) $\eta = 1/5$ (b) $\eta = 3/4$

**Fig. 7**. Visualizes the fuzzy results of $\underline{\vartheta}(y,\xi), \overline{\vartheta}(y,\xi)$ at different values of fractional order $\eta$, uncertainty $\xi$ and space variable $y$ of Example 2.

## Data availability

The data used to support the findings of this study are included within the article.

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## References

1. Baleanu, D., Diethelm, K., Scalas, E. & Trujillo, J. J. *Fractional Calculus: Models and Numerical Methods* Vol. 3 (World Scientific, Singapore, 2012).

2. Gorenflo, R. & Mainardi, F. Fractional calculus. In: Fractals and Fractional Calculus in Continuum Mechanics, pp. 223–276. Springer, Verlag Wien (1997)
3. Miller, K. S. & Ross, B. *An Introduction to the Fractional Calculus and Fractional Differential Equations* (Wiley, New York, NY, USA, 1993).
4. Abbas, S. & Benchohra, M. Fractional order Riemann-Liouville integral equations with multiple time delay. *Appl. Math. E-Notes* **12**, 79–87 (2012).
5. Atangana, A. & Bildik, N. Existence and numerical solution of the Volterra fractional integral equations of the second kind. Mathematical Problems in Engineering **2013** (2013)
6. Lakshmikantham, V. Theory of fractional functional differential equations. *Nonlinear Anal. Theory Methods Appl.* **69**(10), 3337–3343 (2008).
7. Ibrahim, R. W. & Momani, S. Upper and lower bounds of solutions for fractional integral equations. *Surv. Math. Appl.* **2**, 145–156 (2007).
8. Zheng, B. Explicit bounds derived by some new inequalities and applications in fractional integral equations. *J. Inequalities Appl.* **2014**(1), 1–12 (2014).
9. Alinezhad, M. & Allahviranloo, T. On the solution of fuzzy fractional optimal control problems with the Caputo derivative. *Inf. Sci.* **421**, 218–236 (2017).
10. Hukuhara, M. Integration des applications mesurables dont la valeur est un compact convexe. *Funkcialaj Ekvacioj* **10**(3), 205–223 (1967).
11. Agarwal, R. P., Lakshmikantham, V. & Nieto, J. J. On the concept of solution for fractional differential equations with uncertainty. *Nonlinear Anal. Theory Methods Appl.* **72**(6), 2859–2862 (2010).
12. Arshad, S. & Lupulescu, V. On the fractional differential equations with uncertainty. *Nonlinear Anal. Theory Methods Appl.* **74**(11), 3685–3693 (2011).
13. Allahviranloo, T., Armand, A., Gouyandeh, Z. & Ghadiri, H. Existence and uniqueness of solutions for fuzzy fractional Volterra-Fredholm integro-differential equations. *Journal of Fuzzy Set Valued Analysis* **2013**, 1–9 (2013).
14. Abu Arqub, O. Adaptation of reproducing kernel algorithm for solving fuzzy Fredholm-Volterra integrodifferential equations. *Neural Comput. Appl.* **28**(7), 1591–1610 (2017).
15. Agarwal, R. P., Arshad, S., O'Regan, D. & Lupulescu, V. Fuzzy fractional integral equations under compactness type condition. *Fract. Calc. Appl. Anal.* **15**(4), 572–590 (2012).
16. Stefanini, L. & Bede, B. Generalized Hukuhara differentiability of interval-valued functions and interval differential equations. *Nonlinear Anal. Theory Methods Appl.* **71**(3–4), 1311–1328 (2009).
17. Allahviranloo, T., Armand, A. & Gouyandeh, Z. Fuzzy fractional differential equations under generalized fuzzy Caputo derivative. *J. Intell. Fuzzy Syst.* **26**(3), 1481–1490 (2014).
18. Allahviranloo, T., Armand, A., Gouyandeh, Z. & Ghadiri, H. Existence and uniqueness of solutions for fuzzy fractional Volterra-Fredholm integro-differential equations. *Journal of Fuzzy Set Valued Analysis* **2013**, 1–9 (2013).
19. Vu, H. & Van Hoa, N. Applications of contractive-like mapping principles to fuzzy fractional integral equations with the kernel $\psi$ -functions. *Soft Comput.* **24**(24), 18841–18855 (2020).
20. Ahmad, N. et al. On analysis of the fuzzy fractional order Volterra-Fredholm integro-differential equation. *Alex. Eng. J.* **60**(1), 1827–1838 (2021).
21. Hoa, N. On the stability for implicit uncertain fractional integral equations with fuzzy concept. *Iran. J. Fuzzy Syst.* **18**(1), 185–201 (2021).
22. Vu, H., Rassias, J. & Van Hoa, N. Ulam-Hyers-Rassias stability for fuzzy fractional integral equations. *Iran. J. Fuzzy Syst.* **17**(2), 17–27 (2020).
23. Vu, H. & Van Hoa, N. Hyers-Ulam stability of fuzzy fractional Volterra integral equations with the kernel $\psi$- function via successive approximation method. *Fuzzy Sets Syst.* **419**, 67–98 (2021).
24. Khodadadi, E. & Çelik, E. The variational iteration method for fuzzy fractional differential equations with uncertainty. *J. Fixed Point Theory Appl.* **2013**(1), 1–7 (2013).
25. Mazandarani, M. & Kamyad, A. V. Modified fractional Euler method for solving fuzzy fractional initial value problem. *Commun. Nonlinear Sci. Numer. Simul.* **18**(1), 12–21 (2013).
26. Ngo, V.H. Fuzzy fractional functional integral and differential equations. *Fuzzy Sets Syst.* **280**(C), 58–90 (2015)
27. Van Hoa, N., Lupulescu, V. & O'Regan, D. Solving interval-valued fractional initial value problems under caputo gH-fractional differentiability. *Fuzzy Sets Syst.* **309**, 1–34 (2017).
28. Van Ngo, H., Lupulescu, V. & O'Regan, D. A note on initial value problems for fractional fuzzy differential equations. *Fuzzy Sets Syst.* **347**, 54–69 (2018).
29. Armand, A., Allahviranloo, T., Abbasbandy, S. & Gouyandeh, Z. The fuzzy generalized Taylor's expansion with application in fractional differential equations. *Iran. J. Fuzzy Syst.* **16**(2), 57–72 (2019).
30. Bica, A.M., Ziari, S. & Satmari, Z. Numerical method for fractional fuzzy integral equations (2021)
31. Haq, E. U., Hassan, Q. M. U., Ahmad, J. & Ehsan, K. Fuzzy solution of system of fuzzy fractional problems using a reliable method. *Alex. Eng. J.* **61**(4), 3051–3058 (2022).
32. Bidari, A., Dastmalchi Saei, F., Baghmisheh, M. & Allahviranloo, T. A new Jacobi Tau method for fuzzy fractional Fredholm nonlinear integro-differential equations. *Soft Comput.* **25**, 5855–5865 (2021).
33. Bica, A. M., Ziari, S. & Satmari, Z. An iterative method for solving linear fuzzy fractional integral equation. *Soft Comput.* **26**(13), 6051–6062 (2022).
34. Alijani, Z. & Kangro, U. Numerical solution of a linear fuzzy Volterra integral equation of the second kind with weakly singular kernels. *Soft Comput.* **26**(22), 12009–12022 (2022).
35. Kumar, S., Nieto, J. J. & Ahmad, B. Chebyshev spectral method for solving fuzzy fractional Fredholm-Volterra integro-differential equation. *Math. Comput. Simul.* **192**, 501–513 (2022).
36. Haq, E. U., Hassan, Q. M. U., Ahmad, J. & Ehsan, K. Fuzzy solution of system of fuzzy fractional problems using a reliable method. *Alex. Eng. J.* **61**(4), 3051–3058 (2022).
37. Agilan, K. & Parthiban, V. Existence results for fuzzy fractional Volterra integro differential equations. In: *AIP Conference Proceedings*, vol. 2516, p. 180002 (AIP Publishing LLC, 2022).
38. Agilan, K. & Parthiban, V. Initial and boundary value problem of fuzzy fractional-order nonlinear Volterra integro-differential equations. *J. Appl. Math. Comput.* **69**(2), 1765–1793 (2023).
39. Agilan, K. & Parthiban, V. Numerical solution of fuzzy fractional volterra integro differential equations with boundary conditions. *Physica Scripta* **99**(3), 035257 (2024).
40. Deng, T., Huang, J., Wang, Y. & Li, H. A generalized fuzzy barycentric Lagrange interpolation method for solving two-dimensional fuzzy fractional Volterra integral equations. *Numer. Algorithms* **98**(2), 743–766 (2025).
41. Ghazouani, A.E., Elomari, M. & Melliani, S. Existence, uniqueness, and UH-stability results for nonlinear fuzzy fractional Volterra–Fredholm integro-differential equations. *J. Nonlinear Complex Data Sci.* **25**, 457-477 (2025)
42. Sharif, A. A., Hamood, M. M. & Hamoud, A. A. A novel study for a class of nonlinear fuzzy fractional Volterra-Fredholm integro-differential equations. *Discontinuity Nonlinearity Complex.* **14**(02), 407–415 (2025).
43. Zahid, M., Ud Din, F., Shah, K. & Abdeljawad, T. Fuzzy fixed point approach to study the existence of solution for Volterra type integral equations using fuzzy sehgal contraction. *PLoS One* **19**(6), 0303642 (2024).

44. Shah, K., Abdeljawad, T. & Abdalla, B. On a coupled system under coupled integral boundary conditions involving non-singular differential operator. *AIMS Math* **8**(4), 9890–9910 (2023).
45. Shah, K., Amin, R., Ali, G., Mlaiki, N. & Abdeljawad, T. Algorithm for the solution of nonlinear variable-order Pantograph fractional integro-differential equations using haar method. *Fractals* **30**(08), 2240225 (2022).
46. Agilan, K. & Parthiban, V. Existence of solutions of fuzzy fractional Panto-graph equations. *Int. J. Math. Comput. Sci.* **15**, 1117–1122 (2020).
47. Van Hoa, N., Allahviranloo, T. & Pedrycz, W. A new approach to the fractional Abel k- integral equations and linear fractional differential equations in a fuzzy environment. *Fuzzy Sets Syst.* **481**, 108895 (2024).
48. Muhammad, G., Akram, M., Hussain, N. & Allahviranloo, T. Fuzzy Langevin fractional delay differential equations under granular derivative. *Inf. Sci.*. **681**, 121250 (2024)
49. Wang, G., Feng, M., Zhao, X. & Yuan, H. Fuzzy fractional delay integro-differential equation with the generalized Atangana-Baleanu fractional derivative. *Demonstratio Mathematica* **57**(1), 20240008 (2024).
50. El Ghazouani, A., Abdou Amir, F. I., Elomari, M. & Melliani, S. Fuzzy neutral fractional integro-differential equation existence and stability results involving the caputo fractional generalized Hukuhara derivative. *J. Nonlinear Complex Data Sci.* **25**(1), 53–78 (2024).
51. Ahmad, N. et al. On analysis of the fuzzy fractional order Volterra-Fredholm integro-differential equation. *Alex. Eng. J.* **60**(1), 1827–1838 (2021).
52. Padmapriya, V., Kaliyappan, M. & Parthiban, V. Solution of fuzzy fractional integro-differential equations using Adomian decomposition method. *Journal of Informatics & Mathematical Sciences* **9**(3), 501-507 (2017)
53. Savla, S. & Sharmila, R. Solving linear and nonlinear fuzzy fractional Volterra-Fredholm integro differential equations using Shehu Adomian decomposition method. *Ind. J. Sci. Technol.* **17**(12), 1222–1230 (2024).
54. Vu, H., Hoa, N. V. & An, T. V. Stability for initial value problems of fuzzy Volterra integro-differential equation with fractional order derivative. *J. Intell. Fuzzy Syst.* **37**(4), 5669–5688 (2019).
55. Vinh An, T. & Van Hoa, N. Extremal solutions of fuzzy fractional Volterra integral equations involving the generalized kernel functions by the monotone iterative technique. *J. Intell. Fuzzy Syst.* **38**(4), 5143–5155 (2020).
56. Shabestari, M. R. M., Ezzati, R. & Allahviranloo, T. Numerical solution of fuzzy fractional integro-differential equation via two-dimensional Legendre wavelet method. *J. Intell. Fuzzy Syst.* **34**(4), 2453–2465 (2018).
57. Hamoud, A., Mohammad, N. M. & Ghadle, K. Existence and uniqueness results for Volterra-Fredholm integro differential equations. *Adv. Theory Nonlinear Anal. Appl.* **4**(4), 361–372 (2020).


## Author contributions

K,A., V.P., K.N., and M.A.S. conceived of the presented idea. KA developed the theory and performed the computations. K,A., V.P., K.N., and M.A.S. verified the analytical methods. M.A.S. encouraged K.A., and V.P. to investigate and supervised the findings of this work. All authors discussed the results and contributed to the final manuscript.

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## Declarations

## Competing interests

The authors declare no competing interests.

## Additional information

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