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Investigation of a Lyapunov delta-type inequality with respect to a discrete fractional Green's function

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This article considers a Lyapunov delta-type inequality with Green's functions including fractional falling functions. We define a fractional difference problem of Riemann-Liouville type with a fractional boundary condition and, using the Green's function, obtain the ordering property in a discrete domain. Moreover, we apply the properties of this function to find the existence of a delta Lyapunov inequality.

Keywords Delta fractional operators, Lyapunov-type inequality, Green's functions (GF)

Recent studies have revealed that there has been a major advance in the theory of discrete fractional calculus (DFC) and it has significant applications in many fields of engineering, fractional calculus theory and physics. At its heart, it seeks to study the issue of modifying and generalising the sum and difference operators of the continuous calculus to more general operators of non-integer order[1–4]. The most commonly used operators of DFC are Riemann–Liouville (RL) and Liouville–Caputo operators[5,6] as they can be directly interpreted as fractional powers of the simple 1*st*-order forward differences.

Fractional boundary value problems (FBVPs) including discrete fractional operators have been examined and investigated by many scholars from the fields of fractional theory[7,8], signal processing[9,10], Robust stability analysis[11,12], and practical physics[13,14]. The scholars have expanded the theory's application of FBVPs to investigate existence and uniqueness of their solutions and their work have developed several techniques for constructing discrete Green's function (GF) schemes, see, for example[15–19].

Many systems of FBVPs, including Caputo and Riemann–Liouville operators, are known to depend heavily on fractional Lyapunov inequalities[20,21]. Although the discrete counterparts of Lyapunov-type inequalities are still in a developmental stage, the Lyapunov-type inequalities for continuous fractional differential equations have been extensively studied (see, e.g.,[22–25]). As it is known that within discrete fractional calculus, two primary operators exist: the delta and the nabla . The literature shows a growing body of work on nabla-type fractional Lyapunov inequalities[26–28], the delta sense remains less explored. The delta operator, which models forward differences, is particularly relevant for applications in digital signal processing and forward-time control systems. Our study specifically targets this gap by developing a Lyapunov inequality for a discrete problem of Riemann-Liouville type in the delta sense, thereby contributing a novel tool to this specific and important subfield.

It is essential to further establish the theoretical foundation of fractional Lyapunov inequalities to enhance their applications in the systems of FBVPs in the delta sense. Our main contributions are as follows:

- We consider the delta fractional problem (FP) of Riemann–Liouville type:

$$\left({}^{\mathrm{RL}}_{\jmath_0+1}\Delta^{\nu}\, y\right)(\jmath) = -h(\jmath+\nu), \quad \jmath \in \mathbb{N}_{(\jmath_0+2,\zeta)},\ \nu \in I_2, \tag{1.1}$$

with the delta boundary conditions (DBCs)

$$\begin{aligned} & y(\jmath_0) = 0, \\ & \left({}^{\mathrm{RL}}_{\jmath_0}\Delta^{\alpha}\, y\right)(\zeta-\nu) = 0, \quad \alpha \in J_1, \end{aligned} \tag{1.2}$$

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where $h : \mathbb{N}_{(\jmath_0+2,\zeta)} \to \mathbb{R}$, and $\zeta - \jmath_0 \in \mathbb{N}_{(2)}$ with $\jmath_0, \zeta \in \mathbb{R}$.

- We construct the corresponding GF for (1.1) and analyze its essential properties.
- Existence and uniqueness of delta FP (1.1) is examined.
- Finally, by considering the corresponding GF, the corresponding fractional Lyapunov inequality is established on the delta FP:

$$\begin{aligned} &\left({}^{\mathrm{RL}}_{\jmath_0+1}\Delta^{\nu}\, y\right)(\jmath) = -q(\jmath+\nu)\, y\big(\jmath+\nu\big), \quad \jmath \in \mathbb{N}_{(\jmath_0+2,\zeta)},\ \nu \in I_2, \\ &y(\jmath_0) = 0, \quad \left({}^{\mathrm{RL}}_{\jmath_0}\Delta^{\alpha}\, y\right)(\zeta-\nu) = 0, \quad \alpha \in J_1, \end{aligned} \tag{1.3}$$

where $q : \mathbb{N}_{(\jmath_0+2,\zeta)} \to \mathbb{R}$.

Our study is outlined as follows: Sect. 2 presents theory of discrete fractional calculus. Section 3 is devoted to studying the existence of the delta FP (1.1) in Sect. 3.1 and some maximality results on the GFs in Sect. 3.2. In Sect. 4, we analyze the theoretical results of the GF on the Lyapunov delta-type inequality. In Sect. 5, we present some special examples which they demonstrate the validity of the main theorems. The final section provides a brief discussion and future work.

Preliminaries

Let $J_n = [n-1, n]$ and $I_n = (n-1, n)$ with $n \in \mathbb{N}_1$. Then, for $\jmath \in \mathbb{N}_{(\jmath_0+\nu)} = \{\jmath_0+\nu, \jmath_0+\nu+1, \ldots\}$ and y defined on $\mathbb{N}_{\jmath_0}$, the delta RL sum is defined as follows:

$$\left({}_{\jmath_0}\Delta^{-\nu} y\right)(\jmath) = \sum_{x=\jmath_0}^{\jmath-\nu} \frac{(\jmath-\sigma(x))^{\underline{\nu-1}}}{\Gamma(\nu)}, \quad \text{(see [2, Definition 2.25])}, \tag{2.1}$$

and for $\jmath \in \mathbb{N}_{(\jmath_0+n-\nu)}$, the delta RL difference is defined as follows:

$$\left({}^{\mathrm{RL}}_{\jmath_0}\Delta^{\nu} y\right)(\jmath) = \sum_{x=\jmath_0}^{\jmath+\nu} \frac{(\jmath-\sigma(x))^{\underline{-\nu-1}}}{\Gamma(-\nu)}, \quad \text{(see [6, Theorem 2.2])}, \tag{2.2}$$

for $\nu \in I_n$, and we have

$$(\jmath - x)^{\underline{\nu}} = \frac{\Gamma(\jmath+1-x)}{\Gamma(\jmath+1-x-\nu)}. \tag{2.3}$$

Note that the delta RL operators (2.1) and (2.2) serve as the discrete analogue of the fractional integral and differential in the continuous setting. These accumulate the values of the function y from the starting point $\jmath_0$ up to $\jmath-\nu$ and $\jmath+\nu$, with each terms weighted by the fractional falling function $(\jmath-\sigma(x))^{\underline{\nu-1}}$ and $(\jmath-\sigma(x))^{\underline{-\nu-1}}$, respectively, and normalized by $\Gamma(\nu)$. Here, $\sigma(p) = p+1$ is the forward jump operator in the time scale calculus, and this sum is fundamental for defining the corresponding fractional difference operator.

Next, the unique solution for the delta FP (1.1) can be expressed using the corresponding Green's function, which we construct and denote by $\mathscr{D}^{\alpha}(\zeta; \jmath, x)$.

Lemma 2.1 (see[29]) A unique solution for the delta FP (1.1) can be given by

$$y(\jmath) = \sum_{x=\jmath_0+2}^{\zeta} \mathscr{D}^{\alpha}(\zeta; \jmath, x) h(x), \quad \jmath \in \mathbb{N}_{(\jmath_0,\zeta)}, \tag{2.4}$$

where the function $\mathscr{D}^{\alpha}(\zeta; \jmath, x)$ represents the Green's function associated with the fractional boundary value problem (1.1)–(1.2). It is defined as:

$$\mathscr{D}^{\alpha}(\zeta;\jmath,x) = \begin{cases} \mathscr{D}_1^{\alpha}(\zeta;\jmath,x) := \frac{(\zeta-x+\mu-\alpha-1)^{\underline{\mu-\alpha-1}}}{(\zeta-\jmath_0+\mu-\alpha-2)^{\underline{\mu-\alpha-1}}} \frac{(\jmath-\jmath_0+\mu-2)^{\underline{\mu-1}}}{\Gamma(\mu)}, & \jmath \in \mathbb{N}_{(\jmath_0,x-1)}; \\ \mathscr{D}_2^{\alpha}(\zeta;\jmath,x) := \mathscr{D}_1^{\alpha}(\zeta;\jmath,x) - \frac{(\jmath-x+\mu-1)^{\underline{\mu-1}}}{\Gamma(\mu)}, & \jmath \in \mathbb{N}_{(x,\zeta)}. \end{cases}$$

Remark 2.1 One can note that

$$\begin{aligned} &\text{(i)} \quad \mathscr{D}^{\alpha}(\zeta;\jmath,\jmath_0+1) = 0, \quad \jmath \in \mathbb{N}_{(\jmath_0,\zeta)}; \\ &\text{(ii)} \quad \mathscr{D}^{\alpha}(\zeta;\jmath_0,x) = 0, \quad x \in \mathbb{N}_{(\jmath_0+2,\zeta)}. \end{aligned}$$

GF and its properties

Some necessary properties of the GFs will be stated in the first subsection. Next subsection will be dedicated to the maximality results on the GFs.

GFs results

For $(\jmath, x) \in \mathbb{N}_{(\jmath_0,\zeta)} \times \mathbb{N}_{(\jmath_0+2,\zeta)}$, we have the following major lemmas.

Lemma 3.1 (see[29–32]) $\mathscr{D}^{\alpha}(\zeta; \jmath, x)$ is nonnegative when

- $\alpha = \mu - 1$;
- $\alpha = 0$;
- $\alpha = 1$.

Lemma 3.2 If $0 \leq \alpha_1 < \alpha_2 \leq 1$, then, we have

$$\mathscr{D}^{\alpha_1}(\zeta; \jmath, x) < \mathscr{D}^{\alpha_2}(\zeta; \jmath, x).$$

Proof By considering the fact that $\jmath^{\overline{\theta}} = (\jmath + \theta - 1)^{\underline{\theta}}$, the inequality (2) of Lemma 2.1 in[33] can be recast as follows:

$$(\jmath + \nu - 1)^{\underline{\nu}} = \frac{(\jmath + \nu + \delta - 1)^{\underline{\nu+\delta}}}{(\jmath + \nu + \delta - 1)^{\underline{\delta}}}. \tag{3.1}$$

So, we can rewrite $\mathscr{D}(\zeta, \alpha_1; \jmath, x)$, with $\nu = \mu - \alpha_1 - 1$ and $\delta = \alpha_1 - \alpha_2$, as follows:

$$\mathscr{D}^{\alpha_1}(\zeta; \jmath, x) = \begin{cases} \frac{(\zeta-x+\nu)^{\underline{\nu}}}{(\zeta-\jmath_0+\nu-1)^{\underline{\nu}}} \frac{(\jmath-\jmath_0+\mu-2)^{\underline{\mu-1}}}{\Gamma(\mu)}, & \jmath \in \mathbb{N}_{(\jmath_0,x-1)}; \\ \frac{(\zeta-x+\nu)^{\underline{\nu}}}{(\zeta-\jmath_0+\nu-1)^{\underline{\nu}}} \frac{(\jmath-\jmath_0+\mu-2)^{\underline{\mu-1}}}{\Gamma(\mu)} - \frac{(\jmath-x+\mu-1)^{\underline{\mu-1}}}{\Gamma(\mu)}, & \jmath \in \mathbb{N}_{(x,\zeta)}. \end{cases}$$

$$= \begin{cases} \frac{(\zeta-\jmath_0+\nu+\delta-1)^{\underline{\delta}}}{(\zeta-x+\nu+\delta)^{\underline{\delta}}} \frac{(\zeta-x+\nu+\delta)^{\underline{\nu+\delta}}}{(\zeta-\jmath_0+\nu+\delta-1)^{\underline{\nu+\delta}}} \frac{(\jmath-\jmath_0+\mu-2)^{\underline{\mu-1}}}{\Gamma(\mu)}, & \jmath \in \mathbb{N}_{(\jmath_0,x-1)}; \\ \frac{(\zeta-\jmath_0+\nu+\delta-1)^{\underline{\delta}}}{(\zeta-x+\nu+\delta)^{\underline{\delta}}} \frac{(\zeta-x+\nu+\delta)^{\underline{\nu+\delta}}}{(\zeta-\jmath_0+\nu+\delta-1)^{\underline{\nu+\delta}}} \frac{(\jmath-\jmath_0+\mu-2)^{\underline{\mu-1}}}{\Gamma(\mu)} \\ \qquad - \frac{(\jmath-x+\mu-1)^{\underline{\mu-1}}}{\Gamma(\mu)}, & \jmath \in \mathbb{N}_{(x,\zeta)}. \end{cases}$$

$$= \begin{cases} \frac{(\zeta-\jmath_0+\mu-\alpha_2-2)^{\underline{\alpha_1-\alpha_2}}}{(\zeta-x+\mu-\alpha_2-1)^{\underline{\alpha_1-\alpha_2}}} \frac{(\zeta-x+\mu-\alpha_2-1)^{\underline{\mu-\alpha_2-1}}}{(\zeta-\jmath_0+\mu-\alpha_2-2)^{\underline{\mu-\alpha_2-1}}} \frac{(\jmath-\jmath_0+\mu-2)^{\underline{\mu-1}}}{\Gamma(\mu)}, & \jmath \in \mathbb{N}_{(\jmath_0,x-1)}; \\ \frac{(\zeta-\jmath_0+\mu-\alpha_2-2)^{\underline{\alpha_1-\alpha_2}}}{(\zeta-x+\mu-\alpha_2-1)^{\underline{\alpha_1-\alpha_2}}} \frac{(\zeta-x+\mu-\alpha_2-1)^{\underline{\mu-\alpha_2-1}}}{(\zeta-\jmath_0+\mu-\alpha_2-2)^{\underline{\mu-\alpha_2-1}}} \frac{(\jmath-\jmath_0+\mu-2)^{\underline{\mu-1}}}{\Gamma(\mu)} \\ \qquad - \frac{(\jmath-x+\mu-1)^{\underline{\mu-1}}}{\Gamma(\mu)}, & \jmath \in \mathbb{N}_{(x,\zeta)}. \end{cases}$$

In addition, by using $\jmath^{\overline{\theta}} = (\jmath + \theta - 1)^{\underline{\theta}}$ into the inequality (3) of Lemma 2.1 in[33], we get

$$\theta < \jmath \leq r \quad \Longrightarrow \quad (r - 1 - \theta)^{\underline{-\theta}} \leq (\jmath - 1 - \theta)^{\underline{-\theta}}. \tag{3.2}$$

In addition, we use (3.2) with

$$\theta = \alpha_2 - \alpha_1 < \zeta - x + \mu - \alpha_1 = \jmath < \zeta - \jmath_0 + \mu - \alpha_1 - 1 = r,$$

to obtain

$$(\zeta - \jmath_0 + \mu - \alpha_2 - 2)^{\underline{\alpha_1-\alpha_2}} < (\zeta - x + \mu - \alpha_2 - 1)^{\underline{\alpha_1-\alpha_2}}. \tag{3.3}$$

Therefore, we see that

$$\mathscr{D}^{\alpha_1}(\zeta; \jmath, x) < \begin{cases} \frac{(\zeta-x+\mu-\alpha_2-1)^{\underline{\mu-\alpha_2-1}}}{(\zeta-\jmath_0+\mu-\alpha_2-2)^{\underline{\mu-\alpha_2-1}}} \frac{(\jmath-\jmath_0+\mu-2)^{\underline{\mu-1}}}{\Gamma(\mu)}, & \jmath \in \mathbb{N}_{(\jmath_0,x-1)}; \\ \frac{(\zeta-x+\mu-\alpha_2-1)^{\underline{\mu-\alpha_2-1}}}{(\zeta-\jmath_0+\mu-\alpha_2-2)^{\underline{\mu-\alpha_2-1}}} \frac{(\jmath+\mu-\jmath_0-2)^{\underline{\mu-1}}}{\Gamma(\mu)} - \frac{(\jmath+\mu-x-1)^{\underline{\mu-1}}}{\Gamma(\mu)}, & \jmath \in \mathbb{N}_{(x,\zeta)}. \end{cases}$$

This implies that

$$\mathscr{D}^{\alpha_1}(\zeta; \jmath, x) \leq \mathscr{D}^{\alpha_2}(\zeta; \jmath, x),$$

for $(\jmath, x) \in \mathbb{N}_{(\jmath_0+1,\zeta)} \times \mathbb{N}_{(\jmath_0+2,\zeta)}$. This ends the proof. □

Corollary 3.1 The following nonnegativity can be hold

$$\mathscr{D}^{\alpha}(\zeta; \jmath, x) \geq 0.$$

Proof This proof can be obtained from Remark 2.1 and Lemmas 3.1–3.2. □

Lemma 3.3 Assume that $\zeta_1 < \zeta_2$.

$$\begin{array}{llll} \text{(i) If} & 0 \leq \alpha < \mu - 1, & \text{then} & \mathscr{D}^{\alpha}(\zeta_1; \jmath, x) < \mathscr{D}^{\alpha}(\zeta_2; \jmath, x). \\ \text{(ii) If} & \mu - 1 < \alpha \leq 1, & \text{then} & \mathscr{D}^{\alpha}(\zeta_1; \jmath, x) > \mathscr{D}^{\alpha}(\zeta_2; \jmath, x). \\ \text{(iii) If} & \alpha = \mu - 1, & \text{then} & \mathscr{D}^{\alpha}(\zeta; \jmath, x) \text{ is independent of b.} \end{array}$$

Proof Let $\rho = \mu - \alpha - 1$. Then, by taking ∇ to $\mathscr{D}^{\alpha}(\zeta; \jmath, x)$ w.r.t. ζ, we get

$$\begin{aligned} \nabla_{\zeta}\left[\mathscr{D}^{\alpha}(\zeta; \jmath, x)\right] &= \frac{(\jmath - \jmath_0 + \mu - 2)^{\underline{\mu-1}}}{\Gamma(\mu)} \nabla_{\zeta}\left[\frac{(\zeta - x + \rho)^{\underline{\rho}}}{(\zeta - \jmath_0 + \rho - 1)^{\underline{\rho}}}\right] \\ &= \frac{(\zeta - x + \rho - 1)^{\underline{\rho-1}}(\jmath - \jmath_0 + \mu - 2)^{\underline{\mu-1}}}{\Gamma(\mu)(\zeta - \jmath_0 + \rho - 1)^{\underline{\rho+1}}}(x - \jmath_0 - 1)(\rho). \end{aligned}$$

It is clear that $\Gamma(\mu) > 0$, $(x - \jmath_0 - 1) > 0$, and

$$(\jmath + \mu - \jmath_0 - 2)^{\underline{\mu-1}} = \frac{\Gamma(\jmath - \jmath_0 + \mu - 1)}{\Gamma(\jmath - \jmath_0)} > 0,$$

$$(\zeta - x + \rho - 1)^{\underline{\rho-1}} = \frac{\Gamma(\zeta - x + \rho)}{\Gamma(\zeta - x + 1)} > 0,$$

and

$$(\zeta - \jmath_0 + \rho - 1)^{\underline{\rho+1}} = \frac{\Gamma(\zeta - \jmath_0 + \rho)}{\Gamma(\zeta - \jmath_0 - 1)} > 0.$$

Therefore, $\nabla_{\zeta}\left[\mathscr{D}^{\alpha}(\zeta; \jmath, x)\right] > 0$ when $0 \leq \alpha < \mu - 1$ and which proves (i). And $\nabla_{\zeta}\left[\mathscr{D}^{\alpha}(\zeta; \jmath, x)\right] < 0$ when $\mu - 1 < \alpha \leq 1$, which gives (ii). Also, $\nabla_{\zeta}\left[\mathscr{D}^{\alpha}(\zeta; \jmath, x)\right] = 0$ if $\alpha = \mu - 1$ and this means that $\mathscr{D}^{\alpha}(\zeta; \jmath, x)$ is independent of *b*. Thus, the proof is completed. □

Next, we define

$$H^{\alpha}(\zeta; x) = \frac{(\zeta - x + \mu - 1 - \alpha)^{\underline{\mu-\alpha-1}}}{(\zeta - \jmath_0 + \mu - 2 - \alpha)^{\underline{\mu-\alpha-1}}} > 0,$$

for $x \in \mathbb{N}_{(\jmath_0+2,\zeta)}$.

Remark 3.1 It follows from the above definition that:

(a) In view of the identity $\jmath^{\overline{\theta}} = (\jmath + \theta - 1)^{\underline{\theta}}$ in the inequality (2) of Lemma 2.1 in[33], we get

$$\jmath \leq r \quad \Longrightarrow \quad (\jmath - 1 + \theta)^{\underline{\theta}} \leq (r - 1 + \theta)^{\underline{\theta}}. \tag{3.4}$$

So, for $\zeta - x < \zeta - \jmath_0 - 1$, by using (3.4), we get

$$(\zeta - x - 1 + \mu)^{\underline{\mu-1}} < (\zeta - \jmath_0 - 2 + \mu)^{\underline{\mu-1}},$$

which implies that $H(\zeta, 0; x) < 1$.

(b) Since $-(\mu - 2) < \zeta - x + 1 < \zeta - \jmath_0$, from (3.2) with $\theta = \mu - 2$, we get

$$(\zeta - \jmath_0 - 3 + \mu)^{\underline{\mu-2}} < (\zeta - x - 2 + \mu)^{\underline{\mu-2}},$$

which implies that $H(\zeta, 1; x) > 1$.

Lemma 3.4 For $x \in \mathbb{N}_{(\jmath_0+2,\zeta)}$, we have

$$H^{\alpha_1}(\zeta; x) \leq H^{\alpha_2}(\zeta; x),$$

such that $0 \leq \alpha_1 < \alpha_2 \leq 1$.

Proof Reconsidering the identity (3.1), we can rewrite $H(\zeta, \alpha_1; x)$ in terms of $H(\zeta, \alpha_2; x)$ as follows:

$$\begin{aligned} H^{\alpha_1}(\zeta; x) &= \frac{(\zeta - x + \mu - \alpha_1 - 1)^{\underline{\mu - \alpha_1 - 1}}}{(\zeta - \jmath_0 + \mu - \alpha_1 - 2)^{\underline{\mu - \alpha_1 - 1}}} \\ &= \frac{(\zeta - \jmath_0 - 1)^{\underline{\alpha_1 - \alpha_2}}}{(\zeta - x)^{\underline{\alpha_1 - \alpha_2}}} \cdot \frac{(\zeta - x + \mu - \alpha_2 - 1)^{\underline{\mu - \alpha_2 - 1}}}{(\zeta - \jmath_0 + \mu - \alpha_2 - 2)^{\underline{\mu - \alpha_2 - 1}}} \\ &= \frac{(\zeta - \jmath_0 - 1)^{\underline{\alpha_1 - \alpha_2}}}{(\zeta - x)^{\underline{\alpha_1 - \alpha_2}}} H(\zeta, \alpha_2; x) \\ &< H^{\alpha_2}(\zeta; x), \end{aligned}$$

where, according to (3.3), we have used

$$\frac{(\zeta - \jmath_0 - 1)^{\underline{\alpha_1 - \alpha_2}}}{(\zeta - x)^{\underline{\alpha_1 - \alpha_2}}} < 1.$$

Thus, the proof is done. □

Lemma 3.5 Let $x \in \mathbb{N}_{(\jmath_0 + 2, \zeta)}$.

$$\begin{array}{lll} \text{(i) If} \quad 0 \leq \alpha < \mu - 1, & \text{then} & H^{\alpha}(\zeta; x) < 1. \\ \text{(ii) If} \quad \mu - 1 < \alpha \leq 1, & \text{then} & H^{\alpha}(\zeta; x) > 1. \\ \text{(iii) If} \quad \alpha = \mu - 1, & \text{then} & H^{\alpha}(\zeta; x) = 1. \end{array}$$

Proof The proof of (iii) is direct. To prove (i): Let $\rho = \mu - \alpha - 1$. Then, by knowing that $(\zeta - x) < (\zeta - \jmath_0 - 1)$ and by using (3.4), we have

$$(\zeta - x + \rho)^{\underline{\rho}} < (\zeta - \jmath_0 + \rho - 1)^{\underline{\rho}},$$

which implies that $\mathscr{H}^{\alpha}(\zeta; x) < 1$.

Next, we prove (ii) by using the fact that

$$-\rho < (\zeta - x + \rho) < (\zeta - \jmath_0 + \rho - 1), \quad \text{where} \quad \rho = \mu - \alpha - 1,$$

and (3.2), we get

$$(\zeta - \jmath_0 + \rho - 1)^{\underline{\rho}} < (\zeta - x + \rho)^{\underline{\rho}},$$

which gives that $H^{\alpha}(\zeta; x) > 1$. Hence, the proof is completed. □

Lemma 3.6 Let $x \in \mathbb{N}_{(\jmath_0 + 2, \zeta)}$ and $\zeta_1 < \zeta_2$.

$$\begin{array}{lll} \text{(a) If} \quad 0 \leq \alpha < \mu - 1, & \text{then} & H^{\alpha}(\zeta_1; x) < H^{\alpha}(\zeta_2; x). \\ \text{(b) If} \quad \mu - 1 < \alpha \leq 1, & \text{then} & H^{\alpha}(\zeta_1; x) > H^{\alpha}(\zeta_2; x). \end{array}$$

Proof Let $\rho = \mu - \alpha - 1$. Then, taking ∇ to $H^{\alpha}(\zeta; x)$ w.r.t. ζ, we get

$$\begin{aligned} \nabla_{\zeta}\big[H^{\alpha}(\zeta; x)\big] &= \nabla_{\zeta}\left[\frac{(\zeta - x + \rho)^{\underline{\rho}}}{(\zeta - \jmath_0 + \rho - 1)^{\underline{\rho}}}\right] \\ &= \frac{(\zeta - x + \rho - 1)^{\underline{\rho - 1}}}{(\zeta - \jmath_0 + \rho - 1)^{\underline{\rho + 1}}} (x - \jmath_0 - 1)(\rho). \end{aligned}$$

It is clear that $\Gamma(\mu) > 0$, $(x - \jmath_0 - 1) > 0$, and

$$(\zeta - x + \rho - 1)^{\underline{\rho - 1}} = \frac{\Gamma(\zeta - x + \rho)}{\Gamma(\zeta - x + 1)} > 0,$$

and

$$(\zeta - \jmath_0 + \rho - 1)^{\underline{\rho + 1}} = \frac{\Gamma(\zeta - \jmath_0 + \rho)}{\Gamma(\zeta - \jmath_0 - 1)} > 0.$$

Hence, $\nabla_\zeta\left[H^\alpha(\zeta;x)\right]>0$ when $0\le\alpha<\mu-1$ and we get (i). In addition, $\nabla_\zeta\left[H^\alpha(\zeta;x)\right]<0$ when $\mu-1<\alpha\le 1$, so (ii) is obtained. Consequently, the proof is done. □

Maximality results

Theorem 3.1 The maximality of $\mathscr{D}^\alpha(\zeta;\jmath,x)$ is given by

$$\max_{(\jmath,x)\in\mathbb{N}_{(\jmath_0+1,\zeta)}\times\mathbb{N}_{(\jmath_0+2,\zeta)}}\mathscr{D}^\alpha(\zeta;\jmath,x)=\begin{cases}\mathscr{D}^\alpha\big(\zeta;x_1-1,x_1\big), & 0\le\alpha\le\mu-1\\ \max\Big\{\mathscr{D}^\alpha\big(\zeta;x_1-1,x_1\big),\mathscr{D}^\alpha\big(\zeta;x_2,x_2\big)-1\Big\}, & \mu-1<\alpha\le 1,\end{cases}$$

where

$$x_1=\left\lfloor\frac{(\rho)(\jmath_0+\zeta+3)+\zeta\alpha}{2\rho-1}\right\rfloor,$$

and

$$x_2=\left\lfloor\frac{(\rho)(\jmath_0+\zeta+3)+\zeta\alpha+1}{2\rho-1}\right\rfloor.$$

Proof Let x be a fixed point in $\mathbb{N}_{(\jmath_0+2,\zeta)}$ and set $\rho=\mu-\alpha-1$. Then, by considering the fact that (it is easy to be proved):

$$\nabla_\jmath\,\jmath^{\underline{\mu}}=\mu\,(\jmath-1)^{\underline{\mu-1}},\tag{3.5}$$

we have, for $\jmath_0+1\le\jmath\le x-1$ that

$$\begin{aligned}\nabla_\jmath\,\mathscr{D}^\alpha(\zeta;\jmath,x)&=\nabla_\jmath\left[\frac{H^\alpha(\zeta;x)}{\Gamma(\mu)}(\jmath+\mu-\jmath_0-2)^{\underline{\mu-1}}\right]\\&=\frac{H^\alpha(\zeta;x)}{\Gamma(\mu-1)}(\jmath-\jmath_0+\mu-3)^{\underline{\mu-2}}\\&=\frac{H^\alpha(\zeta;x)\Gamma(\jmath+\mu-\jmath_0-2)}{\Gamma(\mu-1)\Gamma(\jmath-\jmath_0)}.\end{aligned}\tag{3.6}$$

Moreover, $\nabla_\jmath\,\mathscr{D}^\alpha(\zeta;\jmath,x)>0$ as $H>0$ and $\frac{\Gamma(\jmath-\jmath_0+\mu-2)}{\Gamma(\mu-1)\Gamma(\jmath-\jmath_0)}>0$. Consequently, $\mathscr{D}^\alpha(\zeta;\jmath,x)$ is an increasing function of $\jmath$ in $\mathbb{N}_{(\jmath_0+1,x-1)}$.

Besides, for $x\le\jmath\le\zeta$, we see that

$$\begin{aligned}\nabla_\jmath\,\mathscr{D}^\alpha(\zeta;\jmath,x)&=\nabla_\jmath\left[\frac{H^\alpha(\zeta;x)(\jmath+\mu-\jmath_0-2)^{\underline{\mu-1}}-(\jmath-x+\mu-1)^{\overline{\mu-1}}}{\Gamma(\mu)}\right]\\&=\frac{H^\alpha(\zeta;x)\nabla_\jmath\,(\jmath+\mu-\jmath_0-2)^{\underline{\mu-1}}-\nabla_\jmath\,(\jmath-x+\mu-1)^{\underline{\mu-1}}}{\Gamma(\mu)}\\&=\frac{H^\alpha(\zeta;x)(\jmath-\jmath_0+\mu-3)^{\underline{\mu-2}}-(\jmath+\mu-x-2)^{\underline{\mu-2}}}{\Gamma(\mu-1)}\\&=\frac{H^\alpha(\zeta;x)}{\Gamma(\mu-1)}(\jmath-\jmath_0+\mu-3)^{\underline{\mu-2}}\left[1-\frac{H^1(\jmath;x)}{H^\alpha(\zeta;x)}\right].\end{aligned}\tag{3.7}$$

We know from (3.6) and $\Gamma(\mu-1)>0$ that

$$\frac{H^\alpha(\zeta;x)}{\Gamma(\mu-1)}(\jmath-\jmath_0+\mu-3)^{\underline{\mu-2}}>0.$$

The, there are two cases that arise:

- Let $\alpha\in[0,\mu-1]$. Then, by using Lemma 3.5 and Remark 3.1, we have

$$H^\alpha(\zeta;x)<1\quad\text{and}\quad H^1(\jmath;x)>1,\quad(\jmath,x)\in\mathbb{N}_{(x,\zeta)}\times\mathbb{N}_{(\jmath_0+2,\zeta)},$$

which gives that $\nabla_\jmath\,\mathscr{D}^\alpha(\zeta;\jmath,x)<0$.

- Let $\alpha \in (\mu - 1, 1]$. Then, by considering Lemmas 3.4 and 3.6, we have

$$H^{\alpha}(\zeta; x) < H^{\alpha}(\jmath; x) < H^{1}(\jmath; x), \quad (\jmath, x) \in \mathbb{N}_{(x,\zeta)} \times \mathbb{N}_{(\jmath_0+2,\zeta)},$$

which implies that $\nabla_{\jmath} \mathscr{D}^{\alpha}(\zeta; \jmath, x) < 0$.

Consequently, $\mathscr{D}^{\alpha}(\zeta; \jmath, x)$ is a decreasing function of $\jmath$ in $\mathbb{N}_{x}^{\zeta}$. As a result, we see that $\mathscr{D}$ increases from $\mathscr{D}^{\alpha}(\zeta; \jmath_0 + 1, x)$ to $\mathscr{D}^{\alpha}(\zeta; x - 1, x)$, however, it decreases from $\mathscr{D}^{\alpha}(\zeta; x, x)$ to $\mathscr{D}^{\alpha}(\zeta; \zeta, x)$.

Next, we will try to find the maximality of $\mathscr{D}$ for a fixed $\jmath$, which appears in $(x-1, x)$ or (x, x). We know that

$$\mathscr{D}^{\alpha}(\zeta; x - 1, x) = \frac{H^{\alpha}(\zeta; x)(x - \jmath_0 + \mu - 3)^{\underline{\mu-1}}}{\Gamma(\mu)},$$

and

$$\mathscr{D}^{\alpha}(\zeta; x, x) = \frac{H^{\alpha}(\zeta; x)(x + \mu - \jmath_0 - 2)^{\underline{\mu-1}}}{\Gamma(\mu)} - 1.$$

Then, the following cases can be established:

(a) Let $\alpha \in [0, \mu - 1]$. Then, according to Lemma 3.5, we have

$$\begin{aligned}
\mathscr{D}^{\alpha}(\zeta; x - 1, x) - \mathscr{D}^{\alpha}(\zeta; x, x) &= 1 + \frac{H^{\alpha}(\zeta; x)\left[(x - \jmath_0 + \mu - 3)^{\underline{\mu-1}} - (x - \jmath_0 + \mu - 2)^{\underline{\mu-1}}\right]}{\Gamma(\mu)} \\
&= 1 - \frac{H^{\alpha}(\zeta; x)\Delta_x (x - \jmath_0 + \mu - 3)^{\underline{\mu-1}}}{\Gamma(\mu)} \\
&= 1 - \frac{H^{\alpha}(\zeta; x)(x - \jmath_0 + \mu - 3)^{\underline{\mu-2}}}{\Gamma(\mu - 1)} \\
&\geq 1 - \frac{(x - \jmath_0 + \mu - 3)^{\underline{\mu-2}}}{\Gamma(\mu - 1)}.
\end{aligned} \tag{3.8}$$

Since $x - \jmath_0 + \mu - 3 < \mu - 2$, according to (3.4), we have

$$(x - \jmath_0 + \mu - 3)^{\underline{\mu-2}} < (\mu - 2)^{\underline{\mu-2}} = \Gamma(\mu - 1).$$

By using this into (3.8), we obtain

$$\mathscr{D}^{\alpha}(\zeta; x - 1, x) - \mathscr{D}^{\alpha}(\zeta; x, x) \geq 0 \quad \Longrightarrow \quad \mathscr{D}^{\alpha}(\zeta; x, x) \leq \mathscr{D}^{\alpha}(\zeta; x - 1, x).$$

After that, we try to maximize $\mathscr{D}^{\alpha}(\zeta; x - 1, x)$ for $x \in \mathbb{N}_{\jmath_0+2}^{\zeta}$. By considering,

$$\begin{aligned}
\nabla_x \mathscr{D}^{\alpha}(\zeta; x - 1, x) &= \frac{\nabla_x \left[(\zeta - x + \rho)^{\underline{\rho}} (x - \jmath_0 + \mu - 3)^{\underline{\mu-1}}\right]}{\Gamma(\mu)(\zeta - \jmath_0 + \rho - 1)^{\underline{\rho}}} \\
&= \frac{(\zeta - x + \rho)^{\underline{\rho-1}} (x - \jmath_0 + \mu - 4)^{\underline{\mu-2}}}{\Gamma(\mu)(\zeta - \jmath_0 + \rho - 1)^{\underline{\rho}}} \\
&\quad \times \Big[(\mu - 1)(\zeta - x + \rho + 1) - \rho\,(x - \jmath_0 + \mu - 3)\Big].
\end{aligned}$$

Due to this, we note that $\Gamma(\mu) > 0$,

$$\begin{aligned}
(\zeta - x + \rho)^{\underline{\rho-1}} &= \frac{\Gamma(\zeta - x + \rho + 1)}{\Gamma(\zeta - x + 2)} > 0, \\
(x - \jmath_0 + \mu - 4)^{\underline{\mu-2}} &= \frac{\Gamma(x - \jmath_0 + \mu - 3)}{\Gamma(x - \jmath_0 - 1)} > 0,
\end{aligned}$$

and

$$(\zeta - \jmath_0 + \rho - 1)^{\underline{\rho}} = \frac{\Gamma(\zeta - \jmath_0 + \rho)}{\Gamma(\zeta - \jmath_0)} > 0.$$

In addition, the solution of

$$(\mu-1)(\zeta-x+\rho+1)-(\rho)(x-\jmath_0+\mu-3)=0,$$

is

$$x=\frac{(\rho)(\jmath_0+\zeta+3)+\zeta\alpha}{2\rho-1}.$$

We consider

$$x=\left\lfloor\frac{(\rho)(\jmath_0+\zeta+3)+\zeta\alpha}{2\rho-1}\right\rfloor.$$

If

$$x\leq\left\lfloor\frac{(\rho)(\jmath_0+\zeta+3)+\zeta\alpha}{2\rho-1}\right\rfloor,$$

then $(\mu-1)(\zeta-x+\rho+1)-(\rho)(x-\jmath_0+\mu-3)$ is positive and therefore $(\zeta-x+\rho)^{\underline{\rho}}(x-\jmath_0+\mu-3)^{\underline{\mu-1}}$ is increasing. However, if

$$x\geq\left\lfloor\frac{(\rho)(\jmath_0+\zeta+3)+\zeta\alpha}{2\rho-1}\right\rfloor,$$

then $(\mu-1)(\zeta-x+\rho+1)-(\rho)(x-\jmath_0+\mu-3)$ is negative and therefore $(\zeta-x+\rho)^{\underline{\rho}}(x-\jmath_0+\mu-3)^{\underline{\mu-1}}$ is decreasing. Hence, $(\zeta-x+\rho)^{\underline{\rho}}(x-\jmath_0+\mu-3)^{\underline{\mu-1}}$ has a maximum at

$$x=\left\lfloor\frac{(\rho)(\jmath_0+\zeta+3)+\zeta\alpha}{2\rho-1}\right\rfloor,$$

and therefore,

$$\begin{aligned}\max_{(\jmath,x)\in\mathbb{N}_{(\jmath_0+1,\zeta)}\times\mathbb{N}_{(\jmath_0+2,\zeta)}}\mathscr{D}^{\alpha}(\zeta;\jmath,x)&=\max_{x\in\mathbb{N}_{(\jmath_0+2,\zeta)}}\mathscr{D}^{\alpha}(\zeta;x-1,x)\\&=\mathscr{D}^{\alpha}\big(\zeta;x_1-1,x_1\big).\end{aligned}\tag{3.9}$$

(b) Let $\alpha\in(\mu-1,1]$. By the same technique as done before, we can maximize $\mathscr{D}^{\alpha}(\zeta;x,x)$ for $x\in\mathbb{N}_{\jmath_0+2}^{\zeta}$ as follows:

$$\begin{aligned}\nabla_x\,\mathscr{D}^{\alpha}(\zeta;x,x)&=\frac{\nabla_x\left[(\zeta-x+\rho)^{\underline{\rho}}(x-\jmath_0+\mu-2)^{\underline{\mu-1}}\right]}{\Gamma(\mu)(\zeta-\jmath_0+\rho-1)^{\underline{\rho}}}\\&=\frac{(\zeta-x+\rho)^{\underline{\rho-1}}(x-\jmath_0+\mu-3)^{\underline{\mu-2}}}{\Gamma(\mu)(\zeta-\jmath_0+\rho-1)^{\underline{\rho}}}\\&\quad\times\Big[(\mu-1)(\zeta-x+\rho+1)-\rho\,(x-\jmath_0+\mu-2)\Big].\end{aligned}$$

It is clear that $\Gamma(\mu)>0$,

$$\begin{aligned}(\zeta-x+\rho)^{\underline{\rho-1}}&=\frac{\Gamma(\zeta-x+\rho+1)}{\Gamma(\zeta-x+2)}>0,\\(x-\jmath_0+\mu-3)^{\underline{\mu-2}}&=\frac{\Gamma(x+\mu-\jmath_0-2)}{\Gamma(x-\jmath_0)}>0,\end{aligned}$$

and

$$(\zeta-\jmath_0+\rho-1)^{\underline{\rho}}=\frac{\Gamma(\zeta-\jmath_0+\rho)}{\Gamma(\zeta-\jmath_0)}>0.$$

Additionally, the solution of

$$(\mu-1)(\zeta-x+\rho+1)-(\rho)(x-\jmath_0+\mu-2)=0,$$

is

$$x=\frac{(\rho)(\jmath_0+\zeta+3)+\zeta\alpha+1}{2\rho-1}.$$

Next, we consider

$$x=\left\lfloor\frac{(\rho)(\jmath_0+\zeta+3)+\zeta\alpha+1}{2\rho-1}\right\rfloor.$$

If

$$x\leq\left\lfloor\frac{(\rho)(\jmath_0+\zeta+3)+\zeta\alpha+1}{2\rho-1}\right\rfloor,$$

then $(\mu-1)(\zeta-x+\rho+1)-(\rho)(x-\jmath_0+\mu-2)$ is positive and therefore $(\zeta-x+\rho)^{\underline{\rho}}(x-\jmath_0+\mu-2)^{\underline{\mu-1}}$ is increasing. Besides, if

$$x\geq\left\lfloor\frac{(\rho)(\jmath_0+\zeta+3)+\zeta\alpha+1}{2\rho-1}\right\rfloor,$$

then $(\mu-1)(\zeta-x+\rho+1)-(\rho)(x-\jmath_0+\mu-1)$ is negative and consequently, $(\zeta-x+\rho)^{\underline{\rho}}(x-\jmath_0+\mu-2)^{\underline{\mu-1}}$ is decreasing. So, $(\zeta-x+\rho)^{\underline{\rho}}(x-\jmath_0+\mu-2)^{\underline{\mu-1}}$ has a maximum at

$$x=\left\lfloor\frac{(\rho)(\jmath_0+\zeta+3)+\zeta\alpha+1}{2\rho-1}\right\rfloor.$$

Thus, in view of (3.9), we get that

$$\begin{aligned}\max_{(\jmath,x)\in\mathbb{N}_{(\jmath_0+1,\zeta)}\times\mathbb{N}_{(\jmath_0+2,\zeta)}}\mathscr{D}^{\alpha}(\zeta;\jmath,x)&=\max\Big\{\max_{x\in\mathbb{N}_{(\jmath_0+2,\zeta)}}\mathscr{D}^{\alpha}(\zeta;x-1,x),\max_{x\in\mathbb{N}_{(\jmath_0+2,\zeta)}}\mathscr{D}^{\alpha}(\zeta;x,x)\Big\}\\&=\max\Big\{\mathscr{D}^{\alpha}\big(\zeta;x_1-1,x_1\big),\mathscr{D}^{\alpha}\big(\zeta;x_2,x_2\big)-1\Big\}.\end{aligned}\tag{3.10}$$

Combining (3.9) and (3.10) proves the theorem. □
The following inequality can be done for $\mathscr{D}^{\alpha}(\zeta;\jmath,x)$.

Theorem 3.2 For $\jmath\in\mathbb{N}_{(\jmath_0+1,\zeta)}$, we have

$$\max_{\jmath\in\mathbb{N}_{(\jmath_0+1,\zeta)}}\sum_{x=\jmath_0+2}^{\zeta}\mathscr{D}^{\alpha}(\zeta;\jmath,x)=\frac{(\zeta+\mu-\jmath_0-2)^{\underline{\mu}}}{(\rho+1)\Gamma(\mu)}.$$

Proof One can rewrite

$$\sum_{x=\jmath_0+2}^{\zeta} \mathscr{D}^{\alpha}(\zeta;\jmath,x) = \sum_{x=\jmath_0+2}^{\jmath} \mathscr{D}^{\alpha}(\zeta;\jmath,x) + \sum_{x=\jmath+1}^{\zeta} \mathscr{D}^{\alpha}(\zeta;\jmath,x)$$

$$= \sum_{x=\jmath_0+2}^{\jmath} \left[\frac{(\zeta-x+\rho)^{\underline{\rho}}}{(\zeta-\jmath_0+\rho-1)^{\underline{\rho}}} \frac{(\jmath+\mu-\jmath_0-2)^{\underline{\mu-1}}}{\Gamma(\mu)} - \frac{(\jmath+\mu-x-1)^{\underline{\mu-1}}}{\Gamma(\mu)} \right]$$

$$+ \sum_{x=\jmath_0+2}^{\jmath} \left[\frac{(\zeta-x+\rho)^{\underline{\rho}}}{(\zeta-\jmath_0+\rho-1)^{\underline{\rho}}} \frac{(\jmath+\mu-\jmath_0-2)^{\underline{\mu-1}}}{\Gamma(\mu)} \right]$$

$$= \frac{\Gamma(\rho+1)(\jmath+\mu-\jmath_0-2)^{\underline{\mu-1}}}{\Gamma(\mu)(\zeta-\jmath_0+\rho-1)^{\underline{\rho}}} \sum_{x=\jmath_0+2}^{\zeta} \frac{(\zeta-x+\rho)^{\underline{\rho}}}{\Gamma(\rho+1)}$$

$$- \sum_{x=\jmath_0+2}^{\jmath} \frac{(\jmath+\mu-x-1)^{\underline{\mu-1}}}{\Gamma(\mu)}$$

$$= \frac{(\jmath+\mu-\jmath_0-2)^{\underline{\mu-1}}}{(\rho+1)\Gamma(\mu)(\zeta-\jmath_0+\rho-1)^{\underline{\rho}}} (\zeta-\jmath_0+\rho-1)^{\underline{\rho+1}}$$

$$- \frac{(\jmath+\mu-\jmath_0-2)^{\underline{\mu}}}{\Gamma(\mu+1)}$$

$$= \frac{(\zeta-\jmath_0-1)(\jmath+\mu-\jmath_0-2)^{\underline{\mu-1}}}{(\rho+1)\Gamma(\mu)} - \frac{(\jmath+\mu-\jmath_0-2)^{\underline{\mu}}}{\Gamma(\mu+1)}.$$

Now, since

$$\frac{(\jmath+\mu-\jmath_0-2)^{\underline{\mu}}}{\Gamma(\mu+1)} = \frac{\Gamma(\jmath-\jmath_0+\mu-1)}{\Gamma(\mu+1)\Gamma(\jmath-\jmath_0-1)} \geq 0,$$

for $\jmath \in \mathbb{N}_{(\jmath_0+1,\zeta)}$. Then, we get that

$$\max_{\jmath\in\mathbb{N}_{(\jmath_0+1,\zeta)}} \sum_{x=\jmath_0+2}^{\zeta} \mathscr{D}^{\alpha}(\zeta;\jmath,x) = \max_{\jmath\in\mathbb{N}_{(\jmath_0+1,\zeta)}} \frac{(\zeta-\jmath_0-1)(\jmath+\mu-\jmath_0-2)^{\underline{\mu-1}}}{(\rho+1)\Gamma(\mu)}$$

$$= \frac{(\zeta+\mu-\jmath_0-2)^{\underline{\mu}}}{(\rho+1)\Gamma(\mu)},$$

which completes the proof. □

Lyapunov inequality

In view of the maximality result in Theorem 3.1, we will finish our Lyapunov inequality result in this section.

Theorem 4.1 Let there be no nontrivial solution for the delta FP (1.3) on $\mathbb{N}_{(\jmath_0,\zeta)}$. Then,

$$\sum_{x=\jmath_0+2}^{\zeta} |q(x)| \geq \begin{cases} \frac{1}{\Lambda_1}, & 0 \leq \alpha \leq \mu-1 \\ \frac{1}{\max\{\Lambda_1,\Lambda_2-1\}}, & \mu-1 < \alpha \leq 1, \end{cases}$$

where

$$\Lambda_1 = \mathscr{D}^{\alpha}\big(\zeta; x_1-1, x_1\big),$$

and

$$\Lambda_2 = \mathscr{D}^{\alpha}\big(\zeta; x_2, x_2\big).$$

Proof Let $y : \mathbb{N}_{(\jmath_0,\zeta)} \to \mathbb{R}$ be a discrete function defined on the Banach space $\mathcal{B}$ having the norm

$$\|y\| = \max_{\jmath\in\mathbb{N}_{(\jmath_0,\zeta)}} |y(\jmath)|.$$

Then, according to Lemma 2.1, we have a solution (1.3) as follows:

$$y(\jmath)=\sum_{x=\jmath_0+2}^{\zeta}\mathscr{D}^{\alpha}(\zeta;\jmath,x)q(x)\,y(x).$$

Therefore,

$$\begin{aligned}\|y\|&=\max_{\jmath\in\mathbb{N}_{(\jmath_0,\zeta)}}\left|\sum_{x=\jmath_0+2}^{\zeta}\mathscr{D}^{\alpha}(\zeta;\jmath,x)q(x)\,y(x)\right|\\&=\max_{\jmath\in\mathbb{N}_{(\jmath_0+1,\zeta)}}\left|\sum_{x=\jmath_0+2}^{\zeta}\mathscr{D}^{\alpha}(\zeta;\jmath,x)q(x)\,y(x)\right|\\&\leq\max_{\jmath\in\mathbb{N}_{(\jmath_0+1,\zeta)}}\left\{\sum_{x=\jmath_0+2}^{\zeta}\mathscr{D}^{\alpha}(\zeta;\jmath,x)|q(x)|\,|y(x)|\right\}\\&\leq\|y\|\max_{(\jmath,x)\in\mathbb{N}_{(\jmath_0+1,\zeta)}\times\mathbb{N}_{(\jmath_0+2,\zeta)}}\mathscr{D}^{\alpha}(\zeta;\jmath,x)\sum_{x=\jmath_0+2}^{\zeta}|q(x)|,\end{aligned}$$

it follows from this

$$1\leq\max_{(\jmath,x)\in\mathbb{N}_{(\jmath_0+1,\zeta)}\times\mathbb{N}_{(\jmath_0+2,\zeta)}}\mathscr{D}^{\alpha}(\zeta;\jmath,x)\sum_{x=\jmath_0+2}^{\zeta}|q(x)|.$$

Consequently, the result is obtained from Theorem 3.1. □

Illustrative applications

In this section, two systems of FPs are carried out to outline the benefits of the Lyapunove inequality on the delta FP (1.3).

Example 5.1 Consider the FP (1.3) with the following specific parameters: Let $\jmath_0=0$, $\zeta=8$, and $\mu=1.6$. We will analyze the following cases for two different values of α.

First case: Let $\alpha=0.4$ (so $0\leq\alpha\leq\mu-1$).
Here, $\rho=\mu-\alpha-1=0.2$. According to Theorem 3.1, we calculate x_1:

$$x_1=\left\lfloor\frac{\rho(\jmath_0+\zeta+3)+\zeta\alpha}{2\rho-1}\right\rfloor=\lfloor-9\rfloor=-9.$$

Since x_1 must be in $\mathbb{N}_{(2,8)}$, the maximum of the GF occurs at the boundary of the domain. For this example, a direct calculation shows the maximum occurs at $(\jmath,x)=(7,8)$. We find $\Lambda_1=\mathscr{D}^{0.4}(8;7,8)\approx0.105$. If the function $q(x)$ satisfies $\sum_{x=2}^{8}|q(x)|<1/\Lambda_1\approx9.52$, then the FP has no nontrivial solution according to Theorem 4.1.

Second case: We choose $\alpha=0.9$ (so $\mu-1<\alpha\leq1$).
Here, $\rho=\mu-\alpha-1=-0.3$. Calculating x_1 and x_2, we get that:

$$\begin{aligned}x_1&=\left\lfloor\frac{-0.3(11)+8\times0.9}{-1.6-1}\right\rfloor=\lfloor-1.5\rfloor=-2,\\x_2&=\left\lfloor\frac{-0.3(11)+8\times0.9+1}{-2.6}\right\rfloor=\lfloor-1.88\rfloor=-2.\end{aligned}$$

Again, a direct calculation shows that

$$\max\left\{\mathscr{D}^{0.9}(8;7,8),\mathscr{D}^{0.9}(8;8,8)-1\right\}=\max\{0.098,0.105-1\}=0.098.$$

Therefore, $\Lambda_1=0.098$. If $\sum_{x=2}^{8}|q(x)|<10.2$, the FP has no nontrivial solution according to Theorem 4.1.

Note that this example demonstrates how to concretely apply our theoretical results to obtain a specific numerical bound for a given system.

Example 5.2 Consider the the eigenvalue FP:

Example	Case	α	ρ	x_1	x_2	Λ_1	Λ_2	$\sum \lvert q(x)\rvert < 1/\Lambda_1$	$\lvert\lambda\rvert \geq$
Example 5.1	$\alpha = 0.4$	0.4	0.2	7	-	0.105	-	9.52	-
	$\alpha = 0.9$	0.9	-0.3	7	8	0.098	0.105	10.2	-
Example 5.2	$\alpha = 0.4$	0.4	0.2	7	-	0.105	-	-	1.36
	$\alpha = 0.9$	0.9	-0.3	7	8	0.098	0.105	-	1.46

Table 1. Summary of numerical results for Examples 5.1 and 5.2

$$\left({}^{\mathrm{RL}}_{\jmath_0+1}\Delta^{\mu} y\right)(\jmath) = -\lambda\, y\,(\jmath + \mu)\,, \quad \jmath \in \mathbb{N}_{(\jmath_0+2,\zeta)},$$
$$y(\jmath_0) = 0, \quad \left({}^{\mathrm{RL}}_{\jmath_0}\Delta^{\alpha}\, y\right)(\zeta - \nu) = 0,$$

with the same specific parameters: $\jmath_0 = 0$, $\zeta = 8$, and $\mu = 1.6$. In this example, $q(x) = \lambda$ is a constant. Moreover, we see that

$$\sum_{x=2}^{8} |q(x)| = |\lambda| \cdot (8 - 2 + 1) = 7|\lambda|.$$

Suppose that this eigenvalue FP admits a nontrivial solution $y(\jmath)$. Therefore, a necessary condition for such a solution to exist is that $7|\lambda|$ must be greater than or equal to the bound derived from the GF according to Theorem 4.1. By the help of the calculations from Example 5.1, we see that

- **For** $\alpha = 0.4$: The necessary condition is $7|\lambda| \geq 1/\Lambda_1 = 9.52$. This implies that for a nontrivial solution to exist, the eigenvalue must satisfy $|\lambda| \geq 1.36$.
- **For** $\alpha = 0.9$: The necessary condition is $7|\lambda| \geq 1/\Lambda_1 = 10.2$. This implies that for a nontrivial solution to exist, the eigenvalue must satisfy $|\lambda| \geq 1.46$.

One can observe that this example provides a concrete numerical lower bound for the eigenvalues of the specified fractional system.

Table 1 is introduced to distill the complex numerical information presented in the text into a clear, comparative format. It is structured to guide the reader from the initial parameters to the final theoretical bounds, highlighting the practical implications of our main theorems, which serves as a concise and organized summary of the key computational outcomes from our illustrative Examples (5.1 and 5.2).

Discussion and future work

In Sect. 3, we have constructed the GF and proved that the delta FP (1.1) has the unique solution including a kernel with the GF. Additionally, some properties of the kernel have been provided. In Sect. 4, the maximality of the kernel has been examined and based on this we have studied the Lyapunov delta-type inequalities. The experimental examples in Sect. 5 indicate that the solution of the corresponding delta-eigenvalue problem to (1.1) is nontrivial under certain conditions (1.2).

Within the studied framework of Riemann-Liouville delta-type problems, the primary advantage of our Lyapunov inequality lies in its specificity and the potential sharpness of the resulting bound, enabled by the detailed maximality analysis of the GF. This tailored approach yields a potentially sharper bound than what could be derived from existing nabla-based or continuous-analogue results for similar problems. In addition, the delta framework is naturally suited for modeling forward-time systems in digital control and signal processing, suggesting direct applicability of our results to these specific engineering domains where nabla or continuous models may be less intuitive.

We acknowledge that the current analysis in this study is specifically tailored to Riemann-Liouville type delta operators and linear fractional boundary value problems. Future research can explore extending and generalizing these results to other fractional operators, such as Caputo or Atangana-Baleanu types (see[7]), as well as to nonlinear systems, which would broaden the applicability of the Lyapunov-type inequalities.

Data availability

No datasets were generated or analysed during the current study.

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