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A new epidemic model of sexually transmittable diseases: a fractional numerical approach

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This study aims at investigating the dynamics of sexually transmitted infectious disease (STID), which is serious health concern. In so doing, the integer order STID model is progressed in to the time-delayed non-integer order STID model by introducing the Caputo fractional derivatives in place of integer order derivatives and including the delay factors in the susceptible and infectious compartments. Moreover, unique existence of the solution for the underlying model is ensured by establishing some benchmark results. Likewise, the positivity and boundedness of the solutions for the projected model is explored. The basic reproduction number is R_0 is found out for the model. The time-delayed non-integer order STID model holds two steady states, namely, the STID free and endemic steady state. The model stability is carried out at the steady states. The non-standard finite difference (NSFD) technique is hybridized with the Grunwald Letnikov (GL) approximation for finding the numerical solutions of the time-delayed non-integer order STID model. The boundedness and non-negativity of the numerical scheme is confirmed. The simulated graphs are presented with the help of an appropriate test example. These graphs show that the proposed numerical algorithm provides the positive bounded solutions. The article is ended with productive outcomes of the study.

Keywords New epidemic model, Delayed fractional differential equations, STIDs, Voltera Lyapunov function, LaSalle principal, GL non-standard finite difference schemes

Sexually transmitted infectious diseases (STIDs) basically propagate through sexual contacts. These disease are a serious public health matter that influence the social, economic and physical health of the infected individuals. Some common STIDs are mycoplasma genitalium (MG), Hepatitis B and C, Herpes simplex virus, human papillomavirus and many more in the sequence. The World Health Organization (WHO) estimates that 376 million new cases of STIDs appear each year, making indicating a terrific public health stumper with deep social outcomes. The main source of transmission are oral, anal and vaginal contact, fluid exchanges such as blood, vaginal secretions and semen. Multiple sexual partners and unprotected sexual effective activities can make the transmission of STIDs very fast [1]. The teenagers and young adults compromise the major portion of the infected individuals with low health facilities and environment, are in danger of getting the infection. By addressing the social cultural and behavioral issues that consequently lead to the development of STIDs, the infections may be controlled and prevented effectively [2]. Advancements in molecular diagnosis, therapeutic approaches and public health policies have culminated the medical treatment facilities. However, stigma of being guilty, antibiotic resistance and inaccessible medical support impacted the attempts to control and eradicate the STIDs [3]. In epidemic models, the frequent occurring of the infection affects the basic key reproductive number, remarkably. It is a matter of fact that nonlinear incidences may give rise intrigue or chaotic behaviors, contrary to the conventional incidences, which are frequently used in the classical epidemic models [4–8]. Many researchers studied the effect of nonlinear occurrences in the propagation of STIDs. The integer order models do not capture the nonlinear and complex dynamics of the STIDs. Because, the classical derivatives have the local behavior and

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they do not incorporate the past state of the system to describe the current or future state of the system. So, the fractional order epidemic models are inevitable to describe the complex nature of the infection dynamics. Since, the fractional derivatives used in these models have non local nature and they consider the past state, memory effect to depict the present and future state of the infection. This improvement makes it possible to analyze STIDs propagation in more detail and with greater precision [9].

In various areas of research, non-integer order and delayed non-integer order systems are significant for modeling intricate dynamic processes. They have made significant contributions to the scientific and technical analysis of real events in several fields to develop prediction models for systems with memory, temporal lag, or non-local coupling [18–23].

The mathematical models presented in this paper are based on delay differential equations (DDE) that are extremely helpful if the description of the considered systems includes time dependency not only on the present state of the system but also on the system's previous state. There are several classifications of DDEs which have a different type of delay, neutral, time-dependent, stochastic, state-dependent and constant delay. Such DDEs find widespread use in a numeral branches, biology, network related to electrical fields, living sciences, ecological sciences, mechanics, and populace mobility [24–26].

While delays plays a vital role in real scenarios systems and the learning of DDEs has fascinated attention. These types of differential equations have the discrete and short term memory effects, they cannot incorporate the long term. This particular study is a relatively new field of exploration. A generalization of the DDE to any non-integer order is the FDDE. For modeling real-world challenges, FDDEs have continuous and long term memory effect. Many, time-delayed non-integer order mathematical systems are used to incorporate the real world wonder in many fields of sciences and engineering. Furthermore, several approximation techniques are applied to solve the mathematical systems [27–33].

In this work, some factors associated with viral propagation in the time-delayed non-integer order STIDs model are investigated. The way in which infections spread is not easy and cannot be described by an integer order model, particularly when estimating the present state based on previous movement. However, the fractional order derivatives are more effective in determining the memory effect factors and histories of state variables so it can contribute in scoop out the knowledge of the disease phenomenon. It enables a better, a more effective way to study the spread of diseases.

Hence, the type of incidence rates that may be used to describe the COVID-19 pandemic, in part, as fractional and stable, is the harmonic mean type incidence rates [35]. Examples of deterministic and fractional cases involve dealing with the spread of the computer virus [36]. Fractional order epidemic models [37] include vaccination and include asymptomatic carriers. In [36] both fractional and deterministic approach for modelling computer viruses propagation is used. Harmonic mean type incidence rates are, therefore, useful in the construction of any epidemic models for theoretical and numerical analysis, including those presented in this paper and those studied in [38] for rabies. By analysis with the real time data and fractional modelling particularly using the ABC operator mentioned in [39], the important information of the COVID-19 epidemic dynamics are ensured. The non-linear dengue epidemic system is studied by applying the fractional approach [40] and also the Cutaneous Leishmania system [41]. Majee et al. [42] proposed a fractional order waning immunity delayed epidemic model under optimal control to demonstrate the impact of delay factors on epidemic dynamics. Comparable, fractional and fractal–fractional modeling strategies applied to such equations have been utilized for describing and managing actions such as alcohol dependence [43]. Research has also been conducted on the synchronous transmission of (for instance) HBV and COVID-19, to demonstrate the importance of efficient control measures [45]. Thus, the fractional order models have been also used apart from the epidemiology, examined the psychological and social parameters such as employees' negative attitude towards work [45], passengers' perception of safety in unstable train systems [46] and the corruption indexes dynamics [47].

The pattern of this paper is structured as follows to systematically address the research objectives. Firstly, Section "Cornerstones" is dedicated to presenting the mathematical model. Within this section, emphasis is placed on exploring system positivity, boundedness, as well as the existence of endemic equilibrium points, and local stability. Through rigorous examination, the intricacies of infection propagation dynamics within the population are elucidated. Subsequently, Section "Finite difference technique" is devoted to conducting numerical simulations that serve to validate the findings derived from the analysis of the mathematical model. These simulations provide empirical evidence supporting the theoretical analysis developed earlier. By leveraging computational techniques, the robustness and efficacy of the proposed model are assessed, thereby enhancing our understanding of infection spread within the context of delayed fractional order STIDs.

Cornerstones

Here we discussed some basic definitions of fractional calculus for better understanding.

Caputo fractional derivative [12,15]

Let $\mathrm{y}(\tau)$ satisfies some smoothness conditions in every finite interval (0,t) with $\mathrm{t} \leq \tau$. Then

$$ {}_{0}^{c}\mathrm{D}_{\mathrm{t}}^{\phi}\ \mathrm{y}\,(\mathrm{t}) = \frac{1}{\Gamma\,(\mathrm{m}-\phi)} \int_{0}^{\mathrm{t}} (\mathrm{t}-\tau)^{-\phi-1+\mathrm{m}}\, \frac{\mathrm{d}^{\mathrm{m}}}{\mathrm{d}\tau^{\mathrm{m}}}\ \mathrm{y}\,(\tau)\,\mathrm{d}\tau,\ \ \mathrm{m}-1<\phi<\mathrm{m}. $$

Mittage-Leffler function [11]

The Mittage-Leffler function is represented by $E_{\alpha,\beta}$, described as below

$$E_{\alpha,\beta}(s)=\sum_{k=0}^{\infty}\frac{s^k}{\Gamma(\alpha k+\beta)},\alpha\in\mathbb{R}^+,s\in C.$$

Mittage-Leffler with two variables [11]

The two variables form of Mittage–Leffler function is defined as,

$$E_{\alpha,\beta}(s)=\sum_{k=0}^{\infty}\frac{s^k}{\Gamma(\alpha k+\beta)},\alpha,\beta\in\mathbb{R}^+,s\in C,\text{and }E_{\alpha,\beta}(s)=s\cdot E_{\alpha,\alpha+\beta}(s)+\frac{1}{\Gamma(\beta)}.$$

Unraveling of the model

The parameters and state variables used in the transformed model are enlisted below in Table 1.

By replacing Caputo derivatives instead of ordinary derivative in [13] for ascertaining the claims as above. The fractional delayed model will adopt the form as explained in system (1)–(5).

$${}_0^{\mathrm{c}}\mathrm{D}_{\mathrm{t}}^{\phi}\,\mathrm{S}(\mathrm{t})=\mathrm{Z}^{\phi}-\mathrm{S}\cdot\mathrm{g}(\mathrm{t}-\tau,\mathrm{I})\cdot\mathrm{e}^{-\mu\tau}-\left(\mathrm{w}_1^{\phi}+\mathrm{d}^{\phi}\right)\cdot\mathrm{S},\tag{1}$$

$${}_0^{\mathrm{c}}\mathrm{D}_{\mathrm{t}}^{\phi}\,\mathrm{I}(\mathrm{t})=\mathrm{S}\cdot\mathrm{g}(\mathrm{t}-\tau,\mathrm{I})\cdot\mathrm{e}^{-\mu\tau}+\mathrm{v}_1^{\phi}\mathrm{T}-\mathrm{m}^{\phi}\mathrm{I},\tag{2}$$

$${}_0{}^{\mathrm{c}}\mathrm{D}_{\mathrm{t}}^{\Phi}\,\mathrm{T}(\mathrm{t})=\tau_2{}^{\Phi}\mathrm{I}-\mathrm{n}^{\Phi}\mathrm{T},\tag{3}$$

$${}_0{}^{\mathrm{c}}\mathrm{D}_{\mathrm{t}}^{\Phi}\,\mathrm{A}(\mathrm{t})=\tau_1{}^{\Phi}\mathrm{I}-\mathrm{d}^{\Phi}\mathrm{A}+\mathrm{v}_2{}^{\Phi}\mathrm{T},\tag{4}$$

$${}_0{}^{\mathrm{c}}\mathrm{D}_{\mathrm{t}}^{\Phi}\,\mathrm{R}(\mathrm{t})=\mathrm{w}_1{}^{\Phi}\mathrm{S}-\mathrm{d}^{\Phi}\mathrm{R}.\tag{5}$$

where, $g(\mathrm{t}-\tau,\mathrm{I})=\frac{I.\alpha(\mathrm{t}-\tau)}{1+I.\beta(\mathrm{t}-\tau)}$, with initial conditions

$$S(t)=\mathrm{S}_0\geq 0,I(t)=\mathrm{I}_0\geq 0,A(t)=\mathrm{A}_0\geq 0,T(t)=\mathrm{T}_0\geq 0,R(t)=\mathrm{R}_0\geq 0.$$

Feasible region

The feasible region is defined as, for any time $t\geq 0$,

$$\Omega=\left\{(S,T,I,A,R)\in\mathbb{R}_+^5:S+T+I+A+R\leq\frac{Z^{\phi}}{d^{\phi}},S\geq 0,T\geq 0,A\geq 0,I\geq 0,R\geq 0\right\}.\tag{6}$$

Lemma 2.6 *For any initial positive values, the system (1)-(5) is positive invariant in* $\mathbb{R}_+^5$ [14].

Proof Consider the Eq. (1),

${}_0{}^{\mathrm{c}}\mathrm{D}_{\mathrm{t}}^{\Phi}S|_{S=0}=\mathrm{Z}^{\Phi}\geq 0$, similarly from Eq. (2) we have,
${}_0{}^{\mathrm{c}}\mathrm{D}_{\mathrm{t}}^{\Phi}I|_{I=0}=\mathrm{v}_1{}^{\Phi}\mathrm{T}\geq 0$, after considering Eq. (3) we reach at,
${}_0{}^{\mathrm{c}}\mathrm{D}_{\mathrm{t}}^{\Phi}T|_{T=0}=\tau_2{}^{\Phi}\mathrm{I}\geq 0$, from Eq. (4) we get,
${}_0{}^{\mathrm{c}}\mathrm{D}_{\mathrm{t}}^{\Phi}A|_{A=0}=\tau_1{}^{\Phi}\mathrm{I}+\mathrm{v}_2{}^{\Phi}\mathrm{T}\geq 0$, and finally from Eq. (5) we conclude as,
${}_0{}^{\mathrm{c}}\mathrm{D}_{\mathrm{t}}^{\Phi}R|_{R=0}=\mathrm{w}_1{}^{\Phi}\mathrm{S}\geq 0$. As desired.

Symbol	State variables and parameter rates
S	Susceptible
I	Infected
R	Recovered
T	Treatment
A	Awareness
Z	Recruitment
d	Natural mortality
τ_1	Infected individuals safe sex practice
τ_2	Infected receive anti-viral medicines
ω_1	Susceptible individuals safe sex practice
ν_1	Treatment defaulters
ν_2	Treated individuals adhere to safe sex practices
μ	Delay parameter death rate
τ	Delay parameter
ϕ	Fractional derivative order

Table 1. Symbols for state variables and parameter rates.

Lemma 2.7 *For positive initial conditions, the system (1)–(5) is bounded for all t ∈ [0, t_m)* [14].

Proof Consider system (1)–(5) as,

$$
\begin{aligned}
{}_0{}^cD_t^{\phi} N(t) &= {}_0{}^cD_t^{\phi} S(t) + {}_0{}^cD_t^{\phi} I(t) + {}_0{}^cD_t^{\phi} A(t) + {}_0{}^cD_t^{\phi} T(t) + {}_0{}^cD_t^{\phi} R(t),\\
{}_0{}^cD_t^{\phi} N(t) &= \mathrm{Z}^{\phi} - Sg(\mathrm{t}-\tau, I)\, e^{-\mu\tau} - \left({w_1}^{\phi} + d^{\phi}\right) S + Sg(\mathrm{t}-\tau, I)\, e^{-\mu\tau}\\
&\quad + {v_1}^{\phi} T - m^{\phi} I + {\tau_2}^{\phi} I - n^{\phi} T + {\tau_1}^{\phi} I - d^{\phi} A + {v_2}^{\phi} T + {w_1}^{\phi} S - d^{\phi} R,\\
{}_0{}^cD_t^{\phi} N(t) &= \mathrm{Z}^{\phi} - d^{\phi}(S + I + T + A + R),\\
{}_0{}^cD_t^{\phi} N(t) &= \mathrm{Z}^{\phi} - d^{\phi}(N),\\
{}_0{}^cD_t^{\phi} N(t) + d^{\phi}(N) &= \mathrm{Z}^{\phi}.
\end{aligned}
$$

By applying Laplace transform operator we get,

$$
\begin{aligned}
s^{\phi}\mathcal{L}\{N(t)\} - s^{\phi-1}N(0) + d^{\phi}\mathcal{L}\{N(t)\} &= \mathrm{Z}^{\phi}\left\{\frac{1}{s}\right\},\\
\left(s^{\phi} + d^{\phi}\right)\mathcal{L}\{N(t)\} &= \frac{\mathrm{Z}^{\phi}}{s} + s^{\phi-1}.N(0),\\
\mathcal{L}\{N(t)\} &= \frac{\mathrm{Z}^{\phi}}{s\left(s^{\phi} + d^{\phi}\right)} + \frac{N(0).s^{\phi-1}}{\left(s^{\phi} + d^{\phi}\right)}.
\end{aligned}
$$

Now applying Laplace inverse operator on above expression we reach at,

$$
\begin{aligned}
\mathcal{L}^{-1}\mathcal{L}\{N(t)\} &= \mathcal{L}^{-1}\left\{\frac{\mathrm{Z}^{\phi}}{s\left(s^{\phi} + d^{\phi}\right)} + \frac{N(0).s^{\phi-1}}{\left(s^{\phi} + d^{\phi}\right)}\right\},\\
N(t) &= \mathcal{L}^{-1}\left\{\frac{\mathrm{Z}^{\phi}.s^{\phi-(1+\phi)}}{\left(s^{\phi} + d^{\phi}\right)}\right\} + \mathcal{L}^{-1}\left\{\frac{N(0).s^{\phi-1}}{\left(s^{\phi} + d^{\phi}\right)}\right\},\\
N(t) &= \mathrm{Z}^{\phi}.\frac{1}{d^{\phi}}.d^{\phi}.t^{\phi}.E_{\phi,1+\phi}\left(-d^{\phi}t^{\phi}\right) + N(0)\,t^{0}E_{\phi,1}\left(-d^{\phi}t^{\phi}\right).
\end{aligned}
$$

Let $M = max\{N(0), \frac{\mathrm{Z}^{\phi}}{d^{\phi}}\}$ then we have,

$$
N(t) \le Md^{\phi}.t^{\phi}.E_{\phi,1+\phi}\left(-d^{\phi}t^{\phi}\right) + Mt^{0}E_{\phi,1}\left(-d^{\phi}t^{\phi}\right).
$$

Since, $E_{\phi,\beta}(z) = z.E_{\phi,\phi+\beta}(z) + \frac{1}{\Gamma(\beta)}$,

$$
\begin{aligned}
N(t) &\le M\left\{d^{\phi}.t^{\phi}.E_{\phi,1+\phi}\left(-d^{\phi}t^{\phi}\right) + \frac{1}{\Gamma(1)} + \left(-d^{\phi}t^{\phi}\right)E_{\phi,1+\phi}\left(-d^{\phi}t^{\phi}\right)\right\},\\
N(t) &\le M.\frac{1}{\Gamma(1)},
\end{aligned}
$$

$N(t) \le M$. As required.

Lemma 2.8 *Consider the system: ${}_{t_0}{}^{C}D_t^{b}x(t) = g(t,x), t_0 > 0$, with initial condition $x(t_0) = x_{t_0}$, where, $\alpha \in (0,1], g : [t_0,\infty)\times\Omega \to \mathbb{R}, \Omega \subseteq C^1[t_0,\infty)$, if local Lipschitz condition is satisfied by g(t, x), with respect to x, then, there exists a solution on $[t_0,\infty)\times\Omega$ which is unique.*

Theorem 2.9 *For every time t, the system (1)–(5) exist a solution which is unique.*

Proof For the existence and uniqueness of the system, take into consideration the region $\Omega\times[t_0, Y)$. Where,

$$
\Omega = \left\{(S,I,T,A,R) \in R^5, S,I,T,A,R \in C\prime[t_0,\infty)\, and \|S\|, \|I\|, \|T\|, \|A\|, \|R\| \le \frac{\mathrm{Z}^{\phi}}{d^{\phi}}\right\} and y < +\infty.
$$

Let $\delta(S) = \mathrm{Z}^{\phi} - S.g(\mathrm{t}-\tau, I).e^{-\mu\tau} - \left({w_1}^{\phi} + d^{\phi}\right)S$,

$$
\begin{aligned}
\|\,\delta(S_1) - \delta(S_2)\,\| &= \|\,\mathrm{Z}^{\phi} - S_1 g(\mathrm{t}-\tau, I).e^{-\mu\tau} - \left({w_1}^{\phi} + d^{\phi}\right)S_1 - \left(\mathrm{Z}^{\phi} - S_2 g(\mathrm{t}-\tau, I)\,e^{-\mu\tau} - \left({w_1}^{\phi} + d^{\phi}\right)S_2\right)\|,\\
\|\,\delta(S_1) - \delta(S_2)\,\| &= \|\,-S_1 g(\mathrm{t}-\tau, I)\,e^{-\mu\tau} - \left({w_1}^{\phi} + d^{\phi}\right)S_1 + S_2 g(\mathrm{t}-\tau, I)\,e^{-\mu\tau} + \left({w_1}^{\phi} + d^{\phi}\right)S_2)\|,
\end{aligned}
$$

$$\| \delta(S_1) - \delta(S_2) \| = \| \left(g(\mathrm{t}-\tau, I) e^{-\mu\tau} + {w_1}^{\phi} + d^{\phi} \right) (S_2 - S_1) \|,$$

$$\| \delta(S_1) - \delta(S_2) \| \leq \| \left(g(\mathrm{t}-\tau, I) e^{-\mu\tau} + {w_1}^{\phi} + d^{\phi} \right) \| \, . \, \| (S_2 - S_1) \| \, .$$

Therefore δ(S) satisfies the local Lipchitz condition (lemma 2.8), moreover, for contraction mapping,

$$\left(g(\mathrm{t}-\tau, I) e^{-\mu\tau} + {w_1}^{\phi} + d^{\phi} \right) < 1,$$

Let, $\mathrm{F}_1 = \left(g(\mathrm{t}-\tau, I) e^{-\mu\tau} + {w_1}^{\phi} + d^{\phi} \right)$, similarly, we have $\mathrm{F}_2 = \mathrm{S}.e^{-\mu\tau} - \mathrm{m}^{\Phi}$, $\mathrm{F}_3 = \mathrm{n}^{\Phi}$, $\mathrm{F}_4 = \mathrm{d}^{\Phi}$, $\mathrm{F}_5 = \mathrm{d}^{\Phi}$.

Also let, $\mathrm{F} = \max\{\mathrm{F}_1, \mathrm{F}_2, \mathrm{F}_3, \mathrm{F}_4, \mathrm{F}_5\}$.

Therefore,

$$\|\delta(S_1) - \delta(S_2)\| \leq \mathrm{F}\|\mathrm{S}_1 - \mathrm{S}_2\|,$$

$$\|\partial(\mathrm{I}_1) - \partial(\mathrm{I}_2)\| \leq \mathrm{F}\|\mathrm{I}_1 - \mathrm{I}_2\|,$$

$$\|\sigma(\mathrm{T}_1) - \sigma(\mathrm{T}_2)\| \leq \mathrm{F}\|\mathrm{T}_1 - \mathrm{T}_2\|,$$

$$\|\pi(\mathrm{A}_1) - \pi(\mathrm{A}_2)\| \leq \mathrm{F}\|\mathrm{A}_1 - \mathrm{A}_2\|,$$

$$\|\gamma(\mathrm{R}_1) - \gamma(\mathrm{R}_2)\| \leq \mathrm{F}\|\mathrm{R}_1 - \mathrm{R}_2\|.$$

For $\mathrm{F} < 1$, $\delta(S), \partial(\mathrm{I}), \sigma(\mathrm{T}), \pi(A), \gamma(\mathrm{R})$ are contraction mappings.

Therefore, $\delta(S), \partial(\mathrm{I}), \sigma(\mathrm{T}), \pi(A) \, and \gamma(\mathrm{R})$ satisfy local Lipshitz conditions. Therefore, according to Banach fixed point theorem, we conclude that, the solution of the time-delayed non-integer order differential equation exists and is unique.

Basic reproductive number (R_0)

In this section, by using next generation method on system (1)–(5) we have, R_0 in the form as,

$$\mathcal{R}_0 = \frac{\alpha n^{\phi} \mathrm{Z}^{\phi} e^{-\mu\tau}}{({w_1}^{\phi} + d^{\phi})(m^{\phi} n^{\phi} - {v_1}^{\phi} {\tau_2}^{\phi})}. \tag{7}$$

Steady states of the model

In this segment, two equilibrium points of the system (1)–(5) are presented.

The infection free steady state is denoted by E^0 and is given below.

$$\mathrm{E0} = (\mathrm{T0},\ \mathrm{S0},\ \mathrm{I0},\ \mathrm{A0},\ \mathrm{R0}) = \left(0, \frac{\mathrm{Z}^{\phi}}{{w_1}^{\phi} + d^{\phi}}, 0, 0, \frac{{\mathrm{w}_1}^{\phi} \mathrm{Z}^{\phi}}{\mathrm{d}^{\phi}({w_1}^{\phi} + d^{\phi})} \right).$$

The disease existence point is represented by $\mathrm{E}^* = (\mathrm{T}^*, \mathrm{S}^*, \mathrm{I}^*, \mathrm{A}^*, \mathrm{R}^*)$ and values of state variables are mentioned as below,

$$\mathrm{T}^* = \frac{{\tau_2}^{\phi} \mathrm{I}^*}{\mathrm{n}^{\phi}},$$

$$\mathrm{A}^* = \frac{{\tau_1}^{\phi} + \frac{{\mathrm{v}_2}^{\phi} {\tau^{\phi}}_2}{\mathrm{n}^{\phi}}}{\mathrm{d}^{\phi}} \mathrm{I}^*,$$

$$\mathrm{R}^* = \frac{{\mathrm{w}_1}^{\phi} \mathrm{S}^*}{\mathrm{d}^{\phi}}, \text{ and}$$

$$\mathrm{S}^* = \frac{(\mathrm{m}^{\phi} - \frac{{\mathrm{v}_1}^{\phi} {\tau^{\phi}}_2}{\mathrm{n}^{\phi}})\mathrm{I}^*}{\mathrm{g}(\mathrm{I}^*) e^{-\mu\tau}}.$$

Theorem 2.12 *The disease free equilibrium E^0 is locally asymptotically stable if $0 < \mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$.*

Proof Consider (J_{E^0}), jacobian matrix with disease free equilibrium point,

$$J_{E^0} = \begin{pmatrix} -({w_1}^{\phi} + d^{\phi}) & \frac{-\eta \mathrm{Z}^{\phi}}{({w_1}^{\phi} + d^{\phi})} & 0 & 0 & 0 \\ 0 & \frac{-\eta \mathrm{Z}^{\phi}}{({w_1}^{\phi} + d^{\phi})} - m^{\phi} & v_1 & 0 & 0 \\ 0 & {\tau_2}^{\phi} & -n^{\phi} & 0 & 0 \\ 0 & {\tau_1}^{\phi} & {v_2}^{\phi} & -d^{\phi} & 0 \\ {w_1}^{\phi} & 0 & 0 & 0 & -d^{\phi} \end{pmatrix}.$$

Since, $det\,(J_{E^0} - \lambda I) = 0$, calculating the eigenvalues of the characteristic equation we have,

$$\left(d^{\phi} + \lambda\right)^2 \left[\left(w_1{}^{\phi} + d^{\phi}\right) + \lambda\right] \left[\left(\frac{-\eta Z^{\phi}}{\left(w_1{}^{\phi} + d^{\phi}\right)} + m^{\phi} + \lambda\right)\left(n^{\phi} + \lambda\right) - v_1{}^{\phi}\tau_2{}^{\phi}\right] = 0.$$

By using Routh–Hurwitz stability criteria, we conclude that roots are negative and real. For more details see [13]. Hence the system (1)-(5) is locally asymptotically stable at E^0.

Theorem 2.13 *If $R_0 > 1$, then the endemic equilibrium E^* is locally asymptotically stable.*

Proof Consider J_{E^*},

$$J_{E^*} = \begin{pmatrix} -g\left(I^*\right)e^{-\mu\tau} - \left(w_1^{\phi} + d^{\phi}\right) & S^* g\prime\left(I^*\right)e^{-\mu\tau} & 0 & 0 & 0 \\ g\left(I^*\right)e^{-\mu\tau} & S^* g'\left(I^*\right)e^{-\mu\tau} - m^{\alpha} & v_1^{\phi} & 0 & 0 \\ 0 & \tau_2^{\phi} & -n^{\alpha} & 0 & 0 \\ 0 & \tau_1^{\phi} & v_2^{\phi} & -d^{\alpha} & 0 \\ w_1^{\phi} & 0 & 0 & 0 & -d^{\alpha} \end{pmatrix}$$

$$det\,(J_{E^*} - \lambda I) = 0,$$

$$(d + \lambda)^2\left(\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3\right) = 0$$

where

$$b_1 = g\left(I^*\right)e^{-\mu\tau} + w_1^{\phi} + d^{\phi} + m^{\alpha} + n^{\alpha} - S^* g\prime\left(I^*\right)e^{-\mu\tau},$$

$$b_2 = \left(m^{\alpha} + n^{\alpha}\right)\left[g\left(I^*\right)e^{-\mu\tau} + w_1^{\alpha} + d^{\alpha}\right] - S^* g\prime\left(I^*\right)e^{-\mu\tau}\left(w_1^{\alpha} + d^{\alpha} + n^{\alpha}\right) + m^{\phi}n^{\phi} - v_1^{\phi}\tau_2^{\phi},$$

$$b_3 = \left(m^{\phi}n^{\phi} - v_1^{\phi}\tau_2^{\phi}\right)\left[g\left(I^*\right)e^{-\mu\tau} + w_1^{\alpha} + d^{\alpha}\right] - n^{\alpha}\left(w_1^{\alpha} + d^{\alpha}\right)S^* g\prime\left(I^*\right)e^{-\mu\tau}.$$

So, by the Routh Hurwitz stability criterion, the E^* is locally asymptotically stable. More details may be seen in [13].

Lemma 2.14 *Let* $\mathrm{p} : [0,\infty) \to \mathbb{R}^+$ *be a continuous function and let* $\mathrm{t}_0 \geq 0$. *Then, for* $\mathrm{t} \geq \mathrm{t}_0$, $\mathrm{w} \in (0,1)$ and $\mathrm{p}^* \in \mathbb{R}^+$, *the following inequality holds* [14,16,17]

$${}_0{}^{\mathrm{c}}\mathrm{D}_{\mathrm{t}}^{\mathrm{w}}\left[\mathrm{p}\left(\mathrm{t}\right) - \mathrm{p}^* - \mathrm{p}^*\ln\frac{\mathrm{p}(\mathrm{t})}{\mathrm{p}^*}\right] \leq \left(1 - \frac{\mathrm{p}^*(\mathrm{t})}{\mathrm{p}}\right){}_0{}^{\mathrm{c}}\mathrm{D}_{\mathrm{t}}^{\mathrm{w}}\mathrm{p}(\mathrm{t}).$$

Theorem 2.15 *The system (1)–(5) is globally asymptotically stable at disease free equilibrium point E^0, if $R_0 < 1$.*

Proof Consider a Voltera type Candidate Lyapunov function as introduced in [14,16,17],

$${}_0{}^{c}D_t^{\varnothing}\sqsubseteq = {}_0{}^{c}D_t^{\varnothing}\left(S - S^* - S^* ln\tfrac{s}{S^*}\right) + {}_0{}^{c}D_t^{\varnothing}\left(R - R^* - R^* ln\tfrac{R}{R^*}\right) + {}_0{}^{c}D_t^{\varnothing}A + {}_0{}^{c}D_t^{\varnothing}I + {}_0{}^{c}D_t^{\varnothing}T \quad ,$$

using above lemma 2.14,

$${}_0{}^{c}D_t^{\varnothing}\sqsubseteq \leq \left(1 - \frac{S^*}{s}\right){}_0{}^{c}D_t^{\varnothing}S + \left(1 - \frac{R^*}{R}\right){}_0{}^{c}D_t^{\varnothing}R + {}_0{}^{c}D_t^{\varnothing}A + {}_0{}^{c}D_t^{\varnothing}I + {}_0{}^{c}D_t^{\varnothing}T,$$

$${}_0{}^{c}D_t^{\varnothing}\sqsubseteq \leq \left(-\frac{S^*}{s}\right){}_0{}^{c}D_t^{\varnothing}S + \left(-\frac{R^*}{R}\right){}_0{}^{c}D_t^{\varnothing}R + {}_0{}^{c}D_t^{\varnothing}S + {}_0{}^{c}D_t^{\varnothing}R + {}_0{}^{c}D_t^{\varnothing}A + {}_0{}^{c}D_t^{\varnothing}I + {}_0{}^{c}D_t^{\varnothing}T,$$

${}_0{}^{c}D_t^{\varnothing}\sqsubseteq \leq \left(-\frac{S^*}{s}\right){}_0{}^{c}D_t^{\varnothing}S + \left(-\frac{R^*}{R}\right){}_0{}^{c}D_t^{\varnothing}R + 0$, using DFE equilibrium points,

$${}_0{}^{c}D_t^{\varnothing}\sqsubseteq (S, R, A, T, I) \leq 0.$$

Hence, by LaSalle's invariance theorem, the system (1)-(5) is globally asymptotically stable.

Theorem 2.16 *The system (1)–(5) is globally asymptotically stable at endemic equilibrium point E^*, if $R_0 > 1$.*

Proof Consider a Voltera type Candidate Lyapunov function [14,16,17],

$$\Psi = K_1\left(S - S^* - S^* ln\frac{s}{S^*}\right) + K_2\left(A - A^* - A^* ln\frac{A}{A^*}\right) + K_3\left(T - T^* - T^* ln\frac{T}{T^*}\right) + K_4\left(I - I^* - I^* ln\frac{I}{I^*}\right),$$

$${}_0{}^{c}D_t^{\varnothing}\Psi = K_{10}{}^{c}D_t^{\varnothing}\left(S - S^* - S^* ln\frac{s}{S^*}\right) + K_{20}{}^{c}D_t^{\varnothing}\left(A - A^* - A^* ln\frac{A}{A^*}\right) + K_{30}{}^{c}D_t^{\varnothing}\left(T - T^* - T^* ln\frac{T}{T^*}\right) + K_{40}{}^{c}D_t^{\varnothing}\left(I - I^* - I^* ln\frac{I}{I^*}\right),$$

$$ {}_0{}^{c}D_t^{\varnothing}\Psi \le K_1\left(1-\frac{S^*}{s}\right){}_0{}^{c}D_t^{\varnothing}S + K_2\left(1-\frac{A^*}{A}\right){}_0{}^{c}D_t^{\varnothing}A + K_3\left(1-\frac{T^*}{T}\right){}_0{}^{c}D_t^{\varnothing}T + K_4\left(1-\frac{I^*}{I}\right){}_0{}^{c}D_t^{\varnothing}I, $$

$$ {}_0^{c}D_t^{\emptyset}\Psi \le K_1\left(\frac{S-S^*}{s}\right)\left(Z^{\phi} - S\cdot g\left(t-\tau,I\right).e^{-\mu\tau} - \left(w_1^{\phi}+d^{\phi}\right)\cdot S\right) + K_2\left(\frac{A-A^*}{A}\right)\left(\tau_1^{\phi}I - d^{\phi}A + v_2^{\phi}T\right) $$

$$ +K_3\left(\frac{T-T^*}{T}\right)\left(\tau_2^{\phi}I - n^{\phi}T\right) + K_4\left(\frac{I-I^*}{I}\right)\left(S\cdot g\left(t-\tau,I\right)\cdot e^{-\mu\tau} + v_1^{\phi}T - m^{\phi}I\right), $$

$$ {}_0^{c}D_t^{\emptyset}\Psi \le K_1\left(S-S^*\right)\left(\frac{Z^{\phi}}{S} - g\left(t-\tau,I\right)\cdot e^{-\mu\tau} - \left(w_1^{\phi}+d^{\phi}\right)\right) + K_2\left(A-A^*\right)\left(\frac{\tau_1^{\phi}I}{A} - d^{\phi} + \frac{v_2^{\phi}T}{A}\right) $$

$$ +K_3\left(T-T^*\right)\left(\frac{\tau_2^{\phi}I}{T} - n^{\phi}\right) + K_4\left(I-I^*\right)\left(\frac{S.g\left(t-\tau,I\right)\cdot e^{-\mu\tau}}{I} + \frac{v_1^{\phi}T}{I} - m^{\phi}\right). $$

Now, we consider the Eq. (1) and treat it as follows,

$$ {}_0{}^{c}\mathrm{D}_t^{\Phi}\mathrm{S}^*\left(\mathrm{t}\right) = \mathrm{Z}^{\Phi} - S^*.\mathrm{g}\left(\mathrm{t}-\tau,\mathrm{I}\right).\mathrm{e}^{-\mu\tau} - \left({w_1}^{\Phi}+\mathrm{d}^{\Phi}\right).S^* = 0, $$

$$ \mathrm{g}\left(\mathrm{t}-\tau,\mathrm{I}\right).\mathrm{e}^{-\mu\tau} - \left({w_1}^{\Phi}+\mathrm{d}^{\Phi}\right) = \frac{\mathrm{Z}^{\Phi}}{S^*}. $$

Similarly from the Eq. (2),

$$ {}_0^{c}\mathrm{D}_t^{\phi}\ \mathrm{I}^*\left(\mathrm{t}\right) = \mathrm{S}\cdot\mathrm{g}\left(\mathrm{t}-\tau,\mathrm{I}^*\right)\cdot\mathrm{e}^{-\mu\tau} + \mathrm{v}_1^{\phi}\mathrm{T} - \mathrm{m}^{\phi}\mathrm{I}^* = 0,\quad \frac{\mathrm{S}\cdot\mathrm{g}\left(\mathrm{t}-\tau,\mathrm{I}^*\right)\cdot\mathrm{e}^{-\mu\tau}}{\mathrm{I}^*} + \frac{\mathrm{v}_1^{\phi}\mathrm{T}}{\mathrm{I}^*} = \mathrm{m}^{\phi}. $$

Also the Eq. (3) gives the following result

$$ {}_0{}^{c}\mathrm{D}_t^{\Phi}\mathrm{T}^*\left(\mathrm{t}\right) = {\tau_2}^{\Phi}\mathrm{I} - \mathrm{n}^{\Phi}\mathrm{T}^* = 0, $$

$$ \frac{{\tau_2}^{\Phi}\mathrm{I}}{\mathrm{T}^*} = \mathrm{n}^{\Phi}. $$

Lastly, the Eq. (4) helps us to reach at the following expression,

$$ {}_0{}^{c}\mathrm{D}_t^{\Phi}\mathrm{A}^*\left(\mathrm{t}\right) = {\tau_1}^{\Phi}\mathrm{I} - \mathrm{d}^{\Phi}\mathrm{A}^* + {\mathrm{v}_2}^{\Phi}\mathrm{T} = 0, $$

$$ \mathrm{d}^{\Phi} = \frac{{\tau_1}^{\Phi}\mathrm{I}}{\mathrm{A}^*} + \frac{{\mathrm{v}_2}^{\Phi}\mathrm{T}}{\mathrm{A}^*}, $$

$$ {}_0{}^{c}D_t^{\varnothing}\Psi \le K_1\left(S-S^*\right)\left(\frac{\mathrm{Z}^{\Phi}}{S} - \frac{\mathrm{Z}^{\Phi}}{S^*}\right) + K_2\left(A-A^*\right)\left(\frac{{\tau_1}^{\Phi}\mathrm{I}}{A} + \frac{{\mathrm{v}_2}^{\Phi}\mathrm{T}}{A} - \frac{{\tau_1}^{\Phi}\mathrm{I}}{A^*} - \frac{{\mathrm{v}_2}^{\Phi}\mathrm{T}}{A^*}\right) + K_3\left(T-T^*\right)\left(\frac{{\tau_2}^{\Phi}\mathrm{I}}{T} - \frac{{\tau_2}^{\Phi}\mathrm{I}}{T^*}\right) $$

$$ + K_4\left(I-I^*\right)\left(\frac{\mathrm{S.g}\left(\mathrm{t}-\tau,\mathrm{I}\right).\mathrm{e}^{-\mu\tau}}{I} + \frac{{\mathrm{v}_1}^{\Phi}\mathrm{T}}{I} - \frac{\mathrm{S.g}\left(\mathrm{t}-\tau,\mathrm{I}^*\right).\mathrm{e}^{-\mu\tau}}{\mathrm{I}^*} + \frac{{\mathrm{v}_1}^{\Phi}\mathrm{T}}{\mathrm{I}^*}\right), $$

$$ {}_0{}^{c}D_t^{\varnothing}\Psi \le -K_1(S-S^*)^2\left(\frac{\mathrm{Z}^{\Phi}}{SS^*}\right) - K_2(A-A^*)^2\left(\frac{{\tau_1}^{\Phi}\mathrm{I}+{\mathrm{v}_2}^{\Phi}\mathrm{T}}{AA^*}\right) $$

$$ - K_3(T-T^*)^2\left(\frac{{\tau_2}^{\Phi}\mathrm{I}}{TT^*}\right) - K_4(I-I^*)^2\left(\frac{\mathrm{S.g}\left(\mathrm{t}-\tau,\mathrm{I}\right).\mathrm{e}^{-\mu\tau}+{\mathrm{v}_1}^{\Phi}\mathrm{T}}{II^*}\right). $$

Here, we set $K_1 = K_2 = K_3 = K_4 = 1$, so we have,

$$ {}_0{}^{c}D_t^{\varnothing}\Psi \le 0. $$

Hence, by LaSalle's invariance theorem, the system (1)–(5) is globally asymptotically stable.

Finite difference technique

Here we will proposed the Grunwald Letnikov finite difference scheme for the above designed model (1)–(5). The Grünwald-Letnikov NSFD scheme presents itself as a low complexity, numerically efficient scheme which preserves main physical properties of fractional systems. Due to these features, the system is very well suited for modeling biological system (1)–(5), where memory effects and positivity constraints are important [10].

By using Grunwald–Letnikov approximation,
${}_0{}^{c}D_t^{\alpha}y\left(t_{n+1}\right) = \frac{1}{\psi(h)^{\alpha}}\left\{y_{n+1} - \sum_{\nu=1}^{n+1}C_{\nu}^{\alpha}y_{n+1-\nu} - r_{n+1}^{\alpha}y_0\right\}$ to Eq. (1), we have,

$$ \frac{1}{\varphi\left(h\right)^{\phi}}\{S_{n+1} - \sum_{v=1}^{n+1}c_v^{\phi}S_{n+1-v} - \gamma_{n+1}^{\phi}S_0\} = \mathrm{Z}^{\phi} - S_{n+1}g\left(\mathrm{t}-\tau,I\right)e^{-\mu\tau} - \left({w_1}^{\phi}+d^{\phi}\right)S_{n+1}, $$

$$S_{n+1}-\sum_{v=1}^{n+1}c_v^{\phi}S_{n+1-v}-\gamma_{n+1}^{\phi}S_0=\varphi(h)^{\phi}\left\{\mathrm{Z}^{\phi}-S_{n+1}g(\mathrm{t}-\tau,I)\,e^{-\mu\tau}-\left({w_1}^{\phi}+d^{\phi}\right)S_{n+1}\right\},$$

$$S_{n+1}\left\{1+\varphi(h)^{\phi}g(\mathrm{t}-\tau,I)\,e^{-\mu\tau}+\varphi(h)^{\phi}\left({w_1}^{\phi}+d^{\phi}\right)\right\}=\varphi(h)^{\phi}\mathrm{Z}^{\phi}+\sum_{v=1}^{n+1}c_v^{\phi}S_{n+1-v}+\gamma_{n+1}^{\phi}S_0,$$

$$S_{n+1}=\frac{\varphi(h)^{\phi}\mathrm{Z}^{\phi}+\sum_{v=1}^{n+1}c_v^{\phi}S_{n+1-v}+\gamma_{n+1}^{\phi}S_0}{1+\varphi(h)^{\phi}g(\mathrm{t}-\tau,I)\,e^{-\mu\tau}+\varphi(h)^{\phi}\left({w_1}^{\phi}+d^{\phi}\right)}. \tag{8}$$

Proceeding in this manner we have the following results,

$$I_{n+1}=\frac{\varphi(h)^{\phi}S_{n+1}g(\mathrm{t}-\tau,I)\,e^{-\mu\tau}+\varphi(h)^{\phi}{v_1}^{\phi}T_n+\sum_{v=1}^{n+1}c_v^{\phi}I_{n+1-v}+\gamma_{n+1}^{\phi}I_0}{1+\varphi(h)^{\phi}{m}^{\phi}}, \tag{9}$$

$$T_{n+1}=\frac{\varphi(h)^{\phi}\left\{\tau_2^{\phi}I_{n+1}\right\}+\sum_{v=1}^{n+1}c_v^{\phi}T_{n+1-v}+\gamma_{n+1}^{\phi}T_0}{1+n^{\phi}\varphi(h)^{\phi}}, \tag{10}$$

$$A_{n+1}=\frac{\varphi(h)^{\phi}{\tau_1}^{\phi}I_{n+1}+\varphi(h)^{\phi}{v_2}^{\phi}T_{n+1}+\sum_{v=1}^{n+1}c_v^{\phi}A_{n+1-v}+\gamma_{n+1}^{\phi}A_0}{1+d^{\phi}\varphi(h)^{\phi}}, \tag{11}$$

$$R_{n+1}=\frac{\varphi(h)^{\phi}{w_1}^{\phi}S_{n+1}+\sum_{v=1}^{n+1}c_v^{\phi}R_{n+1-v}+\gamma_{n+1}^{\phi}R_0}{1+d^{\phi}\varphi(h)^{\phi}}. \tag{12}$$

Lemma 3.1 *Let all the controlled parameters and state variables are positive with* $S_0, I_0, A_0, T_0 and R_0 \geq 0$. *Then,* $S_{n+1}, I_{n+1}, A_{n+1}, T_{n+1}, R_{n+1} \geq 0, \forall n \in Z^+$.

Proof Since, we have from the Eq. (8),

$$S_{n+1}=\frac{\varphi(h)^{\phi}\mathrm{Z}^{\phi}+\sum_{v=1}^{n+1}c_v^{\phi}S_{n+1-v}+\gamma_{n+1}^{\phi}S_0}{1+\varphi(h)^{\phi}g(\mathrm{t}-\tau,I)\,e^{-\mu\tau}+\varphi(h)^{\phi}\left({w_1}^{\phi}+d^{\phi}\right)},$$

For n = 0,

$$S_1=\frac{\varphi(h)^{\phi}\mathrm{Z}^{\phi}+\sum_{v=1}^{1}c_v^{\phi}S_{1-v}+\gamma_1^{\phi}S_0}{1+\varphi(h)^{\phi}g(\mathrm{t}-\tau,I)\,e^{-\mu\tau}+\varphi(h)^{\phi}\left({w_1}^{\phi}+d^{\phi}\right)}.$$

Since S_0 and all the parameters are positive.

Then $S_1 \geq 0$. Similarly it can easily be proved that $I_1, A_1, T_1 and R_1 \geq 0$. Next we suppose that the result holds for $n = \{1,2,3,4,\ldots,n-1\}\, i.e S_n, I_n, A_n, T_n, R_n \geq 0, \forall n = \{1,2,3,4,\ldots,n-1\}$. Moreover for $n \in Z^+$ we have,

$$S_{n+1}=\frac{\varphi(h)^{\phi}\mathrm{Z}^{\phi}+\sum_{v=1}^{n+1}c_v^{\phi}S_{n+1-v}+\gamma_{n+1}^{\phi}S_0}{1+\varphi(h)^{\phi}g(\mathrm{t}-\tau,I)\,e^{-\mu\tau}+\varphi(h)^{\phi}\left({w_1}^{\phi}+d^{\phi}\right)}.$$

Since, all the discretized state variables and parameters are positive. Therefore, $S_{n+1} \geq 0$. Similarly $I_{n+1}, A_{n+1}, T_{n+1}, R_{n+1} \geq 0$. Hence, the proposed numerical scheme preserve the positivity for $n \in Z^+$.

Lemma 3.2 *Suppose that* S_0, I_0, A_0, T_0, R_0 *are finite and positive and* $S_0 + I_0 + A_0 + T_0 + R_0 \leq N_0$. *Moreover, all the parameters involved in this model are positive. Then,* $S_{n+1} + A_{n+1} + T_{n+1} + R_{n+1}$ *are bounded by a real constant* $N_n, for n \in Z^+$.

Proof Since, all the parameters and state variables are positive then there exists a constant N_n, such that $S_{n+1}, I_{n+1}, A_{n+1}, T_{n+1}, R_{n+1} \leq N_n for\, n \in Z^+$.

Adding the Eqs. (8)–(12),

$$\begin{aligned}
&\left\{1+\varphi(h)^{\phi} g\left(\mathrm{t}-\tau, I\right) e^{-\mu \tau}+\varphi(h)^{\phi}\left(w_1{}^{\phi}+d^{\phi}\right)\right\} S_{n+1}+\left\{1+\varphi(h)^{\phi} m^{\phi}\right\} I_{n+1} \\
&\quad+\left\{1+n^{\phi} \varphi(h)^{\phi}\right\} T_{n+1}+\left\{1+d^{\phi} \varphi(h)^{\phi}\right\} A_{n+1}+\left\{1+d^{\phi} \varphi(h)^{\phi}\right\} R_{n+1} \\
&=\varphi(h)^{\phi} \mathrm{Z}^{\phi}+\sum_{v=1}^{n+1} c_v^{\phi} S_{n+1-v}+\gamma_{n+1}^{\phi} S_0+\varphi(h)^{\phi} S_{n+1} g(t, I) e^{-\mu \tau} \\
&\quad+\varphi(h)^{\phi} v_1{}^{\phi} T_n+\sum_{v=1}^{n+1} c_v^{\phi} I_{n+1-v}+\gamma_{n+1}^{\phi} I_0+\varphi(h)^{\phi}\left\{\tau_2^{\phi} I_{n+1}\right\} \\
&\quad+\sum_{v=1}^{n+1} c_v^{\phi} T_{n+1-v}+\gamma_{n+1}^{\phi} T_0+\varphi(h)^{\phi} \tau_1{}^{\phi} I_{n+1}+\varphi(h)^{\phi} v_2{}^{\phi} T_{n+1} \\
&\quad+\sum_{v=1}^{n+1} c_v^{\phi} A_{n+1-v}+\gamma_{n+1}^{\phi} A_0+\varphi(h)^{\phi} w_1{}^{\phi} S_{n+1}+\sum_{v=1}^{n+1} c_v^{\phi} R_{n+1-v}+\gamma_{n+1}^{\phi} R_0,
\end{aligned}$$

$$\begin{aligned}
&\left(1+d^{\phi} \varphi(h)^{\phi}\right) S_{n+1}+\left(1+d^{\phi} \varphi(h)^{\phi}\right) I_{n+1}+\left(1+d^{\phi} \varphi(h)^{\phi}\right) A_{n+1}+\left(1+d^{\phi} \varphi(h)^{\phi}\right) T_{n+1}+\left(1+d^{\phi} \varphi(h)^{\phi}\right) R_{n+1} \\
&\quad=\varphi(h)^{\phi} \mathrm{Z}^{\phi}+\sum_{v=1}^{n+1} c_v^{\phi}\{S_{n+1-v}+I_{n+1-v}+T_{n+1-v}+A_{n+1-v}+R_{n+1-v}\}+\gamma_{n+1}^{\phi}\{S_0+I_0+T_0+A_0+R_0\},
\end{aligned}$$

$$\begin{aligned}
&S_{n+1}+I_{n+1}+A_{n+1}+T_{n+1}+R_{n+1} \\
&\quad=\frac{\varphi(h)^{\phi} \mathrm{Z}^{\phi}+\sum_{v=1}^{n+1} c_v^{\phi}\{S_{n+1-v}+I_{n+1-v}+T_{n+1-v}+A_{n+1-v}+R_{n+1-v}\}+\gamma_{n+1}^{\phi}\{S_0+I_0+T_0+A_0+R_0\}}{1+d^{\phi} \varphi(h)^{\phi}}.
\end{aligned}$$

For $n=0$,

$$S_1+I_1+A_1+T_1+R_1=\frac{\varphi(h)^{\phi} \mathrm{Z}^{\phi}+\sum_{v=1}^{1} c_v^{\phi}\{S_{1-v}+I_{1-v}+T_{1-v}+A_{1-v}+R_{1-v}\}+\gamma_1^{\phi}\{S_0+I_0+T_0+A_0+R_0\}}{1+d^{\phi} \varphi(h)^{\phi}},$$

$$S_1+I_1+A_1+T_1+R_1<N_1$$

For $n=1$,

$$S_2+I_2+A_2+T_2+R_2=\frac{\varphi(h)^{\phi} \mathrm{Z}^{\phi}+\sum_{v=1}^{2} c_v^{\phi}\{S_{2-v}+I_{2-v}+T_{2-v}+A_{2-v}+R_{2-v}\}+\gamma_2^{\phi}\{S_0+I_0+T_0+A_0+R_0\}}{1+d^{\phi} \varphi(h)^{\phi}},$$

$$S_2+I_2+A_2+T_2+R_2<N_2.$$

Now, suppose that the system is bounded for $n \in \{2,3, \ldots, n-1\}$,

$$S_{\mathrm{n}}+I_{\mathrm{n}}+A_{\mathrm{n}}+T_{\mathrm{n}}+R_{\mathrm{n}}<N_{\mathrm{n}}, \forall \mathrm{n} \in \{2,3, \ldots, \mathrm{n}-1\}.$$

Further consider,

$$\begin{aligned}
&S_{n+1}+I_{n+1}+A_{n+1}+T_{n+1}+R_{n+1} \\
&\quad=\frac{\varphi(h)^{\phi} \mathrm{Z}^{\phi}+\sum_{v=1}^{n+1} c_v^{\phi}\{S_{n+1-v}+I_{n+1-v}+T_{n+1-v}+A_{n+1-v}+R_{n+1-v}\}+\gamma_{n+1}^{\phi}\{S_0+I_0+T_0+A_0+R_0\}}{1+\mathrm{d}^{\Phi} \varphi(\mathrm{h})^{\Phi}}.
\end{aligned}$$

Finally, we have

$$\mathrm{S}_{\mathrm{n}+1}+\mathrm{I}_{\mathrm{n}+1}+\mathrm{A}_{\mathrm{n}+1}+\mathrm{T}_{\mathrm{n}+1}+\mathrm{R}_{\mathrm{n}+1}<\mathrm{N}_{\mathrm{n}+1},$$

For $\mathrm{n}=\{0,1,2,3, \ldots \mathrm{n}\}$ as desired.

Numerical simulations and graphical discussions

In this segment graphs are plotted for different values of the parameters given in the below Table 2 to support our claimed feature of the scheme.

Numerical simulations

Here, we present the graphs of all variables against time for both the steady states and the effect of delay factor is also discussed.

In this case, we have illustrated the simulated graphs of the systems' variables when $(R_0>1)$ for better perception and progressive behavior. The figures are drawn there with delay factors. Figure 1a shows the graphs for S(t), Fig. 1b for I(t), Fig. 1c for A(t), Fig. 1d for T(t) and Fig. 1e for R(t). These graphs illustrate the role of ∅ , the fractional order parameter by showing various trajectories which depend on this parameter. It can be seen that, the greater ∅ the graph will converge fastly to the endemic equilibrium point.

Parameter symbol	Parameter rates	Parameter values	Sources
Z	Recruitment	150	[13]
d	Natural mortality	2	[13]
τ_1	Infected individuals safe sex practice	2	[13]
τ_2	Infected receive anti-viral medicines	3	[13]
ω_1	Susceptible individuals safe sex practice	1	[13]
ν_1	Treatment defaulters	1	[13]
ν_2	Treated individuals adhere to safe sex practices	1	[13]
μ	Death parameter	0.02	[13]
τ	Delay parameter	Variable	
S_0	$S(0)$	10	[13]
I_0	$I(0)$	10	[13]
A_0	$A(0)$	10	[13]
T_0	$T(0)$	10	[13]
R_0	$R(0)$	10	[13]
φ	Fractional parameter	0.9, 0.7, 0.5	

Table 2. Different values of the parameters

Figure 2a,b show the dynamical behavior of the S(t) and I(t) respectively at the STIDs free state. While Fig. 2c–e represent the behavior of A(t), T(t) and R(t).

When the persons do not follow the resistor strategies, spend serenely life style and do not take the proper cure, the STIDs danger factor increases. In other words, the number of infected populace decreases remarkably, when the delay-strategies are implemented. Handling to the particular level may reduce the number of infected individuals to a situation where delay factors are taken into account. In addition, the effect of delay factor on STIDs is very significant as shown in the plots, indicating that the "τ" has a crucial role in eradicating STIDs.

The role of delay factor τ upon control of STIDs is shown in Fig. 3. In Fig. 3 multiple graphs are drawn against different values of delay factor. This graph shows that when the value of τ increases the infection goes down and down hence the eradication of disease. As a consequence, the infected class can be remarkably reduced by adopting the maximum possible control factors.

Conclusion

In this study, we successfully transform a classical integer-order disease dynamics model into a fractional-order delayed model for analyzing the dynamics of sexually transmitted infectious diseases (STIDs). The fractional Caputo differential operator is used, along with appropriate delay terms to capture the memory effects and time-dependent interactions inherent in the disease transmission dynamics. The analysis shows that the proposed model admits a unique and positive bounded solution, therefore guaranteeing the biological validity of the model. Furthermore, the model's mathematical stability, robustness, and biological relevance are demonstrated by the fact that the long-term steady states are not dependent on the initial conditions of the state variables. The model identifies two equilibrium states, the disease-free equilibrium and the endemic equilibrium. It is shown that the system is locally and globally stable at these equilibria through stability analysis. We computed the basic reproduction number. Biologically, it is interpreted as a critical threshold indicator. In particular, if $R_0 > 1$ then the infection is expected to persist and spread, whereas if $R_0 < 1$ the disease is in a contending possibility. The increase is due to unsafe sexual behaviors, inadequate public health interventions, and socio-economic factors that increase the risk of infection. However, effective control measures, including regular condom use, health education, and more accessible healthcare can decrease and stave off transmission of STIDs. The fractional order system is complicated and hence there is no way to get exact analytical solutions. To achieve this a hybrid numerical approach the Grünwald Letnikov Non-Standard Finite Difference (GL-NSFD) scheme is proposed to approximate the solutions. This scheme is positive, bounded, and converges to numerical results keeping the biological integrity of the model. Using simulated results we show how the system behaves in the presence and absence of delay terms. Comparative graphical analysis of the impact of the delay factors on the disease progression is presented in terms of higher values of the delay parameter τ. Stabilization and control of disease dynamics are highly sensitive to τ. The findings also highlight the need to take into account memory effects, and delays in modeling sexually transmitted diseases, to better understand temporal and behavioral aspects of disease spread. The proposed fractional order delayed model can be used as a generalizable framework with which to pursue investigation of other infectious diseases and to develop interventions. Further studies would broaden this model to include multi-strain infections, co-infections, and the impact of vaccination or treatment programs on disease control.

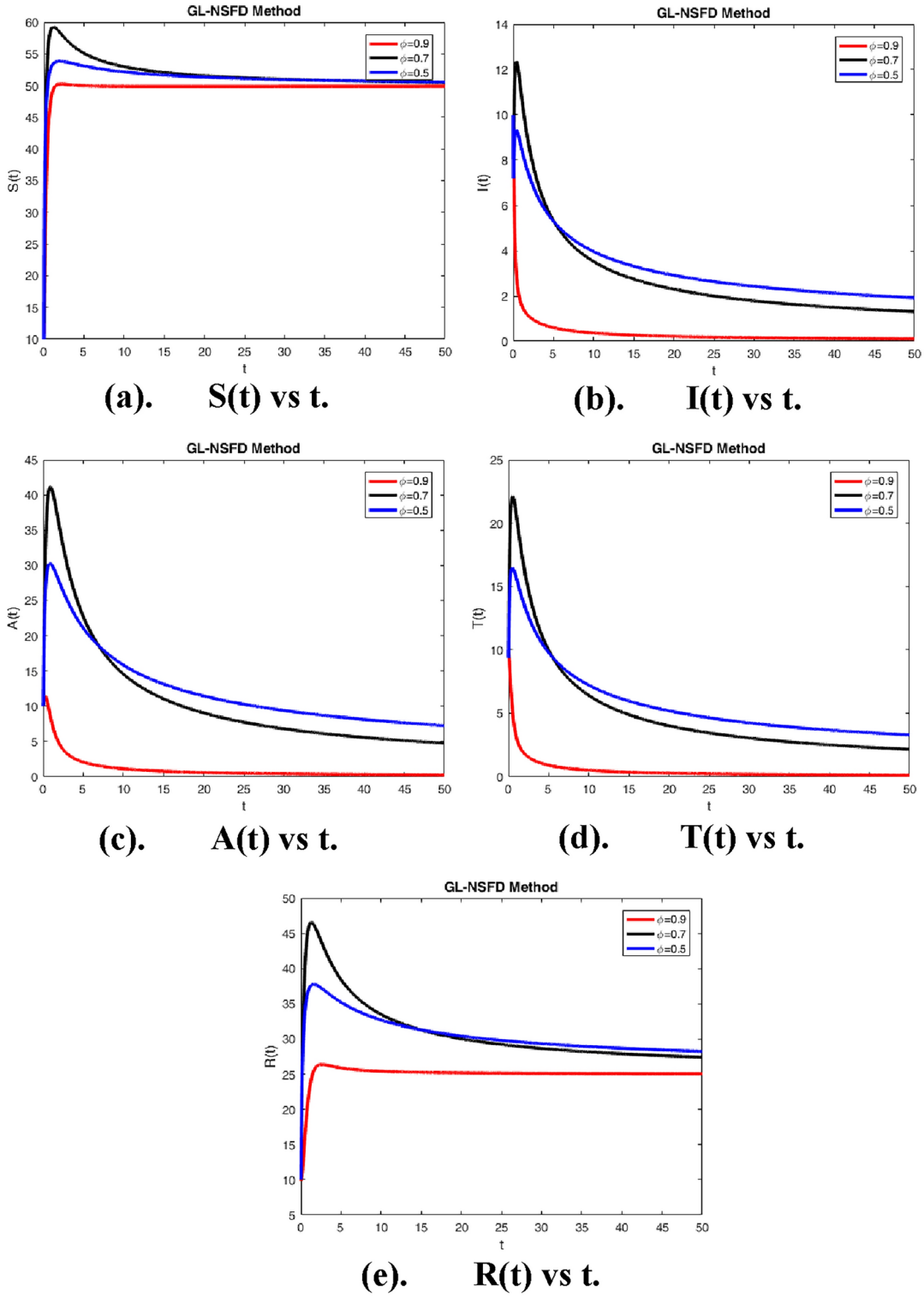

Fig. 1. Graphical dynamics of S(t), I(t), A(t), T(t) and R(t) using suggested approach for disease free equilibrium using values of parameters given in Table 2.

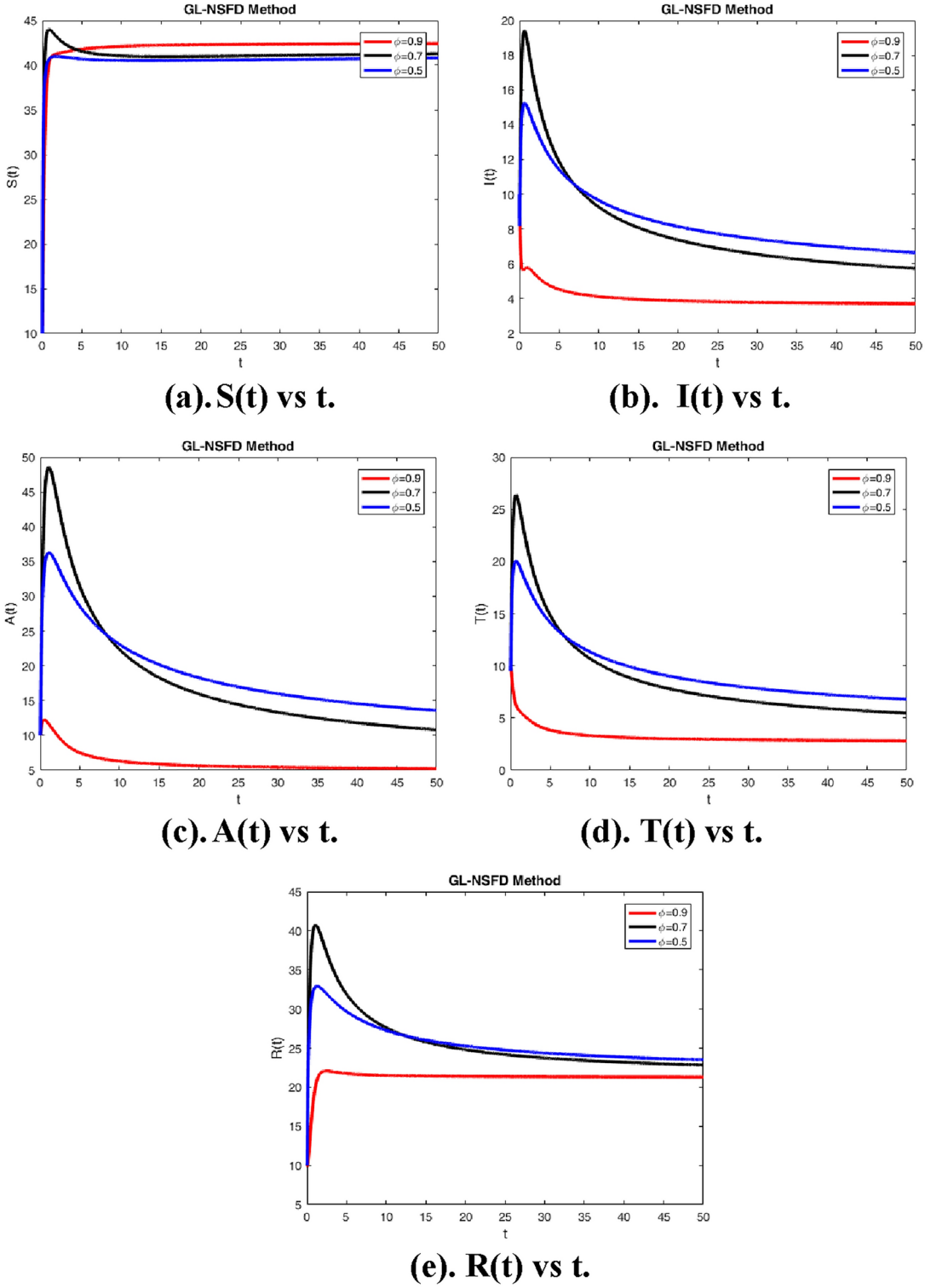

Fig. 2. Graphical dynamics of S(t), I(t), A(t), T(t) and R(t) using suggested approach for endemic equilibrium using values of parameters given in Table 2.

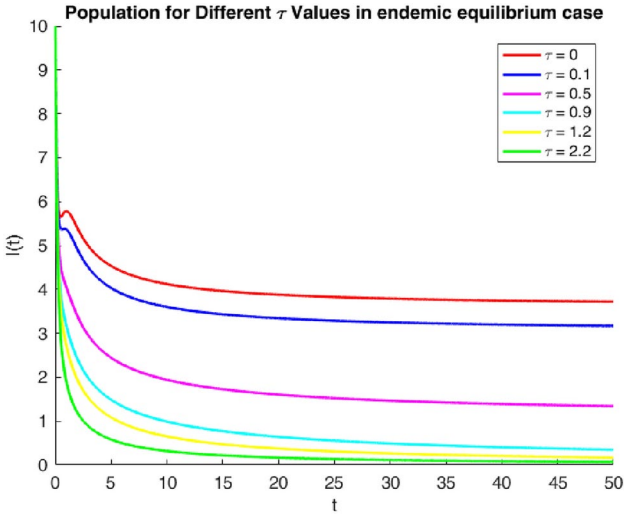

Fig. 3. Effect of delay factor on the dynamics of the proposed STIDs model.

Data availability

The datasets used and analysed during the current study are available from the corresponding author on reasonable request.

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M.R.: Model and analysis, problem solution; M.A.U.R: Problem formulation; A.M.A.: results and discussion; M.R.: plotting and numerical simulations; A.F.A.: analysis; software; Z.I. Software; plotting, discussion; N.A. Modelling, analysis, numerical simulations; Methodology; S.N.: writing; analysis; conclusion, editing; I.K.: methodology; simulations, plotting;

Declarations

Competing interests

The authors declare no competing interests.

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