



OPEN

A flexible extension of the log-logistic model with diverse failure rate shapes and applications

Ahmed Z. Afify^{1✉}, Mohamed E. Mead², Hamada H. Hassan³, Abdussalam Aljadani⁴, Abdulaziz S. Alghamdi⁵ & Enayat M. Abd Elrazik^{1,6}

In this article, we introduce an enhanced version of the log-logistic model, termed the Kumaraswamy alpha-power log-logistic (KAPLL) distribution. The KAPLL model expands upon the traditional log-logistic distribution and several well-established distributions. We investigate the mathematical properties of the KAPLL model, highlighting its ability to effectively model various aging and failure criteria. The KAPLL distribution exhibits remarkable flexibility in modeling various types of hazard rate behaviors. It is capable of accommodating a wide range of shapes, including increasing, decreasing, J-shaped, reversed J-shaped, bathtub-shaped, inverted bathtub-shaped, and even more complex forms such as decreasing–increasing–decreasing failure rates. The KAPLL distribution is characterized by its capacity to exhibit both symmetric and asymmetric shapes in its density function. The proposed KAPLL model overcomes key limitations of existing LL-based generalizations by offering enhanced flexibility in modeling diverse hazard rate shapes and tail behaviors. We estimate the KAPLL parameters using eight classical estimation methods. Comprehensive simulation results are presented and ranked to identify the most effective approach for estimating KAPLL parameters, which we believe will be of great interest to engineers and applied statisticians. To further demonstrate the versatility of the KAPLL distribution, we analyze five real-world datasets from reliability, engineering, biomedical, and environmental sciences, highlighting its flexibility relative to other extensions of the log-logistic model. Likelihood ratio tests conducted across five real datasets confirm that the KAPLL model provides a statistically significant improvement over the baseline log-logistic distribution.

Keywords Bathtub hazard rate, Kumaraswamy alpha-power class, Log-logistic distribution, Maximum likelihood, Order statistics, Real data

The quality of statistical analysis is heavily influenced by the choice of probability distribution. Consequently, significant efforts have been devoted to developing generalized classes of probabilistic distributions, along with relevant statistical methodologies. In practice, these distributions find applications across various fields, including insurance, actuarial science, investment, risk analysis, business and economic research, reliability engineering, chemical engineering, medicine, demography, and sociology, among others.

The statistical literature has introduced several newly generated classes of univariate continuous distributions by incorporating additional shape parameters into baseline models. These extended distributions have garnered the attention of statisticians due to their flexibility and capability to model both monotonic and non-monotonic real-life data. Notable examples include the Marshall–Olkin-G (MO-G)¹, McDonald-G², beta-G³, Kumaraswamy-G⁴, Kumaraswamy Marshal–Olkin-G (KMO-G)⁵, Weibull-G⁶, Weibull Marshall Olkin-G⁷, Marshall–Olkin alpha power-G (MOAP-G)⁸, and Kumaraswamy alpha-power-G (KAP-G)⁹, among others.

The classical log-logistic (LL) distribution, often referred to as the Fisk distribution in the context of income distribution literature^{10–12}, has also been called the Pareto Type III distribution by Arnold¹³, who introduced an additional location parameter. Furthermore, the LL distribution is a special case of the Burr-XII distribution¹⁴

¹Department of Statistics, Mathematics, and Insurance, Benha University, Benha 13511, Egypt. ²Department of Statistics and Insurance, Zagazig University, Zagazig 44519, Egypt. ³Higher Institute of Computer Science and Information Technology, El Shorouk Academy, Shorouk 11837, Egypt. ⁴Department of Management, College of Business Administration in Yanbu, Taibah University, Yanbu Governorate, Saudi Arabia. ⁵Department of Mathematics, College of Science & Arts, King Abdulaziz University, P.O. Box 344, 21911 Rabigh, Saudi Arabia. ⁶Department of Management Information Systems, College of Business Administration in Yanbu, Taibah University, Yanbu Governorate, Saudi Arabia. ✉email: ahmedzafify87@gmail.com

and the Kappa distribution¹⁵, both of which have been applied to streamflow and precipitation data. Additional details about the LL model can be found in¹⁶.

The LL model can be understood as the probability model for a random variable whose logarithm follows a logistic distribution. It serves as an alternative to the log-normal distribution, as it features a hazard rate (HR) function that initially increases before subsequently decreasing.

The LL distribution has been widely applied in survival analysis and reliability engineering due to its mathematical simplicity and closed-form expressions for key functions. However, it suffers from limited flexibility in capturing diverse failure rate shapes, particularly those observed in real-life data such as bathtub-shaped or unimodal HR shapes. These limitations restrict its applicability in complex failure-time scenarios.

In recent years, several researchers have introduced different generalized forms of the LL distribution to enhance its capability and flexibility in modeling time-to-event data. Some notable improvements to the LL model include the alpha-power transformed-LL¹⁷, transmuted LL (TLL)¹⁸, Marshall–Olkin LL (MOLL)¹⁹, McDonald-LL (McLL)²⁰, Kumaraswamy Marshall–Olkin LL (KMOLL)²¹, additive Weibull LL (AWLL)²², extended Weibull LL²³, extended-LL (ExLL)²⁴, Zografos–Balakrishnan LL²⁵, odd Lomax LL²⁶, extended Poisson LL²⁷, extended odd Weibull LL²⁸, skew LL²⁹, cubic transmuted LL³⁰, LL Ailamujia³¹, and generalized Kavya–Manoharan LL³² distributions.

Although numerous generalizations of the LL distribution have been proposed in recent decades—such as the KLL, MOLL, KMOLL, McLL, BLL, AWLL, ExLL, and others—many of these models suffer from limited shape control, parameter redundancy, or complex forms that hinder interpretability or estimation. Most notably, while prior models aim to accommodate various HR shapes, few offer a unified and analytically tractable framework that simultaneously supports a wide spectrum of real-world behaviors, including heavy tails, multimodality, and diverse failure rate patterns (e.g., increasing-decreasing-increasing or inverted bathtub). To address these gaps and the limitations of the traditional LL model, we propose the Kumaraswamy alpha power log-logistic (KAPLL) distribution as a novel and flexible extension of the LL model. The KAPLL distribution introduces a synergistic blend of the Kumaraswamy and alpha power transformations into the LL baseline, allowing for superior control over skewness, tail behavior, and HR dynamics. Unlike existing counterparts, the KAPLL model can produce monotonic (increasing or decreasing), non-monotonic (unimodal, bathtub), and complex hybrid shapes within a concise parameter space. Furthermore, it retains analytical tractability for key functions such as the cumulative distribution function (CDF), probability density function (PDF), and HR function, enabling easier application in real-life survival and reliability analyses. Moreover, in many practical applications, these models show poor fitting when the data exhibit heavy tails or multimodality. On the other hand, the five-parameter KAPLL model that accommodates a wide spectrum of HR shapes, while retaining tractable mathematical properties, making it both theoretically sound and practically applicable. These advantages position the KAPLL distribution as a significant advancement over earlier LL-based distributions, filling crucial gaps in flexibility, interpretability, and practical applicability.

The proposed KAPLL distribution is constructed by applying the KAP transformation introduced by⁹ to the LL distribution. The KAP transformation introduces boundedness, enhances control over skewness, and allows for more flexible tail behavior and hazard rate shapes. These features collectively enable the KAPLL distribution to capture a wider range of failure rate patterns and improve goodness-of-fit in practical applications. The development of the KAPLL model is thus motivated by the need for increased adaptability in modeling time-to-event data across various domains.

Other motivations for introducing the KAPLL distribution include the following: (i) The KAPLL model can effectively represent various HR function (HRF) shapes, including increasing, J-shaped, decreasing, reversed J-shaped, bathtub, modified bathtub, decreasing-increasing-decreasing, and unimodal forms; (ii) The KAPLL distribution encompasses several known lifetime submodels, as outlined in Table 2; (iii) It is particularly suitable for modeling skewed real-life data that may not be adequately represented by other established distributions; (iv) The KAPLL distribution has applications across various fields, including survival analysis, public health, industrial reliability, biomedical studies, and engineering; and (v) Empirical results demonstrate that the KAPLL distribution outperforms many well-known LL distributions in the context of five real-life data examples.

In addition to the theoretical development and practical applications of the KAPLL model, this study provides a comprehensive comparison of eight classical and modern estimation methods for estimating its parameters. Through an extensive Monte Carlo simulation study, the performance of each method is rigorously evaluated under varying sample sizes and parameter settings. This analysis not only highlights the robustness and limitations of each estimator but also serves as a valuable guideline for engineers and applied statisticians when selecting the most appropriate estimation technique for real-world data. By offering practical insights into estimator behavior, this work bridges the gap between theoretical modeling and applied implementation, reinforcing the KAPLL model's utility in diverse reliability and survival analysis contexts.

The rest of the paper is organized into seven sections. The KAPLL distribution is investigated in Section “The KAPLL distribution”. In Section “Properties of the KAPLL distribution”, some key properties of the KAPLL distribution are explored. Inference about the KAPLL parameters is presented in Section “Estimation methods”. Section “Applications to real-world data” provides simulation studies. In Section “Comparative evaluation and discussion”, we present five real-world data applications. Section “Conclusions and future perspectives” gives some conclusions.

The KAPLL distribution

The KAPLL model and its special cases are presented in this section. The CDF of the two-parameter LL model has the form

$$G(x) = \left(1 + \frac{\lambda}{x^\beta}\right)^{-1}, \quad x > 0, \lambda, \beta > 0, \quad (1)$$

where λ and β are the scale and shape parameters, respectively.

The LL PDF reduces to

$$g(x) = \lambda\beta x^{-\beta-1} \left(1 + \frac{\lambda}{x^\beta}\right)^{-2}. \quad (2)$$

The KAPLL distribution is constructed based on the KAP-G family, which is specified by the CDF

$$F(x) = \begin{cases} 1 - \left\{1 - \left[\frac{\alpha^{G(x)} - 1}{\alpha - 1}\right]^a\right\}^b & \text{if } \alpha, a, b > 0, \alpha \neq 1, \\ G(x), & \text{if } \alpha = 1, \end{cases} \quad (3)$$

where α, a and b are shape parameters.

The corresponding PDF of the KAP-G class is expressed by

$$f(x) = \frac{a b \ln \alpha}{\alpha - 1} g(x) \alpha^{G(x)} \left[\frac{\alpha^{G(x)} - 1}{\alpha - 1}\right]^{a-1} \left\{1 - \left[\frac{\alpha^{G(x)} - 1}{\alpha - 1}\right]^a\right\}^{b-1}. \quad (4)$$

Table 1 provides the special sub-families of the KAP-G family. Further information about the KAP-G family can be explored in⁹.

Inserting (1) in Eq. (3), the CDF of the KAPLL distribution follows as

$$F(x) = 1 - \left[1 - \left(\frac{\alpha^{\left(1 + \frac{\lambda}{x^\beta}\right)^{-1}} - 1}{\alpha - 1}\right)^a\right]^b. \quad (5)$$

The PDF corresponding to (5) takes the form

$$f(x) = \frac{ab\lambda\beta \ln \alpha}{\alpha - 1} x^{-\beta-1} \left(1 + \frac{\lambda}{x^\beta}\right)^{-2} \alpha^{\left(1 + \frac{\lambda}{x^\beta}\right)^{-1}} \left(\frac{\alpha^{\left(1 + \frac{\lambda}{x^\beta}\right)^{-1}} - 1}{\alpha - 1}\right)^{a-1} \left[1 - \left(\frac{\alpha^{\left(1 + \frac{\lambda}{x^\beta}\right)^{-1}} - 1}{\alpha - 1}\right)^a\right]^{b-1}. \quad (6)$$

The HRF of the KAPLL distribution reduces to

$$h(x) = \frac{ab\lambda\beta \ln \alpha}{\alpha - 1} x^{-\beta-1} \left(1 + \frac{\lambda}{x^\beta}\right)^{-2} \alpha^{\left(1 + \frac{\lambda}{x^\beta}\right)^{-1}} \left(\frac{\alpha^{\left(1 + \frac{\lambda}{x^\beta}\right)^{-1}} - 1}{\alpha - 1}\right)^{a-1} \left[1 - \left(\frac{\alpha^{\left(1 + \frac{\lambda}{x^\beta}\right)^{-1}} - 1}{\alpha - 1}\right)^a\right]^{-1}.$$

Table 2 provides five important special sub-models of the new KAPLL distribution. Figures 1 and 2 illustrate the flexibility and diverse behavior of the KAPLL distribution through its PDF and HRF, respectively, for various combinations of parameter values with $\lambda = 1$. Figure 1 displays several shapes of the PDF, highlighting the ability of the KAPLL distribution to capture a wide range of distributional forms. These include both symmetric and asymmetric density profiles, as well as unimodal structure. This flexibility is essential in modeling real-world data sets that deviate from classical symmetric assumptions, enabling better fit and inferential accuracy across diverse applications.

Figure 2 presents the corresponding HRF plots under the same parameter settings, showcasing the model's capability to accommodate various failure rate behaviors. The plots include increasing, decreasing, J-shaped, bathtub-shaped, retested J-shaped, decreasing-increasing-decreasing and inverted bathtub-shaped hazard functions. These are critical features in survival and reliability analysis, where the failure rate of a system or component may not follow a monotonic trend. For example, the bathtub-shaped curve, which characterizes early failures, a stable period, and wear-out failures, is effectively captured by the KAPLL model in several of the plotted scenarios.

α	a	b	Sub-class	Authors
α	a	1	Exponentiated alpha power-G (EAP-G)	Kariuket al. ³³
α	1	1	Alpha power-G (AP-G)	Mahdavi and Kundu ³⁴
1	a	b	Kumaraswamy-G (K-G)	Cordeiro and de Castro ⁴
1	a	1	Exponentiated-G (E-G)	Gupta et al. ³⁵
1	1	1	Baseline model	–

Table 1. Special sub-classes of the KAP-G class.

λ	β	α	a	b	Sub-model	Authors
λ	β	α	a	1	EAPLL distribution	Kariuki et al. ³⁶
λ	β	1	a	b	KLL distribution	De Santana et al. ³⁷
λ	β	1	a	1	ELL distribution	Rosaiah et al. ³⁸
λ	β	α	1	1	APLL distribution	Aldahlan ¹⁷
λ	β	1	1	1	LL distribution	Fisk ¹⁰

Table 2. Sub-models of the KAPLL distribution.

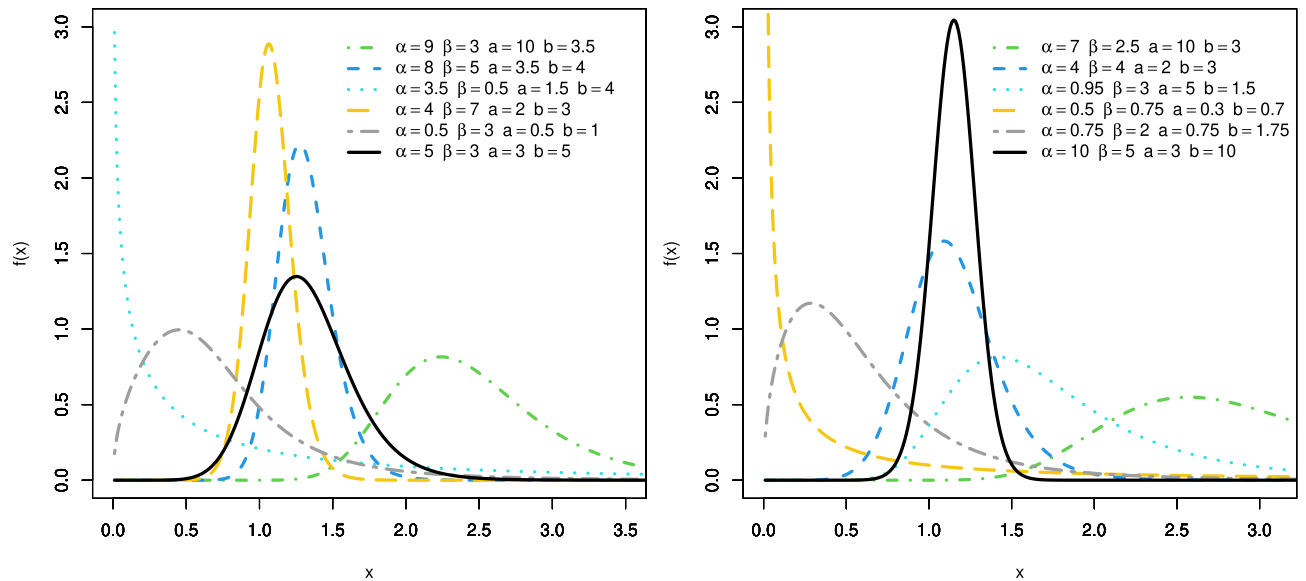


Fig. 1. Some possible density shapes of the KAPLL distribution for various parametric values with $\lambda = 1$.

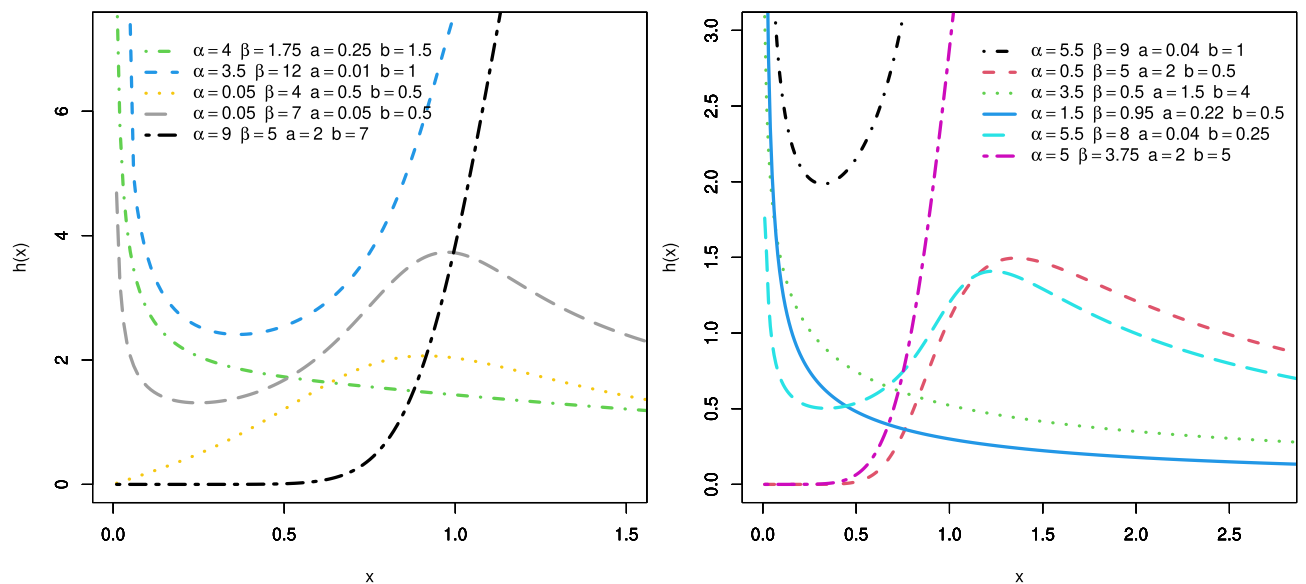


Fig. 2. Some possible failure rate shapes of the KAPLL distribution for various parametric values with $\lambda = 1$.

These visualizations support our claim that the KAPLL distribution is a highly adaptable model capable of fitting data with diverse statistical characteristics. The inclusion of these graphical representations thus reinforces the theoretical properties derived and provides intuitive understanding of the model's practical relevance.

Remark 1 A statistical model is said to be identifiable if distinct parameter values produce distinct probability distributions. For the KAPLL distribution, identifiability can be established by examining the form of the PDF, which depends on a combination of one-to-one transformations involving the parameters α , β , λ , a , and b . Specifically, the transformation $(1 + \lambda/x^\beta)^{-1}$, the exponential component $\alpha^{(1+\lambda/x^\beta)^{-1}}$, and the power transformations controlled by a and b are all one-to-one functions over their domains. Consequently, the joint behavior of these terms makes the overall PDF injective in its parameter vector, supporting theoretical identifiability.

In addition to the analytical justification, we provide empirical support. Figure 1 displays the plots of the KAPLL density for twelve different combinations of parameters. It is clearly observed that each parameter set produces a distinct shape for the density function, confirming that the model is identifiable in practice. These variations are evident in the location, scale, skewness, and tail behavior of the plotted densities.

Properties of the KAPLL distribution

This section provides some key features of the KAPLL model.

Linear representation

Mead et al.⁹ provided a useful mixture representation of the PDF of the KAP-G class. According to⁹, the KAP-G density reduces to

$$f(x) = ab \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j} (\ln \alpha)^{k+1}}{k! [a(1+i) - j]^{-k} (\alpha - 1)^{a(1+i)}} g(x) (G(x))^k \binom{b-1}{i} \binom{a(1+i)-1}{j}.$$

Using the PDF and CDF of the LL model and after some algebra, the KAPLL density takes the form

$$f(x) = ab \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j} (\ln \alpha)^{k+1} [a(1+i) - j]^k}{k! (\alpha - 1)^{a(1+i)}} \lambda \beta x^{-\beta-1} \left(1 + \frac{\lambda}{x^\beta}\right)^{-2} \times \left(1 + \frac{\lambda}{x^\beta}\right)^{-k} \binom{b-1}{i} \binom{a(1+i)-1}{j}.$$

It can also be rewritten simply as follows

$$f(x) = \sum_{k=0}^{\infty} d_k \zeta_{k+1}(x), \quad (7)$$

where $\zeta_{k+1}(x) = (k+1) g(x) (G(x))^k$ is the exponentiated-LL density with power parameter and $(k+1) > 0$, and

$$d_k = ab \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j} (\ln \alpha)^{k+1} [a(1+i) - j]^k}{(k+1)! (\alpha - 1)^{a(1+i)}} \binom{b-1}{i} \binom{a(1+i)-1}{j}.$$

Quantile function, skewness and kurtosis

The quantile function (QF) of the KAPLL distribution follows by inverting Eq. (5) as

$$Q(u) = \left[\frac{-1}{\lambda} \left(1 - \frac{\ln \alpha}{\ln(1 + \xi)} \right) \right]^{-\frac{1}{\beta}}, \quad (8)$$

where $\xi = \left\{ (\alpha - 1) \left[1 - (1 - u)^{\frac{1}{b}} \right] \right\}^{\frac{1}{a}}$, U follows the uniform (0,1) distribution. The effects of the shape parameters on the skewness (SK) and kurtosis (KU) can be studied by using the QF. The Bowley SK (BSK)³⁹ is one of the earliest SK measures, which is defined by $BSK = \frac{Q(3/4) + Q(1/4) - 2*Q(1/2)}{Q(3/4) - Q(1/4)}$. The Moors KU (MKU)⁴⁰ is defined based on octiles as follows $MKU = \frac{Q(3/8) - Q(1/8) + Q(7/8) - Q(5/8)}{Q(6/8) - Q(2/8)}$. The BSK and MKU plots of the KAPLL distribution for some selected choices of α , λ , a and b as functions of the parameter a are displayed in Fig. 3. The plots of the BSK and MKU are obtained for $\lambda = 1$, $\beta = 2$ and $b = 1.5$ and different values of a and β . The plots show that the flexible shapes of the two measures of the KAPLL distribution, which depend on the values of a and β . Furthermore, the KAPLL distribution can be used in modeling positive and negative SK as well as symmetric real-life data.

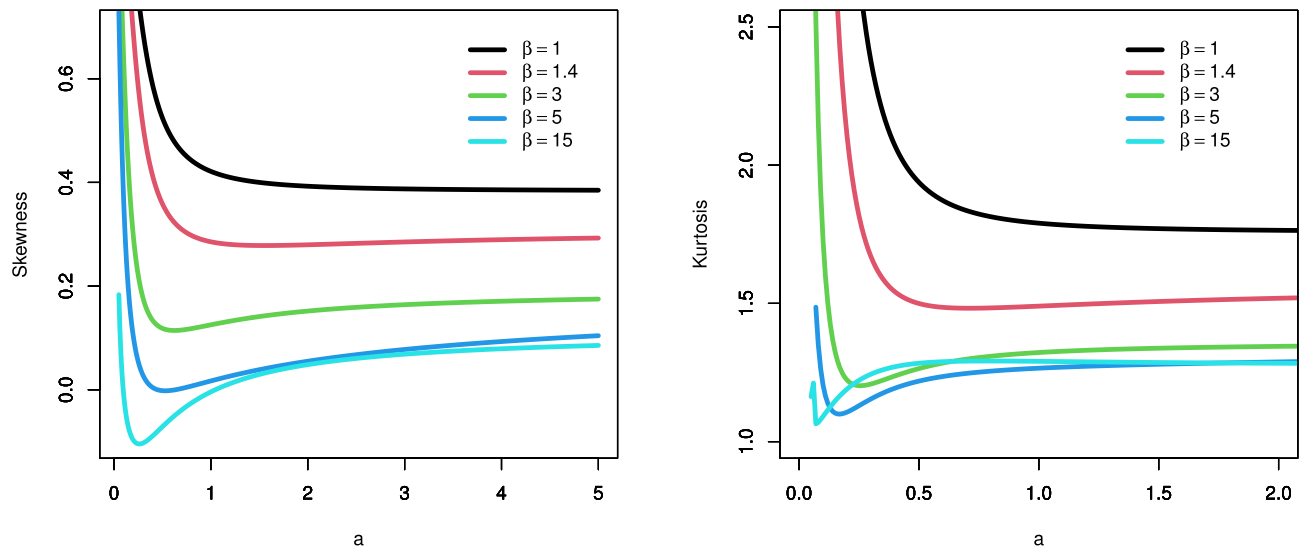


Fig. 3. The plots of BSK and MKU measures of the KAPLL distribution for some parametric values.

Some moments

The r th moment of X can be obtained from Eq. (7) as

$$\mu'_r = E(X^r) = \sum_{k=0}^{\infty} d_k \int_0^{\infty} x^r \zeta_{k+1}(x) dx = \sum_{k=0}^{\infty} d_k \lambda \beta \int_0^{\infty} x^{r-\beta-1} \left(1 + \frac{\lambda}{x^\beta}\right)^{-k-2} dx.$$

After calculating the integration, the r th moment of X follows as

$$\mu'_r = \sum_{k=0}^{\infty} \frac{d_k \lambda^{\frac{r}{\beta}} \Gamma\left(1 - \frac{r}{\beta}\right) \Gamma\left(\frac{r}{\beta} + k + 1\right)}{\Gamma(k + 2)}, \quad \frac{r}{\beta} < 1. \quad (9)$$

The mean of X , say, μ_X , follows from (9) with $r = 1$.

The r th incomplete moment of the KAPLL distribution has the form

$$I_r(t) = \int_0^t x^r f(x) dx = \sum_{k=0}^{\infty} d_k \int_0^t x^r \zeta_{k+1}(x) dx.$$

After some algebra, $I_r(t)$ reduces to

$$I_r(t) = \sum_{k=0}^{\infty} \frac{d_k \lambda \beta t^{r-\beta}}{r - \beta} {}_2F_1\left(\frac{-r}{\beta} + 1, k + 2; \frac{-r}{\beta} + 2; -\lambda t^{-\beta}\right),$$

where ${}_2F_1\left(\frac{-r}{\beta} + 1, k + 2; \frac{-r}{\beta} + 2; -\lambda t^{-\beta}\right)$ is the hyper geometric function.

The mean residual life of the KAPLL distribution at age t has the form

$$\psi(t) = \frac{1 - I_1(t)}{S(t) - t} = \sum_{k=0}^{\infty} \frac{(1 - \beta) - d_k \lambda \beta t^{1-\beta} {}_2F_1\left(\frac{-1}{\beta} + 1, k + 2; \frac{-1}{\beta} + 2; -\lambda t^{-\beta}\right)}{(1 - \beta) [(1 - \eta^a)^b - t]},$$

where $d = \left(1 + \frac{\lambda}{t^\beta}\right)^{-1}$, $\eta = (\alpha^d - 1) / (\alpha - 1)$ and $I_1(t)$ is the first incomplete moment.

The mean inactivity time of the KAPLL distribution takes the form

$$\varphi(t) = t - \frac{I_1(t)}{F(t)} = \frac{\lambda \beta t^{1-\beta} {}_2F_1\left(\frac{-1}{\beta} + 1, k + 2; \frac{-1}{\beta} + 2; -\lambda t^{-\beta}\right)}{(1 - \beta) [(1 - \eta^a)^b - 1]} + t.$$

The Lorenz (L)⁴¹, Bonferroni (B) and Zenga (Z) curves are considered the most important inequality curves and have some applications in insurance, medicine, reliability, and economics.

The L curve is defined for the KAPLL distribution as follows

$$L(p) = \frac{I_1(x_p)}{\mu'_1} = \sum_{k=0}^{\infty} \frac{\lambda^{1-\frac{1}{\beta}} \beta x_p^{r-\beta} \Gamma(k+2) {}_2F_1\left(\frac{-r}{\beta}+1, k+2; \frac{-r}{\beta}+2; -\lambda x_p^{-\beta}\right)}{(r-\beta) \Gamma\left(1-\frac{1}{\beta}\right) \Gamma\left(\frac{1}{\beta}+k+1\right)}.$$

The B and Z inequality curves can be determined, through their relationship with the L curve, by the following formulae⁴²

$$B(p) = \frac{L(p)}{p} \quad \text{and} \quad Z(p) = \frac{L(p) - p}{p[1 - L(p)]}.$$

The moment generating function (MGF) of the KAPLL model is defined by

$$M(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f(x) dx.$$

Based on Eq. (7), we can write

$$M(t) = \sum_{k=0}^{\infty} d_k \int_0^{\infty} e^{tx} \zeta_{k+1}(x) dx.$$

After some algebra, the MGF of the KAPLL follows as

$$M(t) = \sum_{k=0}^{\infty} \frac{d_k t^r \lambda^{\frac{r}{\beta}} \Gamma\left(1 - \frac{r}{\beta}\right) \Gamma\left(\frac{r}{\beta} + k + 1\right)}{k! \Gamma(k+2)}, \quad \frac{r}{\beta} < 1.$$

The values of μ_X are obtained by two ways, for some values of α , λ , β , a and b , based on the numerical integration (NI) and summation (SUM) formula. Table 3 display the values of μ_X using the NI and SUM expressions at truncated M terms, where M refers to the truncated terms from the SUM (9). Table 3 shows that the SUM in (9) converges to the NI of μ_X for all values of α , λ , β , a and b when M increases. Furthermore, the μ_X , variance (VA), SK, and KU of the KAPLL distribution are obtained numerically for some choices of α , λ , β , a and b . The values of the four measures are reported in Table 4. The numerical values in Tables 3 and 4 are computed by the R program⁴³.

Rényi and m-entropies

The entropy of a random variable X is a measure of the uncertain variation.

The Rényi entropy is defined by

$$P_X(m) = \frac{1}{1-m} \log \int_0^{\infty} f^m(x) dx, \quad m > 0, m \neq 1.$$

Using Equation (7), the Rényi entropy of the KAPLL model takes the form

$$P_X(m) = \frac{1}{1-m} \log \left[\sum_{k=0}^{\infty} \frac{-d_k \beta^{-1+m} \lambda^{\frac{1-m}{\beta}} \Gamma\left(\frac{1+\beta m[2+k]-m[1+\beta]}{\beta}\right) \Gamma\left(\frac{-1+m[1+\beta]}{\beta}\right)}{\Gamma(m(2+k))} \right]. \quad (10)$$

The m -entropy, say, $L_X(m)$, is defined by

$$L_X(m) = \frac{1}{m-1} \log \left[1 - \int_0^{\infty} f^m(x) dx \right], \quad m > 0, m \neq 1.$$

Hence, using (10), the m -entropy of the KAPLL model follows as

$$L_X(m) = \frac{1}{q-1} \log \left[1 + \sum_{k=0}^{\infty} \frac{d_k \beta^{-1+m} \lambda^{\frac{1-m}{\beta}} \Gamma\left(\frac{1+\beta m[2+k]-m[1+\beta]}{\beta}\right) \Gamma\left(\frac{-1+m[1+\beta]}{\beta}\right)}{\Gamma(m(2+k))} \right].$$

Order statistics

Let $X_1, X_2, \dots, X_n$ be a random sample of size n and let $X_{1:n}, \dots, X_{n:n}$ be their associated order statistic. Then, the PDF of the i th order statistics, say, $X_{i:n}$, which is denoted by $f_{x_{i:n}}(x)$, reduces to

α	β	a	b	M	SUM	NI
1.3	1.5	0.25	1	5	0.62757	0.85041
				10	0.74042	0.85041
				15	0.78243	0.85041
				20	0.80173	0.85041
	1.6	0.5	1.2	5	0.88404	0.93266
				10	0.96722	0.93266
				15	0.98776	0.93266
				20	0.99172	0.93266
	3	0.75	1.5	5	0.69962	0.83819
				10	0.77568	0.83819
				15	0.79124	0.83819
				20	0.74890	0.83819
1.5	1.5	0.25	1	5	0.74384	0.89073
				10	0.83357	0.89073
				15	0.84326	0.89073
				20	0.86982	0.89073
	1.6	0.5	1.2	5	1.04145	0.97440
				10	1.07580	0.97440
				15	1.06180	0.97440
				20	1.06330	0.97440
	3	0.75	1.5	5	0.79863	0.85839
				10	0.83394	0.85839
				15	0.84299	0.85839
				20	0.84833	0.85839
1.6	1.5	0.25	1	5	0.77493	0.90934
				10	0.89087	0.90934
				15	0.79373	0.90934
				20	1.09557	0.90934
	1.6	0.5	1.2	5	1.08703	0.99367
				10	1.12801	0.99367
				15	1.04761	0.99367
				20	1.19677	0.99367
	3	0.75	1.5	5	0.83086	0.86763
				10	0.85664	0.86763
				15	0.85455	0.86763
				20	0.87206	0.86763

Table 3. The values of μ_x using the SUM and NI formulae for some choices of the parameters at truncated M terms with $\lambda = 1$.

$$f_{x_{i:n}}(x) = \frac{n!f(x)}{(n-i)!(i-1)!} [F(x)]^{i-1} [1-F(x)]^{n-i}. \quad (11)$$

Substituting (5) and (6) in (11), the i th order statistic of the KAPLL distribution follows as

$$f_{x_{i:n}}(x) = \frac{ab\lambda\beta n! \ln \alpha (\alpha - 1)}{(n-i)!(i-1)!} x^{-\beta-1} d^2 \alpha^d \eta^{a-1} [1-\eta^a]^{b(n-i+1)-1} [1-[1-\eta^a]^b]^{i-1}, \quad (12)$$

where $d = (1 + \lambda x^{-\beta})^{-1}$ and $\eta = (\alpha^d - 1) / (\alpha - 1)$. After some algebra, we obtain

$$f_{x_{i:n}}(x) = \sum_{k=0}^{i-1} \sum_{j,m,s=0}^{\infty} \frac{ab(\ln \alpha)^{s+1} (-1)^{k+j+m} [a(j+1)-m]^s}{\beta_{(i,n-i+1)} s! (\alpha-1)^{a(j+1)}} \lambda \beta x^{-\beta-1} d^{s+2} \\ \times \binom{i-1}{k} \binom{b(n+k-i+1)-1}{j} \binom{a(j+1)-1}{m}.$$

The last equation can be expressed as

α	β	λ	a	b	μ_X	VA	SK	KU
1.1	4.2	0.5	0.25	1.5	0.384833	0.081047	1.231206	6.418125
				1.2	0.453045	0.115131	1.649860	13.154210
			0.3	1.5	0.435780	0.084355	1.163740	6.475173
				1.2	0.505004	0.119226	1.645997	13.915290
		1	0.25	1.5	0.453885	0.112741	1.231206	6.418125
				1.2	0.534337	0.160156	1.649860	13.154380
			0.3	1.5	0.513974	0.117343	1.163740	6.475172
				1.2	0.595619	0.165852	1.645996	13.915490
	5	0.5	0.25	1.5	0.431810	0.075244	0.845171	4.377854
				1.2	0.495444	0.099715	1.086700	6.676772
			0.3	1.5	0.482768	0.075693	0.783174	4.467333
				1.2	0.546261	0.099977	1.078139	7.082776
		1	0.25	1.5	0.496020	0.099286	0.845167	4.377850
				1.2	0.569116	0.131576	1.086697	6.676757
			0.3	1.5	0.554555	0.099877	0.783173	4.467333
				1.2	0.627489	0.131920	1.078136	7.082787
1.5	4.2	0.5	0.25	1.5	0.399599	0.087170	1.225434	6.389715
				1.2	0.470311	0.123715	1.643846	13.113560
			0.3	1.5	0.452465	0.090687	1.157994	6.449689
				1.2	0.524208	0.128063	1.640385	13.881880
		1	0.25	1.5	0.471301	0.121259	1.225433	6.389714
				1.2	0.554701	0.172095	1.643845	13.113310
			0.3	1.5	0.533653	0.126151	1.157994	6.449688
				1.2	0.618269	0.178144	1.640385	13.881940
	5	0.5	0.25	1.5	0.445712	0.079996	0.840448	4.361579
				1.2	0.511293	0.105926	1.081703	6.657154
			0.3	1.5	4.452605	0.778322	0.080437	0.498277
				1.2	0.563697	0.106159	1.073299	7.066194
		1	0.25	1.5	0.511989	0.105555	0.840448	4.361579
				1.2	0.587321	0.139770	1.081701	6.657149
			0.3	1.5	0.572369	0.106137	0.778321	4.452604
				1.2	0.647517	0.140078	1.073298	7.066195

Table 4. Findings of some measures of the KAPLL distribution for some values of its parameters.

$$f_{i:n}(x) = \sum_{s=0}^{\infty} P_s h_{s+1}(x),$$

where $h_{s+1}(x) = (s+1)g(x)[G(x)]^s$ is the exponentiated-LL density with power parameter $(s+1) > 0$, and

$$P_s = \sum_{k=0}^{i-1} \sum_{j,m=0}^{\infty} \frac{ab(\ln \alpha)^{s+1} (-1)^{k+j+m} [a(j+1)-m]^s}{\beta_{(i,n-i+1)} (s+1)! (\alpha-1)^{a(j+1)}} \binom{i-1}{k} \binom{b(n+k-i+1)-1}{j} \binom{a(j+1)-1}{m}.$$

Estimation methods

In this section, we use eight methods to estimate the KAPLL parameters namely: the maximum likelihood estimators (MLE), least-squares estimators (LSE), weighted LSE (WLSE), maximum product of spacing estimators (MPSE), percentiles-estimators (PCE), Cramér von Mises estimators (CRVME), Anderson–Darling estimators (ADE), and right tail ADE (RADE). These estimation methods have been widely used in the literature for flexible and complex distributions (see, e.g.,^{44–47}).

Let $x_1, x_2, \dots, x_n$ be a random sample from KAPLL distribution then the logarithm of the likelihood function, say, (ℓ) , becomes

$$\begin{aligned} \ell = & n [\ln a + \ln b + \ln \lambda + \ln \beta] + n \ln \left(\frac{\ln \alpha}{\alpha - 1} \right) + (\beta - 1) \sum_{i=1}^n \ln x_i - \lambda \sum_{i=1}^n x_i^\beta \\ & + \ln(\alpha) \sum_{i=1}^n d_i + (a - 1) \sum_{i=1}^n \ln(\eta_i) + (b - 1) \sum_{i=1}^n \ln(1 - \eta_i^a), \end{aligned}$$

where $d_i = (1 + \lambda x_i^{-\beta})^{-1}$ and $\eta_i = (\alpha^{d_i} - 1) / (\alpha - 1)$.

To obtain the MLE of a, b, α, λ and β , the first derivatives of ℓ are obtained with respect to the parameters. These derivatives are

$$\begin{aligned} \frac{\partial \ell}{\partial a} &= \frac{n}{a} + \sum_{i=1}^n \ln(\eta_i) - (b - 1) \sum_{i=1}^n \eta_i^a (1 - \eta_i^a)^{-1} \ln(\eta_i), \\ \frac{\partial \ell}{\partial b} &= \frac{n}{b} + \sum_{i=1}^n \ln(1 - \eta_i^a), \\ \frac{\partial \ell}{\partial \alpha} &= \frac{n [\alpha - 1 - \alpha \ln(\alpha)]}{\alpha (\alpha - 1) \ln(\alpha)} + \frac{1}{\alpha} \sum_{i=1}^n d_i + (a - 1) \sum_{i=1}^n \left[\frac{(\alpha - 1) d_i \alpha^{d_i - 1} - (\alpha^{d_i} - 1)}{(\alpha - 1) (\alpha^{d_i} - 1)} \right] \\ &\quad - a(b - 1) \sum_{i=1}^n \eta_i^{a-1} (1 - \eta_i^a)^{-1} \left[\frac{(\alpha - 1) d_i \alpha^{d_i - 1} - (\alpha^{d_i} - 1)}{(\alpha - 1)^2} \right], \\ \frac{\partial \ell}{\partial \lambda} &= \frac{n}{\lambda} - \sum_{i=1}^n x_i^\beta - \ln(\alpha) \sum_{i=1}^n \frac{d_i}{x_i^\beta} - (a - 1) \sum_{i=1}^n \frac{d_i^2 \alpha^{d_i} \ln(\alpha)}{\eta_i x_i^\beta (\alpha - 1)} + (b - 1) \\ &\quad \times \sum_{i=1}^n \frac{d_i^2 \alpha^{d_i} \eta_i^{a-1} \ln(\alpha)}{x_i^\beta (\alpha - 1) (1 - \eta_i^a)} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \ell}{\partial \beta} &= \frac{n}{\beta} + \sum_{i=1}^n \ln(x_i) - \lambda \sum_{i=1}^n x_i^\beta \ln(x_i) + \ln(\alpha) \sum_{i=1}^n \lambda d_i^2 x_i^{-\beta} \ln(x_i) \\ &\quad + (a - 1) \sum_{i=1}^n \frac{\lambda d_i^2 \alpha^{d_i} \ln(\alpha) \ln(x_i)}{x_i^\beta (\alpha^{d_i} - 1)} - a(b - 1) \sum_{i=1}^n \frac{\lambda d_i^2 \alpha^{d_i} \ln(\alpha) \ln(x_i) \eta_i^{a-1}}{x_i^\beta (\alpha - 1) (1 - \eta_i^a)}. \end{aligned}$$

The LSE and WLSE of the KAPLL parameters $\alpha, \beta, \lambda, a$ and b can be obtained by minimizing

$$O(\alpha, \beta, \lambda, a, b) = \sum_{i=1}^n \nu_i \left[F(x_{i:n}) - \frac{i}{n+1} \right]^2,$$

where $\nu_i = 1$ in case of the LSE approach and $\nu_i = (n+1)^2(n+2)/[i(n-i+1)]$ in case of the WLSE approach. Furthermore, the LSE and WLSE follow by solving the nonlinear equations

$$\sum_{i=1}^n \left[1 - [1 - (\eta_{i:n})^a]^b - \frac{i}{n+1} \right] \delta_s(x_{i:n}) = 0, \quad s = 1, 2, 3, 4, 5,$$

where

$$\delta_1(x_{i:n}) = \frac{\partial F(x_{i:n})}{\partial \alpha} = ab(\eta_{i:n})^{a-1} [1 - (\eta_{i:n})^a]^{b-1} \frac{(\alpha - 1) d_{i:n} \alpha^{d_{i:n} - 1} - (\alpha^{d_{i:n}} - 1)}{(\alpha - 1)^2}, \quad (13)$$

$$\begin{aligned} \delta_2(x_{i:n}) &= \frac{\partial F(x_{i:n})}{\partial \beta} = ab \lambda d_{i:n}^2 \alpha^{d_{i:n}} x_{i:n}^{-\beta} \ln(x_{i:n}) \ln(\alpha) (\alpha - 1)^{-1} \\ &\quad (\eta_{i:n})^{a-1} [1 - (\eta_{i:n})^a]^{b-1}, \end{aligned} \quad (14)$$

$$\begin{aligned} \delta_3(x_{i:n}) &= \frac{\partial F(x_{i:n})}{\partial \lambda} = -ab d_{i:n}^2 \alpha^{d_{i:n}} \ln(\alpha) x_{i:n}^{-\beta} (\alpha - 1)^{-1} \\ &\quad (\eta_{i:n})^{a-1} [1 - (\eta_{i:n})^a]^{b-1}, \end{aligned} \quad (15)$$

$$\delta_4(x_{i:n}) = \frac{\partial F(x_{i:n})}{\partial a} = b \ln(\eta_{i:n}) (\eta_{i:n})^a [1 - (\eta_{i:n})^a]^{b-1} \quad (16)$$

and

$$\delta_5(x_{i:n}) = \frac{\partial F(x_{i:n})}{\partial b} = -\ln(1 - (\eta_{i:n})^a) [1 - (\eta_{i:n})^a]^b, \quad (17)$$

where $d_{i:n} = (1 + \lambda x_{i:n}^{-\beta})^{-1}$ and $\eta_{i:n} = (\alpha^{d_{i:n}} - 1) / (\alpha - 1)$.

The MPSE is an alternative approach to the MLE^{48,49}. The uniform spacings of a random sample of size n from the KAPLL distribution are defined by

$$H(\alpha, \beta, \lambda, a, b) = \sum_{i=1}^{n+1} \log[D_i],$$

where $D_i = D_i(\alpha, \beta, \lambda, a, b) = F(x_{i:n}) - F(x_{i-1:n})$ be the uniform spacing for $(i = 1, \dots, n+1)$, and $F(x_{0:n}) = 0, F(x_{n+1:n}) = 1$, and $\sum_{i=1}^{n+1} D_i = 1$.

The MPSE of the KAPLL parameters are determined by solving the non-linear equations

$$\sum_{i=1}^{n+1} \frac{1}{D_i} \delta_s(x_{i:n}) - \delta_s(x_{i-1:n}) = 0, \quad s = 1, 2, 3, 4, 5,$$

where $\delta_s(x_{i:n}) = 0$ are defined in (13–17) for $s = 1, 2, 3, 4, 5$.

The CRVME can be obtained based on the difference between the estimates of the CDF and the empirical CDF. The CRVME of the KAPLL parameters minimize the following function

$$C(\alpha, \beta, \lambda, a, b) = \frac{-1}{12n} + \sum_{i=1}^n \left[F(x_{i:n}) - \frac{2i-1}{2n} \right]^2,$$

which also follow by solving

$$\sum_{i=1}^n \left[F(x_{i:n}) - \frac{2i-1}{2n} \right] \delta_s(x_{i:n}) = 0, \quad s = 1, 2, 3, 4, 5.$$

The ADE of the KAPLL parameters are obtained by minimizing

$$A(\alpha, \beta, \lambda, a, b) = -n - \frac{1}{n} + \sum_{i=1}^n (2i-1) [\log F(x_{i:n}) + \log S(x_{i:n})],$$

which can be found as solutions of the system

$$\sum_{i=1}^n (2i-1) \left[\frac{\delta_s(x_{i:n})}{F(x_i)} + \frac{\delta_i(x_{n+1-i:n})}{S(x_{n+1-i:n})} \right] = 0, \quad s = 1, 2, 3, 4, 5.$$

The RADE of the KAPLL parameters $\alpha, \beta, \lambda, a$ and b are obtained by minimizing the following function

$$R(\alpha, \beta, \lambda, a, b) = \frac{1}{n} - 2 \sum_{i=1}^n F(x_{i:n}) - \frac{1}{n} \sum_{i=1}^n (2i-1) \log S(x_{n+1-i:n}).$$

They can also be obtained from the non-linear equations

$$-2 \sum_{i=1}^n \delta_S(x_{i:n}) + \frac{1}{n} \sum_{i=1}^n (2i-1) \delta_S(x_{n+1-i:n}) / S(x_{n+1-i:n}) = 0, \quad s = 1, 2, 3, 4, 5.$$

The PCE of $\alpha, \beta, \lambda, a$ and b can be determined by minimizing the quantity

$$\sum_{i=1}^n \left(x_{i:n} - \left\{ \frac{-1}{\lambda} \left[1 - \frac{\ln \alpha}{\ln(1 + \xi_i)} \right] \right\}^{\frac{-1}{\beta}} \right)^2.$$

Numerical simulations

This section gives detailed simulation results to explore the behavior and performances of the introduced estimation methods in estimating the KAPLL parameters based on the following three measures namely: the mean square

errors (MSE), $MSE = \frac{1}{N} \sum_{i=1}^N (\hat{\theta}_i - \theta)^2$, average absolute biases ($|BIAS|$), $|BIAS| = \frac{1}{N} \sum_{i=1}^N |\hat{\theta}_i - \theta|$, and mean relative errors (MRE), $MRE = \frac{1}{N} \sum_{i=1}^N |\hat{\theta}_i - \theta|/\theta$, where $\theta = (\alpha, \beta, \lambda, a, b)^T$.

Several sample sizes and parameters combinations are considered, i.e., $n = \{50, 100, 250, 500\}$, $\theta = (0.5, 1.5, 0.5, 0.9, 1.2)^T$, $\theta = (0.5, 1.5, 0.5, 1.1, 1.2)^T$, $\theta = (0.5, 1.5, 0.5, 1.3, 1.2)^T$, $\theta = (0.5, 1.5, 0.75, 0.6, 0.3)^T$, $\theta = (0.5, 1.5, 0.75, 0.6, 0.5)^T$, $\theta = (0.5, 1.5, 0.75, 1.3, 0.3)^T$, $\theta = (0.5, 1.5, 1, 0.6, 0.3)^T$, $\theta = (0.5, 1.5, 1, 0.6, 0.5)^T$. We generated $N = 5000$ random samples based on the QF in (8) using the R software⁴³.

Tables 5, 6, 7, 8, 9, 10, 11, 12 and 13 give the findings of simulations including the $|BIAS|$, MSE, and MRE for the eight estimation approaches. The results show that the values of $|BIAS|$, MSE and MRE decrease with the increase of n , indicating that the estimators demonstrate desirable asymptotic behavior in simulation, although no formal proof of consistency is provided. Hence, the introduced estimation methods perform very well in estimating the KAPLL parameters. Generally, based on simulation results, the ordering performance of the estimators is as follow: WLSE, MPSE, ADE, RADE, LSE, CRVME, ADE, and PCE. This order confirms the superiority of the MPS estimation method with overall score of 38.

Applications to real-world data

This section demonstrates the importance and practical applicability of the proposed KAPLL distribution by analyzing five real-world datasets from reliability, engineering, biomedical, and environmental sciences. In particular, these datasets are employed as independent validation tools to rigorously assess the generalizability and empirical relevance of the proposed model beyond the controlled settings of the simulation study. Our objective is to highlight the enhanced flexibility of the KAPLL model compared to existing extensions of the LL distribution. To this end, we focused on two primary criteria for selecting these datasets: their benchmark use in the literature and their practical relevance to real-world reliability, engineering, biomedical, and environmental contexts. All five datasets have been previously analyzed in well-established studies, allowing for meaningful comparisons with existing lifetime models. Additionally, they represent diverse practical scenarios—including mechanical component failure, stress testing of materials, biomedical, and environmental analysis—where flexible distributions are essential. Furthermore, the selected datasets exhibit, as shown from Table 14, a range of skewness values—both negative and positive—highlighting the diverse distributional characteristics that the proposed KAPLL model aims to capture. These datasets were chosen specifically to highlight the modeling flexibility and adaptability of the proposed KAPLL distribution across a variety of realistic reliability applications. The datasets vary in size, with 40, 63, 119, 300, and 576 observations, respectively. This section demonstrates the importance and practical applicability of the proposed KAPLL distribution by analyzing five real-world datasets. Our objective is to showcase the enhanced flexibility of the KAPLL model compared to existing extensions of the LL distribution. To this end, we selected the datasets based on two primary criteria: (1) their frequent use as benchmarks in the literature, and (2) their practical relevance to real-world reliability, engineering, biomedical, and environmental problems. All five datasets have been previously analyzed in well-established studies, enabling meaningful comparisons with existing lifetime models. Moreover, they reflect a variety of practical scenarios—including mechanical component failure, material stress testing, and fracture toughness analysis—where flexible distributions are critical for accurate modeling.

The first dataset is addressed by⁵⁰, and it contains 40 observations of time to failure ($10^3 h$) of turbocharger of one type of engine. The data are: 3.5, 1.6, 5.4, 4.8, 6.0, 7.0, 6.5, 7.3, 8.0, 7.7, 8.4, 3.9, 2.0, 5.0, 6.1, 5.6, 8.1, 6.5, 7.3, 7.1, 7.8, 2.6, 8.4, 4.5, 5.8, 5.1, 6.3, 7.3, 6.7, 7.7, 8.3, 7.9, 8.5, 4.6, 3.0, 5.3, 8.7, 6.0, 9.0, 8.8.

The second dataset is studied by⁵¹, and it consists of 63 observations of strengths of 1.5 cm glass fibers. The data are: 0.93, 0.55, 1.25, 1.49, 1.36, 1.52, 1.61, 1.58, 1.64, 1.73, 1.68, 1.81, 0.74, 2, 1.04, 1.39, 1.27, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.51, 1.54, 1.60, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.50, 1.55, 1.61, 1.62, 1.66, 1.70, 1.77, 1.84, 0.84, 1.24, 1.30, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.7, 1.78, 1.89.

The third dataset contains 119 observations of fracture toughness MPa m^{1/2} data from the material Alumina (Al_2O_3)⁵². The data are: 5.5, 5, 4.9, 6.4, 5.1, 5.2, 5.2, 5, 4.7, 4, 4.5, 4.2, 4.1, 4.56, 5.01, 4.7, 3.13, 3.12, 2.68, 2.77, 2.7, 2.36, 4.38, 5.73, 4.35, 6.81, 1.91, 2.66, 2.61, 1.68, 2.04, 2.08, 2.13, 3.8, 3.73, 3.71, 3.28, 3.9, 4, 3.8, 4.1, 3.9, 4.05, 4, 3.95, 4, 4.5, 4.5, 4.2, 4.55, 4.65, 4.1, 4.25, 4.3, 4.5, 4.7, 5.15, 4.3, 4.5, 4.9, 5, 5.35, 5.15, 5.25, 5.8, 5.85, 5.9, 5.75, 6.25, 6.05, 5.9, 3.6, 4.1, 4.5, 5.3, 4.85, 5.3, 5.45, 5.1, 5.3, 5.2, 5.3, 5.25, 4.75, 4.5, 4.2, 4, 4.15, 4.25, 4.3, 3.75, 3.95, 3.51, 4.13, 5.4, 5, 2.1, 4.6, 3.2, 2.5, 4.1, 3.5, 3.2, 3.3, 4.6, 4.3, 4.3, 4.5, 5.5, 4.6, 4.9, 4.3, 3, 3.4, 3.7, 4.4, 4.9, 4.9, 5.

The fourth dataset comprised 300 adult Egyptian individuals with documented age (ranging from 18 to 85 years) and sex, equally distributed between males and females. All participants underwent high-resolution three-dimensional multidetector computed tomography (MDCT) imaging of the neck at the Radiodiagnosis Department, Ain Shams University Hospital, Egypt, for clinical indications unrelated to neck disorders, and were classified as Group I. Individuals in the preadolescent age group were excluded, as skeletal structures at this stage are not reliable for sex determination due to the absence of secondary sexual characteristics, which only emerge following hormonally driven bone remodeling during puberty. This dataset was studied by⁵³. The data are: 33.7, 36, 38.3, 33.7, 32, 34.4, 38.5, 38.2, 33.2, 32.3, 32.5, 30.5, 32.7, 38.2, 28.4, 29.5, 32, 37, 32, 31.4, 27.7, 30.9, 33.5, 36.3, 29.5, 31.2, 31.4, 33.6, 33, 33.7, 36.8, 28.9, 34.7, 31.9, 32.8, 28.3, 38.1, 36.2, 36.9, 32.5, 37.7, 41, 31.9, 36.3, 40.7, 38.7, 35.5, 36.1, 36.9, 35.9, 35.6, 33.2, 34.8, 31.7, 39.5, 39.9, 34.7, 35.9, 34.3, 39.7, 36, 38.5, 29.9, 35.4, 36.4, 35, 33.5, 36.6, 35, 30.1, 31, 36.4, 34.7, 33.9, 36.1, 39.7, 38.9, 36.6, 32.6, 39.3, 33.8, 34.2, 32.5, 38.8, 36, 36, 35, 37.8, 41.3, 32.4, 38.1, 37.6, 36.3, 36.8, 36.5, 38, 33.4, 36, 43.8, 41.1, 38, 36.7, 35.9, 31.3, 33.5, 37, 33, 33.8, 35.8, 40.7, 40.9, 36.7, 37.4, 33.4, 31.6, 37.5, 36.5, 39.2, 31.1, 35.9, 29.5, 35.7, 37.5, 37.4, 37.5, 37.4, 40, 42.3, 32.5, 35.5, 45.1, 39, 32.6, 33, 41.7, 38.5, 37.3, 34.4, 37.8, 31.7, 37.9, 34.3, 32.2, 34.2, 35.1, 44.6, 33.4, 30.3, 30.5, 29.4, 30.9, 31.7, 28.5, 32.6, 32.6, 33.2, 29.1, 32.1, 26.5, 29.4, 35.3, 29.3, 32.3, 29.4, 33.3, 30.4, 27.9, 25.7, 32.3, 30.4, 27.6, 24.3, 30.8, 29.7, 24.2, 25, 29.7, 30.3, 28.8, 31, 25.9, 30.1, 26.1, 27.3, 27.2, 28.6, 34.6, 27.8, 26.5, 26.8, 29.3, 31.6, 30.1, 30.8, 29.2, 30.9, 30,

<i>n</i>	Est.	Param.	MLE	LSE	WLSE	CRVME	MPSE	PCE	ADE	RADE
50	[BIAS]	$\hat{\alpha}$	13.79918 ^{7}	1.66602 ^{2}	1.39576 ^{1}	2.64101 ^{4}	2.25594 ^{3}	19.15002 ^{8}	2.89846 ^{5}	3.49067 ^{6}
		$\hat{\beta}$	1.00666 ^{5}	1.04565 ^{8}	0.74098 ^{2}	1.03743 ^{6}	1.04350 ^{7}	0.68923 ^{1}	0.98406 ^{4}	0.94772 ^{3}
		$\hat{\lambda}$	5.35467 ^{7}	1.76256 ^{4}	1.05001 ^{1}	2.55171 ^{5}	3.17232 ^{6}	7.02271 ^{8}	1.54795 ^{2}	1.66010 ^{3}
		$\hat{a}$	3.72812 ^{8}	1.03218 ^{3}	0.91647 ^{2}	1.21232 ^{5}	1.49265 ^{7}	0.78424 ^{1}	1.29096 ^{6}	1.17564 ^{4}
		$\hat{b}$	18.45088 ^{8}	2.90396 ^{2}	2.45908 ^{1}	4.55886 ^{5}	4.06692 ^{3}	14.23576 ^{7}	4.38515 ^{4}	4.60979 ^{6}
	MSE	$\hat{\alpha}$	4496.01494 ^{8}	150.92487 ^{2}	54.10721 ^{1}	375.43290 ^{5}	505.07704 ^{6}	4032.36281 ^{7}	285.33144 ^{3}	304.83374 ^{4}
		$\hat{\beta}$	1.66459 ^{5}	1.78519 ^{6}	1.00143 ^{2}	1.84526 ^{7}	2.32696 ^{8}	0.77654 ^{1}	1.65078 ^{4}	1.56039 ^{3}
		$\hat{\lambda}$	994.95771 ^{7}	247.97180 ^{4}	83.23976 ^{2}	569.31107 ^{5}	1382.89690 ^{8}	598.79310 ^{6}	156.68726 ^{3}	57.78546 ^{1}
		$\hat{a}$	38.64495 ^{8}	2.59955 ^{3}	2.75564 ^{1}	3.55463 ^{6}	11.03842 ^{5}	1.89334 ^{7}	4.64761 ^{4}	3.40989 ^{2}
		$\hat{b}$	4133.99120 ^{7}	310.23642 ^{2}	127.70231 ^{1}	1122.53557 ^{4}	664.56571 ^{3}	1559.78305 ^{8}	404.97672 ^{5}	222.30922 ^{6}
	MRE	$\hat{\alpha}$	27.59835 ^{7}	3.33204 ^{2}	2.79152 ^{1}	5.28202 ^{4}	4.51187 ^{3}	38.30004 ^{8}	5.79692 ^{5}	6.98134 ^{6}
		$\hat{\beta}$	0.67110 ^{5}	0.69710 ^{8}	0.49399 ^{2}	0.69162 ^{6}	0.69567 ^{7}	0.45948 ^{1}	0.65604 ^{4}	0.63181 ^{3}
		$\hat{\lambda}$	10.70934 ^{7}	3.52511 ^{4}	2.10002 ^{1}	5.10341 ^{5}	6.34464 ^{6}	14.04542 ^{8}	3.09589 ^{2}	3.32020 ^{3}
		$\hat{a}$	4.14235 ^{8}	1.14686 ^{3}	1.01830 ^{2}	1.34702 ^{5}	1.65850 ^{7}	0.87138 ^{1}	1.43440 ^{6}	1.30627 ^{4}
		$\hat{b}$	15.37573 ^{8}	2.41997 ^{2}	2.04924 ^{1}	3.79905 ^{5}	3.38910 ^{3}	11.86313 ^{7}	3.65429 ^{4}	3.84149 ^{6}
		$\sum Ranks$	106 ^{8}	55 ^{2}	23 ^{1}	78 ^{6}	86 ^{7}	72 ^{5}	62 ^{4}	58 ^{3}
100	[BIAS]	$\hat{\alpha}$	5.05188 ^{7}	0.86692 ^{5}	0.58358 ^{1}	1.07198 ^{6}	0.58784 ^{2}	8.17783 ^{8}	0.77034 ^{3}	0.80811 ^{4}
		$\hat{\beta}$	0.77895 ^{6}	0.85800 ^{8}	0.47962 ^{1}	0.84831 ^{7}	0.70582 ^{3}	0.65163 ^{2}	0.73659 ^{4}	0.75226 ^{5}
		$\hat{\lambda}$	3.37272 ^{7}	0.57680 ^{3}	0.25796 ^{1}	0.88317 ^{6}	0.59993 ^{4}	8.11096 ^{8}	0.37768 ^{2}	0.61071 ^{5}
		$\hat{a}$	2.27357 ^{8}	0.90070 ^{4}	0.55609 ^{1}	1.08044 ^{7}	0.81232 ^{2}	0.89265 ^{3}	0.91134 ^{5}	0.95876 ^{6}
		$\hat{b}$	9.23428 ^{7}	1.68772 ^{3}	1.01127 ^{1}	2.35051 ^{6}	1.68871 ^{4}	9.70375 ^{8}	1.62873 ^{2}	1.80643 ^{5}
	MSE	$\hat{\alpha}$	580.86187 ^{7}	32.98763 ^{5}	2.15039 ^{1}	46.56683 ^{6}	18.61853 ^{4}	975.35707 ^{8}	2.77178 ^{2}	3.94429 ^{3}
		$\hat{\beta}$	0.94633 ^{4}	1.12881 ^{8}	0.43502 ^{1}	1.10448 ^{6}	1.11399 ^{7}	0.61952 ^{2}	0.85686 ^{3}	0.96658 ^{5}
		$\hat{\lambda}$	1476.04710 ^{7}	4.69822 ^{3}	0.47992 ^{1}	43.22758 ^{5}	47.69396 ^{6}	2540.74150 ^{8}	0.83818 ^{2}	5.83434 ^{4}
		$\hat{a}$	16.75731 ^{8}	1.92415 ^{2}	1.02403 ^{1}	2.89343 ^{6}	3.07291 ^{7}	2.17721 ^{3}	2.35934 ^{5}	2.31088 ^{4}
		$\hat{b}$	2074.14772 ^{8}	16.33372 ^{3}	4.18200 ^{1}	132.32043 ^{6}	38.67804 ^{5}	1769.44317 ^{7}	10.21161 ^{2}	17.03038 ^{4}
	MRE	$\hat{\alpha}$	10.10375 ^{7}	1.73384 ^{5}	1.16717 ^{1}	2.14395 ^{6}	1.17567 ^{2}	16.35567 ^{8}	1.54067 ^{3}	1.61623 ^{4}
		$\hat{\beta}$	0.51930 ^{6}	0.57200 ^{8}	0.31975 ^{1}	0.56554 ^{7}	0.47055 ^{3}	0.43442 ^{2}	0.49106 ^{4}	0.50151 ^{5}
		$\hat{\lambda}$	6.74544 ^{7}	1.15359 ^{3}	0.51592 ^{1}	1.76634 ^{6}	1.19987 ^{4}	16.22193 ^{8}	0.75537 ^{2}	1.22141 ^{5}
		$\hat{a}$	2.52619 ^{8}	1.00078 ^{4}	0.61788 ^{1}	1.20049 ^{7}	0.90257 ^{2}	0.99184 ^{3}	1.01260 ^{5}	1.06528 ^{6}
		$\hat{b}$	7.69523 ^{7}	1.40644 ^{3}	0.84273 ^{1}	1.95876 ^{6}	1.40726 ^{4}	8.08646 ^{8}	1.35727 ^{2}	1.50536 ^{5}
		$\sum Ranks$	104 ^{8}	67 ^{4}	15 ^{1}	93 ^{7}	59 ^{3}	86 ^{6}	46 ^{2}	70 ^{5}
Continued										

<i>n</i>	Est.	Param.	MLE	LSE	WLSE	CRVME	MPSE	PCE	ADE	RADE
250	BIAS	$\hat{\alpha}$	1.26404 {7}	0.52345 {3}	0.33111 {2}	0.52551 {4}	0.26806 {1}	2.83250 {8}	0.67427 {6}	0.66050 {5}
		$\hat{\beta}$	0.48791 {3}	0.61734 {8}	0.29139 {1}	0.61173 {6}	0.40703 {2}	0.61572 {7}	0.51946 {4}	0.53884 {5}
		$\hat{\lambda}$	0.29854 {5}	0.31398 {7}	0.14820 {1}	0.30620 {6}	0.19171 {2}	2.91575 {8}	0.24833 {3}	0.25654 {4}
		$\hat{a}$	0.91192 {8}	0.69673 {4}	0.33248 {1}	0.74385 {5}	0.42524 {2}	0.90212 {7}	0.66131 {3}	0.79576 {6}
		$\hat{b}$	1.85281 {7}	1.22654 {5}	0.55765 {1}	1.29630 {6}	0.82871 {2}	3.92978 {8}	1.13651 {3}	1.14696 {4}
	MSE	$\hat{\alpha}$	10.97826 {7}	1.54446 {5}	0.46527 {2}	0.74536 {3}	0.22027 {1}	301.36595 {8}	1.63710 {6}	1.53825 {4}
		$\hat{\beta}$	0.38304 {3}	0.55064 {6}	0.16251 {1}	0.55696 {7}	0.37921 {2}	0.55759 {8}	0.44808 {4}	0.46189 {5}
		$\hat{\lambda}$	0.20222 {5}	0.30962 {7}	0.05199 {1}	0.24957 {6}	0.10480 {2}	421.23930 {8}	0.11836 {3}	0.12082 {4}
		$\hat{a}$	3.14928 {8}	1.17028 {3}	0.42246 {1}	1.31932 {4}	0.82446 {2}	2.37599 {7}	1.32609 {5}	1.59916 {6}
		$\hat{b}$	18.44476 {7}	3.96596 {3}	1.36041 {1}	4.12561 {4}	6.10806 {6}	451.58806 {8}	4.21853 {5}	3.49327 {2}
	MRE	$\hat{\alpha}$	2.52808 {7}	1.04689 {3}	0.66222 {2}	1.05103 {4}	0.31685 {1}	5.66500 {8}	1.34854 {6}	1.32100 {5}
		$\hat{\beta}$	0.32527 {3}	0.41156 {8}	0.19426 {2}	0.40782 {6}	0.15024 {1}	0.41048 {7}	0.34631 {4}	0.35923 {5}
		$\hat{\lambda}$	0.59709 {5}	0.62796 {7}	0.29641 {2}	0.61240 {6}	0.19880 {1}	5.83151 {8}	0.49666 {3}	0.51309 {4}
		$\hat{a}$	1.01324 {8}	0.77414 {4}	0.36942 {1}	0.82650 {5}	0.47249 {2}	1.00236 {7}	0.73479 {3}	0.88418 {6}
		$\hat{b}$	1.54400 {7}	1.02212 {5}	0.46470 {1}	1.08025 {6}	0.69059 {2}	3.27481 {8}	0.94710 {3}	0.95580 {4}
		$\sum Ranks$	90 {7}	78 {5.5}	20 {1}	78 {5.5}	29 {2}	115 {8}	61 {3}	69 {4}
500	BIAS	$\hat{\alpha}$	0.67606 {7}	0.46508 {3}	0.22501 {2}	0.48212 {5}	0.15842 {1}	8.00000 {8}	0.50971 {6}	0.46621 {4}
		$\hat{\beta}$	0.34748 {3}	0.46420 {6}	0.21281 {1}	0.46803 {7}	0.22536 {2}	0.58812 {8}	0.36087 {4}	0.39258 {5}
		$\hat{\lambda}$	0.28825 {7}	0.21510 {6}	0.11010 {2}	0.21209 {5}	0.09940 {1}	1.93546 {8}	0.20566 {3}	0.21114 {4}
		$\hat{a}$	0.47357 {4}	0.55642 {7}	0.22413 {2}	0.55128 {6}	0.22366 {1}	0.99824 {8}	0.40954 {3}	0.53338 {5}
		$\hat{b}$	0.85521 {5}	0.94331 {6}	0.35714 {2}	1.00397 {7}	0.35428 {1}	2.40966 {8}	0.71385 {3}	0.74353 {4}
	MSE	$\hat{\alpha}$	3.38281 {7}	0.54509 {3}	0.20604 {2}	0.63743 {4}	0.08541 {1}	34.31505 {8}	0.85354 {6}	0.67213 {5}
		$\hat{\beta}$	0.19538 {3}	0.32266 {6}	0.09283 {1}	0.32385 {7}	0.12981 {2}	0.49723 {8}	0.21925 {4}	0.24787 {5}
		$\hat{\lambda}$	0.18995 {7}	0.09508 {6}	0.02531 {2}	0.09490 {5}	0.02384 {1}	621.04050 {8}	0.07490 {3}	0.07848 {4}
		$\hat{a}$	0.90991 {7}	0.74891 {5}	0.16155 {1}	0.69816 {4}	0.19002 {2}	2.50944 {8}	0.50329 {3}	0.76939 {6}
		$\hat{b}$	5.06531 {7}	2.27457 {5}	0.53478 {1}	2.63117 {6}	0.84308 {2}	354.43454 {8}	1.77273 {4}	1.57918 {3}
	MRE	$\hat{\alpha}$	1.35211 {7}	0.93015 {3}	0.45002 {2}	0.96423 {5}	0.31685 {1}	2.48736 {8}	1.01943 {6}	0.93241 {4}
		$\hat{\beta}$	0.23165 {3}	0.30947 {6}	0.14187 {1}	0.31202 {7}	0.15024 {2}	0.39208 {8}	0.24058 {4}	0.26172 {5}
		$\hat{\lambda}$	0.57649 {7}	0.43021 {6}	0.22019 {2}	0.42418 {5}	0.19880 {1}	3.87092 {8}	0.41132 {3}	0.42229 {4}
		$\hat{a}$	0.52619 {4}	0.61824 {7}	0.24904 {2}	0.61253 {6}	0.24851 {1}	1.10916 {8}	0.45504 {3}	0.59264 {5}
		$\hat{b}$	0.71267 {5}	0.78609 {6}	0.29762 {2}	0.83664 {7}	0.29523 {1}	2.00805 {8}	0.59488 {3}	0.61961 {4}
		$\sum Ranks$	83 {6}	81 {5}	25 {2}	86 {7}	20 {1}	120 {8}	58 {3}	67 {4}

Table 5. Numerical values of the KAPLL distribution for $\theta = (0.5, 1.5, 0.5, 0.9, 1.2)^T$.

33.2, 30.5, 33.1, 35.8, 30.2, 29.7, 26.8, 29.5, 32.8, 30.3, 34.2, 27, 30.3, 36.3, 33.5, 31.7, 31.1, 31.5, 26.6, 31.3, 31.2, 31.2, 30.3, 26.2, 29.4, 34.5, 29.5, 28.6, 34.2, 40.5, 34.1, 36.6, 35.2, 36.2, 30.7, 38.3, 34, 38.8, 29.7, 31.2, 26.1, 33.3, 30.5, 28.9, 26.2, 30, 34.3, 25.9, 27.8, 28.5, 34.8, 26.5, 24.4, 28.5, 28.9, 30.9, 24.6, 30.5, 31.2, 33.8, 29.5, 34.8, 26.7, 29.5, 33, 33.4, 26.3, 26.1, 35, 31.9, 34.4, 26.8, 31, 29.5, 25.4, 30.8, 29.3, 24.9, 29.2, 33.9, 33.3, 26.6, 23.3, 37.4, 31, 31, 27.2, 31.9, 29, 40.1, 31.2, 32.9, 29, 29.9, 32.9, 26.9, 33.6, 29.5, 26.4, 28.9, 32.7, 38, 34.3

The fifth dataset represents wind speed data extracted from a comprehensive meteorological dataset recorded at 15-minute intervals by the USDA-ARS Conservation and Production Laboratory (CPRL), Soil and Water Management Research Unit (SWMRU) research weather station located in Bushland, Texas. Although the original dataset includes several meteorological variables, only the wind speed (m/s) variable is analyzed in this study. A subsample covering six consecutive days in early 2019 was selected, resulting in a total of 576 observations. The measurements were collected under standardized reference conditions using calibrated sensors installed at a height of 2 meters, with quality control and gap-filling procedures applied as described in the original data source. The dataset is publicly available at: <https://doi.org/10.15482/USDA.ADC/1526433>. The data are: 10.590, 10.560, 10.800, 10.190, 9.570, 9.100, 9.920, 10.010, 9.820, 9.330, 9.290, 8.740, 9.320, 8.930, 8.000, 8.720, 9.010, 9.640, 9.170, 9.500, 9.700, 9.370, 9.520, 9.130, 9.030, 9.490, 8.890, 7.860, 7.790, 8.330, 7.010, 7.120, 6.370, 5.890, 7.230, 7.030, 6.420, 6.600, 7.000, 6.850, 6.290, 6.790, 6.070, 5.910, 5.620, 5.620, 5.740, 5.150, 5.340, 4.880, 4.690, 4.590, 4.750, 4.780, 4.530, 4.310, 4.200, 4.050, 3.790, 3.840, 3.380, 3.670, 3.250, 2.850, 2.820, 2.680,

<i>n</i>	Est.	Param.	MLE	LSE	WLSE	CRVME	MPSE	PCE	ADE	RADE
50	BIAS	$\hat{\alpha}$	7.33456 ^{7}	1.38926 ^{2}	1.35976 ^{1}	1.65595 ^{3}	1.80146 ^{5}	12.65589 ^{8}	1.75620 ^{4}	1.99641 ^{6}
		$\hat{\beta}$	1.04280 ^{7}	1.00917 ^{5}	0.74365 ^{2}	0.97049 ^{4}	1.08162 ^{8}	0.70165 ^{1}	1.03252 ^{6}	0.96964 ^{3}
		$\hat{\lambda}$	3.61486 ^{7}	0.97180 ^{3}	0.43525 ^{1}	0.99036 ^{4}	2.51147 ^{6}	6.45657 ^{8}	0.67095 ^{2}	1.35638 ^{5}
		$\hat{a}$	4.69408 ^{8}	1.13577 ^{2}	1.17080 ^{3}	1.29105 ^{4}	1.66163 ^{7}	0.89241 ^{1}	1.51592 ^{6}	1.30182 ^{5}
		$\hat{b}$	12.81277 ^{8}	2.33602 ^{2}	1.74975 ^{1}	2.65505 ^{3}	3.31281 ^{5}	11.33044 ^{7}	2.83263 ^{4}	3.54498 ^{6}
	MSE	$\hat{\alpha}$	1980.28316 ^{7}	36.56780 ^{1}	104.03092 ^{4}	93.10813 ^{3}	310.88334 ^{6}	2028.52414 ^{8}	154.01348 ^{5}	63.97997 ^{2}
		$\hat{\beta}$	1.91463 ^{6}	1.67370 ^{5}	1.01067 ^{2}	1.65201 ^{3}	2.78783 ^{8}	0.82660 ^{1}	1.92654 ^{7}	1.67047 ^{4}
		$\hat{\lambda}$	1052.96209 ^{8}	15.17457 ^{4}	2.20518 ^{1}	9.24062 ^{3}	956.98240 ^{7}	384.60779 ^{6}	6.84232 ^{2}	24.20128 ^{5}
		$\hat{a}$	57.20212 ^{8}	2.70028 ^{2}	4.02568 ^{1}	3.57443 ^{3}	12.12977 ^{6}	2.24387 ^{7}	5.68728 ^{4}	3.72829 ^{5}
		$\hat{b}$	3051.10202 ^{7}	39.12638 ^{2}	25.55038 ^{1}	43.72462 ^{3}	393.18296 ^{5}	944.54632 ^{8}	103.58815 ^{4}	110.37023 ^{6}
	MRE	$\hat{\alpha}$	14.66912 ^{7}	2.77852 ^{2}	2.71952 ^{1}	3.31190 ^{3}	3.60293 ^{5}	25.31178 ^{8}	3.51240 ^{4}	3.99281 ^{6}
		$\hat{\beta}$	0.69520 ^{7}	0.67278 ^{5}	0.49577 ^{2}	0.64699 ^{4}	0.72108 ^{8}	0.46777 ^{1}	0.68835 ^{6}	0.64643 ^{3}
		$\hat{\lambda}$	7.22972 ^{7}	1.94360 ^{3}	0.87051 ^{1}	1.98072 ^{4}	5.02294 ^{6}	12.91315 ^{8}	1.34189 ^{2}	2.71275 ^{5}
		$\hat{a}$	4.26735 ^{8}	1.03252 ^{2}	1.06436 ^{3}	1.17368 ^{4}	1.51057 ^{7}	0.81129 ^{1}	1.37810 ^{6}	1.18347 ^{5}
		$\hat{b}$	10.67731 ^{8}	1.94668 ^{2}	1.45812 ^{1}	2.21254 ^{3}	2.76067 ^{5}	9.44203 ^{7}	2.36053 ^{4}	2.95415 ^{6}
		$\sum Ranks$	111 ^{8}	42 ^{2}	29 ^{1}	51 ^{3}	96 ^{7}	73 ^{6}	68 ^{4}	70 ^{5}
100	BIAS	$\hat{\alpha}$	2.42640 ^{7}	0.55106 ^{3}	0.50483 ^{2}	0.77253 ^{6}	0.49271 ^{1}	5.12919 ^{8}	0.66276 ^{4}	0.68811 ^{5}
		$\hat{\beta}$	0.80877 ^{6}	0.87406 ^{8}	0.52866 ^{1}	0.83991 ^{7}	0.69908 ^{3}	0.66093 ^{2}	0.77118 ^{4}	0.77679 ^{5}
		$\hat{\lambda}$	1.33128 ^{7}	0.43880 ^{3}	0.22592 ^{1}	0.51526 ^{6}	0.46614 ^{4}	5.07549 ^{8}	0.31129 ^{2}	0.48645 ^{5}
		$\hat{a}$	2.77457 ^{8}	1.07371 ^{4}	0.72560 ^{1}	1.22572 ^{7}	0.97362 ^{2}	0.98439 ^{3}	1.12954 ^{6}	1.12948 ^{5}
		$\hat{b}$	5.64302 ^{7}	1.47045 ^{3}	1.00659 ^{1}	1.81188 ^{6}	1.49176 ^{4}	6.78444 ^{8}	1.44124 ^{2}	1.58196 ^{5}
	MSE	$\hat{\alpha}$	151.72662 ^{7}	1.24826 ^{2}	1.22870 ^{1}	13.83277 ^{6}	6.53826 ^{5}	418.91065 ^{8}	1.77298 ^{3}	4.15017 ^{4}
		$\hat{\beta}$	1.08838 ^{5}	1.22056 ^{8}	0.54272 ^{1}	1.10947 ^{6}	1.20524 ^{7}	0.67294 ^{2}	0.98338 ^{3}	1.08278 ^{4}
		$\hat{\lambda}$	344.87276 ^{7}	1.05437 ^{3}	0.11262 ^{1}	2.43180 ^{5}	13.43283 ^{6}	829.52179 ^{8}	0.21779 ^{2}	2.26268 ^{4}
		$\hat{a}$	22.61881 ^{8}	2.37882 ^{2}	1.59389 ^{1}	3.23610 ^{6}	4.03338 ^{7}	2.43539 ^{3}	3.15933 ^{5}	2.84484 ^{4}
		$\hat{b}$	930.33900 ^{8}	6.34267 ^{3}	3.47107 ^{1}	17.68316 ^{5}	25.63750 ^{6}	728.96137 ^{7}	5.82179 ^{2}	13.98925 ^{4}
	MRE	$\hat{\alpha}$	4.85280 ^{7}	1.10212 ^{3}	1.00967 ^{2}	1.54506 ^{6}	0.98542 ^{1}	10.25839 ^{8}	1.32551 ^{4}	1.37622 ^{5}
		$\hat{\beta}$	0.53918 ^{6}	0.58271 ^{8}	0.35244 ^{1}	0.41773 ^{7}	0.46606 ^{3}	0.44062 ^{2}	0.51412 ^{4}	0.51786 ^{5}
		$\hat{\lambda}$	2.66257 ^{7}	0.87760 ^{3}	0.45183 ^{1}	1.03053 ^{6}	0.93229 ^{4}	10.15098 ^{8}	0.62258 ^{2}	0.97290 ^{5}
		$\hat{a}$	2.52233 ^{8}	0.97610 ^{4}	0.65963 ^{1}	1.11429 ^{7}	0.88511 ^{2}	0.89490 ^{3}	1.02685 ^{6}	1.02680 ^{5}
		$\hat{b}$	4.70252 ^{7}	1.22538 ^{3}	0.83883 ^{1}	1.50990 ^{6}	1.24313 ^{4}	5.65370 ^{8}	1.20103 ^{2}	1.31830 ^{5}
		$\sum Ranks$	105 ^{8}	60 ^{4}	17 ^{1}	92 ^{7}	59 ^{3}	86 ^{6}	51 ^{2}	70 ^{5}
Continued										

n	Est.	Param.	MLE	LSE	WLSE	CRVME	MPSE	PCE	ADE	RADE
250	BIAS	$\hat{\alpha}$	0.74436 {7}	0.42494 {3}	0.27098 {2}	0.43157 {4}	0.25931 {1}	1.34578 {8}	0.48593 {6}	0.46979 {5}
		$\hat{\beta}$	0.48119 {3}	0.64179 {8}	0.29941 {1}	0.62659 {7}	0.41092 {2}	0.61360 {6}	0.53331 {4}	0.54595 {5}
		$\hat{\lambda}$	0.29436 {7}	0.23069 {3}	0.13571 {1}	0.24185 {5}	0.17811 {2}	1.48411 {8}	0.23247 {4}	0.25769 {6}
		$\hat{a}$	1.22361 {8}	0.90018 {4}	0.45350 {1}	0.94375 {5}	0.53277 {2}	0.98477 {7}	0.85536 {3}	0.96788 {6}
		$\hat{b}$	1.54671 {7}	1.13231 {5}	0.56658 {1}	1.20242 {6}	0.75828 {2}	2.18539 {8}	1.01671 {3}	1.02101 {4}
	MSE	$\hat{\alpha}$	3.33489 {7}	0.42465 {4}	0.23743 {2}	0.41778 {3}	0.19265 {1}	42.08727 {8}	0.78241 {6}	0.70966 {5}
		$\hat{\beta}$	0.37727 {2}	0.60976 {8}	0.17786 {1}	0.58674 {7}	0.42207 {3}	0.55548 {6}	0.48271 {5}	0.47553 {4}
		$\hat{\lambda}$	0.20610 {7}	0.11413 {4}	0.03550 {1}	0.14141 {6}	0.08150 {2}	78.50242 {8}	0.09260 {3}	0.12883 {5}
		$\hat{a}$	5.41023 {8}	1.73323 {3}	0.73747 {1}	1.93751 {4}	1.27584 {2}	2.70132 {7}	2.03777 {5}	2.10666 {6}
		$\hat{b}$	11.42179 {7}	3.07742 {4}	1.27643 {1}	3.39453 {5}	5.27248 {6}	74.89245 {8}	3.03455 {3}	2.57829 {2}
	MRE	$\hat{\alpha}$	1.48871 {7}	0.84987 {3}	0.54195 {2}	0.86313 {4}	0.31171 {1}	2.69155 {8}	0.97186 {6}	0.93958 {5}
		$\hat{\beta}$	0.32079 {3}	0.42786 {8}	0.19960 {2}	0.41773 {7}	0.14498 {1}	0.40906 {6}	0.35554 {4}	0.36397 {5}
		$\hat{\lambda}$	0.58873 {7}	0.46137 {3}	0.27143 {2}	0.48370 {5}	0.18713 {1}	2.96822 {8}	0.46495 {4}	0.51537 {6}
		$\hat{a}$	1.11237 {8}	0.81835 {4}	0.41228 {1}	0.85795 {5}	0.48433 {2}	0.89525 {7}	0.77760 {3}	0.87989 {6}
		$\hat{b}$	1.28893 {7}	0.94359 {5}	0.47215 {1}	1.00202 {6}	0.63190 {2}	1.82116 {8}	0.84726 {3}	0.85084 {4}
		$\sum Ranks$	95 {7}	69 {4}	20 {1}	79 {6}	30 {2}	111 {8}	62 {3}	74 {5}
500	BIAS	$\hat{\alpha}$	0.45700 {7}	0.38068 {4}	0.18916 {2}	0.42425 {6}	0.15585 {1}	8.00000 {8}	0.39933 {5}	0.34425 {3}
		$\hat{\beta}$	0.35195 {3}	0.47907 {7}	0.22324 {2}	0.47757 {6}	0.21748 {1}	0.58784 {8}	0.37349 {4}	0.40889 {5}
		$\hat{\lambda}$	0.27922 {7}	0.17447 {4}	0.10275 {2}	0.16728 {3}	0.09357 {1}	0.92382 {8}	0.18674 {5}	0.20692 {6}
		$\hat{a}$	0.65039 {4}	0.74067 {7}	0.28280 {2}	0.74066 {6}	0.27836 {1}	1.07300 {8}	0.55749 {3}	0.69247 {5}
		$\hat{b}$	0.75424 {5}	0.88013 {6}	0.34005 {1}	0.95158 {7}	0.34495 {2}	1.56713 {8}	0.68952 {3}	0.69752 {4}
	MSE	$\hat{\alpha}$	0.61419 {7}	0.33102 {4}	0.09468 {2}	0.43840 {5}	0.07462 {1}	4.95213 {8}	0.45935 {6}	0.32327 {3}
		$\hat{\beta}$	0.20023 {3}	0.34194 {7}	0.10455 {1}	0.33449 {6}	0.13594 {2}	0.50041 {8}	0.23760 {4}	0.26315 {5}
		$\hat{\lambda}$	0.17778 {7}	0.04559 {4}	0.01777 {1}	0.04449 {3}	0.02196 {2}	88.16119 {8}	0.05950 {5}	0.07808 {6}
		$\hat{a}$	1.76911 {7}	1.27425 {6}	0.24527 {1}	1.21132 {5}	0.31193 {2}	2.91211 {8}	0.90346 {3}	1.14540 {4}
		$\hat{b}$	2.89955 {7}	1.84740 {5}	0.39680 {1}	2.21361 {6}	0.90606 {2}	49.11612 {8}	1.52244 {4}	1.23049 {3}
	MRE	$\hat{\alpha}$	0.91399 {7}	0.76135 {4}	0.37831 {2}	0.84849 {6}	0.31171 {1}	1.33575 {8}	0.79866 {5}	0.68850 {3}
		$\hat{\beta}$	0.23463 {3}	0.31938 {7}	0.14883 {2}	0.31838 {6}	0.14498 {1}	0.39189 {8}	0.24900 {4}	0.27259 {5}
		$\hat{\lambda}$	0.55845 {7}	0.34894 {4}	0.20550 {2}	0.33456 {3}	0.18713 {1}	1.84763 {8}	0.37348 {5}	0.41384 {6}
		$\hat{a}$	0.59127 {4}	0.67334 {7}	0.25709 {2}	0.67333 {6}	0.25305 {1}	0.97545 {8}	0.50681 {3}	0.62952 {5}
		$\hat{b}$	0.62853 {5}	0.73344 {6}	0.28337 {1}	0.79298 {7}	0.28746 {2}	1.30594 {8}	0.57460 {3}	0.58127 {4}
		$\sum Ranks$	83 {7}	82 {6}	24 {2}	81 {5}	21 {1}	120 {8}	62 {3}	67 {4}

Table 6. Numerical values of the KAPLL distribution for $\theta = (0.5, 1.5, 0.5, 1.1, 1.2)^T$.

2.930, 2.580, 2.720, 1.950, 1.670, 1.320, 1.290, 0.720, 0.740, 0.410, 0.530, 0.850, 0.970, 1.210, 1.900, 1.600, 1.690, 1.360, 0.720, 0.960, 0.660, 0.390, 0.870, 0.990, 0.900, 1.500, 1.900, 2.170, 2.230, 2.880, 2.860, 2.910, 2.760, 2.130, 1.400, 1.400, 1.750, 1.820, 2.010, 2.370, 2.600, 2.550, 2.530, 2.250, 2.390, 2.110, 1.770, 1.790, 1.490, 1.370, 1.310, 1.930, 1.780, 1.670, 1.390, 1.070, 1.050, 1.270, 1.550, 2.390, 2.710, 2.600, 2.050, 1.840, 2.210, 2.580, 2.870, 3.360, 3.390, 3.280, 3.620, 3.300, 3.180, 3.070, 3.440, 3.000, 1.930, 0.920, 1.650, 1.890, 1.190, 1.950, 1.470, 1.400, 0.970, 1.170, 1.110, 0.810, 0.870, 0.690, 1.570, 4.050, 4.290, 4.500, 4.270, 4.290, 4.000, 3.740, 3.380, 3.200, 2.870, 1.830, 1.500, 1.470, 1.560, 2.450, 2.740, 2.510, 2.390, 2.550, 2.050, 2.490, 2.590, 2.940, 3.250, 3.170, 2.580, 2.580, 2.360, 2.780, 2.390, 2.390, 3.010, 2.910, 2.320, 1.310, 1.310, 0.610, 1.350, 1.280, 1.280, 1.200, 1.950, 1.860, 1.490, 1.060, 0.740, 1.310, 1.790, 1.850, 1.540, 1.770, 1.950, 1.970, 1.790, 1.490, 2.030, 2.490, 3.070, 2.570, 2.630, 2.840, 2.750, 2.790, 3.050, 2.250, 2.160, 2.530, 3.190, 3.700, 2.890, 2.520, 2.620, 3.350, 3.620, 2.930, 3.790, 4.000, 4.090, 4.230, 3.810, 3.980, 3.650, 3.600, 3.580, 3.710, 3.710, 4.080, 3.960, 4.160, 4.450, 4.940, 5.460, 5.680, 4.950, 5.450, 5.550, 5.070, 5.490, 5.640, 5.560, 5.890, 6.510, 6.120, 5.330, 5.780, 4.320, 2.570, 2.570, 2.630, 2.130, 0.940, 1.020, 1.640, 1.640, 1.750, 1.840, 1.570, 1.680, 1.790, 1.340, 1.340, 1.430, 1.480, 1.530, 1.620, 1.770, 1.730, 1.000, 1.530, 1.330, 1.110, 0.500, 1.130, 1.830, 1.950, 1.850, 1.490, 1.190, 2.220, 2.140, 1.830, 1.550, 1.640, 2.010, 3.440, 3.010, 3.270, 3.520, 3.260, 3.180, 3.260, 3.960, 4.230, 4.350, 4.050, 4.120, 4.360, 3.810, 3.610, 4.490, 4.910, 4.260, 3.570, 2.960, 2.930, 2.030, 2.830, 2.900, 2.280, 2.650, 2.220, 2.280, 2.450, 2.440, 1.750, 1.250, 1.550, 1.950, 2.300, 3.180, 3.030, 2.650, 2.570, 2.580, 2.180, 2.030, 1.930, 1.740, 1.950, 1.540, 1.900, 1.640, 0.990, 1.460, 1.070, 1.520, 1.400, 1.490,

<i>n</i>	Est.	Param.	MLE	LSE	WLSE	CRVME	MPSE	PCE	ADE	RADE
50	BIAS	$\hat{\alpha}$	3.93312 ^{7}	1.01714 ^{4}	0.83242 ^{1}	1.04594 ^{5}	0.86109 ^{2}	9.57785 ^{8}	1.00265 ^{3}	1.47971 ^{6}
		$\hat{\beta}$	1.05376 ^{6}	0.99863 ^{5}	0.80202 ^{2}	0.96798 ^{3}	1.08363 ^{8}	0.73295 ^{1}	1.06518 ^{7}	0.99262 ^{4}
		$\hat{\lambda}$	2.29374 ^{7}	0.79990 ^{3}	0.45629 ^{1}	0.87379 ^{4}	0.89547 ^{5}	5.78546 ^{8}	0.55845 ^{2}	1.10547 ^{6}
		$\hat{a}$	5.74078 ^{8}	1.22598 ^{2}	1.35092 ^{4}	1.33741 ^{3}	1.80774 ^{7}	1.01636 ^{1}	1.67183 ^{6}	1.39469 ^{5}
		$\hat{b}$	9.03582 ^{7}	2.13614 ^{3}	1.49371 ^{1}	2.36181 ^{5}	2.10756 ^{2}	9.88831 ^{8}	2.18338 ^{4}	2.91530 ^{6}
	MSE	$\hat{\alpha}$	455.05529 ^{7}	8.94097 ^{2}	16.38925 ^{4}	8.02337 ^{1}	20.95470 ^{5}	1474.37252 ^{8}	10.68622 ^{3}	28.19877 ^{6}
		$\hat{\beta}$	2.08534 ^{6}	1.71142 ^{4}	1.21994 ^{2}	1.68813 ^{3}	2.97266 ^{8}	0.90974 ^{1}	2.10723 ^{7}	1.82623 ^{5}
		$\hat{\lambda}$	361.13516 ^{7}	4.96876 ^{2}	7.60626 ^{4}	6.46392 ^{3}	45.53682 ^{6}	393.36590 ^{8}	3.55248 ^{1}	14.78641 ^{5}
		$\hat{a}$	83.78229 ^{8}	2.81482 ^{2}	4.69174 ^{1}	3.39328 ^{3}	12.59144 ^{6}	2.49430 ^{7}	6.07295 ^{4}	3.87087 ^{5}
		$\hat{b}$	1081.38525 ^{7}	20.79202 ^{4}	12.22512 ^{1}	25.87663 ^{5}	77.44083 ^{2}	871.60132 ^{8}	32.42482 ^{3}	73.77608 ^{6}
	MRE	$\hat{\alpha}$	7.86624 ^{7}	2.03427 ^{4}	1.66485 ^{1}	2.09187 ^{5}	1.72217 ^{2}	19.15571 ^{8}	2.00531 ^{3}	2.95943 ^{6}
		$\hat{\beta}$	0.70251 ^{6}	0.66575 ^{5}	0.53468 ^{2}	0.64532 ^{3}	0.72242 ^{8}	0.48863 ^{1}	0.71012 ^{7}	0.66174 ^{4}
		$\hat{\lambda}$	4.58748 ^{7}	1.59980 ^{3}	0.91258 ^{1}	1.74757 ^{4}	1.79095 ^{5}	11.57092 ^{8}	1.11690 ^{2}	2.21094 ^{6}
		$\hat{a}$	4.41598 ^{8}	0.94306 ^{2}	1.03917 ^{4}	1.02878 ^{3}	1.39057 ^{7}	0.78181 ^{1}	1.28602 ^{6}	1.07284 ^{5}
		$\hat{b}$	7.52985 ^{7}	1.78011 ^{3}	1.24476 ^{1}	1.96818 ^{5}	1.75630 ^{2}	8.24026 ^{8}	1.81948 ^{4}	2.42942 ^{6}
		$\sum Ranks$	106 ^{8}	46 ^{2}	34 ^{1}	53 ^{3}	80 ^{7}	77 ^{5}	65 ^{4}	79 ^{6}
100	BIAS	$\hat{\alpha}$	1.24153 ^{7}	0.50907 ^{3}	0.36319 ^{1}	0.59168 ^{5}	0.42588 ^{2}	3.56225 ^{8}	0.55255 ^{4}	0.59552 ^{6}
		$\hat{\beta}$	0.80638 ^{6}	0.89233 ^{8}	0.56662 ^{1}	0.85182 ^{7}	0.70281 ^{3}	0.67294 ^{2}	0.77663 ^{4}	0.79657 ^{5}
		$\hat{\lambda}$	0.65956 ^{7}	0.37923 ^{3}	0.22310 ^{1}	0.43336 ^{6}	0.38522 ^{4}	4.31181 ^{8}	0.31835 ^{2}	0.42725 ^{5}
		$\hat{a}$	3.32806 ^{8}	1.20256 ^{4}	0.95967 ^{1}	1.32971 ^{7}	1.16938 ^{3}	1.08864 ^{2}	1.28651 ^{6}	1.24808 ^{5}
		$\hat{b}$	3.66364 ^{7}	1.40726 ^{5}	0.98523 ^{1}	1.61606 ^{6}	1.22079 ^{2}	6.32767 ^{8}	1.28787 ^{3}	1.37300 ^{4}
	MSE	$\hat{\alpha}$	16.68662 ^{7}	1.23050 ^{4}	0.53242 ^{1}	3.04661 ^{6}	0.77863 ^{2}	217.14918 ^{8}	1.13992 ^{3}	2.36896 ^{5}
		$\hat{\beta}$	1.15750 ^{4}	1.32129 ^{8}	0.63375 ^{1}	1.19286 ^{6}	1.29367 ^{7}	0.69095 ^{2}	1.01849 ^{3}	1.16068 ^{5}
		$\hat{\lambda}$	25.40878 ^{7}	0.89816 ^{4}	0.09431 ^{1}	1.36695 ^{6}	0.84141 ^{3}	1234.52459 ^{8}	0.21901 ^{2}	0.95744 ^{5}
		$\hat{a}$	30.35583 ^{8}	2.65161 ^{2}	2.61829 ^{1}	3.31920 ^{5}	6.05689 ^{7}	2.96839 ^{3}	3.64692 ^{6}	3.06725 ^{4}
		$\hat{b}$	84.47239 ^{7}	6.46549 ^{3}	3.18586 ^{1}	10.41527 ^{6}	9.08248 ^{5}	2590.68728 ^{8}	4.39740 ^{2}	7.84346 ^{4}
	MRE	$\hat{\alpha}$	2.48307 ^{7}	1.01815 ^{3}	0.72637 ^{1}	0.65308 ^{6}	0.29742 ^{2}	1.14904 ^{8}	0.57935 ^{4}	0.55220 ^{5}
		$\hat{\beta}$	0.53758 ^{6}	0.59489 ^{8}	0.37775 ^{1}	0.56788 ^{7}	0.46854 ^{3}	0.44862 ^{2}	0.35670 ^{4}	0.36972 ^{5}
		$\hat{\lambda}$	1.31911 ^{7}	0.75846 ^{3}	0.44620 ^{1}	0.86673 ^{6}	0.77043 ^{4}	8.62363 ^{8}	0.63671 ^{2}	0.85450 ^{5}
		$\hat{a}$	2.56005 ^{8}	0.92505 ^{4}	0.73821 ^{1}	1.02285 ^{7}	0.89952 ^{3}	0.83741 ^{2}	0.98962 ^{6}	0.96006 ^{5}
		$\hat{b}$	3.05303 ^{7}	1.17272 ^{5}	0.82103 ^{1}	1.34672 ^{6}	1.01733 ^{2}	5.27306 ^{8}	1.07322 ^{3}	1.14417 ^{4}
		$\sum Ranks$	103 ^{8}	67 ^{4}	15 ^{1}	91 ^{7}	52 ^{2}	85 ^{6}	54 ^{3}	73 ^{5}
Continued										

n	Est.	Param.	MLE	LSE	WLSE	CRVME	MPSE	PCE	ADE	RADE
250	BIAS	$\hat{\alpha}$	0.45154 {7}	0.36368 {4}	0.21404 {1}	0.36893 {5}	0.24152 {2}	1.07309 {8}	0.35666 {3}	0.38764 {6}
		$\hat{\beta}$	0.48107 {3}	0.64276 {8}	0.32441 {1}	0.63639 {7}	0.39594 {2}	0.63393 {6}	0.53506 {4}	0.55458 {5}
		$\hat{\lambda}$	0.28746 {7}	0.21215 {4}	0.13752 {1}	0.21120 {3}	0.18756 {2}	1.21756 {8}	0.22315 {5}	0.26008 {6}
		$\hat{a}$	1.62851 {8}	1.05386 {4}	0.57234 {1}	1.09009 {7}	0.60794 {2}	1.05833 {5}	0.98189 {3}	1.08570 {6}
		$\hat{b}$	1.33138 {7}	1.03592 {5}	0.56036 {1}	1.09034 {6}	0.66999 {2}	1.89716 {8}	0.87844 {3}	0.91333 {4}
	MSE	$\hat{\alpha}$	0.73770 {7}	0.28090 {3}	0.09846 {1}	0.29619 {4}	0.16049 {2}	23.65790 {8}	0.37411 {5}	0.46541 {6}
		$\hat{\beta}$	0.37105 {2}	0.62392 {8}	0.21024 {1}	0.61582 {7}	0.43261 {3}	0.61439 {6}	0.50622 {5}	0.50418 {4}
		$\hat{\lambda}$	0.19568 {7}	0.07986 {2}	0.03581 {1}	0.08064 {3}	0.08487 {4}	80.70630 {8}	0.08833 {5}	0.14179 {6}
		$\hat{a}$	9.33550 {8}	2.13198 {3}	1.06111 {1}	2.34296 {5}	1.74541 {2}	2.85968 {7}	2.39320 {6}	2.25930 {4}
		$\hat{b}$	7.32543 {7}	2.45706 {4}	1.14393 {1}	2.69180 {5}	3.92688 {6}	60.56412 {8}	2.08155 {3}	1.90151 {2}
	MRE	$\hat{\alpha}$	0.90308 {7}	0.72736 {4}	0.42807 {2}	0.73785 {5}	0.29742 {1}	2.14617 {8}	0.71332 {3}	0.77529 {6}
		$\hat{\beta}$	0.32072 {3}	0.42850 {8}	0.21627 {2}	0.42426 {7}	0.13509 {1}	0.39663 {8}	0.25141 {4}	0.27933 {5}
		$\hat{\lambda}$	0.57491 {7}	0.32904 {4}	0.19195 {1}	0.32292 {3}	0.20629 {2}	1.10149 {8}	0.35804 {5}	0.40727 {6}
		$\hat{a}$	0.63300 {5}	0.68519 {6}	0.26245 {2}	0.69799 {7}	0.23609 {1}	0.83696 {8}	0.52349 {3}	0.63186 {4}
		$\hat{b}$	0.58632 {5}	0.68100 {6}	0.27224 {2}	0.73277 {7}	0.25848 {1}	1.01292 {8}	0.52224 {3}	0.53229 {4}
		$\sum Ranks$	95 {7}	70 {4}	18 {1}	80 {6}	34 {2}	107 {8}	60 {3}	76 {5}
500	BIAS	$\hat{\alpha}$	0.35885 {7}	0.30070 {5}	0.14756 {1}	0.32654 {6}	0.14871 {2}	8.00000 {8}	0.28968 {4}	0.27610 {3}
		$\hat{\beta}$	0.34982 {3}	0.48291 {6}	0.21401 {2}	0.48327 {7}	0.20264 {1}	0.59495 {8}	0.37712 {4}	0.41900 {5}
		$\hat{\lambda}$	0.27474 {7}	0.16452 {4}	0.09598 {1}	0.16146 {3}	0.10314 {2}	0.55075 {8}	0.17902 {5}	0.20364 {6}
		$\hat{a}$	0.82290 {5}	0.89074 {6}	0.34118 {2}	0.90739 {7}	0.30692 {1}	1.08804 {8}	0.68054 {3}	0.82142 {4}
		$\hat{b}$	0.70358 {5}	0.81720 {6}	0.32668 {2}	0.87932 {7}	0.31018 {1}	1.21551 {8}	0.62668 {3}	0.63875 {4}
	MSE	$\hat{\alpha}$	0.29257 {7}	0.21993 {5}	0.04636 {1}	0.24856 {6}	0.06675 {2}	1.96818 {8}	0.19921 {4}	0.17010 {3}
		$\hat{\beta}$	0.19410 {3}	0.34633 {7}	0.09831 {1}	0.34434 {6}	0.13022 {2}	0.52823 {8}	0.24468 {4}	0.27192 {5}
		$\hat{\lambda}$	0.17655 {7}	0.04480 {4}	0.01452 {1}	0.04181 {3}	0.02604 {2}	1.42854 {8}	0.05935 {5}	0.08624 {6}
		$\hat{a}$	2.55423 {7}	1.65318 {5}	0.39050 {1}	1.68069 {6}	0.41505 {2}	2.82497 {8}	1.21969 {3}	1.40095 {4}
		$\hat{b}$	2.39326 {7}	1.48191 {5}	0.35758 {1}	1.74645 {6}	0.59674 {2}	3.97401 {8}	1.12441 {4}	0.91611 {3}
	MRE	$\hat{\alpha}$	0.71769 {7}	0.60141 {5}	0.29512 {1}	0.65308 {6}	0.29742 {2}	1.14904 {8}	0.57935 {4}	0.55220 {3}
		$\hat{\beta}$	0.23321 {3}	0.32194 {6}	0.14267 {2}	0.32218 {7}	0.13509 {1}	0.39663 {8}	0.25141 {4}	0.27933 {5}
		$\hat{\lambda}$	0.54947 {7}	0.32904 {4}	0.19195 {1}	0.32292 {3}	0.20629 {2}	1.10149 {8}	0.35804 {5}	0.40727 {6}
		$\hat{a}$	0.63300 {5}	0.68519 {6}	0.26245 {2}	0.69799 {7}	0.23609 {1}	0.83696 {8}	0.52349 {3}	0.63186 {4}
		$\hat{b}$	0.58632 {5}	0.68100 {6}	0.27224 {2}	0.73277 {7}	0.25848 {1}	1.01292 {8}	0.52224 {3}	0.53229 {4}
		$\sum Ranks$	85 {6}	80 {5}	21 {1}	87 {7}	24 {2}	120 {8}	58 {3}	65 {4}

Table 7. Numerical values of the KAPLL distribution for $\theta = (0.5, 1.5, 0.5, 1.3, 1.2)^T$.

1.680, 1.390, 0.710, 1.100, 0.990, 0.370, 0.510, 0.620, 1.170, 0.970, 0.690, 0.710, 0.400, 0.770, 0.990, 1.080, 1.050, 1.570, 2.110, 2.290, 2.240, 2.090, 1.900, 2.160, 2.250, 1.880, 1.940, 1.790, 1.380, 1.660, 1.900, 2.260, 2.560, 2.690, 3.090, 3.440, 3.320, 3.430, 3.840, 3.660, 2.720, 2.570, 2.200, 1.900, 1.790, 1.930, 2.010, 1.760, 1.430, 1.150, 1.230, 1.750, 1.990, 2.180, 2.050, 2.740, 3.420, 3.130, 2.670, 3.320, 3.600, 3.520, 2.960, 3.020, 3.680, 3.680, 3.600, 3.130, 2.540, 2.780, 2.950, 3.000, 2.940, 3.610, 4.280, 4.680, 5.010, 5.360, 5.360, 4.700, 5.170, 4.920, 5.120, 4.110, 5.090, 6.060, 5.870, 5.730, 6.070, 5.540, 5.370, 4.930, 4.910, 4.860, 4.590, 4.180, 3.650, 3.740, 2.800, 2.920, 3.050, 2.930, 3.070, 2.670, 2.210, 2.820, 2.760, 2.680, 2.420, 3.000, 3.060, 3.220, 3.420, 3.410, 3.540, 3.420, 3.340, 3.340, 3.560, 4.100, 3.340, 3.350, 4.010, 3.810, 4.520, 5.360, 5.728, 5.957, 4.963, 4.780, 3.966, 4.138, 3.928, 4.480, 5.243, 5.162, 4.681, 5.241, 5.265, 3.824, 3.782, 3.646, 3.913, 3.463, 3.274, 3.169, 3.609, 3.217, 3.267, 4.124, 4.654, 4.716, 4.056, 4.198, 3.854, 3.744, 3.764, 3.340, 3.539, 3.792, 3.480, 3.267, 2.980, 3.523, 4.555, 4.876, 4.954, 4.473, 4.863, 4.638, 4.485, 4.887, 5.889, 5.840, 6.566, 5.105, 5.478, 6.173, 7.829, 7.721, 7.546, 7.863, 8.140, 8.590, 9.010, 7.907, 5.997, 6.118, 5.842, 5.954, 6.111, 5.570, 4.627, 3.836, 3.148, 3.445, 3.483, 3.538, 2.752, 2.612, 2.618, 2.200, 2.810, 3.516, 4.258, 4.700, 5.358, 5.013, 5.081, 4.970, 4.517, 6.434, 6.619, 7.068, 6.562, 6.873, 6.312, 6.648, 6.907, 9.350, 9.910, 10.270.

The descriptive statistics summarized in Table 14 underscore the diversity among the five datasets in terms of size (n), central tendency, variability, and distributional shape. Notably, the skewness values range from negative to positive, indicating that the datasets are not symmetrically distributed. This variation reflects the complexity

<i>n</i>	Est.	Param.	MLE	LSE	WLSE	CRVME	MPSE	PCE	ADE	RADE
50	[BIAS]	$\hat{\alpha}$	2.10982 ^{7}	0.98476 ^{6}	0.60669 ^{2}	0.61406 ^{3}	0.47519 ^{1}	4.01192 ^{8}	0.70275 ^{5}	0.69017 ^{4}
		$\hat{\beta}$	0.68973 ^{6}	0.85650 ^{8}	0.62423 ^{3}	0.82091 ^{7}	0.29652 ^{1}	0.37823 ^{2}	0.68909 ^{5}	0.65208 ^{4}
		$\hat{\lambda}$	9.74554 ^{8}	1.09309 ^{6}	0.64656 ^{3}	0.75128 ^{4}	0.51383 ^{1}	3.89186 ^{7}	0.64320 ^{2}	0.97694 ^{5}
		$\hat{a}$	3.79455 ^{8}	0.84377 ^{2}	0.77113 ^{1}	0.98346 ^{5}	1.28942 ^{7}	1.08595 ^{6}	0.94879 ^{4}	0.94841 ^{3}
		$\hat{b}$	3.53050 ^{8}	0.40299 ^{5}	0.27779 ^{1}	0.36626 ^{4}	0.35944 ^{3}	1.01159 ^{7}	0.31448 ^{2}	0.40860 ^{6}
	MSE	$\hat{\alpha}$	667.02712 ^{7}	107.70264 ^{6}	3.54587 ^{4}	1.53320 ^{2}	0.62266 ^{1}	3606.82461 ^{8}	3.84992 ^{5}	2.23235 ^{3}
		$\hat{\beta}$	0.67637 ^{4}	1.12874 ^{8}	0.61105 ^{3}	0.99738 ^{7}	0.84821 ^{6}	0.26576 ^{1}	0.69775 ^{5}	0.59647 ^{2}
		$\hat{\lambda}$	83232.95657 ^{8}	158.05868 ^{6}	1.69126 ^{3}	2.38192 ^{4}	0.68193 ^{1}	584.32791 ^{7}	1.16142 ^{2}	13.69024 ^{5}
		$\hat{a}$	63.15764 ^{8}	1.85323 ^{6}	2.15647 ^{1}	2.50965 ^{3}	10.07516 ^{5}	7.37178 ^{7}	2.84686 ^{2}	2.58046 ^{4}
		$\hat{b}$	8417.58478 ^{7}	4.47107 ^{6}	0.35712 ^{2}	0.45203 ^{3}	1.16293 ^{1}	24.13930 ^{8}	0.40043 ^{5}	0.75442 ^{4}
	MRE	$\hat{\alpha}$	4.21964 ^{7}	1.96952 ^{6}	1.21338 ^{2}	1.22811 ^{3}	0.95038 ^{1}	8.02383 ^{8}	1.40551 ^{5}	1.38033 ^{4}
		$\hat{\beta}$	0.45982 ^{5}	0.57100 ^{8}	0.41615 ^{2}	0.54728 ^{7}	0.49022 ^{6}	0.25216 ^{1}	0.45939 ^{4}	0.43472 ^{3}
		$\hat{\lambda}$	12.99406 ^{8}	1.45745 ^{6}	0.86208 ^{3}	1.00171 ^{4}	0.68510 ^{1}	5.18915 ^{7}	0.85761 ^{2}	1.30258 ^{5}
		$\hat{a}$	6.32425 ^{8}	1.40628 ^{2}	1.28522 ^{1}	1.63910 ^{5}	2.14903 ^{7}	1.80991 ^{6}	1.58132 ^{4}	1.58068 ^{3}
		$\hat{b}$	11.76834 ^{8}	1.34328 ^{5}	0.92595 ^{1}	1.22088 ^{4}	1.19812 ^{3}	3.37196 ^{7}	1.04828 ^{2}	1.36199 ^{6}
		$\sum Ranks$	108 ^{8}	81 ^{6}	32 ^{1}	65 ^{5}	51 ^{2}	88 ^{7}	54 ^{3}	61 ^{4}
100	[BIAS]	$\hat{\alpha}$	0.84301 ^{7}	0.43550 ^{5}	0.30507 ^{2}	0.46623 ^{6}	0.28281 ^{1}	1.62325 ^{8}	0.41301 ^{4}	0.40497 ^{3}
		$\hat{\beta}$	0.53143 ^{5}	0.66938 ^{7}	0.45487 ^{2}	0.69243 ^{8}	0.52057 ^{3}	0.35033 ^{1}	0.52768 ^{4}	0.53825 ^{6}
		$\hat{\lambda}$	0.49169 ^{6}	0.47443 ^{5}	0.42226 ^{3}	0.49851 ^{7}	0.29409 ^{1}	2.63782 ^{8}	0.39593 ^{2}	0.45934 ^{4}
		$\hat{a}$	1.68292 ^{8}	0.72696 ^{5}	0.50268 ^{1}	0.88526 ^{7}	0.70873 ^{4}	0.69930 ^{3}	0.65472 ^{2}	0.77654 ^{6}
		$\hat{b}$	0.32922 ^{7}	0.23814 ^{4}	0.15888 ^{1}	0.29142 ^{6}	0.16946 ^{2}	0.92702 ^{8}	0.19033 ^{3}	0.23877 ^{5}
	MSE	$\hat{\alpha}$	13.37040 ^{7}	0.56205 ^{3}	0.20691 ^{2}	0.73231 ^{6}	0.14081 ^{1}	170.17219 ^{8}	0.60274 ^{5}	0.56324 ^{4}
		$\hat{\beta}$	0.42093 ^{5}	0.67254 ^{7}	0.33392 ^{2}	0.68990 ^{8}	0.44338 ^{6}	0.26207 ^{1}	0.39759 ^{3}	0.40575 ^{4}
		$\hat{\lambda}$	0.67064 ^{7}	0.45537 ^{3}	0.47893 ^{4}	0.59990 ^{5}	0.18468 ^{1}	188.09102 ^{8}	0.35698 ^{2}	0.60456 ^{6}
		$\hat{a}$	16.12036 ^{8}	1.35335 ^{2}	0.94222 ^{1}	1.90221 ^{5}	4.64823 ^{7}	2.35848 ^{6}	1.55865 ^{3}	1.81776 ^{4}
		$\hat{b}$	0.62858 ^{7}	0.18464 ^{3}	0.08822 ^{1}	0.26222 ^{6}	0.19429 ^{5}	21.60443 ^{8}	0.12108 ^{2}	0.18856 ^{4}
	MRE	$\hat{\alpha}$	1.68602 ^{7}	0.87101 ^{5}	0.61014 ^{2}	0.93246 ^{6}	0.56561 ^{1}	3.24651 ^{8}	0.82603 ^{4}	0.80994 ^{3}
		$\hat{\beta}$	0.35428 ^{5}	0.44625 ^{7}	0.30325 ^{2}	0.46162 ^{8}	0.34705 ^{3}	0.23356 ^{1}	0.35179 ^{4}	0.35883 ^{6}
		$\hat{\lambda}$	0.65559 ^{6}	0.63257 ^{5}	0.56302 ^{3}	0.66468 ^{7}	0.39212 ^{1}	3.51709 ^{8}	0.52791 ^{2}	0.61245 ^{4}
		$\hat{a}$	2.80486 ^{8}	1.21160 ^{5}	0.83780 ^{1}	1.47544 ^{7}	1.18121 ^{4}	1.16549 ^{3}	1.09120 ^{2}	1.29423 ^{6}
		$\hat{b}$	1.09739 ^{7}	0.79381 ^{4}	0.52960 ^{1}	0.97141 ^{6}	0.56488 ^{2}	3.09005 ^{8}	0.63445 ^{3}	0.79589 ^{5}
		$\sum Ranks$	100 ^{8}	70 ^{4.5}	28 ^{1}	98 ^{7}	42 ^{2}	87 ^{6}	45 ^{3}	70 ^{4.5}
Continued										

<i>n</i>	Est.	Param.	MLE	LSE	WLSE	CRVME	MPSE	PCE	ADE	RADE
250	BIAS	$\hat{\alpha}$	0.37957 {7}	0.28330 {5}	0.18161 {2}	0.26826 {4}	0.16323 {1}	0.65826 {8}	0.31424 {6}	0.25495 {3}
		$\hat{\beta}$	0.34004 {4}	0.46961 {8}	0.28146 {1}	0.45493 {7}	0.29652 {3}	0.28988 {2}	0.37506 {5}	0.39717 {6}
		$\hat{\lambda}$	0.37470 {7}	0.35558 {5}	0.21668 {2}	0.36921 {6}	0.16002 {1}	0.64644 {8}	0.30441 {4}	0.25259 {3}
		$\hat{a}$	0.51041 {5}	0.51452 {6}	0.27239 {2}	0.52402 {7}	0.23736 {1}	0.42427 {4}	0.36967 {3}	0.57541 {8}
		$\hat{b}$	0.13292 {4}	0.14792 {6}	0.08819 {2}	0.13899 {5}	0.07459 {1}	0.42310 {8}	0.11971 {3}	0.15186 {7}
	MSE	$\hat{\alpha}$	2.40148 {7}	0.15180 {4}	0.05285 {2}	0.12138 {3}	0.04961 {1}	26.45866 {8}	0.27786 {6}	0.21749 {5}
		$\hat{\beta}$	0.19176 {3}	0.32962 {8}	0.13360 {1}	0.31452 {7}	0.16404 {2}	0.24845 {6}	0.20346 {4}	0.24100 {5}
		$\hat{\lambda}$	0.45584 {7}	0.29471 {5}	0.12495 {2}	0.31329 {6}	0.05716 {1}	4.91916 {8}	0.24773 {4}	0.15484 {3}
		$\hat{a}$	1.61523 {8}	0.74439 {4}	0.27168 {1}	0.77097 {5}	0.34305 {2}	0.78066 {6}	0.45301 {3}	1.07124 {7}
		$\hat{b}$	0.05766 {6}	0.05575 {5}	0.02094 {2}	0.04668 {4}	0.01329 {1}	1.62186 {8}	0.03775 {3}	0.06527 {7}
	MRE	$\hat{\alpha}$	0.75914 {7}	0.56661 {5}	0.36321 {2}	0.53652 {4}	0.19566 {1}	1.31653 {8}	0.62848 {6}	0.50989 {3}
		$\hat{\beta}$	0.22669 {3}	0.31308 {8}	0.18764 {2}	0.30329 {7}	0.11117 {1}	0.19325 {5}	0.18314 {4}	0.20869 {6}
		$\hat{\lambda}$	0.49960 {7}	0.47410 {5}	0.28891 {2}	0.49228 {6}	0.13870 {1}	0.86192 {8}	0.40587 {4}	0.33678 {3}
		$\hat{a}$	0.85068 {5}	0.85753 {6}	0.45398 {2}	0.87337 {7}	0.39560 {1}	0.70711 {4}	0.61611 {3}	0.95901 {8}
		$\hat{b}$	0.44308 {4}	0.49305 {6}	0.29398 {2}	0.46331 {5}	0.24863 {1}	1.41032 {8}	0.39902 {3}	0.50619 {7}
		$\sum Ranks$	85 {6}	86 {7}	27 {2}	83 {5}	19 {1}	97 {8}	62 {3}	81 {4}
500	BIAS	$\hat{\alpha}$	0.20066 {4}	0.23162 {6}	0.11984 {2}	0.24449 {7}	0.09783 {1}	8.00000 {8}	0.21114 {5}	0.17249 {3}
		$\hat{\beta}$	0.22994 {3}	0.35988 {8}	0.18305 {2}	0.34879 {7}	0.16676 {1}	0.28890 {5}	0.27471 {4}	0.31303 {6}
		$\hat{\lambda}$	0.28490 {5}	0.31362 {7}	0.13005 {2}	0.30834 {6}	0.10402 {1}	0.41688 {8}	0.23302 {4}	0.16351 {3}
		$\hat{a}$	0.22302 {3}	0.35847 {6}	0.13626 {2}	0.34017 {5}	0.13410 {1}	0.42478 {7}	0.22849 {4}	0.42549 {8}
		$\hat{b}$	0.07570 {3}	0.09947 {5}	0.04874 {2}	0.10226 {6}	0.04565 {1}	0.36592 {8}	0.07598 {4}	0.10468 {7}
	MSE	$\hat{\alpha}$	0.07033 {4}	0.08190 {6}	0.02626 {2}	0.09573 {7}	0.01669 {1}	0.57406 {8}	0.08039 {5}	0.05798 {3}
		$\hat{\beta}$	0.10282 {3}	0.19632 {7}	0.05904 {1}	0.18247 {6}	0.06310 {2}	0.26862 {8}	0.11585 {4}	0.15634 {5}
		$\hat{\lambda}$	0.30601 {7}	0.23048 {5}	0.04296 {2}	0.24301 {6}	0.01887 {1}	0.92476 {8}	0.17077 {4}	0.05198 {3}
		$\hat{a}$	0.16721 {4}	0.36527 {6}	0.05193 {2}	0.32355 {5}	0.04398 {1}	0.93583 {8}	0.14753 {3}	0.62543 {7}
		$\hat{b}$	0.01696 {4}	0.02240 {5}	0.00542 {2}	0.02409 {6}	0.00527 {1}	1.62186 {8}	0.01310 {3}	0.02663 {7}
	MRE	$\hat{\alpha}$	0.40132 {4}	0.46324 {6}	0.23969 {2}	0.48898 {7}	0.19566 {1}	0.68197 {8}	0.42227 {5}	0.34498 {3}
		$\hat{\beta}$	0.15330 {3}	0.23992 {8}	0.12203 {2}	0.23252 {7}	0.11117 {1}	0.19260 {5}	0.18314 {4}	0.20869 {6}
		$\hat{\lambda}$	0.37986 {5}	0.41816 {7}	0.17340 {2}	0.41112 {6}	0.13870 {1}	0.55584 {8}	0.31070 {4}	0.21801 {3}
		$\hat{a}$	0.37170 {3}	0.59746 {6}	0.22710 {2}	0.56695 {5}	0.22351 {1}	0.70797 {7}	0.38082 {4}	0.70915 {8}
		$\hat{b}$	0.25234 {3}	0.33155 {5}	0.16246 {2}	0.34088 {6}	0.15217 {1}	1.21973 {8}	0.25328 {4}	0.34893 {7}
		$\sum Ranks$	58 {3}	93 {7}	29 {2}	92 {6}	16 {1}	112 {8}	61 {4}	79 {5}

Table 8. Numerical values of the KAPLL distribution for $\theta = (0.5, 1.5, 0.75, 0.6, 0.3)^T$.

often encountered in real-world reliability and engineering data. Therefore, it is crucial to employ a flexible distribution capable of capturing such heterogeneity.

The proposed KAPLL model is particularly well-suited for this purpose, as it can accommodate a wide range of skewness and kurtosis levels, making it a strong candidate for modeling these datasets effectively.

The fits of the KAPLL distributions is compared with some existing LL distributions namely: the alpha power transformed LL (APTLL)¹⁷, KMOLL²¹, McL²⁰, AWLL²², alpha power LL (APLL)⁵⁴, TLL¹⁸, and LL distributions.

The performance of the fitted distributions is explored using some measures including the Akaike information criterion (AIC), consistent Akaike IC (CAIC), Hannan–Quinn IC (HQIC), Bayesian IC (BIC), $(-\ell)$, where ℓ the maximized log-likelihood, as well as some statistics include the Cramér–Von Mises (W^*), Anderson–Darling (A^*), Kolmogorov–Smirnov (KS) and the KS p -value (PV).

All numerical results in this section are calculated using the R software. Table 15 provides goodness-of-fit measures of the competing models for the five analyzed datasets. The results in this table indicate the superior fitting performance of the proposed KAPLL distribution over other existing LL extensions. The ML estimates and standard errors (SEs) for the models' parameters are reported in Tables 16 and 17, for all datasets. Overall, the KAPLL distribution have the lowest values for goodness-of-fit criteria among all LL fitted models. Then, it could be chosen as the most adequate model for fitting the five datasets.

<i>n</i>	Est.	Param.	MLE	LSE	WLSE	CRVME	MPSE	PCE	ADE	RADE
50	BIAS	$\hat{\alpha}$	2.90857 ^{7}	1.22842 ^{4}	1.00526 ^{2}	1.79282 ^{5}	0.83292 ^{1}	4.89560 ^{8}	1.09398 ^{3}	1.90487 ^{6}
		$\hat{\beta}$	0.82589 ^{4}	0.99989 ^{8}	0.66345 ^{2}	0.92154 ^{7}	0.91364 ^{6}	0.49144 ^{1}	0.84217 ^{5}	0.82534 ^{3}
		$\hat{\lambda}$	3.13493 ^{7}	1.05086 ^{4}	0.85252 ^{1}	1.47774 ^{5}	1.01798 ^{3}	7.55158 ^{8}	0.92569 ^{2}	2.93827 ^{6}
		$\hat{a}$	2.87125 ^{8}	0.79138 ^{3}	0.66428 ^{2}	0.97617 ^{6}	1.16550 ^{7}	0.50244 ^{1}	0.92428 ^{5}	0.91783 ^{4}
		$\hat{b}$	2.53118 ^{8}	0.77439 ^{3}	0.55727 ^{1}	0.99909 ^{5}	0.94477 ^{4}	2.46564 ^{7}	0.74759 ^{2}	1.21091 ^{6}
	MSE	$\hat{\alpha}$	531.40926 ^{8}	49.64205 ^{4}	21.67096 ^{2}	224.83942 ^{5}	42.40955 ^{3}	355.90411 ^{7}	11.90844 ^{1}	278.31952 ^{6}
		$\hat{\beta}$	1.04045 ^{3}	1.75791 ^{8}	0.84114 ^{2}	1.40113 ^{6}	1.52433 ^{7}	0.39204 ^{1}	1.20348 ^{5}	1.07041 ^{4}
		$\hat{\lambda}$	2290.39863 ^{8}	36.72915 ^{3}	8.90711 ^{1}	188.77097 ^{5}	81.90253 ^{4}	518.44224 ^{6}	17.30250 ^{2}	1049.53033 ^{7}
		$\hat{a}$	34.96166 ^{8}	1.70296 ^{3}	1.63230 ^{1}	2.57092 ^{5}	8.06744 ^{4}	1.12744 ^{7}	2.71833 ^{2}	2.33411 ^{6}
		$\hat{b}$	228.98944 ^{7}	6.02364 ^{4}	1.95820 ^{2}	25.45254 ^{5}	8.57837 ^{1}	66.15547 ^{8}	3.77918 ^{3}	56.07007 ^{6}
	MRE	$\hat{\alpha}$	5.81713 ^{7}	2.45683 ^{4}	2.01051 ^{2}	3.58564 ^{5}	1.66585 ^{1}	9.79119 ^{8}	2.18795 ^{3}	3.80973 ^{6}
		$\hat{\beta}$	0.55059 ^{4}	0.66659 ^{8}	0.44230 ^{2}	0.61436 ^{7}	0.60909 ^{6}	0.32763 ^{1}	0.56144 ^{5}	0.55023 ^{3}
		$\hat{\lambda}$	4.17991 ^{7}	1.40114 ^{4}	1.13669 ^{1}	1.97032 ^{5}	1.35731 ^{3}	10.06877 ^{8}	1.23425 ^{2}	3.91770 ^{6}
		$\hat{a}$	4.78541 ^{8}	1.31896 ^{3}	1.10714 ^{2}	1.62695 ^{6}	1.94250 ^{7}	0.83740 ^{1}	1.54047 ^{5}	1.52972 ^{4}
		$\hat{b}$	5.06236 ^{8}	1.54879 ^{3}	1.11454 ^{1}	1.99817 ^{5}	1.88953 ^{4}	4.93129 ^{7}	1.49518 ^{2}	2.42183 ^{6}
		$\sum Ranks$	103 ^{8}	65 ^{3}	24 ^{1}	82 ^{7}	67 ^{4}	72 ^{5}	50 ^{2}	77 ^{6}
100	BIAS	$\hat{\alpha}$	1.07088 ^{7}	0.62198 ^{4}	0.44930 ^{2}	0.66856 ^{6}	0.34291 ^{1}	2.63029 ^{8}	0.65040 ^{5}	0.59401 ^{3}
		$\hat{\beta}$	0.64445 ^{5}	0.72886 ^{7}	0.46458 ^{2}	0.73665 ^{8}	0.66613 ^{6}	0.44619 ^{1}	0.57810 ^{3}	0.62353 ^{4}
		$\hat{\lambda}$	0.71471 ^{7}	0.56536 ^{4}	0.49202 ^{2}	0.55993 ^{3}	0.38894 ^{1}	4.39275 ^{8}	0.57973 ^{5}	0.64209 ^{6}
		$\hat{a}$	1.34586 ^{8}	0.70769 ^{5}	0.43194 ^{1}	0.79781 ^{7}	0.52980 ^{2}	0.67406 ^{4}	0.60647 ^{3}	0.75193 ^{6}
		$\hat{b}$	0.89239 ^{7}	0.48625 ^{5}	0.33148 ^{1}	0.61587 ^{6}	0.43611 ^{3}	1.06133 ^{8}	0.40707 ^{2}	0.45342 ^{4}
	MSE	$\hat{\alpha}$	25.94600 ^{7}	2.04009 ^{5}	0.95218 ^{2}	1.37902 ^{3}	0.25008 ^{1}	98.43325 ^{8}	2.16820 ^{6}	1.68689 ^{4}
		$\hat{\beta}$	0.61986 ^{4}	0.90178 ^{8}	0.41049 ^{2}	0.88131 ^{7}	0.82344 ^{6}	0.31821 ^{1}	0.55808 ^{3}	0.62913 ^{5}
		$\hat{\lambda}$	1.89292 ^{7}	0.63732 ^{3}	0.62856 ^{2}	0.89926 ^{5}	0.39868 ^{1}	221.29767 ^{8}	0.73840 ^{4}	0.95177 ^{6}
		$\hat{a}$	7.88808 ^{8}	1.38746 ^{3}	0.79083 ^{1}	1.66226 ^{6}	1.48858 ^{4}	2.46499 ^{7}	1.32767 ^{2}	1.62387 ^{5}
		$\hat{b}$	5.37620 ^{7}	0.74403 ^{4}	0.55886 ^{1}	1.18140 ^{5}	1.67507 ^{6}	15.27756 ^{8}	0.68384 ^{2}	0.74061 ^{3}
	MRE	$\hat{\alpha}$	2.14177 ^{7}	1.24396 ^{4}	0.89860 ^{2}	1.33711 ^{6}	0.68583 ^{1}	5.26057 ^{8}	1.30079 ^{5}	1.18803 ^{3}
		$\hat{\beta}$	0.42963 ^{5}	0.48590 ^{7}	0.30972 ^{2}	0.49110 ^{8}	0.44409 ^{6}	0.29746 ^{1}	0.38540 ^{3}	0.41569 ^{4}
		$\hat{\lambda}$	0.95295 ^{7}	0.75381 ^{4}	0.65602 ^{2}	0.74657 ^{3}	0.51858 ^{1}	5.85700 ^{8}	0.77297 ^{5}	0.85612 ^{6}
		$\hat{a}$	2.24310 ^{8}	1.17949 ^{5}	0.71989 ^{1}	1.32968 ^{7}	0.88300 ^{2}	1.12344 ^{4}	1.01078 ^{3}	1.25322 ^{6}
		$\hat{b}$	1.78478 ^{7}	0.97250 ^{5}	0.66297 ^{1}	1.23174 ^{6}	0.87223 ^{3}	2.12266 ^{8}	0.81414 ^{2}	0.90684 ^{4}
		$\sum Ranks$	101 ^{8}	73 ^{5}	24 ^{1}	86 ^{6}	44 ^{2}	90 ^{7}	53 ^{3}	69 ^{4}
Continued										

n	Est.	Param.	MLE	LSE	WLSE	CRVME	MPSE	PCE	ADE	RADE
250	[BIAS]	$\hat{\alpha}$	0.56287 {7}	0.44686 {4}	0.23867 {2}	0.44941 {5}	0.19324 {1}	1.38295 {8}	0.49578 {6}	0.42810 {3}
		$\hat{\beta}$	0.40230 {5}	0.48239 {8}	0.25857 {1}	0.47287 {7}	0.38439 {2}	0.39727 {4}	0.38443 {3}	0.42399 {6}
		$\hat{\lambda}$	0.54412 {7}	0.42275 {4}	0.27435 {2}	0.41514 {3}	0.18875 {1}	2.23589 {8}	0.47022 {5}	0.51823 {6}
		$\hat{a}$	0.40746 {4}	0.44588 {5}	0.20477 {1}	0.47617 {6}	0.25153 {2}	0.87951 {8}	0.31189 {3}	0.50607 {7}
		$\hat{b}$	0.25904 {4}	0.30711 {6}	0.14639 {1}	0.31505 {7}	0.16286 {2}	0.39706 {8}	0.22334 {3}	0.28871 {5}
	MSE	$\hat{\alpha}$	3.68958 {7}	0.52678 {3}	0.10008 {2}	0.53647 {4}	0.05861 {1}	19.59192 {8}	0.80459 {6}	0.60588 {5}
		$\hat{\beta}$	0.24612 {3}	0.39810 {8}	0.13075 {1}	0.35900 {7}	0.28482 {5}	0.28003 {4}	0.24287 {2}	0.28842 {6}
		$\hat{\lambda}$	0.71841 {7}	0.30884 {4}	0.18572 {2}	0.29863 {3}	0.11549 {1}	115.47011 {8}	0.46093 {5}	0.65328 {6}
		$\hat{a}$	0.61835 {4}	0.62015 {5}	0.15031 {1}	0.70440 {6}	0.29737 {3}	4.28709 {8}	0.28427 {2}	0.78647 {7}
		$\hat{b}$	0.30876 {6}	0.29085 {5}	0.08705 {2}	0.31608 {7}	0.07992 {1}	1.48625 {8}	0.16834 {3}	0.27586 {4}
	MRE	$\hat{\alpha}$	1.12574 {7}	0.89372 {4}	0.47734 {2}	0.89883 {5}	0.26955 {1}	2.76590 {8}	0.99156 {6}	0.85620 {3}
		$\hat{\beta}$	0.26820 {5}	0.32160 {8}	0.17238 {2}	0.31525 {7}	0.16130 {1}	0.26485 {4}	0.25629 {3}	0.28266 {6}
		$\hat{\lambda}$	0.72549 {7}	0.56366 {4}	0.36579 {2}	0.55352 {3}	0.14785 {1}	2.98118 {8}	0.62696 {5}	0.69097 {6}
		$\hat{a}$	0.67911 {4}	0.74314 {5}	0.34128 {1}	0.79362 {6}	0.41922 {2}	1.46584 {8}	0.51982 {3}	0.84345 {7}
		$\hat{b}$	0.51808 {4}	0.61422 {6}	0.29277 {1}	0.63010 {7}	0.32571 {2}	0.79412 {8}	0.44667 {3}	0.57742 {5}
	$\sum Ranks$		81 {5}	79 {4}	23 {1}	83 {7}	26 {2}	108 {8}	58 {3}	82 {6}
500	[BIAS]	$\hat{\alpha}$	0.35357 {4}	0.36374 {6}	0.17285 {2}	0.37030 {7}	0.13478 {1}	8.00000 {8}	0.35495 {5}	0.31465 {3}
		$\hat{\beta}$	0.30556 {4}	0.34570 {7}	0.17627 {1}	0.32521 {5}	0.24195 {2}	0.37764 {8}	0.28503 {3}	0.32710 {6}
		$\hat{\lambda}$	0.52884 {7}	0.36884 {3}	0.18028 {2}	0.38021 {4}	0.11089 {1}	1.97140 {8}	0.39543 {5}	0.42642 {6}
		$\hat{a}$	0.23268 {4}	0.28645 {6}	0.12057 {1}	0.27005 {5}	0.15481 {2}	0.61507 {8}	0.20101 {3}	0.31954 {7}
		$\hat{b}$	0.14161 {4}	0.19432 {7}	0.08525 {1}	0.19033 {6}	0.10283 {2}	0.57555 {8}	0.13726 {3}	0.17547 {5}
	MSE	$\hat{\alpha}$	0.19244 {4}	0.30280 {6}	0.05171 {2}	0.35495 {7}	0.02902 {1}	51.12266 {8}	0.19634 {5}	0.16377 {3}
		$\hat{\beta}$	0.14285 {4}	0.20492 {7}	0.06365 {1}	0.18131 {6}	0.11682 {2}	0.24380 {8}	0.12794 {3}	0.16612 {5}
		$\hat{\lambda}$	0.67324 {7}	0.24360 {3}	0.07777 {2}	0.26126 {4}	0.03325 {1}	84.23066 {8}	0.33924 {5}	0.42776 {6}
		$\hat{a}$	0.12854 {4}	0.24103 {6}	0.03816 {1}	0.21001 {5}	0.04518 {2}	1.70006 {8}	0.09966 {3}	0.31236 {7}
		$\hat{b}$	0.04448 {4}	0.11285 {6}	0.01859 {1}	0.11422 {7}	0.02323 {2}	7.94497 {8}	0.04441 {3}	0.07889 {5}
	MRE	$\hat{\alpha}$	0.70715 {4}	0.72747 {6}	0.34570 {2}	0.74060 {7}	0.26955 {1}	2.45739 {8}	0.70989 {5}	0.62931 {3}
		$\hat{\beta}$	0.20371 {4}	0.23047 {7}	0.11752 {1}	0.21680 {5}	0.16130 {2}	0.25176 {8}	0.19002 {3}	0.21807 {6}
		$\hat{\lambda}$	0.70511 {7}	0.49178 {3}	0.24037 {2}	0.50695 {4}	0.14785 {1}	2.62853 {8}	0.52725 {5}	0.56856 {6}
		$\hat{a}$	0.38780 {4}	0.47742 {6}	0.20095 {1}	0.45008 {5}	0.25801 {2}	1.02512 {8}	0.33502 {3}	0.53256 {7}
		$\hat{b}$	0.28322 {4}	0.38863 {7}	0.17051 {1}	0.38067 {6}	0.20565 {2}	1.15109 {8}	0.27452 {3}	0.35095 {5}
	$\sum Ranks$		69 {4}	86 {7}	21 {1}	83 {6}	24 {2}	120 {8}	57 {3}	80 {5}

Table 9. Numerical values of the KAPLL distribution for $\theta = (0.5, 1.5, 0.75, 0.6, 0.5)^T$.

Furthermore, the visual fitting performance of the proposed KAPLL distribution is comprehensively assessed in Figs. 4 and 5. These figures present a detailed graphical comparison between the empirical data and the fitted KAPLL model through various diagnostic plots. Specifically, each panel includes the histogram of the observed dataset overlaid with the estimated PDF, the empirical and fitted CDF, and the probability–probability (PP) plots. These plots collectively illustrate how well the KAPLL model captures the overall distributional behavior of the data.

In addition, Figs. 6, 7 and 8 show the histogram-based visual comparison of the KAPLL model against several other competing lifetime distributions, including known extensions of the LL distribution. For each dataset, the histogram is overlaid with the fitted density curves from all candidate models. This comparative visualization allows for a clear, direct assessment of goodness-of-fit among the competing models.

From the visual evidence provided in all these figures, it is apparent that the KAPLL distribution consistently aligns more closely with the empirical data than the alternative models. Specifically, the KAPLL model's density curves match the peaks, tails, and spread of the histograms more accurately; its CDF trace follows the empirical curves with minimal deviation; and the PP plots demonstrate that the fitted values are nearly perfectly aligned with the 45-degree reference line, indicating strong agreement between the observed and theoretical distributions. These visual diagnostics confirm the superior flexibility and accuracy of the KAPLL model in

<i>n</i>	Est.	Param.	MLE	LSE	WLSE	CRVME	MPSE	PCE	ADE	RADE
50	[BIAS]	$\hat{\alpha}$	9.67712 ^{7}	2.01966 ^{2}	2.04697 ^{3}	4.09986 ^{6}	1.98052 ^{1}	12.30644 ^{8}	2.63589 ^{4}	4.01571 ^{5}
		$\hat{\beta}$	0.85793 ^{5}	0.96952 ^{8}	0.65025 ^{3}	0.94520 ^{7}	0.38832 ^{1}	0.54574 ^{2}	0.89161 ^{6}	0.84504 ^{4}
		$\hat{\lambda}$	13.26830 ^{8}	2.22478 ^{3}	1.34355 ^{1}	4.79735 ^{6}	2.37869 ^{4}	12.32516 ^{7}	2.22258 ^{2}	3.25133 ^{5}
		$\hat{a}$	2.73706 ^{8}	0.74476 ^{3}	0.57849 ^{2}	0.98187 ^{6}	1.03022 ^{7}	0.34142 ^{1}	0.83439 ^{4}	0.86610 ^{5}
		$\hat{b}$	10.14909 ^{8}	1.76173 ^{2}	1.22796 ^{1}	3.48303 ^{6}	2.19672 ^{4}	6.16541 ^{7}	2.14781 ^{3}	2.79630 ^{5}
	MSE	$\hat{\alpha}$	2489.56419 ^{7}	455.16735 ^{3}	148.30146 ^{1}	3094.44862 ^{8}	166.27420 ^{2}	1730.53234 ^{6}	619.52017 ^{4}	823.44777 ^{5}
		$\hat{\beta}$	1.12950 ^{3}	1.59376 ^{7}	0.80926 ^{2}	1.46266 ^{6}	1.87058 ^{8}	0.49722 ^{1}	1.39820 ^{5}	1.26993 ^{4}
		$\hat{\lambda}$	10115.99764 ^{8}	747.92924 ^{4}	113.79328 ^{1}	5210.34968 ^{7}	262.16742 ^{2}	1682.08638 ^{6}	953.08910 ^{5}	621.68754 ^{3}
		$\hat{a}$	30.12420 ^{8}	1.71278 ^{4}	1.26915 ^{1}	2.75748 ^{7}	6.65465 ^{2}	0.44780 ^{5}	2.27188 ^{6}	2.19017 ^{3}
		$\hat{b}$	2657.06258 ^{7}	324.98946 ^{2}	34.83131 ^{3}	1916.90763 ^{6}	118.81959 ^{1}	364.05678 ^{8}	423.79549 ^{4}	313.62395 ^{5}
	MRE	$\hat{\alpha}$	19.35425 ^{7}	4.03932 ^{2}	4.09395 ^{3}	8.19973 ^{6}	3.96104 ^{1}	24.61288 ^{8}	5.27179 ^{4}	8.03142 ^{5}
		$\hat{\beta}$	0.57195 ^{4}	0.64635 ^{8}	0.43350 ^{2}	0.63013 ^{6}	0.64565 ^{7}	0.36383 ^{1}	0.59441 ^{5}	0.56336 ^{3}
		$\hat{\lambda}$	17.69107 ^{8}	2.96637 ^{3}	1.79141 ^{1}	6.39646 ^{6}	3.17158 ^{4}	16.43355 ^{7}	2.96344 ^{2}	4.33510 ^{5}
		$\hat{a}$	4.56177 ^{8}	1.24127 ^{3}	0.96414 ^{2}	1.63645 ^{6}	1.71703 ^{7}	0.56903 ^{1}	1.39065 ^{4}	1.44350 ^{5}
		$\hat{b}$	14.49870 ^{8}	2.51676 ^{2}	1.75423 ^{1}	4.97576 ^{6}	3.13817 ^{4}	8.80773 ^{7}	3.06831 ^{3}	3.99472 ^{5}
		$\sum Ranks$	105 ^{8}	57 ^{2}	26 ^{1}	95 ^{7}	61 ^{3}	68 ^{6}	62 ^{4}	66 ^{5}
100	[BIAS]	$\hat{\alpha}$	2.16364 ^{6}	0.94478 ^{5}	0.75327 ^{2}	3.10311 ^{7}	0.43409 ^{1}	6.11863 ^{8}	0.91021 ^{4}	0.84820 ^{3}
		$\hat{\beta}$	0.66456 ^{5}	0.75758 ^{7}	0.47592 ^{2}	0.78285 ^{8}	0.67493 ^{6}	0.46826 ^{1}	0.61663 ^{3}	0.64549 ^{4}
		$\hat{\lambda}$	1.36379 ^{6}	0.83453 ^{5}	0.64038 ^{3}	3.32039 ^{7}	0.50292 ^{1}	8.94560 ^{8}	0.62523 ^{2}	0.74755 ^{4}
		$\hat{a}$	1.25376 ^{8}	0.70241 ^{5}	0.37232 ^{2}	0.75603 ^{7}	0.50498 ^{3}	0.30254 ^{1}	0.59676 ^{4}	0.71469 ^{6}
		$\hat{b}$	2.16218 ^{6}	0.99282 ^{5}	0.60716 ^{1}	2.56580 ^{7}	0.79736 ^{4}	3.61195 ^{8}	0.75797 ^{2}	0.77455 ^{3}
	MSE	$\hat{\alpha}$	70.62486 ^{6}	16.09093 ^{5}	6.95665 ^{4}	1498.39858 ^{8}	2.91614 ^{1}	545.87552 ^{7}	4.86195 ^{2}	5.04971 ^{3}
		$\hat{\beta}$	0.67290 ^{4}	0.92739 ^{7}	0.44046 ^{2}	1.00213 ^{8}	0.87348 ^{6}	0.36643 ^{1}	0.64481 ^{3}	0.74530 ^{5}
		$\hat{\lambda}$	91.75183 ^{6}	24.55164 ^{4}	25.63545 ^{5}	2177.14525 ^{8}	5.55434 ^{3}	695.88672 ^{7}	1.93923 ^{1}	4.65979 ^{2}
		$\hat{a}$	6.62882 ^{8}	1.49417 ^{5}	0.52842 ^{2}	1.60826 ^{7}	1.42185 ^{4}	0.46663 ^{1}	1.23847 ^{3}	1.55006 ^{6}
		$\hat{b}$	34.15734 ^{6}	9.99693 ^{5}	5.62349 ^{3}	774.90471 ^{8}	6.17170 ^{4}	128.14934 ^{7}	2.40886 ^{2}	2.38248 ^{1}
	MRE	$\hat{\alpha}$	4.32727 ^{6}	1.88957 ^{5}	1.50654 ^{2}	6.20621 ^{7}	0.86819 ^{1}	12.23727 ^{8}	1.82041 ^{4}	1.69639 ^{3}
		$\hat{\beta}$	0.44304 ^{5}	0.50505 ^{7}	0.31728 ^{2}	0.52190 ^{8}	0.44995 ^{6}	0.31217 ^{1}	0.41108 ^{3}	0.43033 ^{4}
		$\hat{\lambda}$	1.81839 ^{6}	1.11271 ^{5}	0.85384 ^{3}	4.42719 ^{7}	0.67056 ^{1}	11.92747 ^{8}	0.83364 ^{2}	0.99673 ^{4}
		$\hat{a}$	2.08960 ^{8}	1.17068 ^{5}	0.62053 ^{2}	1.26006 ^{7}	0.84163 ^{3}	0.50423 ^{1}	0.99461 ^{4}	1.19115 ^{6}
		$\hat{b}$	3.08883 ^{6}	1.41831 ^{5}	0.86737 ^{1}	3.66542 ^{7}	1.13909 ^{4}	5.15993 ^{8}	1.08282 ^{2}	1.10650 ^{3}
		$\sum Ranks$	92 ^{7}	80 ^{6}	36 ^{1}	111 ^{8}	48 ^{3}	75 ^{5}	41 ^{2}	57 ^{4}
Continued										

n	Est.	Param.	MLE	LSE	WLSE	CRVME	MPSE	PCE	ADE	RADE
250	BIAS	$\hat{\alpha}$	0.82587 {7}	0.62340 {4}	0.31406 {2}	0.68447 {5}	0.22139 {1}	2.73846 {8}	0.70599 {6}	0.61480 {3}
		$\hat{\beta}$	0.42239 {4}	0.51508 {8}	0.25675 {1}	0.50376 {7}	0.38832 {2}	0.42636 {5}	0.40156 {3}	0.43625 {6}
		$\hat{\lambda}$	0.55364 {7}	0.37957 {4}	0.24858 {2}	0.36084 {3}	0.20235 {1}	4.82410 {8}	0.41619 {5}	0.50569 {6}
		$\hat{a}$	0.43256 {6}	0.43004 {5}	0.19663 {1}	0.46549 {8}	0.23577 {2}	0.33515 {4}	0.31286 {3}	0.46273 {7}
		$\hat{b}$	0.59968 {6}	0.55272 {5}	0.24388 {1}	0.60430 {7}	0.26822 {2}	1.32062 {8}	0.42652 {3}	0.48014 {4}
	MSE	$\hat{\alpha}$	5.69923 {7}	1.20227 {3}	0.45545 {2}	1.63393 {5}	0.07931 {1}	134.99496 {8}	2.73710 {6}	1.58262 {4}
		$\hat{\beta}$	0.28333 {3}	0.43183 {8}	0.13268 {1}	0.39463 {7}	0.29262 {4}	0.29600 {5}	0.27111 {2}	0.31955 {6}
		$\hat{\lambda}$	0.67957 {7}	0.22384 {4}	0.13277 {2}	0.20119 {3}	0.11365 {1}	336.23169 {8}	0.31536 {5}	0.52774 {6}
		$\hat{a}$	0.78395 {8}	0.56911 {5}	0.13984 {1}	0.66643 {6}	0.15116 {2}	0.51435 {4}	0.31648 {3}	0.67523 {7}
		$\hat{b}$	3.12773 {7}	0.94707 {5}	0.26579 {2}	1.21731 {6}	0.26268 {1}	20.27756 {8}	0.81260 {4}	0.80625 {3}
	MRE	$\hat{\alpha}$	1.65174 {7}	1.24680 {4}	0.62811 {2}	1.36895 {5}	0.30583 {1}	5.47691 {8}	1.41198 {6}	1.22960 {3}
		$\hat{\beta}$	0.28160 {4}	0.34339 {8}	0.17116 {2}	0.33584 {7}	0.16279 {1}	0.28424 {5}	0.26770 {3}	0.29083 {6}
		$\hat{\lambda}$	0.73819 {7}	0.50610 {4}	0.33144 {2}	0.48112 {3}	0.16406 {1}	6.43213 {8}	0.55492 {5}	0.67426 {6}
		$\hat{a}$	0.72093 {6}	0.71673 {5}	0.32772 {1}	0.77581 {8}	0.39295 {2}	0.55858 {4}	0.52143 {3}	0.77122 {7}
		$\hat{b}$	0.85668 {6}	0.78960 {5}	0.34840 {1}	0.86329 {7}	0.38318 {2}	1.88660 {8}	0.60932 {3}	0.68592 {4}
		$\sum Ranks$	92 {7}	77 {4}	23 {1}	87 {6}	24 {2}	99 {8}	60 {3}	78 {5}
500	BIAS	$\hat{\alpha}$	0.40975 {4}	0.47575 {6}	0.18043 {2}	0.49396 {7}	0.15292 {1}	8.00000 {8}	0.41072 {5}	0.40580 {3}
		$\hat{\beta}$	0.30412 {4}	0.37011 {7}	0.17144 {1}	0.35221 {6}	0.24418 {2}	0.39017 {8}	0.29132 {3}	0.31158 {5}
		$\hat{\lambda}$	0.51293 {7}	0.31844 {3}	0.17756 {2}	0.33591 {4}	0.12305 {1}	3.12398 {8}	0.34707 {5}	0.41005 {6}
		$\hat{a}$	0.22530 {4}	0.28002 {7}	0.10931 {1}	0.27400 {5}	0.15033 {2}	0.30692 {8}	0.19189 {3}	0.27683 {6}
		$\hat{b}$	0.23501 {4}	0.35597 {7}	0.12407 {1}	0.35175 {6}	0.16184 {2}	0.72900 {8}	0.22660 {3}	0.27655 {5}
	MSE	$\hat{\alpha}$	0.40440 {5}	0.71226 {6}	0.07205 {2}	0.83106 {7}	0.03838 {1}	56.69572 {8}	0.37205 {3}	0.38928 {4}
		$\hat{\beta}$	0.13965 {4}	0.22695 {7}	0.06367 {1}	0.21156 {6}	0.11189 {2}	0.25908 {8}	0.13940 {3}	0.16335 {5}
		$\hat{\lambda}$	0.59414 {7}	0.15478 {3}	0.06434 {2}	0.17191 {4}	0.03926 {1}	172.80713 {8}	0.22486 {5}	0.33706 {6}
		$\hat{a}$	0.14118 {4}	0.22698 {6}	0.03087 {1}	0.21407 {5}	0.04387 {2}	0.29298 {8}	0.09359 {3}	0.24602 {7}
		$\hat{b}$	0.20236 {4}	0.40840 {6}	0.04266 {1}	0.42089 {7}	0.06778 {2}	6.26779 {8}	0.15151 {3}	0.23461 {5}
	MRE	$\hat{\alpha}$	0.81950 {4}	0.95149 {6}	0.36086 {2}	0.98793 {7}	0.30583 {1}	3.45655 {8}	0.82143 {5}	0.81161 {3}
		$\hat{\beta}$	0.20275 {4}	0.24674 {7}	0.11429 {1}	0.23480 {6}	0.16279 {2}	0.26011 {8}	0.19422 {3}	0.20772 {5}
		$\hat{\lambda}$	0.68390 {7}	0.42458 {3}	0.23675 {2}	0.44788 {4}	0.16406 {1}	4.16530 {8}	0.46276 {5}	0.54673 {6}
		$\hat{a}$	0.37550 {4}	0.46670 {7}	0.18218 {1}	0.45667 {5}	0.25054 {2}	0.51153 {8}	0.31981 {3}	0.46138 {6}
		$\hat{b}$	0.33573 {4}	0.50853 {7}	0.17724 {1}	0.50251 {6}	0.23120 {2}	1.04143 {8}	0.32371 {3}	0.39507 {5}
		$\sum Ranks$	70 {4}	88 {7}	21 {1}	85 {6}	24 {2}	120 {8}	55 {3}	77 {5}

Table 10. Numerical values of the KAPLL distribution for $\theta = (0.5, 1.5, 0.75, 1.3, 0.3)^T$.

capturing the complex characteristics of the datasets, thereby validating its efficacy as a robust and generalizable extension of the LL family.

To validate the proposed KAPLL distribution beyond controlled simulations, we used five independent real-world datasets as external validation benchmarks. These datasets were chosen because they represent diverse domains (mechanical failure, material strength, and fracture toughness), cover a range of sample sizes (40–576), and exhibit different distributional shapes (positive skewness, heavy tails). By analyzing these datasets, we aimed to test the generalizability and robustness of the KAPLL model under practical conditions.

Table 18 presents the parameter estimates of the KAPLL distribution obtained using the eight estimation methods, along with the corresponding KS statistics and their associated PV for the five real datasets. Taken together, these analyses confirm that the datasets serve as effective validation tools: the KAPLL consistently adapts across domains, and the observed fit patterns are consistent with those obtained in the simulation study, thereby validating both the model and its estimation strategies. Based on the KS statistics and p -values, the CRVME method provides the best fit for Dataset 1, while the ML method performs best for Dataset 2. For Datasets 3, 4 and 5, the LS method yields the most favorable results. Overall, all estimation methods produce strong and consistent performance across the datasets, reinforcing the robustness and flexibility of the KAPLL model. Furthermore, the estimates derived from real data are largely consistent with the findings from the simulation study, where the WLSE method was most effective on average. Although WLSE is not always the

<i>n</i>	Est.	Param.	MLE	LSE	WLSE	CRVME	MPSE	PCE	ADE	RADE
50	[BIAS]	$\hat{\alpha}$	1.47507 ^{7}	1.35017 ^{6}	0.60770 ^{2}	0.83142 ^{3}	0.58597 ^{1}	2.19882 ^{8}	0.86140 ^{4}	1.08296 ^{5}
		$\hat{\beta}$	0.73413 ^{6}	0.87595 ^{8}	0.63505 ^{3}	0.82633 ^{7}	0.29422 ^{1}	0.40347 ^{2}	0.70372 ^{5}	0.65429 ^{4}
		$\hat{\lambda}$	0.80595 ^{2}	1.33009 ^{7}	0.89118 ^{4}	0.89758 ^{5}	0.60297 ^{1}	3.81477 ^{8}	0.80754 ^{3}	1.15516 ^{6}
		$\hat{a}$	4.21671 ^{8}	0.81180 ^{2}	0.78065 ^{1}	0.95423 ^{6}	1.27450 ^{7}	0.87367 ^{3}	0.95152 ^{5}	0.92921 ^{4}
		$\hat{b}$	0.66267 ^{7}	0.41288 ^{6}	0.27785 ^{1}	0.36366 ^{4}	0.33930 ^{3}	1.26333 ^{8}	0.31985 ^{2}	0.40640 ^{5}
	MSE	$\hat{\alpha}$	113.47375 ^{6}	245.53194 ^{8}	2.34354 ^{2}	2.44269 ^{3}	1.11265 ^{1}	144.57115 ^{7}	5.46429 ^{4}	6.16705 ^{5}
		$\hat{\beta}$	0.75280 ^{5}	1.16945 ^{8}	0.63200 ^{3}	1.01361 ^{7}	0.84466 ^{6}	0.30605 ^{1}	0.72891 ^{4}	0.60183 ^{2}
		$\hat{\lambda}$	1.57522 ^{2}	223.18141 ^{7}	4.21676 ^{5}	2.46128 ^{4}	1.03667 ^{1}	500.75830 ^{8}	1.66193 ^{3}	15.06015 ^{6}
		$\hat{a}$	77.40649 ^{6}	1.64067 ^{7}	2.17354 ^{1}	2.28303 ^{3}	9.20477 ^{5}	3.38021 ^{8}	2.86580 ^{2}	2.46340 ^{4}
		$\hat{b}$	2.95225 ^{7}	6.20152 ^{6}	0.34010 ^{2}	0.44953 ^{3}	0.92717 ^{1}	52.54912 ^{8}	0.41140 ^{4}	0.71895 ^{5}
	MRE	$\hat{\alpha}$	2.95014 ^{7}	2.70034 ^{6}	1.21539 ^{2}	1.66283 ^{3}	1.17193 ^{1}	4.39764 ^{8}	1.72280 ^{4}	2.16593 ^{5}
		$\hat{\beta}$	0.48942 ^{5}	0.58396 ^{8}	0.42337 ^{2}	0.55089 ^{7}	0.49012 ^{6}	0.26898 ^{1}	0.46915 ^{4}	0.43619 ^{3}
		$\hat{\lambda}$	0.80595 ^{2}	1.33009 ^{7}	0.89118 ^{4}	0.89758 ^{5}	0.60297 ^{1}	3.81477 ^{8}	0.80754 ^{3}	1.15516 ^{6}
		$\hat{a}$	7.02785 ^{8}	1.35300 ^{2}	1.30108 ^{1}	1.59038 ^{6}	2.12417 ^{7}	1.45612 ^{3}	1.58586 ^{5}	1.54868 ^{4}
		$\hat{b}$	2.20889 ^{7}	1.37626 ^{6}	0.92615 ^{1}	1.21220 ^{4}	1.13101 ^{3}	4.21109 ^{8}	1.06618 ^{2}	1.35467 ^{5}
		$\sum Ranks$	86 ^{6}	89 ^{8}	34 ^{1}	70 ^{5}	51 ^{2}	87 ^{7}	55 ^{3}	68 ^{4}
100	[BIAS]	$\hat{\alpha}$	1.10249 ^{7}	0.56864 ^{4}	0.31303 ^{1}	0.60828 ^{5}	0.32779 ^{2}	1.41597 ^{8}	0.46566 ^{3}	0.62576 ^{6}
		$\hat{\beta}$	0.53028 ^{4}	0.70489 ^{7}	0.47865 ^{2}	0.71237 ^{8}	0.52511 ^{3}	0.34542 ^{1}	0.53286 ^{5}	0.53934 ^{6}
		$\hat{\lambda}$	0.61146 ^{6}	0.59804 ^{5}	0.54349 ^{3}	0.62656 ^{7}	0.34043 ^{1}	2.69888 ^{8}	0.54220 ^{2}	0.59756 ^{4}
		$\hat{a}$	1.71458 ^{8}	0.72309 ^{4}	0.50748 ^{1}	0.87052 ^{7}	0.66570 ^{3}	0.73914 ^{5}	0.66122 ^{2}	0.75501 ^{6}
		$\hat{b}$	0.32276 ^{7}	0.23969 ^{5}	0.16642 ^{1}	0.28636 ^{6}	0.16672 ^{2}	0.88621 ^{8}	0.19188 ^{3}	0.23874 ^{4}
	MSE	$\hat{\alpha}$	64.02910 ^{7}	1.25178 ^{4}	0.22012 ^{2}	1.41248 ^{5}	0.20341 ^{1}	80.52489 ^{8}	0.79765 ^{3}	2.11849 ^{6}
		$\hat{\beta}$	0.42283 ^{5}	0.75950 ^{8}	0.36905 ^{2}	0.74388 ^{7}	0.44824 ^{6}	0.26565 ^{1}	0.40734 ^{3}	0.40799 ^{4}
		$\hat{\lambda}$	0.92763 ^{6}	0.70433 ^{3}	0.72169 ^{4}	0.79417 ^{5}	0.23896 ^{1}	331.39594 ^{8}	0.66874 ^{2}	0.93658 ^{7}
		$\hat{a}$	16.87478 ^{8}	1.32053 ^{2}	0.96302 ^{1}	1.80628 ^{5}	3.63644 ^{6}	3.84563 ^{7}	1.58257 ^{3}	1.68163 ^{4}
		$\hat{b}$	0.53473 ^{7}	0.18147 ^{4}	0.09872 ^{1}	0.24515 ^{6}	0.16152 ^{3}	28.55340 ^{8}	0.12042 ^{2}	0.18824 ^{5}
	MRE	$\hat{\alpha}$	2.20498 ^{7}	1.13729 ^{4}	0.62606 ^{1}	1.21656 ^{5}	0.65557 ^{2}	2.83195 ^{8}	0.93132 ^{3}	1.25152 ^{6}
		$\hat{\beta}$	0.35352 ^{4}	0.46993 ^{7}	0.31910 ^{2}	0.47492 ^{8}	0.35007 ^{3}	0.23028 ^{1}	0.35524 ^{5}	0.35956 ^{6}
		$\hat{\lambda}$	0.61146 ^{6}	0.59804 ^{5}	0.54349 ^{3}	0.62656 ^{7}	0.34043 ^{1}	2.69888 ^{8}	0.54220 ^{2}	0.59756 ^{4}
		$\hat{a}$	2.85764 ^{8}	1.20516 ^{4}	0.84580 ^{1}	1.45087 ^{7}	1.10951 ^{3}	1.23190 ^{5}	1.10203 ^{2}	1.25835 ^{6}
		$\hat{b}$	1.07587 ^{7}	0.79895 ^{5}	0.55472 ^{1}	0.95452 ^{6}	0.55574 ^{2}	2.95403 ^{8}	0.63959 ^{3}	0.79581 ^{4}
		$\sum Ranks$	97 ^{8}	71 ^{4}	26 ^{1}	94 ^{7}	39 ^{2}	92 ^{6}	43 ^{3}	78 ^{5}
Continued										

n	Est.	Param.	MLE	LSE	WLSE	CRVME	MPSE	PCE	ADE	RADE
250	BIAS	$\hat{\alpha}$	0.26812 {3}	0.30941 {6}	0.18630 {2}	0.27686 {4}	0.18017 {1}	0.42371 {8}	0.31139 {7}	0.29818 {5}
		$\hat{\beta}$	0.33627 {4}	0.49570 {8}	0.29639 {3}	0.47771 {7}	0.29422 {1}	0.29446 {2}	0.38212 {5}	0.39836 {6}
		$\hat{\lambda}$	0.44028 {6}	0.43630 {5}	0.26810 {2}	0.45685 {7}	0.18179 {1}	0.56347 {8}	0.38745 {4}	0.34445 {3}
		$\hat{a}$	0.55082 {7}	0.52289 {5}	0.27959 {2}	0.53319 {6}	0.23904 {1}	0.43160 {4}	0.38191 {3}	0.57117 {8}
		$\hat{b}$	0.13236 {4}	0.15056 {6}	0.09119 {2}	0.14224 {5}	0.07435 {1}	0.35848 {8}	0.12340 {3}	0.15214 {7}
	MSE	$\hat{\alpha}$	0.19310 {4}	0.23497 {5}	0.05472 {1}	0.15695 {3}	0.05922 {2}	1.37516 {8}	0.26105 {6}	0.37526 {7}
		$\hat{\beta}$	0.19078 {3}	0.36537 {8}	0.14267 {1}	0.34303 {7}	0.16323 {2}	0.25713 {6}	0.21245 {4}	0.24228 {5}
		$\hat{\lambda}$	0.55121 {7}	0.39393 {5}	0.18005 {2}	0.42076 {6}	0.07436 {1}	15.22493 {8}	0.35373 {4}	0.27357 {3}
		$\hat{a}$	2.21197 {8}	0.75770 {4}	0.28897 {1}	0.77491 {5}	0.40756 {2}	0.84513 {6}	0.49940 {3}	1.07326 {7}
		$\hat{b}$	0.05924 {6}	0.05547 {5}	0.02230 {2}	0.04790 {4}	0.01343 {1}	0.48880 {8}	0.04139 {3}	0.06575 {7}
	MRE	$\hat{\alpha}$	0.53625 {3}	0.61883 {6}	0.37261 {2}	0.55373 {4}	0.22470 {1}	0.84742 {8}	0.62279 {7}	0.59636 {5}
		$\hat{\beta}$	0.22418 {4}	0.33047 {8}	0.19760 {3}	0.31847 {7}	0.11139 {1}	0.19631 {2}	0.25474 {5}	0.26557 {6}
		$\hat{\lambda}$	0.44028 {6}	0.43630 {5}	0.26810 {2}	0.45685 {7}	0.11369 {1}	0.56347 {8}	0.38745 {4}	0.34445 {3}
		$\hat{a}$	0.91804 {7}	0.87148 {5}	0.46599 {2}	0.88866 {6}	0.39841 {1}	0.71933 {4}	0.63652 {3}	0.95195 {8}
		$\hat{b}$	0.44121 {4}	0.50185 {6}	0.30398 {2}	0.47414 {5}	0.24785 {1}	1.19492 {8}	0.41133 {3}	0.50713 {7}
		$\sum Ranks$	76 {4}	87 {6.5}	29 {2}	83 {5}	18 {1}	96 {8}	64 {3}	87 {6.5}
500	BIAS	$\hat{\alpha}$	0.22421 {5}	0.23118 {6}	0.13038 {2}	0.24395 {7}	0.11235 {1}	8.00000 {8}	0.21054 {4}	0.17752 {3}
		$\hat{\beta}$	0.22833 {3}	0.36684 {8}	0.19173 {2}	0.36103 {7}	0.16708 {1}	0.29401 {5}	0.27700 {4}	0.31467 {6}
		$\hat{\lambda}$	0.32169 {5}	0.40167 {8}	0.16680 {2}	0.37895 {6}	0.11369 {1}	0.39481 {7}	0.29942 {4}	0.23304 {3}
		$\hat{a}$	0.23835 {4}	0.36776 {6}	0.14127 {2}	0.34591 {5}	0.13397 {1}	0.44996 {8}	0.22884 {3}	0.41975 {7}
		$\hat{b}$	0.07456 {3}	0.10155 {5}	0.05089 {2}	0.10379 {6}	0.04671 {1}	0.39989 {8}	0.07615 {4}	0.10456 {7}
	MSE	$\hat{\alpha}$	0.72911 {7}	0.08826 {5}	0.03018 {2}	0.11029 {6}	0.02238 {1}	1.84928 {8}	0.08583 {4}	0.06527 {3}
		$\hat{\beta}$	0.10082 {3}	0.20417 {7}	0.06473 {2}	0.19448 {6}	0.06419 {1}	0.27770 {8}	0.11921 {4}	0.15784 {5}
		$\hat{\lambda}$	0.35164 {7}	0.34549 {6}	0.07306 {2}	0.32677 {5}	0.02424 {1}	1.60735 {8}	0.24969 {4}	0.10124 {3}
		$\hat{a}$	0.26772 {4}	0.38560 {6}	0.05788 {2}	0.32624 {5}	0.04805 {1}	1.14848 {8}	0.15310 {3}	0.59417 {7}
		$\hat{b}$	0.01650 {4}	0.02386 {5}	0.00570 {1}	0.02450 {6}	0.00576 {2}	0.74513 {8}	0.01328 {3}	0.02642 {7}
	MRE	$\hat{\alpha}$	0.44842 {5}	0.46237 {6}	0.26076 {2}	0.48791 {7}	0.22470 {1}	0.84215 {8}	0.42108 {4}	0.35504 {3}
		$\hat{\beta}$	0.15222 {3}	0.24456 {8}	0.12782 {2}	0.24069 {7}	0.11139 {1}	0.19600 {5}	0.18467 {4}	0.20978 {6}
		$\hat{\lambda}$	0.32169 {5}	0.40167 {8}	0.16680 {2}	0.37895 {6}	0.11369 {1}	0.39481 {7}	0.29942 {4}	0.23304 {3}
		$\hat{a}$	0.39725 {4}	0.61294 {6}	0.23545 {2}	0.57651 {5}	0.22328 {1}	0.74993 {8}	0.38140 {3}	0.69959 {7}
		$\hat{b}$	0.24853 {3}	0.33852 {5}	0.16964 {2}	0.34595 {6}	0.15571 {1}	1.33298 {8}	0.25382 {4}	0.34853 {7}
		$\sum Ranks$	65 {4}	95 {7}	29 {2}	90 {6}	16 {1}	112 {8}	56 {3}	77 {5}

Table 11. Numerical values of the KAPLL distribution for $\theta = (0.5, 1.5, 1, 0.6, 0.3)^T$.

top performer in the real data analysis, the observed variability is expected, as these results are based on single-sample evaluations, unlike the broader simulations that span various sample sizes and parameter configurations. These findings collectively confirm the practical utility of the KAPLL distribution and the adaptability of its estimation procedures to diverse data conditions.

Comparative evaluation and discussion

This comparative evaluation across validation datasets further strengthens the case for the KAPLL distribution. Unlike purely illustrative examples, these datasets act as independent benchmarks, confirming that the proposed model is not overfitted to simulated data but performs well in diverse empirical contexts.

The performance of the proposed KAPLL model was evaluated against several recent extensions of the LL distribution, including the APTLL, KMOLL, McL, AWLL, APPL, TLL, and classical LL models. The comparison, based on criteria such as AIC, CAIC, HQIC, BIC, $-\ell$, W^* , A^* , KS and PV statistics, consistently showed that the proposed model provided superior goodness-of-fit across multiple real-life datasets. This improvement can be attributed to the added flexibility introduced by the KAP transformation, which enable the model to capture complex hazard rate shapes—including bathtub and unimodal patterns—more effectively than existing models. Additionally, the KAPLL model retains closed-form expressions for its density and distribution functions, which is a practical advantage over more computationally intensive alternatives. These findings confirm that

<i>n</i>	Est.	Param.	MLE	LSE	WLSE	CRVME	MPSE	PCE	ADE	RADE
50	[BIAS]	$\hat{\alpha}$	2.46873 ^{7}	1.50327 ^{4}	1.07512 ^{2}	2.24531 ^{5}	1.05461 ^{1}	5.68023 ^{8}	1.27365 ^{3}	2.28022 ^{6}
		$\hat{\beta}$	0.83438 ^{4}	1.02485 ^{8}	0.68106 ^{3}	0.95433 ^{7}	0.38875 ^{1}	0.48673 ^{2}	0.85267 ^{6}	0.84835 ^{5}
		$\hat{\lambda}$	4.72668 ^{7}	1.20624 ^{4}	1.07947 ^{1}	1.79835 ^{5}	1.18788 ^{3}	10.35418 ^{8}	1.11514 ^{2}	2.72375 ^{6}
		$\hat{a}$	2.78063 ^{8}	0.76499 ^{3}	0.66556 ^{2}	0.93139 ^{6}	1.13931 ^{7}	0.57598 ^{1}	0.90956 ^{4}	0.91439 ^{5}
		$\hat{b}$	2.92586 ^{8}	0.78496 ^{3}	0.56769 ^{1}	1.04694 ^{5}	0.92752 ^{4}	2.85140 ^{7}	0.74643 ^{2}	1.12961 ^{6}
	MSE	$\hat{\alpha}$	228.97008 ^{5}	74.73521 ^{3}	19.49588 ^{1}	480.10332 ^{8}	94.92917 ^{4}	334.95326 ^{6}	21.91302 ^{2}	437.03864 ^{7}
		$\hat{\beta}$	1.03780 ^{3}	1.86183 ^{8}	0.88015 ^{2}	1.55943 ^{6}	1.61369 ^{7}	0.38582 ^{1}	1.23939 ^{5}	1.14230 ^{4}
		$\hat{\lambda}$	8210.41973 ^{8}	50.93842 ^{3}	21.32715 ^{1}	271.40051 ^{5}	122.51676 ^{4}	990.32258 ^{7}	23.78871 ^{2}	607.71077 ^{6}
		$\hat{a}$	35.03124 ^{8}	1.56229 ^{3}	1.66976 ^{1}	2.35482 ^{6}	7.25773 ^{4}	1.82019 ^{7}	2.58486 ^{2}	2.28277 ^{5}
		$\hat{b}$	824.43492 ^{7}	7.90810 ^{4}	2.19108 ^{2}	41.89879 ^{5}	8.41371 ^{1}	116.62293 ^{8}	4.51998 ^{3}	39.14577 ^{6}
	MRE	$\hat{\alpha}$	4.93746 ^{7}	3.00653 ^{4}	2.15024 ^{2}	4.49062 ^{5}	2.10921 ^{1}	11.36046 ^{8}	2.54730 ^{3}	4.56044 ^{6}
		$\hat{\beta}$	0.55625 ^{3}	0.68323 ^{8}	0.45404 ^{2}	0.63622 ^{7}	0.63086 ^{6}	0.32448 ^{1}	0.56845 ^{5}	0.56557 ^{4}
		$\hat{\lambda}$	4.72668 ^{7}	1.20624 ^{4}	1.07947 ^{1}	1.79835 ^{5}	1.18788 ^{3}	10.35418 ^{8}	1.11514 ^{2}	2.72375 ^{6}
		$\hat{a}$	4.63439 ^{8}	1.27499 ^{3}	1.10926 ^{2}	1.55232 ^{6}	1.89885 ^{7}	0.95997 ^{1}	1.51593 ^{4}	1.52398 ^{5}
		$\hat{b}$	5.85172 ^{8}	1.56992 ^{3}	1.13538 ^{1}	2.09388 ^{5}	1.85505 ^{4}	5.70280 ^{7}	1.49286 ^{2}	2.25921 ^{6}
		$\sum Ranks$	99 ^{8}	62 ^{3}	24 ^{1}	86 ^{7}	63 ^{4}	75 ^{5}	50 ^{2}	81 ^{6}
100	[BIAS]	$\hat{\alpha}$	1.07208 ^{7}	0.77388 ^{5}	0.41171 ^{2}	0.82799 ^{6}	0.37952 ^{1}	3.55031 ^{8}	0.72975 ^{4}	0.72623 ^{3}
		$\hat{\beta}$	0.64446 ^{5}	0.75245 ^{7}	0.48024 ^{2}	0.76151 ^{8}	0.68140 ^{6}	0.40954 ^{1}	0.58448 ^{3}	0.63475 ^{4}
		$\hat{\lambda}$	0.88780 ^{7}	0.69234 ^{4}	0.61278 ^{2}	0.67647 ^{3}	0.43630 ^{1}	6.32103 ^{8}	0.74036 ^{5}	0.85385 ^{6}
		$\hat{a}$	1.27283 ^{8}	0.69867 ^{4}	0.41716 ^{1}	0.78040 ^{7}	0.54511 ^{2}	0.71922 ^{5}	0.59991 ^{3}	0.74615 ^{6}
		$\hat{b}$	0.84778 ^{7}	0.48846 ^{5}	0.29979 ^{1}	0.60746 ^{6}	0.42882 ^{3}	1.34441 ^{8}	0.41222 ^{2}	0.45605 ^{4}
	MSE	$\hat{\alpha}$	22.31918 ^{7}	7.84381 ^{6}	0.61650 ^{2}	2.19473 ^{3}	0.35059 ^{1}	271.09529 ^{8}	3.40561 ^{5}	2.97883 ^{4}
		$\hat{\beta}$	0.62929 ^{4}	0.96223 ^{7}	0.45664 ^{2}	0.96562 ^{8}	0.83703 ^{6}	0.27979 ^{1}	0.58119 ^{3}	0.65746 ^{5}
		$\hat{\lambda}$	2.89763 ^{7}	0.82032 ^{2}	1.09764 ^{4}	1.05818 ^{3}	0.49223 ^{1}	610.41828 ^{8}	1.19403 ^{5}	1.66506 ^{6}
		$\hat{a}$	6.98005 ^{8}	1.33234 ^{3}	0.72800 ^{1}	1.57330 ^{4}	1.57802 ^{5}	2.37358 ^{7}	1.26787 ^{2}	1.57882 ^{6}
		$\hat{b}$	4.48509 ^{7}	0.76355 ^{4}	0.42530 ^{1}	1.11715 ^{5}	1.57909 ^{6}	49.34355 ^{8}	0.71176 ^{2}	0.75368 ^{3}
	MRE	$\hat{\alpha}$	2.14417 ^{7}	1.54775 ^{5}	0.82342 ^{2}	1.65598 ^{6}	0.75905 ^{1}	7.10062 ^{8}	1.45951 ^{4}	1.45246 ^{3}
		$\hat{\beta}$	0.42964 ^{5}	0.50163 ^{7}	0.32016 ^{2}	0.50767 ^{8}	0.45427 ^{6}	0.27303 ^{1}	0.38965 ^{3}	0.42317 ^{4}
		$\hat{\lambda}$	0.88780 ^{7}	0.69234 ^{4}	0.61278 ^{2}	0.67647 ^{3}	0.43630 ^{1}	6.32103 ^{8}	0.74036 ^{5}	0.85385 ^{6}
		$\hat{a}$	2.12139 ^{8}	1.16445 ^{4}	0.69527 ^{1}	1.30066 ^{7}	0.90852 ^{2}	1.19870 ^{5}	0.99985 ^{3}	1.24358 ^{6}
		$\hat{b}$	1.69555 ^{7}	0.97691 ^{5}	0.59957 ^{1}	1.21492 ^{6}	0.85763 ^{3}	2.68882 ^{8}	0.82444 ^{2}	0.91210 ^{4}
		$\sum Ranks$	101 ^{8}	72 ^{5}	26 ^{1}	83 ^{6}	45 ^{2}	92 ^{7}	51 ^{3}	70 ^{4}
Continued										

n	Est.	Param.	MLE	LSE	WLSE	CRVME	MPSE	PCE	ADE	RADE
250	BIAS	$\hat{\alpha}$	0.54621 {7}	0.46629 {4}	0.26668 {2}	0.46766 {5}	0.20313 {1}	2.16331 {8}	0.51787 {6}	0.46558 {3}
		$\hat{\beta}$	0.40508 {5}	0.48611 {8}	0.26969 {1}	0.47957 {7}	0.38875 {4}	0.38195 {2}	0.38668 {3}	0.42177 {6}
		$\hat{\lambda}$	0.68615 {7}	0.55401 {4}	0.35997 {2}	0.53779 {3}	0.19201 {1}	4.15394 {8}	0.60633 {5}	0.68281 {6}
		$\hat{a}$	0.40668 {4}	0.44402 {5}	0.21086 {1}	0.48021 {6}	0.25007 {2}	0.55649 {8}	0.31699 {3}	0.50003 {7}
		$\hat{b}$	0.25910 {4}	0.30269 {6}	0.14913 {1}	0.31372 {7}	0.16224 {2}	0.76898 {8}	0.23147 {3}	0.28774 {5}
	MSE	$\hat{\alpha}$	3.78127 {7}	0.62432 {3}	0.12188 {2}	0.69081 {4}	0.06487 {1}	114.98991 {8}	1.10701 {6}	0.84401 {5}
		$\hat{\beta}$	0.25117 {3}	0.39963 {8}	0.13788 {1}	0.37206 {7}	0.29037 {6}	0.25328 {4}	0.24794 {2}	0.28902 {5}
		$\hat{\lambda}$	1.08780 {6}	0.56228 {4}	0.31544 {2}	0.52554 {3}	0.12261 {1}	274.42554 {8}	0.74532 {5}	1.09131 {7}
		$\hat{a}$	0.65997 {5}	0.60117 {4}	0.14377 {1}	0.71359 {6}	0.21295 {2}	1.28579 {8}	0.30044 {3}	0.76844 {7}
		$\hat{b}$	0.32634 {7}	0.27528 {5}	0.07706 {2}	0.30584 {6}	0.07612 {1}	10.83367 {8}	0.19746 {3}	0.27193 {4}
	MRE	$\hat{\alpha}$	1.09243 {7}	0.93258 {4}	0.53336 {2}	0.93532 {5}	0.28938 {1}	4.32663 {8}	1.03574 {6}	0.93116 {3}
		$\hat{\beta}$	0.27005 {5}	0.32407 {8}	0.17980 {2}	0.31971 {7}	0.16238 {1}	0.25464 {3}	0.25779 {4}	0.28118 {6}
		$\hat{\lambda}$	0.68615 {7}	0.55401 {4}	0.35997 {2}	0.53779 {3}	0.11303 {1}	4.15394 {8}	0.60633 {5}	0.68281 {6}
		$\hat{a}$	0.67781 {4}	0.74003 {5}	0.35143 {1}	0.80035 {6}	0.41678 {2}	0.92748 {8}	0.52831 {3}	0.83339 {7}
		$\hat{b}$	0.51820 {4}	0.60538 {6}	0.29827 {1}	0.62743 {7}	0.32447 {2}	1.53795 {8}	0.46295 {3}	0.57549 {5}
		$\sum Ranks$	82 {6}	78 {4}	23 {1}	82 {6}	28 {2}	105 {8}	60 {3}	82 {6}
500	BIAS	$\hat{\alpha}$	0.34139 {4}	0.37230 {6}	0.19367 {2}	0.38140 {7}	0.14469 {1}	8.00000 {8}	0.34746 {5}	0.31042 {3}
		$\hat{\beta}$	0.30145 {4}	0.34737 {7}	0.18065 {1}	0.32520 {5}	0.24357 {2}	0.36135 {8}	0.28151 {3}	0.33093 {6}
		$\hat{\lambda}$	0.62313 {7}	0.48516 {3}	0.26110 {2}	0.49756 {4}	0.11303 {1}	1.45986 {8}	0.51943 {5}	0.55246 {6}
		$\hat{a}$	0.23341 {4}	0.28740 {6}	0.11910 {1}	0.27032 {5}	0.15609 {2}	0.57342 {8}	0.19806 {3}	0.31712 {7}
		$\hat{b}$	0.13862 {4}	0.19450 {7}	0.08456 {1}	0.19011 {6}	0.10401 {2}	0.38428 {8}	0.13584 {3}	0.17567 {5}
	MSE	$\hat{\alpha}$	0.19516 {5}	0.33961 {6}	0.06017 {2}	0.41146 {7}	0.03274 {1}	20.82571 {8}	0.18623 {3}	0.18712 {4}
		$\hat{\beta}$	0.13963 {4}	0.20622 {7}	0.06497 {1}	0.18109 {6}	0.11810 {2}	0.22528 {8}	0.12649 {3}	0.17203 {5}
		$\hat{\lambda}$	0.91856 {7}	0.42759 {3}	0.16814 {2}	0.46390 {4}	0.03722 {1}	40.43600 {8}	0.58600 {5}	0.69728 {6}
		$\hat{a}$	0.13128 {4}	0.23670 {6}	0.03610 {1}	0.20947 {5}	0.04661 {2}	1.07637 {8}	0.09867 {3}	0.29810 {7}
		$\hat{b}$	0.04268 {3}	0.10968 {6}	0.01709 {1}	0.11099 {7}	0.02372 {2}	1.74608 {8}	0.04400 {4}	0.07739 {5}
	MRE	$\hat{\alpha}$	0.68278 {4}	0.74460 {6}	0.38735 {2}	0.76280 {7}	0.28938 {1}	1.67945 {8}	0.69493 {5}	0.62084 {3}
		$\hat{\beta}$	0.20096 {4}	0.23158 {7}	0.12044 {1}	0.21680 {5}	0.16238 {2}	0.24090 {8}	0.18768 {3}	0.22062 {6}
		$\hat{\lambda}$	0.62313 {7}	0.48516 {3}	0.26110 {2}	0.49756 {4}	0.11303 {1}	1.45986 {8}	0.51943 {5}	0.55246 {6}
		$\hat{a}$	0.38902 {4}	0.47900 {6}	0.19850 {1}	0.45053 {5}	0.26016 {2}	0.95570 {8}	0.33011 {3}	0.52854 {7}
		$\hat{b}$	0.27725 {4}	0.38901 {7}	0.16912 {1}	0.38023 {6}	0.20803 {2}	0.76855 {8}	0.27167 {3}	0.35135 {5}
		$\sum Ranks$	69 {4}	86 {7}	21 {1}	83 {6}	24 {2}	120 {8}	56 {3}	81 {5}

Table 12. Numerical values of the KAPLL distribution for $\theta = (0.5, 1.5, 1, 0.6, 0.5)^T$.

the proposed model not only enhances flexibility but also improves empirical performance, making it a valuable addition to the family of generalized LL distributions.

Furthermore, we conducted an ablation study to evaluate the performance of the proposed KAPLL model in comparison to its nested sub-models, as listed in Table 2. The models were assessed using several goodness-of-fit criteria, including the $-\ell$, W^* , A^* , KS statistics, and the corresponding PV. We intentionally excluded information-based criteria such as AIC, CAIC, BIC, and HQIC from this comparison, as these metrics are influenced by the number of model parameters, which could bias the evaluation in favor of simpler models. Instead, we focused on distributional goodness-of-fit measures that assess how well the models capture the empirical behavior of the data. The comparison was carried out across all five real datasets. As presented in Table 19, the KAPLL model consistently achieved lower goodness-of-fit statistics and higher p -values, indicating a superior fit relative to its sub-models.

Figures 9, 10, 11, 12 and 13 display the quantile-quantile (QQ) plots comparing the empirical quantiles of the observed data with the theoretical quantiles of the KAPLL model and its nested sub-models across the five datasets. A model that fits well will produce points that closely follow the 45-degree reference line. As evident from the plots, the KAPLL distribution consistently shows the closest alignment to this line in all cases, visually confirming its superior goodness-of-fit and flexibility relative to its sub-models.

θ^T	n	MLE	LSE	WLSE	CRVME	MPSE	PCE	ADE	RADE
$\alpha = 0.5, \beta = 1.5, \lambda = 0.5, a = 0.9, b = 1.2$	50	8	2	1	6	7	5	4	3
	100	8	4	1	7	3	6	2	5
	250	7	5.5	1	5.5	2	8	3	4
	500	6	5	2	7	1	8	3	4
$\alpha = 0.5, \beta = 1.5, \lambda = 0.5, a = 1.1, b = 1.2$	50	8	2	1	3	7	6	4	5
	100	8	4	1	7	3	6	2	5
	250	7	4	1	6	2	8	3	5
	500	7	6	2	5	1	8	3	4
$\alpha = 0.5, \beta = 1.5, \lambda = 0.5, a = 1.3, b = 1.3$	50	8	2	1	3	7	5	4	6
	100	8	4	1	7	2	6	3	5
	250	7	4	1	6	2	8	3	5
	500	6	5	1	7	2	8	3	4
$\alpha = 0.5, \beta = 1.5, \lambda = 0.75, a = 0.6, b = 0.3$	50	8	6	1	5	2	7	3	4
	100	8	4.5	1	7	2	6	3	4.5
	250	6	7	2	5	1	8	3	4
	500	3	7	2	6	1	8	4	5
$\alpha = 0.5, \beta = 1.5, \lambda = 0.75, a = 0.6, b = 0.5$	50	8	3	1	7	4	5	2	6
	100	8	5	1	6	2	7	3	4
	250	5	4	1	7	2	8	3	6
	500	4	7	1	6	2	8	3	5
$\alpha = 0.5, \beta = 1.5, \lambda = 0.75, a = 1.3, b = 0.3$	50	8	2	1	7	3	6	4	5
	100	7	6	1	8	3	5	2	4
	250	7	4	1	6	2	8	3	5
	500	4	7	1	6	2	8	3	5
$\alpha = 0.5, \beta = 1.5, \lambda = 1, a = 0.6, b = 0.3$	50	6	8	1	5	2	7	3	4
	100	8	4	1	7	2	6	3	5
	250	4	6.5	2	5	1	8	3	6.5
	500	4	7	2	6	1	8	3	5
$\alpha = 0.5, \beta = 1.5, \lambda = 1, a = 0.6, b = 0.5$	50	8	3	1	7	4	5	2	6
	100	8	5	1	6	2	7	3	4
	250	6	4	1	6	2	8	3	6
	500	4	7	1	6	2	8	3	5
$\sum Ranks$		212	154.5	38	193.5	81	223	96	154
Overall rank		7	5	1	6	2	8	3	4

Table 13. The ordering performance of all estimators for combinations of parameters

	n	μ_X	Median	SD	VA	SK	KU
Data 1	40	6.2525	6.50	1.955530	3.824096	− 0.6886558	− 0.241834
Data 2	63	1.5068	1.59	0.324126	0.105058	− 0.9220262	1.103138
Data 3	119	4.3254	4.38	1.018495	1.037332	− 0.4220524	0.149670
Data 4	300	32.9150	32.80	4.158355	17.291910	0.1668816	− 0.332076
Data 5	576	3.5140	3.01	2.179863	4.751801	4.7518010	1.156017

Table 14. Descriptive statistics for the five real-life datasets.

To justify the necessity of the proposed KAPLL distribution, which extends the classical LL model by introducing three additional shape parameters (α , a , and b), we conducted a rigorous ablation study using the LRT. This test compares the KAPLL model against its nested sub-model, the LL distribution, across five real-world datasets. Since the KAPLL distribution is a generalization of LL distribution, the LRT statistic follows a chi-square distribution with 3 degrees of freedom. The LRT statistic is calculated as $w = 2(\ell_{\text{KAPLL}} - \ell_{\text{LL}})$, where ℓ_{KAPLL} and ℓ_{LL} are the maximized log-likelihoods of the KAPLL and LL models, respectively. As shown in Table 20, the LRT statistics are highly significant across all five datasets, with very small p -values, indicating strong evidence in favor of the KAPLL model.

These findings provide strong statistical evidence that the added complexity of the KAPLL model translates into meaningful improvements in fit, validating both its theoretical advancement and practical utility in

Distribution	AIC	CAIC	BIC	HQIC	W*	A*	$-\ell$	KS	PV
Dataset 1									
KAPLL	167.85890	169.62360	176.30330	170.91210	0.01941	0.15087	78.92943	0.07307	0.98318
LL	181.41330	181.73760	184.79100	182.63450	0.21418	1.40722	88.70663	0.14369	0.38072
APLL	209.75120	210.07550	213.12890	210.97240	0.45929	2.73731	102.87560	0.26291	0.00793
TLL	424.59910	425.26580	429.66580	426.43110	0.29479	1.86508	209.29960	0.71682	0.00000
APTLL	183.39570	184.06230	188.46230	185.22760	0.21332	1.40194	88.69784	0.14407	0.37745
KMOLL	174.94480	176.70950	183.38920	177.99800	0.07693	0.57260	82.47240	0.10770	0.74232
McLL	183.85530	185.62000	192.29970	186.90850	0.19024	1.27427	86.92764	0.12411	0.56878
AWLL	176.95100	179.49650	187.08430	180.61490	0.07699	0.57304	82.47551	0.10770	0.74231
Dataset 2									
KAPLL	27.68430	28.73693	38.39997	31.89882	0.01775	0.12224	8.84215	0.04469	0.99961
LL	49.57993	49.77993	53.86619	51.26574	0.49694	2.74897	22.78996	0.15371	0.10187
APLL	64.63768	64.83768	68.92394	66.32349	0.73896	4.09825	30.31884	0.22029	0.00442
TLL	495.06690	495.47370	501.49630	497.59560	0.69029	3.77390	244.53350	0.71750	0.00000
APTLL	51.41486	51.82164	57.84426	53.94358	0.49183	2.71991	22.70743	0.14979	0.11834
KMOLL	28.08611	29.13874	38.80178	32.30064	0.01807	0.12722	9.04306	0.05363	0.99349
McLL	43.98007	45.03270	54.69574	48.19459	0.31606	1.72577	16.99003	0.18998	0.02118
AWLL	42.41368	43.91368	55.27249	47.47111	0.23724	1.30370	15.20684	0.15224	0.10784
Dataset 3									
KAPLL	345.03000	345.56100	358.92560	350.67260	0.04197	0.26565	167.51500	0.04633	0.96039
LL	360.67000	360.77340	366.22820	362.92700	0.35495	2.23138	178.33500	0.08032	0.42650
APLL	456.01840	456.12180	461.57660	458.27540	0.96857	5.58224	226.00920	0.23852	0.00000
TLL	1142.62400	1142.83300	1150.96200	1146.01000	0.55031	3.27610	568.31220	0.71297	0.00000
APTLL	362.60310	362.81180	370.94050	365.98860	0.35212	2.21646	178.30150	0.07881	0.45075
KMOLL	345.70010	346.23110	359.59570	351.34270	0.05179	0.31448	167.85010	0.05364	0.88336
McLL	345.61430	346.14530	359.50990	351.25690	0.05363	0.32622	167.80710	0.05591	0.85091
AWLL	349.04620	349.79620	365.72090	355.81720	0.07612	0.46281	168.52310	0.06662	0.66640
Dataset 4									
KAPLL	1713.42200	1713.62600	1731.94100	1720.83300	0.05771	0.38980	851.71080	0.04204	0.66406
LL	2150.37200	2150.41200	2157.78000	2153.33700	0.07734	0.56860	1073.18600	0.28760	0.00000
APLL	2858.35800	2858.39800	2865.76500	2861.32200	0.08269	0.61983	1427.17900	0.44998	0.00000
TLL	1722.23500	1722.31600	1733.34600	1726.68200	0.15225	1.02731	858.11750	0.05205	0.39073
APTLL	1722.23500	1722.31600	1733.34600	1726.68200	0.15225	1.02730	858.11750	0.05205	0.39073
KMOLL	1713.75400	1713.95800	1732.27300	1721.16500	0.05992	0.40195	851.87690	0.04233	0.65566
McLL	1713.51900	1713.72300	1732.03800	1720.93000	0.05966	0.38997	851.75960	0.04248	0.65101
AWLL	1742.39500	1742.68200	1764.61800	1751.28900	0.31129	1.88895	865.19770	0.05706	0.28271
Dataset 5									
KAPLL	2350.58500	2350.69000	2372.36500	2359.07900	0.03279	0.35649	1170.29200	0.02575	0.83958
LL	2366.13600	2366.15700	2374.84800	2369.53400	0.18117	1.20096	1181.06800	0.03638	0.43104
APLL	2401.04800	2401.06900	2409.76100	2404.44600	0.63964	3.86817	1198.52400	0.06408	0.01764
TLL	5063.91900	5063.96100	5076.98800	5069.01600	0.10583	0.71790	2528.96000	0.68000	0.00000
APTLL	2368.13600	2368.17800	2381.20400	2373.23200	0.18117	1.20096	1181.06800	0.03638	0.43104
KMOLL	2382.33100	2382.43600	2404.11100	2390.82500	0.46581	3.34421	1186.16500	0.05880	0.03725
McLL	2350.65000	2350.75500	2372.43000	2359.14400	0.03321	0.35872	1170.32500	0.02611	0.82711
AWLL	2383.62600	2383.77400	2409.76300	2393.81900	0.45517	3.27307	1185.81300	0.05796	0.04171

Table 15. Goodness-of-fit measures for the five datasets of the KAPLL distribution and other competing LL models.

modeling complex data patterns. Thus, the validation datasets not only highlight the superiority of the KAPLL model over its sub-models but also establish its empirical relevance across varied real-world scenarios.

Conclusions and future perspectives

In this paper, a flexible five-parameter model called the Kumaraswamy alpha-power log-logistic (KAPLL) distribution is introduced. The hazard function of the KAPLL model provides monotonic and non-monotonic failure rates, as well as its density can be symmetric and asymmetric. The KAPLL distribution exhibits various types of hazard rate behaviors including increasing, decreasing, J-shaped, reversed J-shaped, bathtub-shaped,

Distribution	Estimates and SEs					
Dataset 1						
KAPLL ($\alpha, \beta, \lambda, a, b$)	9658.12900	6.19488	999785.20000	0.43938	30.83448	
	6823.38144	0.35984	6803.29770	0.11247	43.97041	
LL (β, λ)	4.84162	6996.47772				
	0.70121	9447.15447				
APLL (β, λ)	8.74463	680523.30000				
	0.10925	16777.78000				
TLL (α, β, λ)	979989.90000	0.14731	1.00000			
	8388.79200	0.01632	0.39324			
APTLL (α, a, b)	1.93292	5.82120	4.81336			
	5.62217	1.72831	0.69531			
KMOLL ($\alpha, \beta, \lambda, a, b$)	28.71171	22.67472	5.66345	0.68373	2087.31877	
	241.16458	984.37198	26.24854	3.16798	26900.79379	
McLL ($\alpha, \beta, \lambda, a, b$)	5.83242	77.11220	1.10483	79.95686	0.18127	
	2.63637	3.15352	0.72072	51.18584	0.12886	
AWLL ($\alpha, \lambda, a, b, c, d$)	1.89384	41.17856	0.01666	4.73299	998.90351	2.04481
	26.33612	341.11273	74.60581	76.26542	32032.35327	28.43561
Dataset 2						
KAPLL ($\alpha, \beta, \lambda, a, b$)	1436.63900	26.01191	99993.32000	0.11148	0.53881	
	20.76099	2.59224	5.18317	0.02883	0.24718	
LL (β, λ)	7.92598	28.52572				
	0.87357	12.74795				
APLL (β, λ)	5.04477	263.37020				
	0.40022	222.39458				
TLL (α, β, λ)	376985.90000	0.14224	0.99999			
	11868.46000	0.01256	0.31326			
APTLL (α, a, b)	3.06807	1.42544	7.81452			
	4.99846	0.14176	0.92053			
KMOLL ($\alpha, \beta, \lambda, a, b$)	1.62207	0.99388	47.42360	0.05883	0.27869	
	1.21255	34.91133	15.51009	0.03362	0.10505	
McLL ($\alpha, \beta, \lambda, a, b$)	1.01073	0.04710	133.51100	3977.39000	345.44880	
	0.25336	0.01798	81.29733	4894.88900	326.84480	
AWLL ($\alpha, \lambda, a, b, c, d$)	2.03601	5.37781	0.00485	4.73315	999.36750	2.83922
	25.66087	17.84901	11.77119	61.95393	19163.01736	35.78454

Table 16. The ML estimates (first line) and SEs (second line) for datasets 1 and 2.

inverted bathtub-shaped, and even more complex forms such as decreasing–increasing–decreasing failure rates. Its mathematical features are explored. Furthermore, the KAPLL parameters are estimated by eight approaches of estimation. A simulation study is performed to compare these approaches and showed that the weighted least-squares (WLS) method outperforms all considered estimation methods. Hence, our results, confirms the superiority of the WLS approach in estimating the KAPLL parameters. Finally, the practical importance of the KAPLL distribution is addressed through the analysis of five real-life datasets from reliability, engineering, biomedical, and environmental sciences. Goodness-of-fit statistics and graphical assessments illustrated that the KAPLL model consistently provides a superior fit compared to other log-logistic-based models. Furthermore, this superiority was substantiated through a comprehensive ablation study and likelihood ratio testing. Compared to its nested sub-models, the KAPLL distribution consistently achieved better goodness-of-fit across all datasets. These findings highlight the practical value of the additional shape parameters in enhancing model flexibility and capturing complex data behaviors. Despite its flexibility and strong empirical performance, the KAPLL distribution has some limitations due to its five-parameter structure. In particular, parameter estimation can be challenging for small or low-variability datasets, and the risk of overfitting may be higher than that of simpler models. Users should weigh these considerations carefully when applying the model.

Building upon the foundation of the KAPLL distribution, several promising directions can be pursued to expand its scope and practical relevance across diverse domains. Potential avenues for future research include: Enhancing existing parameter estimation techniques, such as maximum likelihood and Bayesian methods, particularly in the presence of censored data, to increase the model's reliability and precision in real-world applications. Investigating non-parametric or semi-parametric adaptations of the model, which could offer greater flexibility and broader applicability. Further integrating Bayesian frameworks for both inference and model selection, which may provide deeper insights and comparative advantages over traditional methods.

Distribution	Estimates					
Dataset 3						
KAPLL ($\alpha, \beta, \lambda, a, b$)	220.44540	5.49738	4511.03345	0.63990	4.74243	
	503.42917	1.45103	9165.94237	0.25529	4.26687	
LL (β, λ)	7.06485	31046.87530				
	0.24683	10727.58000				
APLL (β, λ)	2.25132	13611220.00000				
	0.07740	8388.60800				
TLL (α, β, λ)	176885.90000	0.16637	0.99999			
	5939.24000	0.01070	0.22795			
APTLL (α, a, b)	2.15546	4.09800	7.01220			
	3.68724	0.48910	0.59645			
KMOLL ($\alpha, \beta, \lambda, a, b$)	7.31997	0.00758	14.43501	0.28314	1.01216	
	1.59190	0.00812	5.42979	0.15190	0.74382	
McLL ($\alpha, \beta, \lambda, a, b$)	11.84917	5.33969	0.34948	1.44853	1.29014	
	8.91872	0.60862	0.29000	2.01147	3.25343	
AWLL ($\alpha, \lambda, a, b, c, d$)	2.22794	11.62858	117.62741	2.66350	13.85730	1.73335
	38.95822	202.30712	12143.47378	46.57716	932.56018	30.31895
Dataset 4						
KAPLL ($\alpha, \beta, \lambda, a, b$)	23.31405	2.41533	75.29050	71.22478	30.37861	
	149.69154	0.51116	247.35434	176.85543	24.97003	
LL (β, λ)	3.89209	680523.30008				
	0.0240808	4205.44583				
APLL (β, λ)	0.88554	1311209.00000				
	0.01947	4194.32000				
TLL (α, β, λ)	32.71872	13.55423	0.00001			
	0.85115	0.64420	0.33768			
APTLL (α, a, b)	1.000012	32.71868	13.554218			
	1.169706	1.4328704	0.6441981			
KMOLL ($\alpha, \beta, \lambda, a, b$)	43.62890813	0.06814131	2.7119242	25.83916478	24.02102188	
	27.48347369	0.05012497	0.8532488	42.30184451	23.17259184	
McLL ($\alpha, \beta, \lambda, a, b$)	0.8405785	14.6431311	75.8602774	516.7208455	10.5506913	
	0.6805722	93.2899103	221.0265196	1395.983161	49.1655846	
AWLL ($\alpha, \lambda, a, b, c, d$)	11.7042955	53.5315497	37.4434629	0.7169097	10.9982328	6.0223624
	260.76694	791.50854	4645.44321	15.97216	12068.48902	169.45835
Dataset 5						
KAPLL ($\alpha, \beta, \lambda, a, b$)	0.45513	0.42199	0.49178	24.99089	73.18344	
	1.29520	0.15053	0.80706	36.77463	80.95251	
LL (β, λ)	2.725903	19.220727				
	0.09414488	2.43323393				
APLL (β, λ)	2.02530	506.43154				
	0.05523	158.99106				
TLL (α, β, λ)	99098.49000	0.16966	0.99999			
	2102.52900	0.00497	0.10373			
APTLL (α, a, b)	1.00002	2.95768	2.72590			
	1.08451063	0.59363472	0.09414487			
KMOLL ($\alpha, \beta, \lambda, a, b$)	66.380378	2.113243	5.7743319	0.2968885	155.758913	
	114.5620659	19.7509401	6.8833595	0.3539369	186.5446495	
McLL ($\alpha, \beta, \lambda, a, b$)	0.354461	0.1913761	21.6689575	129.685355	14.8414287	
	0.1508124	0.5223339	15.3279925	163.401685	14.2225325	
AWLL ($\alpha, \lambda, a, b, c, d$)	0.4264577	98.096872	52.5357368	4.0373439	198.3542142	4.0372805
	0.7589986	613.7514336	3360.410356	7.2031995	3172.175641	7.18952

Table 17. The ML estimates (first line) and SEs (second line) for datasets 3, 4, and 5.

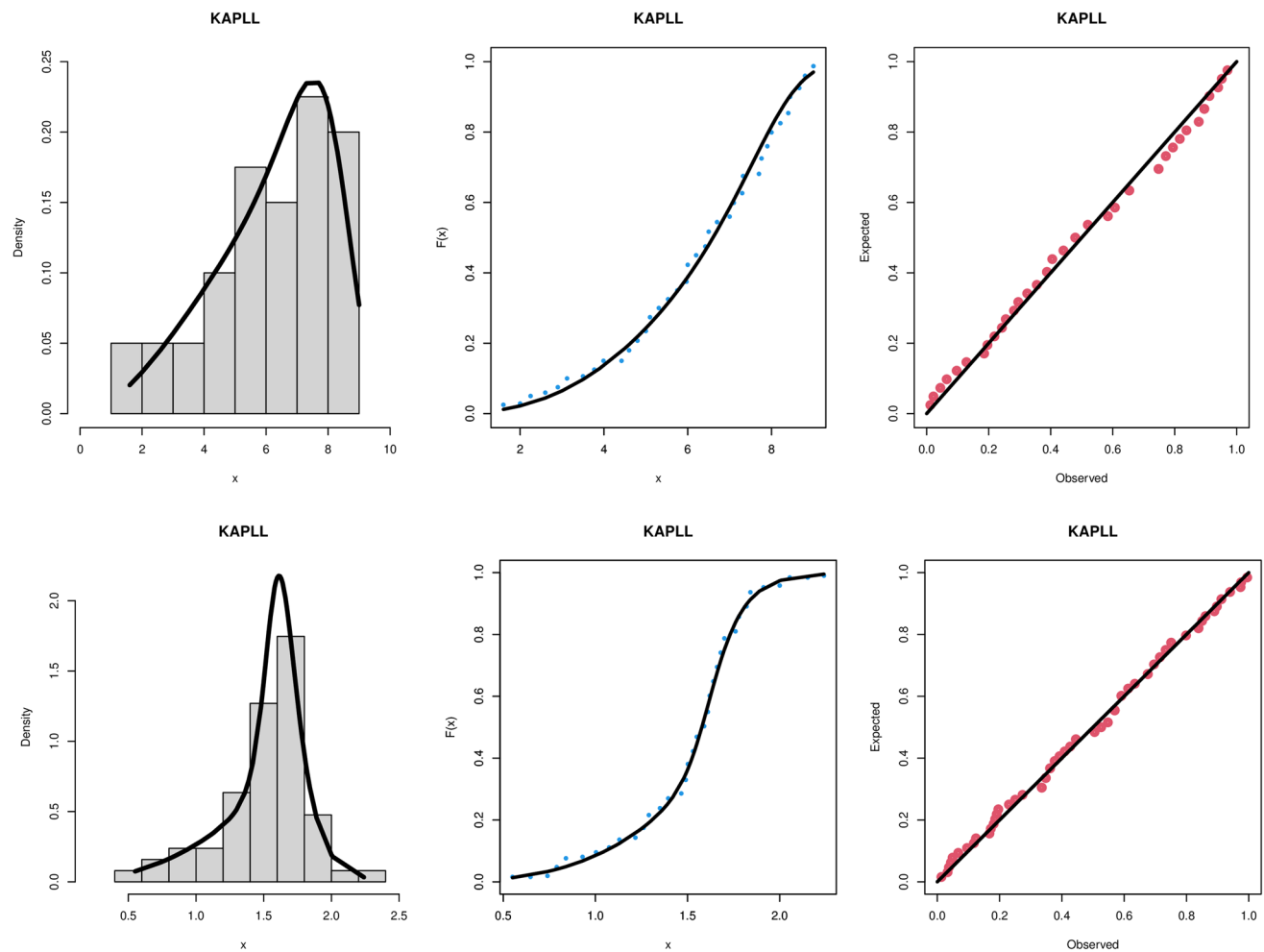


Fig. 4. The fitted PDF, CDF, and PP plots of the KAPLL model for dataset 1 (top panel), and dataset 2 (bottom panel).

Developing a discrete analogue of the KAPLL distribution to facilitate its use in modeling count data, thereby extending its utility to a wider array of applied settings.

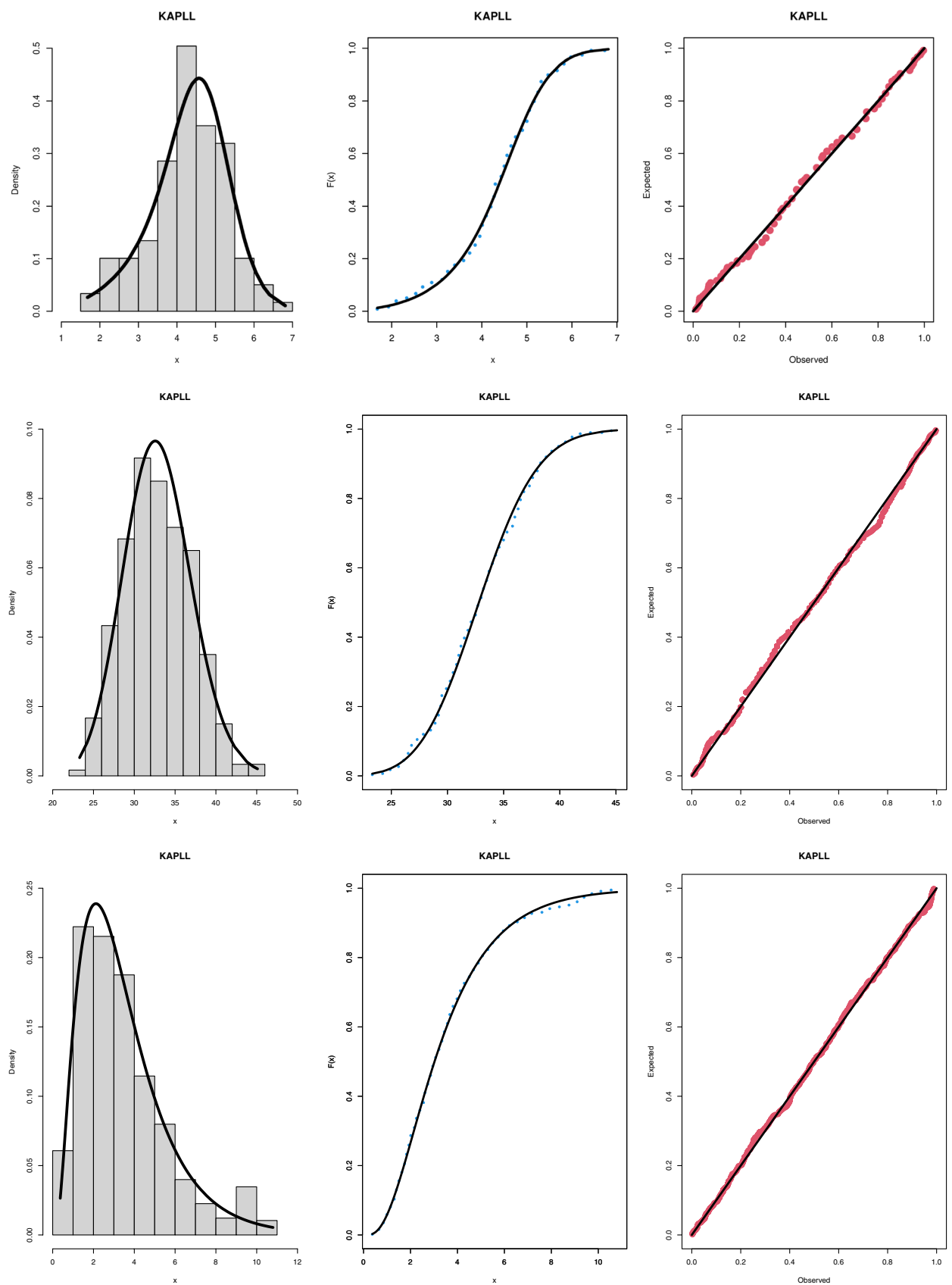


Fig. 5. The fitted PDF, CDF, and PP plots of the KAPLL model for dataset 3 (top panel), dataset 4 (middle panel), and dataset 5 (bottom panel).

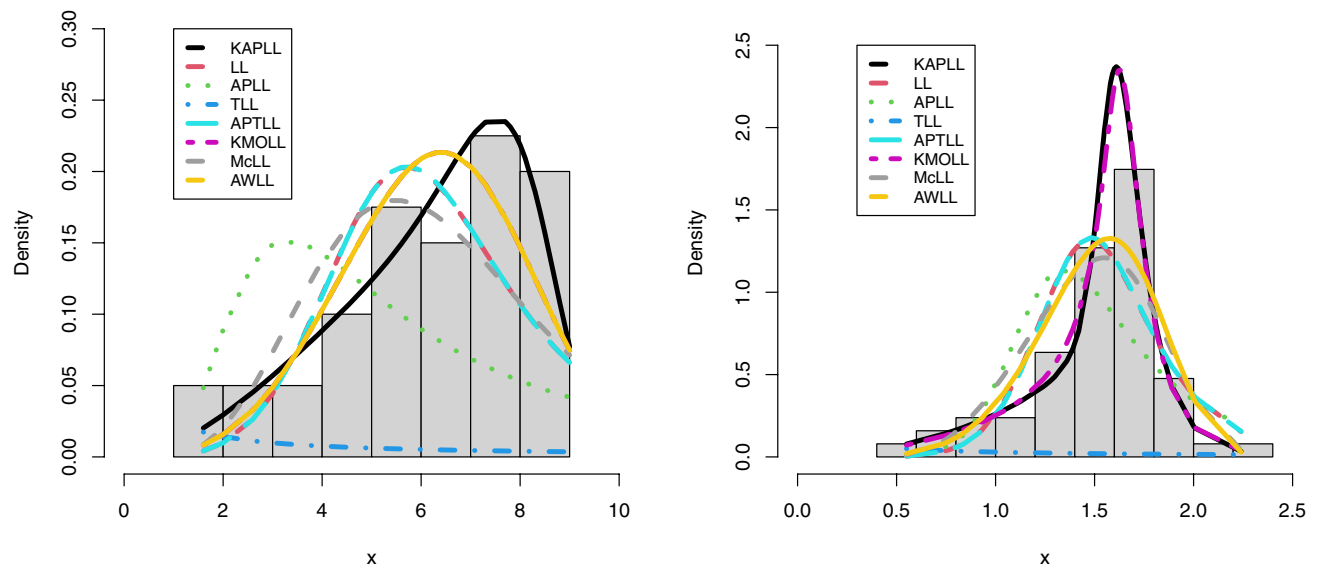


Fig. 6. Plots of histogram for datasets 1 (left panel) and 2 (right panel), and the fitted densities of the KAPLL model and other competing models.

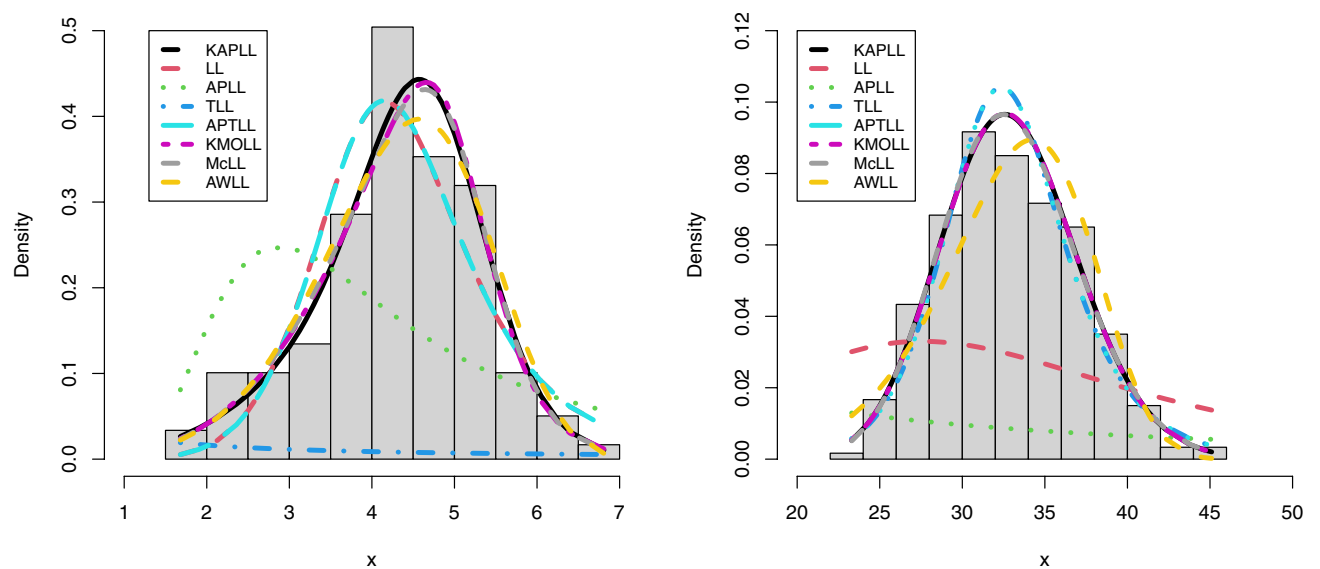


Fig. 7. Plots of histogram for datasets 3 (left panel) and 4 (right panel), and the fitted densities of the KAPLL model and other competing models.

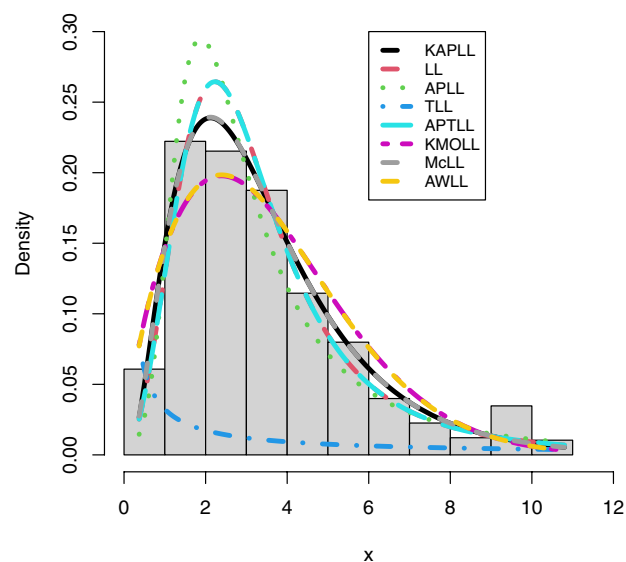


Fig. 8. Plots of histogram for datasets 5 and the fitted densities of the KAPLL model and other competing models.

	α	β	λ	a	b	KS	PV
Dataset 1							
WLSE	9658.176	6.31418	999785.2	0.37130	14.74825	0.05529	0.99970
LSE	9658.176	6.28688	999785.2	0.36797	14.74810	0.05931	0.99896
MLE	9658.129	6.19488	999785.2	0.43938	30.74999	0.07307	0.98318
MPSE	9658.129	6.12636	999785.2	0.42753	30.83327	0.06410	0.99660
CRVME	9658.129	6.11900	999785.2	0.42987	30.83390	0.05376	0.99983
ADE	9658.176	6.34331	999785.2	0.37615	14.75095	0.05859	0.99915
RADE	9658.129	6.16281	999785.2	0.43731	30.83406	0.05938	0.99894
PCE	9658.129	6.08308	999785.2	0.42254	30.79981	0.05607	0.99961
Dataset 2							
WLSE	1436.639	26.23486	99993.32	0.09833	0.48821	0.04700	0.99905
LSE	1436.639	26.07562	99993.32	0.09907	0.50234	0.04885	0.99823
MLE	1436.639	26.01191	99993.32	0.11148	0.53881	0.04469	0.99961
MPSE	1436.639	26.50730	99993.32	0.09632	0.45978	0.04617	0.99930
CRVME	1436.639	26.00942	99993.32	0.10470	0.52595	0.04587	0.99938
ADE	1436.639	26.04734	99993.32	0.10520	0.52341	0.04573	0.99941
RADE	1436.639	26.11433	99993.32	0.10339	0.51358	0.04686	0.99910
PCE	1436.639	26.01628	99993.32	0.09837	0.48961	0.05314	0.99421
Dataset 3							
WLSE	1733.49630	5.76102	4556.323	0.46950	3.22229	0.04262	0.98207
LSE	1733.49650	5.81391	4556.323	0.46490	2.94379	0.04252	0.98254
MLE	220.44540	5.49738	4511.033	0.63990	4.74243	0.04633	0.96039
MPSE	220.44540	5.46146	4511.033	0.62359	4.73342	0.05199	0.90453
CRVME	220.45180	5.67260	4511.033	0.62804	3.54337	0.04311	0.97985
ADE	1733.49640	5.72355	4556.323	0.47774	3.50797	0.04344	0.97825
RADE	220.44480	5.63679	4511.033	0.63217	3.77746	0.04319	0.97948
PCE	220.45100	5.53717	4511.034	0.60343	4.00592	0.05497	0.86477
Dataset 4							
WLSE	22.47070	2.28171	41.79316	83.30084	33.61044	0.03560	0.84163
LSE	21.82941	2.27494	40.64416	82.65706	31.16669	0.03100	0.93528
MLE	23.42137	2.29129	43.44313	84.22156	37.26948	0.04096	0.69560
MPSE	23.31401	2.41512	75.29045	71.22473	30.37840	0.04122	0.68791
CRVME	22.21453	2.41619	74.17018	69.87137	25.29357	0.03200	0.91848
ADE	22.48297	2.28174	41.81348	83.31341	33.71126	0.03587	0.83499
RADE	1.90420	1.77396	52.35557	38.25680	124.27802	0.03391	0.88054
PCE	18.61174	2.32732	68.48394	63.14832	33.56257	0.03867	0.76077
Dataset 5							
WLSE	0.36393	0.40602	0.39714	33.64823	74.57827	0.02280	0.92568
LSE	0.18164	0.39299	0.54802	34.84714	73.00073	0.02208	0.94160
MLE	0.20037	0.39915	0.52362	34.83679	73.02978	0.02456	0.87796
MPSE	0.45441	0.41870	0.49027	24.99085	73.18344	0.02388	0.89773
CRVME	0.18132	0.40089	0.67043	28.67835	70.40277	0.02257	0.93083
ADE	0.17507	0.40915	0.40437	46.12263	56.50213	0.02263	0.92958
RADE	0.13652	0.41335	0.46201	45.46708	50.82757	0.02288	0.92366
PCE	3.27232	0.33737	0.89869	9.46334	767.84825	0.03499	0.48106

Table 18. Parameter estimates of the KAPLL distribution along with KS statistics and corresponding p -values for the five datasets.

Model	ML estimates (first line) and SEs (second line)					W	A	−ℓ	KS	PV
Dataset 1										
KAPLL	9658.12900	6.19488	999785.20000	0.43938	30.83448	0.01941	0.15087	78.92943	0.07307	0.98318
	6823.38144	0.35984	6803.29770	0.11247	43.97041					
EAPLL	179939.60000	7.29782	435733.30000	0.17248		0.10372	0.67660	84.31631	0.09004	0.90195
	4609.87300	0.18158	4982.03100	0.03189						
KLL	2.34786	53161.39587	1.65016	35047.53318		0.07737	0.57556	82.49136	0.10782	0.74109
	0.19131	592.11260	0.10880	18360.23000						
ELL	8.55688	19.00280	0.13576			0.02319	0.19185	79.52812	0.07374	0.98150
	0.09729	0.09742	0.02165							
APLL	1.93292	4.81336	4811.50862			0.21332	1.40194	88.69784	0.14407	0.37745
	7.50218	0.91778	15622.00000							
LL	4.84162	6996.47772				0.21418	1.40722	88.70663	0.14369	0.38072
	0.70121	9447.15447								
Dataset 2										
KAPLL	1436.63900	26.01191	99993.32000	0.11148	0.53881	0.01775	0.12224	8.84215	0.04469	0.99961
	20.76099	2.59224	5.18317	0.02883	0.24718					
EAPLL	2491.30247	18.33016	4557.59726	0.17325		0.02361	0.15886	9.17168	0.05281	0.99467
	25.48549	0.55580	25.42991	0.02335						
KLL	28.30767	1432435.00000	0.12903	0.44704		0.06079	0.35294	10.06389	0.10377	0.50614
	2.05906	1208.43500	0.03259	0.14317						
ELL	1.79002	20.66851	0.22262			0.11363	0.62789	11.91290	0.12931	0.24280
	0.04115	1.24394	0.03718							
APLL	3.06817	7.81451	15.95996			0.49183	2.71991	22.70743	0.14979	0.11834
	4.99850	0.92053	15.36130							
LL	7.92598	28.52572				0.49694	2.74897	22.78996	0.15371	0.10187
	0.87357	12.74795								
Dataset 3										
KAPLL	220.44540	5.49738	4511.03345	0.63990	4.74243	0.04197	0.26565	167.51500	0.04633	0.96039
	503.42917	1.45103	9165.94237	0.25529	4.26687					
EAPLL	188.80120	9.66612	1026831.00000	0.34177		0.04752	0.34897	168.40520	0.05073	0.91941
	386.59798	0.19155	4845.45713	0.08316						
KLL	2.71216	4529.83183	1.85638	2572.55370		0.09369	0.57610	168.74540	0.07255	0.55817
	0.63565	924.22779	0.41222	8957.53149						
ELL	5.21133	14.52152	0.28071			0.05162	0.31326	167.85030	0.05343	0.88610
	0.16832	2.77725	0.07668							
APLL	2.15548	7.01220	19745.67954			0.35212	2.21646	178.30150	0.07881	0.45075
	4.65242	0.76506	38577.28000							
LL	7.06485	31046.87530				0.35495	2.23138	178.33500	0.08032	0.42650
	0.24683	10727.58000								
Dataset 4										
KAPLL	23.3140	2.4153	75.2905	71.2248	30.3786	0.05771	0.38980	851.7108	0.04204	0.66406
	149.6915	0.5112	247.3543	176.8554	24.9700					
EAPLL	29900.0601	6.4182	45915.5005	7746.2828		0.45639	3.13694	888.7469	0.13489	0.00004
	307.7114	0.1235	118.5567	3214.5225						
KLL	2.9062	4511.2885	20.5667	19.6535		0.06055	0.41152	851.9442	0.04340	0.62437
	0.4897	9818.9545	20.3003	12.4187						
ELL	33.0638	14.0205	0.9113			0.15213	1.00920	858.0565	0.05227	0.38541
	0.9965	1.5236	0.2393							
APLL	22702890000.0	7.9436	29521620000.0			0.46512	3.20041	873.9975	0.08171	0.03640
	0.0186	0.0186	0.0000							
LL	3.8921	680523.3001				0.07734	0.56860	1073.1860	0.28760	0.00000
	0.0241	4205.4458								
Dataset 5										
KAPLL	0.4551	0.4220	0.4918	24.9909	73.1834	0.03279	0.35649	1170.2920	0.02575	0.83958
	1.2952	0.1505	0.8071	36.7746	80.9525					
Continued										

Model	ML estimates (first line) and SEs (second line)					<i>W</i>	<i>A</i>	$-\ell$	KS	PV
EAPLL	0.4450	6.3840	102683.1000	0.2124		1.31462	8.35268	1216.8130	0.09969	0.00002
	0.2372	0.1566	35.8677	0.0105						
KLL	0.4069	0.3632	23.7272	95.9248		0.03856	0.40754	1170.3400	0.02657	0.81086
	0.1394	0.3952	24.8463	138.9288						
ELL	3.5418	3.1257	0.7117			0.11607	0.88960	1178.9690	0.04023	0.30872
	0.2934	0.2397	0.1131							
APLL	1.0000	2.7259	19.2205			0.18117	1.20096	1181.0680	0.03638	0.43104
	1.0845	0.0941	10.7023							
LL	2.7259	19.2207				0.18117	1.20096	1181.0680	0.03638	0.43104
	0.0941	2.4332								

Table 19. Goodness-of-fit comparison between the proposed KAPLL model and its sub-models using five real-world datasets.

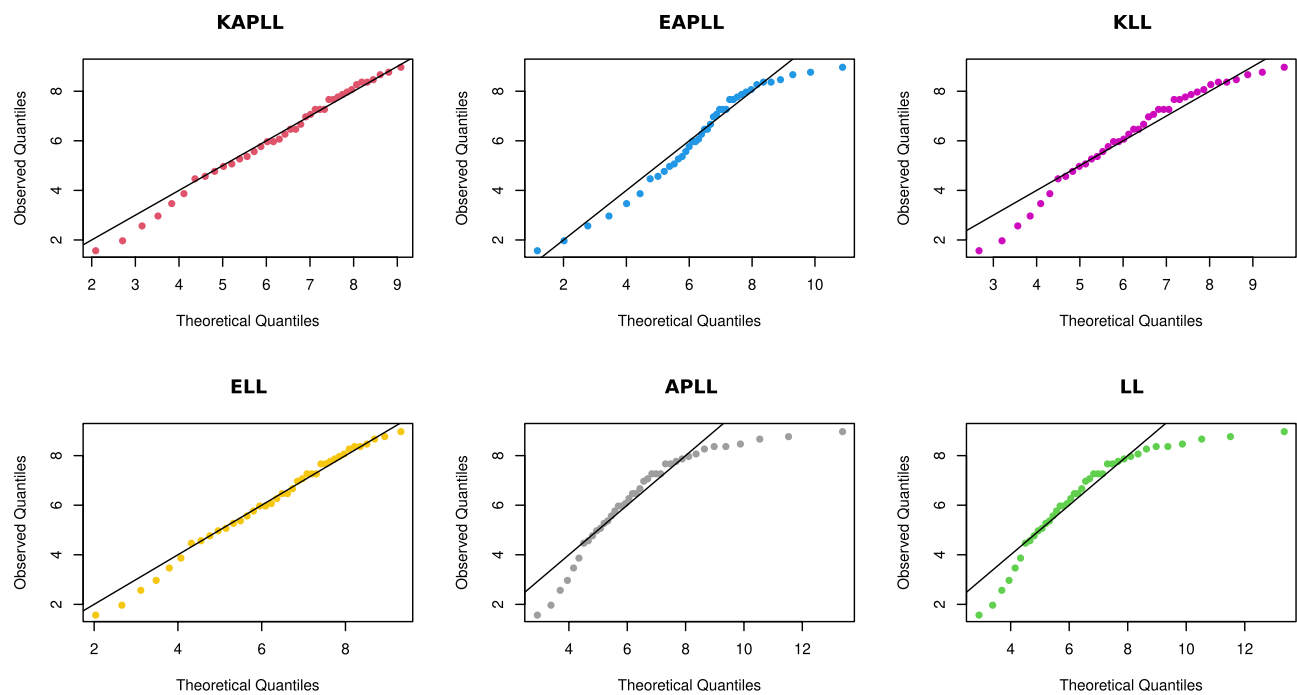


Fig. 9. The QQ plots of the KAPLL model and its sub-models for Dataset 1.

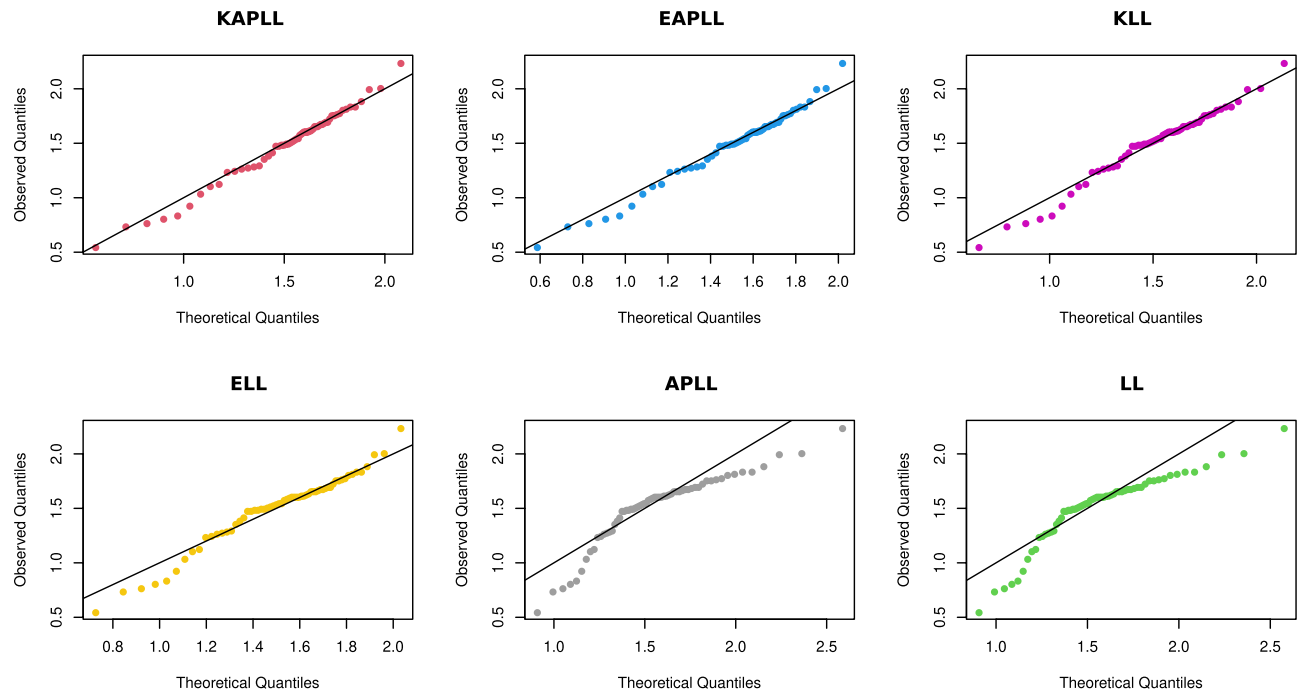


Fig. 10. The QQ plots of the KAPLL model and its sub-models for Dataset 2.

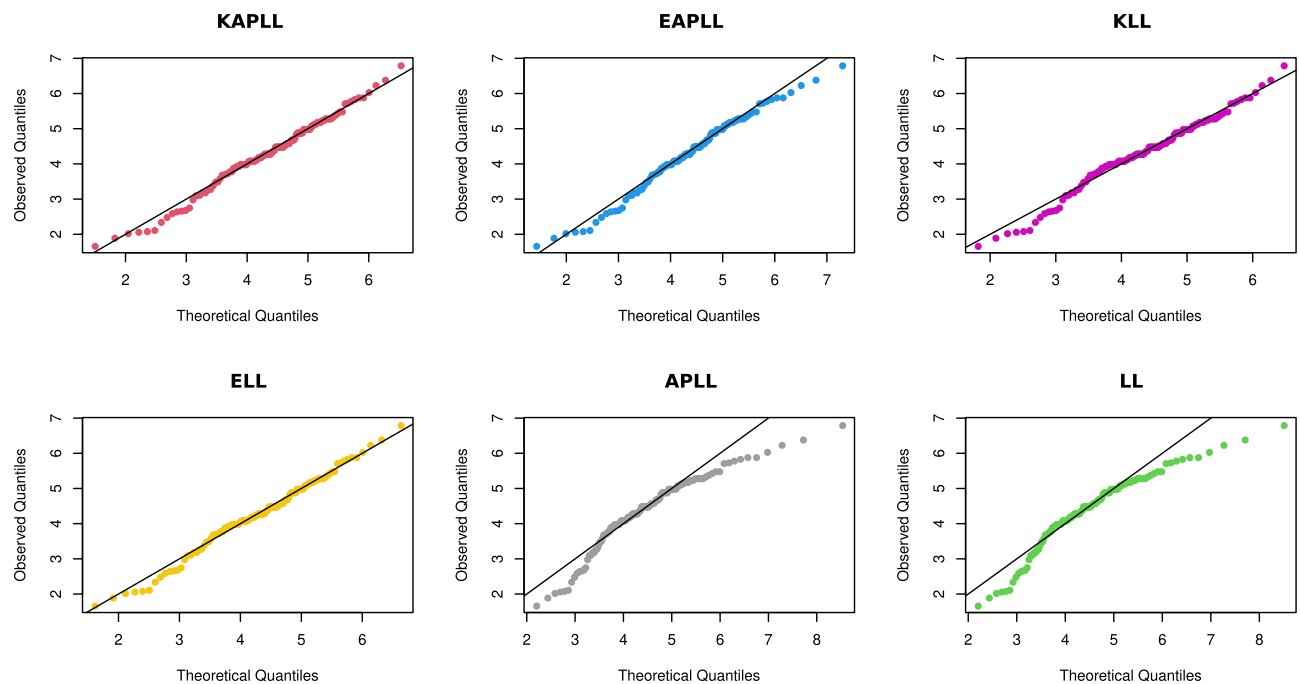


Fig. 11. The QQ plots of the KAPLL model and its sub-models for Dataset 3.

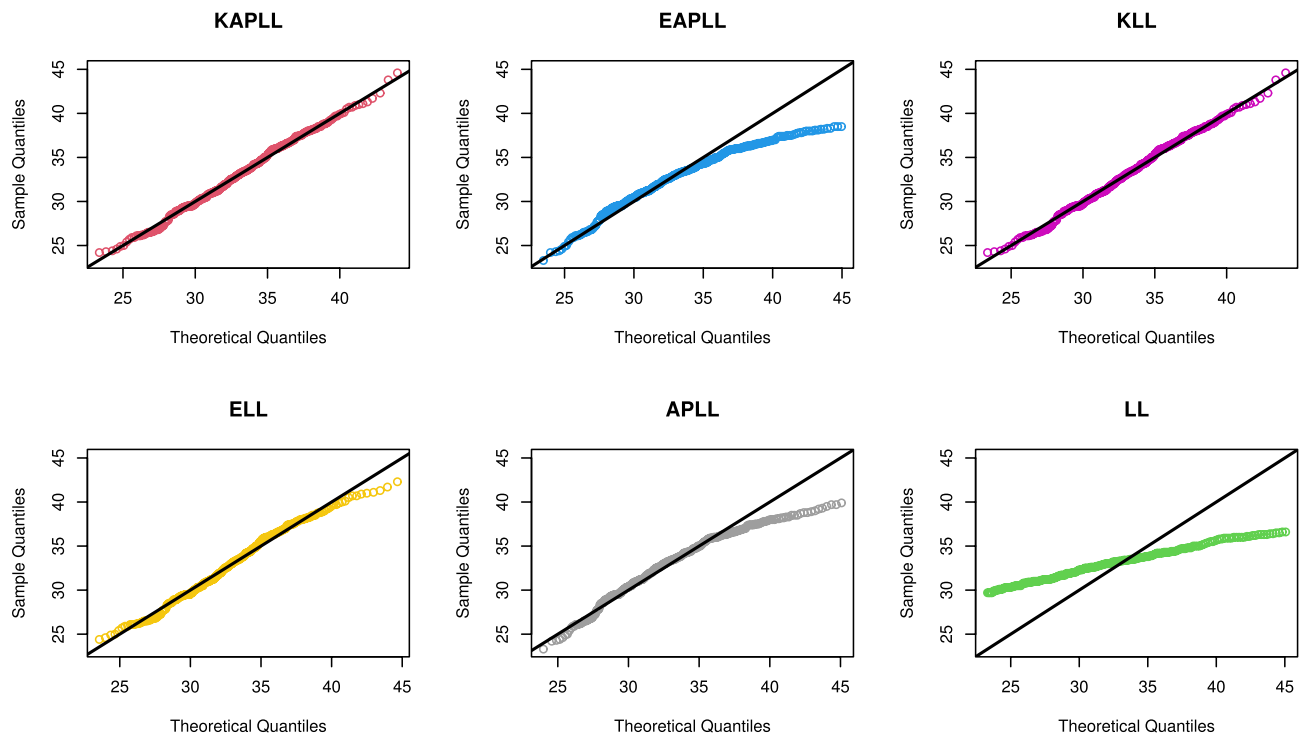


Fig. 12. The QQ plots of the KAPLL model and its sub-models for Dataset 4.

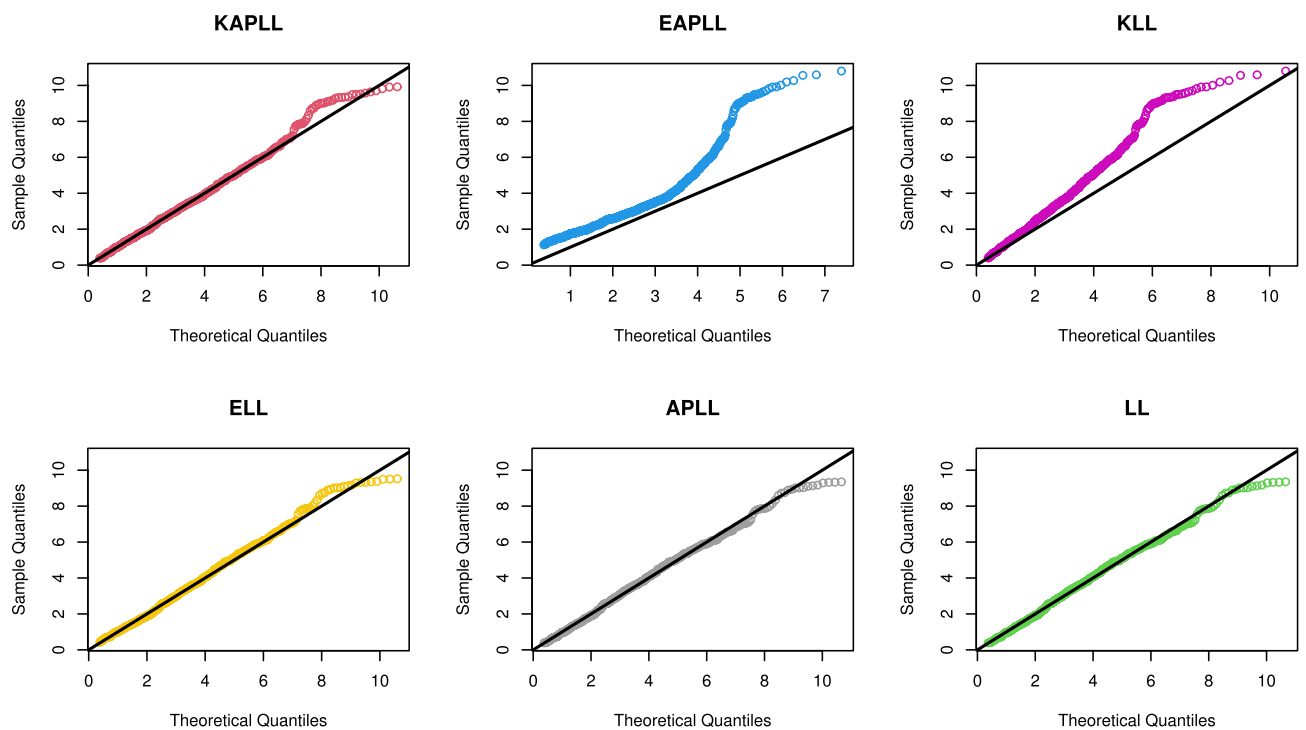


Fig. 13. The QQ plots of the KAPLL model and its sub-models for Dataset 5.

Dataset	Hypotheses	Statistic <i>w</i>	<i>P</i> -value
1	H_0 : KAPLL = LL vs H_1 : H_0 is false	19.5544	0.00021
2	H_0 : KAPLL = LL vs H_1 : H_0 is false	27.8956	0.00019
3	H_0 : KAPLL = LL vs H_1 : H_0 is false	21.6400	0.00008
4	H_0 : KAPLL = LL vs H_1 : H_0 is false	442.9504	0.00010
5	H_0 : KAPLL = LL vs H_1 : H_0 is false	21.2520	0.00002

Table 20. LRT between the KAPLL and LL models.

Data availability

All datasets analyzed during this study are included within the article.

Received: 22 September 2025; Accepted: 29 December 2025
Published online: 24 January 2026

References

1. Marshall, A. W. & Olkin, I. A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. *Biometrika* **84**, 641–652 (1997).

2. Alexander, C., Cordeiro, G. M., Ortega, E. M. & Sarabia, J. M. Generalized beta-generated distributions. *Comput. Stat. Data Anal.* **56**, 1880–1897 (2012).

3. Eugene, N., Lee, C. & Famoye, F. Beta-normal distribution and its applications. *Commun. Stat. Theory Methods* **31**, 497–512 (2002).

4. Cordeiro, G. M. & de Castro, M. A new family of generalized distributions. *J. Stat. Comput. Simul.* **81**, 883–898 (2011).

5. Alizadeh, M. et al. The Kumaraswamy Marshal–Olkin family of distributions. *J. Egypt. Math. Soc.* **23**, 546–557 (2015).

6. Bourguignon, M., Silva, R. B. & Cordeiro, G. M. The Weibull-G family of probability distributions. *J. Data Sci.* **12**, 53–68 (2014).

7. Korkmaz, M. Ç. et al. The Weibull Marshall–Olkin family: Regression model and application to censored data. *Commun. Stat. Theory Methods* **48**, 4171–4194 (2019).

8. Nassar, M., Kumar, D., Dey, S., Cordeiro, G. M. & Afify, A. Z. The Marshall–Olkin alpha power family of distributions with applications. *J. Comput. Appl. Math.* **351**, 41–53 (2019).

9. Mead, M. E., Afify, A. & Butt, N. S. The modified Kumaraswamy Weibull distribution: Properties and applications in reliability and engineering sciences. *Pak. J. Stat. Oper. Res.* **16**, 433–446 (2020).

10. Fisk, P. R. The graduation of income distributions. *Econom.: J. Econom. Soc.* **29**, 171–185 (1961).

11. Dagum, C. A model of income distribution and the conditions of existence of moments of finite order. *Bull. Int. Stat. Inst.* **46**, 199–205 (1975).

12. Shoukri, M. M., Mian, I. U. H. & Tracy, D. S. Sampling properties of estimators of the log-logistic distribution with application to Canadian precipitation data. *Can. J. Stat.* **16**, 223–236 (1988).

13. Arnold, B. C. *Pareto Distrib.* (International Co-operative Publication House, Fairland, Maryland USA, 1983).

14. Burr, I. W. Cumulative frequency functions. *Ann. Math. Stat.* **13**, 215–232 (1942).

15. Mielke, P. W. & Johnson, E. S. Three-parameter kappa distribution maximum likelihood estimates and likelihood ratio tests. *Mon. Weather Rev.* **101**, 701–709 (1973).

16. Kleiber, C. & Kotz, S. *Statistical size distributions in economics and actuarial sciences* (John Wiley & Sons, 2003).

17. Aldahlan, M. A. Alpha power transformed log-logistic distribution with application to breaking stress data. *Adv. Math. Phys.* **2020**, 1–9 (2020).

18. Aryal, G. R. Transmuted log-logistic distribution. *J. Stat. Appl. Probab.* **2**, 11–20 (2013).

19. Gui, W. Marshall–Olkin extended log-logistic distribution and its application in minimification processes. *Appl. Math. Sci.* **7**, 3947–3961 (2013).

20. Tahir, M. H., Mansoor, M., Zubair, M. & Hamedani, G. G. McDonald log-logistic distribution with an application to breast cancer data. *J. Stat. Theory Appl.* **13**, 65–82 (2014).

21. Cakmakyan, S., Ozel, G., Gebaly, Y. M. H. E. & Hamedani, G. G. The Kumaraswamy Marshall–Olkin log-logistic distribution with application. *J. Stat. Theory Appl.* **17**, 59–76 (2018).

22. Hemeda, S. Additive Weibull log logistic distribution: Properties and application. *J. Adv. Res. Appl. Math. Stat.* **3**, 8–15 (2018).

23. Abouelmagd, T. H. M. et al. Extended Weibull log-logistic distribution. *J. Nonlin. Sci. Appl.* **12**, 523–534 (2019).

24. Alfaer, N. M., Gemeay, A. M., Aljohani, H. M. & Afify, A. Z. The extended log-logistic distribution: Inference and actuarial applications. *Mathematics* **9**, 1386 (2021).

25. Ramos, M. W. A., Cordeiro, G. M., Marinho, P. R. D., Dias, C. R. B. & Hamedani, G. G. The Zografos–Balakrishnan log-logistic distribution: Properties and applications. *J. Stat. Theory Appl.* **12**, 225–244 (2013).

26. Cordeiro, G. M., Afify, A. Z., Ortega, E. M., Suzuki, A. K. & Mead, M. E. The odd Lomax generator of distributions: Properties, estimation and applications. *J. Comput. Appl. Math.* **347**, 222–237 (2019).

27. Almamy, J. A. Extended Poisson log-logistic distribution. *Int. J. Stat. Probab* **8**, 56–69 (2019).

28. Afify, A. Z., Hussein, E. A., Alnssyan, B. & Mahran, H. A. The extended log-logistic distribution: Properties, inference, and applications in medicine and geology. *J. Stat. Appl. Probab* **12**, 1155–1580 (2023).

29. Gaire, A. K. & Gurung, Y. B. Skew log-logistic distribution: Properties and application. *Stat. Trans.* **25**, 43–62 (2024).

30. Rahman, M. M., Darwish, J. A., Shahbaz, S. H., Hamedani, G. G. & Shahbaz, M. Q. A new cubic transmuted log-logistic distribution: Properties, applications, and characterizations. *Adv. Appl. Stat.* **91**, 335–361 (2024).

31. Ali, I. et al. On a transmuted distribution based on log-logistic and Ailamujia hazard functions with application to lifetime data. *Lobachevskii J. Math.* **45**, 1557–1569 (2024).

32. Afify, A. Z., Abdelall, Y. Y., AlQadi, H. & Mahran, H. A. The modified log-logistic distribution: Properties and inference with real-life data applications. *Contemp. Math.* **6**, 862–902 (2025).

33. Kariuki, V., Wanjoya, A., Ngesa, O. & Alqawba, M. A flexible family of distributions based on the alpha power family of distributions and its application to survival data. *Pak. J. Stat.* **39**, 237–258 (2023).

34. Mahdavi, A. & Kundu, D. A new method for generating distributions with an application to exponential distribution. *Commun. Stat. Theory Methods* **46**, 6543–6557 (2017).

35. Gupta, R. C., Gupta, P. L. & Gupta, R. D. Modeling failure time data by Lehman alternatives. *Commun. Stat. Theory Methods* **27**, 887–904 (1998).

36. Kariuki, V. et al. Properties, estimation, and applications of the extended log-logistic distribution. *Sci. Rep.* **14**, 1420967 (2024).

37. De Santana, T. V. F., Ortega, E. M., Cordeiro, G. M. & Silva, G. O. The Kumaraswamy-log-logistic distribution. *J. Stat. Theory Appl.* **11**, 265–291 (2012).
38. Rosaiah, K., Kantam, R. R. L. & Kumar, S. Reliability test plans for exponentiated log-logistic distribution. *Econom. Quality Control.* **21**, 279–289 (2006).
39. Kenney, J. F. & Keeping, E. S. *Mathematics of statistics. Princeton New Jersey. Part 1*, 3rd edn, 101–102 (1962).
40. Moors, J. J. A quantile alternative for kurtosis. *J. Royal Stat. Soc. Ser. D: Stat.* **37**, 25–32 (1998).
41. Lorenz, M. O. Methods of measuring the concentration of wealth. *Publ. Am. Stat. Assoc.* **9**, 209–219 (1905).
42. Arcagni, A. & Porro, F. The graphical representation of the inequality. *Dipart. Stat. Metodi Quantitat. Work. Papers.* **37**, 419–437 (2014).
43. R Core Team. R: A language and environment for statistical computing; R foundation for statistical computing: Vienna, Austria, 2025. Available online: <https://www.R-project.org/> (Accessed: 10 June 2025).
44. Ahmad, A., Alsadat, N., Rather, A. A., Meraou, M. A. & El-Din, M. M. A novel statistical approach to COVID-19 variability using the Weibull-inverse Nadarajah Haghighi distribution. *Alexandria Eng. J.* **105**, 950–962 (2024).
45. Ahmad, K. et al. Statistical inference on the exponentiated moment exponential distribution and its discretization. *J. Radiat. Res. Appl. Sci.* **17**, 1–14 (2024).
46. Ahmad, A. et al. Novel family of probability generating distributions: Properties and data analysis. *Phys. Scripta* **99**, 1–19 (2024).
47. Qayoom, D., Rather, A. A., Alsadat, N., Hussam, E. & Gemeay, A. M. A new class of Lindley distribution: System reliability, simulation and applications. *Heliyon* **10**, 1–31 (2024).
48. Cheng, R. C. H. & Amin, N. A. K. Estimating parameters in continuous univariate distributions with a shifted origin. *J. Royal Stat. Soc.: Ser. B (Methodol.)* **45**, 394–403 (1983).
49. Cheng, R. C. H., & Amin, N. A. K. (1979). Maximum product-of-spacings estimation with applications to the lognormal distribution. *Math report*, 791.
50. Xu, K., Xie, M., Tang, L. C. & Ho, S. L. Application of neural networks in forecasting engine systems reliability. *Appl. Soft Comput.* **2**, 255–268 (2003).
51. Smith, R. L. & Naylor, J. A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution. *J. R. Stat. Soc.: Ser. C: Appl. Stat.* **36**, 358–369 (1987).
52. Nadarajah, S. & Kotz, S. On the alternative to the Weibull function. *Eng. Fract. Mech.* **74**, 451–456 (2007).
53. Abdelkader, A. et al. Hyoid bone-based sex discrimination among Egyptians using a multidetector computed tomography: Discriminant function analysis, meta-analysis, and artificial intelligence-assisted study. *Sci. Rep.* **15**, 2680 (2025).
54. Malik, A. S. & Ahmad, S. P. An extension of log-logistic distribution for analyzing survival data. *Pak. J. Stat. Oper. Res.* **16**, 789–801 (2020).

Acknowledgements

This Project was funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, Saudi Arabia under grant no. (IPP: 293-662-2025). The authors, therefore, acknowledge with thanks DSR for technical and financial support.

Author contributions

Conceptualization: Ahmed Z. Afify, Mohamed E. Mead, Abdussalam Aljadani; Data curation: Mohamed E. Mead, Abdulaziz S. Alghamdi, Enayat M. Abd Elrazik; Formal analysis: Mohamed E. Mead, Hamada H. Hassan; Investigation: Hamada H. Hassan, Abdussalam Aljadani; Methodology: Ahmed Z. Afify, Hamada H. Hassan, Enayat M. Abd Elrazik; Resources: Abdussalam Aljadani, Abdulaziz S. Alghamdi; Software: Ahmed Z. Afify, Hamada H. Hassan; Validation: Ahmed Z. Afify, Abdulaziz S. Alghamdi; Visualization: Abdulaziz S. Alghamdi, Enayat M. Abd Elrazik; Writing—original draft: Ahmed Z. Afify, Mohamed E. Mead, Hamada H. Hassan; Writing—review & editing: Ahmed Z. Afify, Abdussalam Aljadani, Enayat M. Abd Elrazik. All authors contributed equally to this work.

Funding

This Project was funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, Saudi Arabia under grant no. (IPP: 293-662-2025). The authors, therefore, acknowledge with thanks DSR for technical and financial support.

Declarations

Competing interests

The authors declare no competing interests.

Additional information

Correspondence and requests for materials should be addressed to A.Z.A.

Reprints and permissions information is available at www.nature.com/reprints.

Publisher's note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Open Access This article is licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License, which permits any non-commercial use, sharing, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if you modified the licensed material. You do not have permission under this licence to share adapted material derived from this article or parts of it. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by-nc-nd/4.0/>.

© The Author(s) 2026