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Reliability Analysis in Stress-Strength Model under Record Values with Practical Verification

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Abstract

This article uses upper record values to estimate the stress-strength reliability parameter, defined as $\Re = P(Z < T)$. We assume that both strength (T) and stress (Z) are independent random variables that follow the inverted exponentiated Pareto distribution with a common second shape parameter. The maximum likelihood and Bayesian estimators of $\Re$ are obtained. Using informative and non-informative priors, the Bayesian estimators are obtained under symmetric and asymmetric loss functions. Two bootstrap-type confidence intervals and highest posterior credible intervals are constructed. Gibbs and Metropolis-Hasting samplers are used to generate Bayesian estimates of reliability $\Re$ based on the suggested loss functions. To investigate the behavior of suggested approaches, extensive simulation studies are carried out using some accuracy measures. Simulation experiment findings validated the consistency of the Bayesian and non-Bayesian estimates of $\Re$. According to specific metrics, Bayesian estimates under symmetric loss function showed more precision than those under asymmetric loss functions. The lengths of credible intervals for Bayesian estimates are less than the bootstrap confidence intervals for different record numbers. The bootstrap-p confidence intervals give more accurate outcomes than bootstrap-t in most cases. The analysis employs two representative datasets. The first includes the timing of goals scored in the final rounds of the European Champions League over two consecutive seasons. The second dataset contains monthly observations of sulfur dioxide concentration in Long Beach, California, spanning the years 1956 to 1974.

Keywords. Inverted exponentiated Pareto distribution; bootstrap confidence intervals; weighted least squares loss function; upper record values; stress-strength model

1. Introduction

The analysis of record statistics is tremendously important in a variety of real-world applications. These applications range from monetary research and monitoring of epidemic diseases to sports analytics, weather forecasting, and reliability engineering. Record values represent the highs and lows of an assembly of random variables [1]. Foster and Stuart [2] explored hypothesis tests to find the distribution of the record values through examination of the sum and differences between the upper and lower records of the series. Consider a sequence of independent and identically distributed (iid) continuous random variables, Z_1, Z_2, ... with cumulative distribution function (CDF) $F(z)$ and probability density function (PDF) $f(z)$. An upper record value (URV), denoted by Z_i, occurs when $Z_i > Z_j$, for all $j < i$. Let R_1, R_2, ..., R_n represent the first n URV from a distribution with CDF $F(z)$ and PDF $f(z)$. The joint PDF of this URV sequence, as suggested by Arnold et al. [3], is given by:

$$f(r_1, r_2, \ldots, r_n) = f(r_n)\prod_{i=1}^{n-1}\frac{f(r_i)}{1-F(r_i)}; \quad -\infty < r_1 < \ldots < r_n < \infty. \tag{1}$$

The stress-strength (SS) reliability model is a significant and fundamental topic in statistical literature. The reliability parameter $\Re$ is defined as the probability that strength T well exceeds Z: $\Re = P(Z < T)$. The SS model is essential for assessing and designing reliable materials in practical engineering and for product life testing. For instance, to guarantee aviation safety, life tests are essential for aircraft components in the aerospace industry. The product is subjected to a variety of environmental challenges during its lifecycle, including temperature, humidity, wind, and pressure. Its capacity to endure these strains is determined by its inherent material strength. The SS model is currently widely used in many different domains, particularly for addressing reliability assessment challenges. Birnbaum [4] was the first to introduce the idea of the SS, and it was later examined in more detail by Birnbaum and McCarty [5]. The standard term 'stress-strength reliability' was later coined by Church and Harris [6], who applied it to the estimation of $\Re$ for normal variates.

Parametric inferences concerning SS reliability models have been a significant area of research. Researchers have investigated these models using a variety of sampling techniques and by choosing different distributions for the random variables of strength (T) and stress (Z). This includes studies that utilize complete samples (e.g., [7]-[9]). On a different note, the statistical inference of the SS reliability models was examined by several authors using record values. The pioneer work of the SS model, particularly from the exponential distribution and its generalization, was introduced by Baklizi [10,11]. Subsequently, Basirat et al. [12] put forward the SS reliability based on proportional hazard rate models. In their work, Nadar and Kizilaslan [13] examined both classical and Bayesian methods for estimating the SS model by utilizing URV from Kumaraswamy's distribution. The Bayesian inferences of the SS Lomax distributions were explored by Mahmoud et al. [14]. Condino et al. [15] created both Bayesian and non-Bayesian estimators for the parameter $\Re$ by working with a proportional reversed hazard family. Inference of SS from Burr X and bathtub-shaped distributions based on URV was covered, respectively, in Refs. [16, 17]. Amin [18] focused on estimating $\Re$ through URV specifically for the Kumaraswamy exponential distribution. Dhanya and Jeevavand [19] used lower record values for the power function distribution model to examine classical and Bayesian inference of $\Re$. Hassan et al. [20, 21] looked into estimating $\Re$ based on the generalized inverse exponential distribution. Estimation of $\Re$ from the proportional reversed hazard family under lower record values was provided by Chaturvedi and Malhotra [22]. Using generalized order statistics, inference of $\Re = P(Z < T)$ from exponential distribution was examined by Khan and Khatoon [23]. Mohamed [24] concentrated on estimating $\Re = P(Z < T)$ using URV, particularly for the gamma Lindley distribution. Moheb et al. [25] provided the Bayesian and non-Bayesian inference of $\Re$ for the inverse Lomax distribution based on lower record values. Yu et al. [26] investigated estimating $\Re$ for the unit Burr III distribution under URV.

In statistical modeling, inverse distributions are crucial for analyzing skewed or heavy-tailed data. The inversion transformation $Y=1/X$ of a random variable X converts a light-tailed distribution into a heavy-tailed one, making it suitable for phenomena with extreme values or sharp peaks near a boundary (Kleiber and Kotz [27]). Furthermore, this type of transformation can also be used to identify heavy tails. As a result, long-tailed and non-monotone hazard rate models need to be developed [28]. To capture these distinctive features, researchers frequently develop new probability models by inverting existing distributions. This process helps reveal underlying patterns in complex data, leading to more accurate predictions and better-informed decisions. In line with this approach, Ghitany et al. [29] introduced a class of inverted exponentiated distributions, including the inverted exponentiated Rayleigh distribution (IERD) and inverted exponentiated exponential distribution (IEED). Among these, the inverted exponentiated Pareto distribution (IEPD) stands out as a particularly flexible and useful model. Its PDF is defined as follows:

$$g\left(t;a_1,b\right)=a_1 b t^{b-1}\ (1+t)^{-(b+1)}\left[1-\left(1+t^{-1}\right)^{-b}\right]^{a_1-1};\ t,a_1,b>0\,, \tag{2}$$

where b and a_1 are the shape parameters. The CDF of the IEPD is given by:

$$G\left(t;a_1,b\right)=1-\left[1-\left(1+t^{-1}\right)^{-b}\right]^{a_1};\ t,a_1,b>0. \tag{3}$$

The IEPD is a flexible tool in many investigations, including fatigue failure analysis, degradation testing, mortality analysis, and evaluation of mechanical and electrical component performance because of its ability to represent non-monotone patterns. Estimation of the IEPD parameters was examined in the case of progressive censoring [30]. Shaikh and Patel [31] focused on parameter estimation for the IEPD using record values. Kumari et al. [32] investigated the multicomponent SS reliability of the IEPD under Type II censoring. Fayomi et al. [33] discussed the multicomponent SS reliability of the IEPD under progressive first failure censoring, while Hassan et al. [34] investigated the estimation of the parameters and some entropy measures for IEPD using a ranked set sampling.

In reliability and lifetime research, estimating the SS reliability is essential. Although the URV is commonly employed to handle incomplete data, there's a significant shortcoming when it comes to estimating the SS reliability for the IEPD. This study aims to address that gap by creating estimators for $\Re=P(Z<T)$ under the assumption that the stress (Z) and strength (T) variables are independent and follow IEPD with a common shape parameter. The main goals of this paper are as follows:

i. The maximum likelihood (ML) and the Bayesian estimators for $\Re$ are derived. The Bayesian estimator is explored using two types of priors: non-informative prior (N-INP) and informative prior (INP).

ii. The Bayesian estimator of $\Re$ is determined under the squared error loss function (SELF), which provides valuable insights. Furthermore, we also present the Bayesian estimator for $\Re$, when considering asymmetric loss functions, such as the minimum expected loss function (MLF) and the weighted SELF (WSELF).

iii. Approximate confidence intervals (ACIs) and Bootstrap confidence intervals (BCIs) for $\Re$ are explored, utilizing both the percentile bootstrap (Boot-p) and bootstrap-t (Boot-t) methods. Also, the highest posterior density (HPD) credible intervals are constructed.

iv. Given the complex nature of the Bayesian estimators, we employ the Markov chain Monte Carlo (MCMC) approach for our analysis. Lastly, we examine the SS model using two data sets of goal-scoring times for the final rounds of the European Champions League for two consecutive years and monthly amounts of sulphur dioxide concentration in Long Beach, California.

The structure of this document is as follows: Section 2 displays the mathematical expression for the SS reliability parameter $\Re = P(Z < T)$ and the associated ML and ACI estimators. The Boot-p and Boot-t methods are presented in Section 3. Section 4 discusses the Bayesian estimators and HPD credible intervals. Details of the numerical simulation results that were acquired in Section 5 to evaluate the proposed approaches using some precision measures. The applicability of the techniques is demonstrated using two real-world datasets in Section 6. A summary of the key findings concludes the paper in Section 7.

2. Expression and Classical Estimation of $\Re = P(Z < T)$

An analytical expression of the SS model's reliability for the IEPD is provided in this section. Additionally, the ML estimator of the SS reliability model $\Re$ is produced under URV.

2.1. Formula and graphical representation of $\Re$

Here, an expression of $\Re = P(Z < T)$ is given assuming both Z and T are independent IEPDs with a common second shape parameter. Let the random variable T represent the strength of the component available to overcome the stress Z applied on that component. Suppose that T and Z follow the IEPD (a_1, b) and the IEPD(a_2, b), respectively, and they are independent. Then, the SS reliability can be expressed as follows:

$$\begin{aligned}\Re = P(Z < T) &= \int_0^{\infty} G_Z(t) g_T(t)\, dt \\ &= 1 - \int_0^{\infty} a_1 b t^{b-1} (1+t)^{-(b+1)} \left[1 - \left(1+t^{-1}\right)^{-b}\right]^{a_1+a_2-1} dt \\ &= \frac{a_2}{a_1 + a_2},\end{aligned} \tag{4}$$

where, $g_T(t)$ is the PDF of the strength random variable (T) and $G_Z(t)$ is the CDF of the stress random variable (Z) at t. Notably, expression (4) depends exclusively on the parameters a_1 and a_2. The 3D plots of the $\Re$ with varying parameter values are illustrated in Figure 1. We can deduct from Figure 1 that; the $\Re$ increases with an increasing of a_1 (for varying choices of a_2).

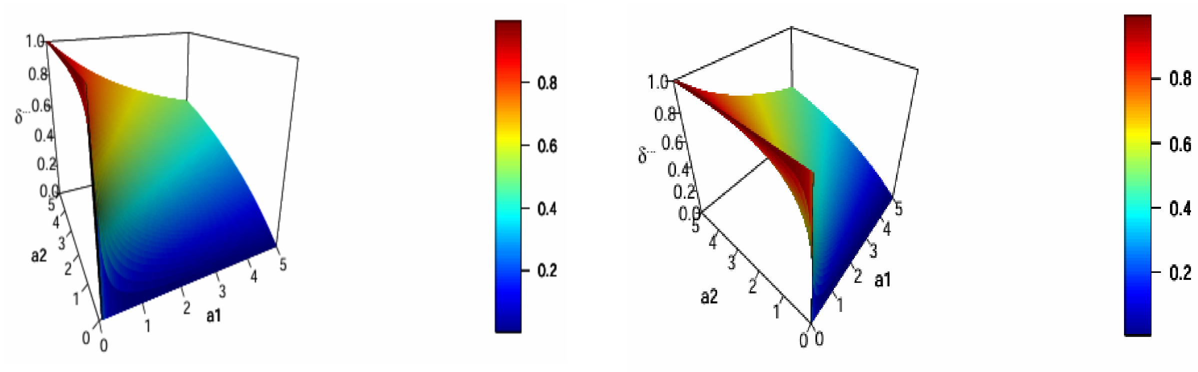

Figure 1: Stress–strength reliability values with different values of parameters.

2.2. Maximum likelihood Estimation

The main goal here is to derive the ML estimator for $\Re$ based on URV on both variables. At first, the ML estimators of the unknown parameters a_1, a_2, and b must be determined. Let $\underline{r} = (r_1, r_2, ..., r_n)$ be defined as the set of the first n URVs on strength $T \sim (a_1, b)$ with PDF given in Equation (2) and CDF given in Equation (3). Also, let $\underline{s} = (s_1, s_2, ..., s_m)$ be represented as the set of the first m URVs on stress $Z \sim (a_2, b)$ with PDF $g(z)$ and CDF $G(z)$. Based on Equation (1), the likelihood function of Z and T is given by:

$$L(\Phi \mid \underline{r}, \underline{s}) = a_1^n a_2^m b^{n+m} \left[1 - \left(1 + \frac{1}{r_n}\right)^{-b}\right]^{a_1} \prod_{k_1=1}^{n} r_{k_1}^{b-1} (1 + r_{k_1})^{-(b+1)} \left[1 - \left(1 + \frac{1}{r_{k_1}}\right)^{-b}\right]^{-1}$$
$$\times \left[1 - \left(1 + \frac{1}{s_m}\right)^{-b}\right]^{a_2} \prod_{k_2=1}^{m} s_{k_2}^{b-1} (1 + s_{k_2})^{-(b+1)} \left[1 - \left(1 + \frac{1}{s_{k_2}}\right)^{-b}\right]^{-1}, \quad (5)$$

where $\Phi = (a_1, a_2, b)^T$ is the set of parameters. The log-likelihood function of Equation (5) is given by:

$$\ell = n \log a_1 + m \log a_2 + (n+m) \log b + (b-1)\left[\sum_{k_1=1}^{n} \log(r_{k_1}) + \sum_{k_2=1}^{m} \log(s_{k_2})\right] + a_1 \log[1 - A_n(b)] + a_2 \log[1 - C_m(b)]$$
$$- \sum_{j_1=1}^{n} \log[1 - A_{k_1}(b)] - (b+1)\left[\sum_{k_2=1}^{m} \log(1 + s_{k_2}) + \sum_{k_1=1}^{n} \log(1 + r_{k_1})\right] - \sum_{k_2=1}^{m} \log[1 - C_{k_2}(b)] \quad (6)$$

where, $A_n(b) = \left(1 + \frac{1}{r_n}\right)^{-b}$, $A_{k_1}(b) = \left(1 + \frac{1}{r_{k_1}}\right)^{-b}$, $C_m(b) = \left(1 + \frac{1}{s_m}\right)^{-b}$, and $C_{k_2}(b) = \left(1 + \frac{1}{s_{k_2}}\right)^{-b}$.

Differentiate the log-likelihood function in Equation (6) for each of the parameters a_1, a_2, and b in order to determine the ML estimators for the unknown parameter vector $\Phi = (a_1, a_2, b)^T$, which we represent as $\hat{\Phi} = (\hat{a}_1, \hat{a}_2, \hat{b})^T$. The resultant partial derivatives are then set to zero. Hence, by resolving the following system of equations, the ML estimators are produced as follows:

$$\frac{\partial \ell}{\partial a_1}=\frac{n}{a_1}-\log\left[1-A_n(b)\right]=0, \tag{7}$$

$$\frac{\partial \ell}{\partial a_2}=\frac{m}{a_2}-\log\left[1-C_m(b)\right]=0, \tag{8}$$

and

$$\begin{aligned}\frac{\partial \ell}{\partial b}=&\frac{n+m}{b}+\frac{a_1 A'_n(b)}{1-A_n(b)}+\left[\sum_{k_1=1}^{n}\log(r_{k_1})+\sum_{k_2=1}^{m}\log(s_{k_2})\right]-\left[\sum_{k_2=1}^{m}\log(1+s_{k_2})+\sum_{k_1=1}^{n}\log(1+r_{k_1})\right]\\&+\sum_{k_1=1}^{n}\frac{A'_{k_1}(b)}{1-A_{k_1}(b)}+\sum_{k_2=1}^{m}\frac{C'_{k_2}(b)}{1-C_{k_2}(b)}+\frac{a_2 C'_m(b)}{1-C_m(b)}=0.\end{aligned} \tag{9}$$

From Equations (7) and (8), the ML estimators of a_1 and a_2 can be obtained as a function of b as follows:

$$\hat{a}_1(\hat{b})=\frac{n}{\log\left(1-A_n(\hat{b})\right)},\ \hat{a}_2(\hat{b})=\frac{m}{\log\left(1-C_m(\hat{b})\right)}. \tag{10}$$

Then inserting Equation (10) in Equation (9) gives:

$$\begin{aligned}&\frac{n+m}{\hat{b}}+\frac{\hat{a}_1(\hat{b})A'_n(\hat{b})}{1-A_n(\hat{b})}+\left[\sum_{k_1=1}^{n}\log(r_{k_1})+\sum_{k_2=1}^{m}\log(s_{k_2})\right]-\left[\sum_{k_2=1}^{m}\log(1+s_{k_2})+\sum_{k_1=1}^{n}\log(1+r_{k_1})\right]\\&+\sum_{k_1=1}^{n}\frac{A'_{k_1}(\hat{b})}{1-A_{k_1}(\hat{b})}+\sum_{k_2=1}^{m}\frac{C'_{k_2}(\hat{b})}{1-C_{k_2}(\hat{b})}+\frac{\hat{a}_1(\hat{b})C'_m(\hat{b})}{1-C_m(\hat{b})}=0,\end{aligned} \tag{11}$$

where,

$$A'_w(b)=-\left(1+\frac{1}{r_w}\right)^{-b}\log\left(1+\frac{1}{r_w}\right),\ w\equiv(k_1,n),\quad C'_{w_1}(b)=-\left(1+\frac{1}{s_{w_1}}\right)^{-b}\log\left(1+\frac{1}{s_{w_1}}\right),\ w_1\equiv(k_2,m).$$

Solving Equation (11) by using an iterative procedure to obtain the ML estimate of b, then inserting $\hat{b}$ in Equation (10) gives the ML estimate of a_1 and a_2, denoted as $\hat{a}_1$ and $\hat{a}_2$. Then, by using the invariance property, the ML estimate of R, represented by $\hat{R}$, is given by:

$$\hat{R}=\frac{\hat{a}_2}{\hat{a}_1+\hat{a}_2}. \tag{12}$$

3.3 Approximate confidence intervals

We begin by discussing the construction of ACIs for the parameters a_1, a_2, and b along with SS parameter R. Since the exact distributions of the ML estimators do not exist in closed form, we rely on the principle of asymptotic normality. This principle states that as the sample size increases, the ML estimators become asymptotically distributed. This reliance on asymptotic behavior enables the construction of ACIs. The calculation requires the observed Fisher information matrix, whose inverse $I^{-1}(\Phi)$ is utilized as the asymptotic variance-covariance matrix. This matrix is presented below:

$$I^{-1}(\Theta) = -\begin{pmatrix} \frac{\partial^2 \ell}{\partial a_1^2} & \frac{\partial^2 \ell}{\partial a_1 \partial a_2} & \frac{\partial^2 \ell}{\partial a_1 \partial b} \\ \frac{\partial^2 \ell}{\partial a_2 \partial a_1} & \frac{\partial^2 \ell}{\partial a_2^2} & \frac{\partial^2 \ell}{\partial a_2 \partial b} \\ \frac{\partial^2 \ell}{\partial b \partial a_1} & \frac{\partial^2 \ell}{\partial b \partial a_2} & \frac{\partial^2 \ell}{\partial b^2} \end{pmatrix}^{-1}_{a_1=\hat{a}_1, a_2=\hat{a}_2, b=\hat{b}} = \begin{pmatrix} \hat{\sigma}_{11}^2 & \hat{\sigma}_{12} & \hat{\sigma}_{13} \\ & \hat{\sigma}_{22}^2 & \hat{\sigma}_{23} \\ & & \hat{\sigma}_{33}^2 \end{pmatrix}$$

where,

$$\frac{\partial^2 \ell}{\partial a_1^2} = \frac{-n}{a_1^2}, \qquad \frac{\partial^2 \ell}{\partial a_1 \partial a_2} = 0, \qquad \frac{\partial^2 \ell}{\partial a_1 \partial b} = \frac{\partial^2 \ell}{\partial b \partial a_2} = \frac{A'_n(b)}{[1 - A_n(b)]},$$

$$\frac{\partial^2 \ell}{\partial a_2^2} = \frac{-m}{a_2^2}, \qquad \frac{\partial^2 \ell}{\partial a_2 \partial a_1} = 0, \qquad \frac{\partial^2 \ell}{\partial a_2 \partial b} = \frac{\partial^2 \ell}{\partial b \partial a_2} = \frac{C'_n(b)}{[1 - C'_n(b)]},$$

$$\frac{\partial^2 \ell}{\partial b^2} = \frac{-(n+m)}{b^2} + \frac{a_1 A''_n(b)}{1 - A_n(b)} + \frac{a_1 (A'_n(b))^2}{[1 - A_n(b)]^2} + \frac{a_2 C''_m(b)}{1 - C_m(b)} + \frac{a_2 (C'_m(b))^2}{[1 - C_m(b)]^2}$$
$$+ \sum_{k_1=1}^{n} \frac{A''_{k_1}(b)}{1 - A_{k_1}(b)} + \sum_{k_1=1}^{n} \frac{(A'_{k_1}(b))^2}{\left[1 - A_{k_1}(b)\right]^2} + \sum_{k_2=1}^{m} \frac{C''_{k_2}(b)}{1 - C_{k_2}(b)} + \sum_{k_2=1}^{m} \frac{(C'_{k_2}(b))^2}{\left[1 - C_{k_2}(b)\right]^2},$$

$$A''_{\omega}(b) = \left(1 + \frac{1}{r_\omega}\right)^{-b} \log^2\left(1 + \frac{1}{r_\omega}\right), \omega \equiv (k_1, n),$$

$$C''_{\omega_1}(b) = \left(1 + \frac{1}{s_{\omega_1}}\right)^{-b} \log^2\left(1 + \frac{1}{s_{\omega_1}}\right), \omega_1 \equiv (k_2, m).$$

The $100(1-\nu)\%$ ACIs under the normal approximation for parameters a_1, a_2, and b are derived directly from the asymptotic variance and are calculated as:

$$a_1 \pm T_{\nu/2}\sigma_{11}, \quad a_2 \pm T_{\nu/2}\sigma_{22}, \quad b \pm T_{\nu/2}\sigma_{33},$$

where $T_{\nu/2}$ is the upper $(\nu/2)$)th percentile point of the standard normal distribution.

Furthermore, building the ACI of SS parameter requires an estimate of the variance of its estimator, $\hat{\vartheta}$. This variance can be approximated using the delta method [35], which leverages the asymptotic normality of the underlying random variable. According to this method, the estimated variance is given by

$$\sigma^2(\hat{\vartheta}) = [B]^T [I^{-1}(\hat{\Theta})][B], \quad B = \left[\frac{\partial \hat{\vartheta}}{\partial a_1}, \frac{\partial \hat{\vartheta}}{\partial a_2}, \frac{\partial \hat{\vartheta}}{\partial b}\right]$$

where $I^{-1}(\Theta)$ is the approximation of the variance-covariance matrix. The detailed derivatives of SS are as follows: $\frac{\partial \hat{\delta}}{\partial a_1}=\frac{-a_2}{(a_1+a_2)^2}$, $\frac{\partial \hat{\delta}}{\partial a_2}=\frac{a_1}{(a_1+a_2)^2}$ and $\frac{\partial \hat{\delta}}{\partial b}=0$.

Consequently, the two-sided $100(1-\nu)\%$ ACIs of δ is determined as follows: $\hat{\delta} \pm T_{\nu/2}\, S(\hat{\delta})$.

3. Bootstrap Confidence Intervals

In this section, we propose to use the two confidence intervals of the δ in the presence of URV based on the parametric bootstrap methods: (i) the percentile bootstrap method (Boot-p) and (ii) the bootstrap-t method (Boot-t) (see Efron [36] and Hall [37]). The steps for computing these two bootstrap confidence intervals are discussed in the following subsections.

3.1. Boot-p confidence Interval

1. Generating data from IEPD (a_1, b) and IEPD (a_2, b), respectively, with initial parameters a_1, a_2, b, and δ from the two originally observed samples of upper records $\underline{r} = (r_1, r_2, \ldots, r_n)$ and $\underline{s} = (s_1, s_2, \ldots, s_m)$.
2. Compute the ML estimates of a_1, a_2, b, and δ using the upper record sample which generated in step 1.
3. Generate a bootstrap sample using $\hat{a}_1, \hat{a}_2, \hat{b}, \hat{\delta}$ and hence applying the supposed upper record sample $\underline{r}^* = (r_1^*, r_2^*, \ldots, r_n^*)$ from IEPD $(\hat{a}_1, \hat{b})$ and a second bootstrap upper record sample $\underline{s}^* = (s_1^*, s_2^*, \ldots, s_m^*)$ from IEPD $(\hat{a}_2, \hat{b})$ to obtain the bootstrap estimates of $\hat{\Theta}^* = (\hat{a}_1^*,\ \hat{a}_2^*,\ \hat{b}^*)$, and $\hat{\delta}^*$.
4. Repeat step (3) B times to have $\hat{\delta}^* = (\hat{\delta}^*_{(1)}, \hat{\delta}^*_{(2)}, \ldots, \hat{\delta}^*_{(B)})$.
5. Let's say that $h_1(t) = P(\hat{\delta}^* \le t)$ be the CDF of $\hat{\delta}^*$. We can specify $\hat{\delta}_B(t) = h_1^{-1}(t)$ for any given t.

   The $100(1-\nu)\%$ bootstrap percentile interval for $\hat{\delta}^*$ is then described as

$$\left[\hat{\delta}_{boot-p}\left(\frac{\nu}{2}\right), \hat{\delta}_{boot-p}\left(1-\frac{\nu}{2}\right)\right]$$

   Namely, simply use the $\nu/2$ and $(1-\nu/2)$ quantiles of the bootstrap sample $\hat{\delta}^*_{(1)}, \hat{\delta}^*_{(2)}, \ldots, \hat{\delta}^*_{(B)}$.

3.2. Boot-t confidence Interval

1. Generating data from IEPD (a_1, b) and IEPD (a_2, b), respectively, with initial parameters a_1, a_2, b, and δ from the two originally observed samples of upper records $\underline{r} = (r_1, r_2, \ldots, r_n)$ and $\underline{s} = (s_1, s_2, \ldots, s_m)$.

2. Compute the ML estimates of a_1, a_2, b, and $\mathcal{R}$ using the upper record sample which generated in step 1.
3. As stated in the boot-p, first we generate bootstrap upper record samples $\underline{r}^* = (r_1^*, r_2^*, \ldots, r_n^*)$ and $\underline{s}^* = (s_1^*, s_2^*, \ldots, s_m^*)$. Using these data, we compute the bootstrap estimates $\hat{\Phi}^* = (\hat{a}_1^*,\ \hat{a}_2^*,\ \hat{b}^*)$, $\hat{\mathcal{R}}^*$ and the following t-statistic:

$$T_b^* = \left(\hat{\mathcal{R}}_b^* - \hat{\mathcal{R}}\right) \Big/ \sqrt{\sigma^2(\hat{\mathcal{R}}_b^*)}, \qquad b = 1, 2, \ldots, B,$$

where $\sigma^2(\hat{\mathcal{R}}^*)$ is the variance of $\hat{\mathcal{R}}^*$. Using the procedure discussed in sub-section 3.3 the two-sided $100(1-\nu)\%$ ACI of $\mathcal{R}$ is determined as follows:

$$\hat{\mathcal{R}} + T^*_{b\ boot-t[\nu/2]}\sqrt{\sigma^2(\hat{\mathcal{R}})} \quad \text{and} \quad \hat{\mathcal{R}} + T^*_{b\ boot-t\left[(1-\nu)/2\right]}\sqrt{\sigma^2(\hat{\mathcal{R}})}\ .$$

4. Bayesian Estimation of $\mathcal{R}$

This section examines the Bayes estimator of $\mathcal{R} = P(Z < T)$ assuming both strength (T) and stress (Z) follow the IEPD based on URVs. Both INP and N-INP are included in this study, which also considers symmetric loss function (SELF) and asymmetric loss functions (MLF and WSELF). Assume that a_1, a_2, and b are independent gamma priors with the following joint informative prior:

$$g(\Phi) \propto a_1^{\alpha_1-1} a_2^{\alpha_2-1} b^{\alpha_3-1} e^{-(\beta_1 a_1 + \beta_2 a_2 + \beta_3 b)}, \alpha_i, \beta_i > 0,\ i = 1, 2, 3. \tag{13}$$

It's interesting to note that when α_i and β_i are close to zero, previous distributions are termed N-INP. The joint posterior distribution of a_1, a_2, and b is determined by combining the likelihood in Equation (5) with the joint prior densities in Equation (13):

$$\pi^*(\Phi \mid \underline{r}, \underline{s}) = K^{-1} a_1^{n+\alpha_1-1} a_2^{m+\alpha_2-1} b^{n+m+\alpha_3-1} e^{-a_1\left(\beta_1 - \log\left[-(1+r_n^{-1})^{-b}\right]\right) - a_2\left(\beta_2 - \log\left[-(1+s_m^{-1})^{-b}\right]\right) - \beta_3 b}$$
$$e^{\left[(b-1)\sum_{k_1=1}^{n}\log(r_{k_1}) - (b+1)\sum_{k_1=1}^{n}\log(1+r_{k_1}) - \sum_{k_1=1}^{n}\log\left[-(1+r_{k_1}^{-1})^{-b}\right]\right]} e^{\left[(b-1)\sum_{k_2=1}^{m}\log(s_{k_2}) - (b+1)\sum_{k_2=1}^{m}\log(1+s_{k_2}) - \sum_{k_2=1}^{m}\log\left[-(1+s_{k_2}^{-1})^{-b}\right]\right]} \tag{14}$$

where

$$K = \int_0^\infty\int_0^\infty\int_0^\infty a_1^{n+\alpha_1-1} a_2^{m+\alpha_2-1} b^{n+m+\alpha_3-1} e^{-a_1\left(\beta_1 - \log\left[-(1+r_n^{-1})^{-b}\right]\right) - a_2\left(\beta_2 - \log\left[-(1+s_m^{-1})^{-b}\right]\right) - \beta_3 b}$$
$$\times e^{\left[(b-1)\sum_{k_1=1}^{n}\log(r_{k_1}) - (b+1)\sum_{k_1=1}^{n}\log(1+r_{k_1}) - \sum_{k_1=1}^{n}\log\left[-(1+r_{k_1}^{-1})^{-b}\right]\right]}$$
$$\times e^{\left[(b-1)\sum_{k_2=1}^{m}\log(s_{k_2}) - (b+1)\sum_{k_2=1}^{m}\log(1+s_{k_2}) - \sum_{k_2=1}^{m}\log\left[-(1+s_{k_2}^{-1})^{-b}\right]\right]} d(a_1, a_2, b),$$

K is the normalizing constant and $d(a_1, a_2, b) = da_1 da_2 db$. The posterior distributions of a_1, a_2, and b are

$$\pi_1^*(a_1 \mid b, \underline{r}) \propto a_1^{n+\alpha_1-1} e^{-a_1\left(\beta_1 - \log\left[-(1+r_n^{-1})^{-b}\right]\right)}, \tag{15}$$

$$p_2^*(a_2|b,\underline{s}) \propto a_2^{m+\alpha_2-1}\, e^{-a_2\left(b_2-\log\left[1-\left(1+s_m^{-1}\right)^{-b}\right]\right)}, \tag{16}$$

and,

$$\begin{aligned} p_3^*(b|a_1,a_2,\underline{r},\underline{s}) \propto{} & b^{n+m+\alpha_3-1}\, e^{-a_1\left(b_1-\log\left[1-\left(1+r_n^{-1}\right)^{-b}\right]\right)-a_2\left(b_2-\log\left[1-\left(1+s_m^{-1}\right)^{-b}\right]\right)-b_3 b} \\ & \times e^{\left[(b-1)\sum_{k_1=1}^{n}\log(r_{k_1})-(b+1)\sum_{k_1=1}^{n}\log(1+r_{k_1})-\sum_{k_1=1}^{n}\log\left[1-\left(1+r_{k_1}^{-1}\right)^{-b}\right]\right]} \\ & \times e^{\left[(b-1)\sum_{k_2=1}^{m}\log(s_{k_2})-(b+1)\sum_{k_2=1}^{m}\log(1+s_{k_2})-\sum_{k_2=1}^{m}\log\left[1-\left(1+s_{k_2}^{-1}\right)^{-b}\right]\right]}. \end{aligned} \tag{17}$$

It's clear from Equation (15) that $p_1^*(a_1|b,\underline{r})$ follows a gamma distribution with parameters $\left(n+\alpha_1, b_1-\log\left[1-\left(1+r_n^{-1}\right)^{-b}\right]\right)$. Also, from Equation (16), it can be observed that $p_2^*(a_2|b,\underline{s})$ follows a gamma distribution with parameters $\left(m+\alpha_2, b_2-\log\left[1-\left(1+s_m^{-1}\right)^{-b}\right]\right)$. So, any technique that generates gamma distributions can easily create samples of a_1 and a_2. The lack of a simple closed-form Equation (17) for $p_3^*(b|a_1,a_2,data)$ prevents from directly sampling from this density using conventional techniques. The Metropolis-Hastings (M-H) algorithm is a useful tool for creating samples from these types of distributions, and it can be applied in this situation. For the proposal distribution in the M-H algorithm, the normal distribution appears to be a good option. The detailed steps of the M-H algorithm are given in Section (5). Three loss functions are taken into consideration for the Bayesian estimator of δ. One of them is the SELF, a symmetric loss function that assigns equal weight to overestimation and underestimation. This is a common choice due to its simplicity and focus on balancing errors. Therefore, the Bayesian estimator of δ under the SELF is given by:

$$\hat{\delta}_{SELF} = E(\delta) = \int_0^\infty\int_0^\infty\int_0^\infty \delta\, p^*(\Phi|\underline{r},\underline{s})\, d(a_1 a_2 b).$$

However, in almost all practical settings, the costs of underestimation and overestimation can differ significantly. Therefore, under such conditions, the use of asymmetric loss functions would be justified. Two such asymmetric loss functions, including WSELF and MLF, would be employed. The MLF, a quadratic loss function presented by Tummala and Sathe [38] as an asymmetric loss function. The MLF modifies the proportion of underestimate and overestimate by δ^2 compared to the symmetric SELF.

$$L_{MLF}(\delta,\hat{\delta}) = \frac{(\hat{\delta}-\delta)^2}{\delta^2}.$$

Therefore, the Bayesian estimator of δ under the MLF is given by:

$$\hat{\delta}_{MLF} = \frac{E(\delta^{-1})}{E(\delta^{-2})} = \frac{\int_0^\infty\int_0^\infty\int_0^\infty \delta^{-1} p^*(\Phi|\underline{r},\underline{s})\, d(a_1 a_2 b)}{\int_0^\infty\int_0^\infty\int_0^\infty \delta^{-2} p^*(\Phi|\underline{r},\underline{s})\, d(a_1 a_2 b)}.$$

The WSELF is a weighted version of symmetric SELF and is an asymmetric loss function [39]. According to Robert [40]. The WSELF modifies the proportion of underestimate and overestimate by δ^{-1} compared to the symmetric SELF.

$$L_{WSELF}(\hat{\delta},\delta)=\frac{(\hat{\delta}-\delta)^2}{\delta}.$$

Therefore, the Bayesian estimator of δ under the WSELF is given by:

$$\hat{\delta}_{WSELF}=\left[E(\delta^{-1})\right]^{-1}=\left[\int_0^\infty\int_0^\infty\int_0^\infty \delta^{-1}\pi^*\left(\Phi \mid \underline{r},\underline{s}\right)d(\alpha_1\alpha_2\beta)\right]^{-1}.$$

Furthermore, the 95% two-sided HPD credible interval for the unknown parameters and any function of them can be calculated as $[\Phi_{0.025N:N},\Phi_{0.975N:N}]$ using Chen and Shao's [41] method.

5. Simulation Study

To validate the performance of the estimators discussed in the preceding sections for different sample sizes and shape parameter values, a Monte Carlo simulation assessment is carried out in this section. The outcomes of the ML estimates (MLEs) and Bayesian estimates (BEs) for the various loss functions are compared in terms of mean squared errors (MSEs) and absolute biases (ABs). Furthermore, we evaluate several BCIs and HPD credible intervals in terms of average interval lengths (AILs) and coverage probability (CP) in %. All the calculations are carried out using R 4.4.2 software, available at https://github.com/tim-lebedkov/packages/releases/download/2025_01/r-4.4.2-R-4.4.2-win.exe. Upper records sizes $(n,m)=(8,8),(8,10),(8,15),(10,8),(10,10),(10,15),(15,8),(15,10)$, and $(15,15)$ are considered for four sets of true parameters and the corresponding real value of δ, in which the common parameter is $b=2$. These sets are as shown below:

$$\begin{array}{llll} Set\ 1: & a_1=1.5 & a_2=0.5 & \delta=0.25 \\ Set\ 2: & a_1=1.5 & a_2=1.5 & \delta=0.50 \\ Set\ 3: & a_1=0.5 & a_2=1.5 & \delta=0.75 \\ Set\ 4: & a_1=9.5 & a_2=4.5 & \delta=0.90 \end{array}$$

We use the boot-p and boot-t procedures to construct the 95% BCIs of δ after calculating the MLEs of δ in each set. Furthermore, we took 1000 boot-p and boot-t replications. A hybrid M-H sampler is used in the MCMC technique to produce the BEs. The MCMC utilized in the BEs under the SELF, WSELF, and MLF for δ and hence computing their HPD intervals. The performance of several estimates obtained from Bayes computation is evaluated using the MCMC simulation approach. Different MCMC sampling techniques are used in the Bayesian paradigm to generate different posterior samples with different sample size values. Both Gibbs sampling and the more general M-H within Gibbs samplers with normal proposal distribution are essential subclasses of MCMC algorithms. The following is a summary of the hybrid M-H and Gibbs sampler:

1. Let the initiative values of a_1,a_2, and b to be $\left(a_1^{0},a_2^{0},b^{0}\right)$; $M=$ Nburn.

2. Set $l=1$.

3. Generate $a_1^{(l)}$ from Gamma $\left(n+\alpha_1,b_1-\log\left[1-(1+r_n^{-1})^{-b^{l-1}}\right]\right)$.

4. Generate $a_2^{(l)}$ from Gamma $\left(m+\alpha_2,b_2-\log\left[1-(1+s_m^{-1})^{-b^{l-1}}\right]\right)$.

5. To generate $b^{(l)}$

- Generate b^* from the normal proposal distribution $N(b^{(l-1)},\sigma^2(\hat{b}))$.
- Evaluate the acceptance probability

$$\nu_b\left(b^{(l-1)},b^*\right)=\min\left[1,\frac{\pi_3^*(b^*|a_1^{(l)},a_2^{(l)},data)}{\pi_3^*(b^{(l-1)}|a_1^{(l)},a_2^{(l)},data)}\right].$$

- Achieve U from $U(0,1)$.
- Confirm the proposal and place $b^{(l)}=b^*$, If $U\le\nu_b\left(b^{(l-1)},b^*\right)$. Elsewhere, deny the proposal and place $b^{(l)}=b^{(l-1)}$.

6. Obtain $a_1^{(l)},a_2^{(l)},b^{(l)}$ and $\vartheta^{(l)}$.
7. Place $l=l+1$.
8. Repeat N times of steps 3–7

Now, calculate the Bayes estimate of ϑ under SELF, WSELF, and MLF as:

$$\hat{\vartheta}_{SELF}=E(\vartheta)=\frac{1}{N-M}\sum_{l=M+1}^{N}\vartheta^{(l)},\quad \hat{\vartheta}_{WSELF}=\left[E(\vartheta^{-1})\right]^{-1}=\left[\frac{1}{N-M}\sum_{l=M+1}^{N}\left(\vartheta^{(l)}\right)^{-1}\right]^{-1},$$

and,

$$\hat{\vartheta}_{MLF}=\frac{E(\vartheta^{-1})}{E(\vartheta^{-2})}=\frac{\left[\frac{1}{N-M}\sum_{l=M+1}^{N}\left(\vartheta^{(l)}\right)^{-1}\right]}{\left[\frac{1}{N-M}\sum_{l=M+1}^{N}\left(\vartheta^{(l)}\right)^{-2}\right]}.$$

where, N is a total iteration of simulation and M is a burn-in period. First, for N-INP, we choose hyper-parameter values $\alpha_1=\alpha_2=\beta_1=\beta_2=0.0001$. Next, the INP were employed to select the hyper-parameters by applying the technique described in Section 4. The first 20% of observations are eliminated after 10000 MCMC sample iterations. Such INP were derived from the MLEs for (α,β) via equating the mean and variance of $(\hat{\alpha}_j,\hat{\beta}_j)$ along with the mean and variance of the considered priors (gamma priors). According to Dey *et al*. [42] and Singh and Tripathi [43], this approach is known as the elicitation of hyper-parameters. In this respect, we propose the following steps to select the values of hyper-parameters

- Set the initial parameter value of (a_1,a_2,b).
- Set $j=1$.
- Based on URVs, the two original samples $\underline{r}=(r_1,r_2,\ldots r_n)$ and $\underline{s}=(s_1,s_2,\ldots s_m)$ were generated from the IEPD (a_1,b) and IEPD (a_2,b) respectively.
- The appropriate means and variances are determined by equating $\hat{a}_1,\hat{a}_2$, and $\hat{b}$ with the gamma prior distributions' mean and variance as:

$$\frac{\alpha_j}{\beta_j}=\frac{1}{l}\sum_{j=1}^{l}\hat{\Phi}^j,\quad \frac{\alpha_j}{\beta_j^2}=\frac{1}{l-1}\sum_{j=1}^{l}\left(\hat{\Phi}^j-\frac{1}{l}\sum_{j=1}^{l}\hat{\Phi}^j\right)^2,$$

where $j=1,2,\ldots,l$, l is the available samples number from the IEPD.

- Now, on solving the above two equations, the estimated hyper-parameters can be written as

$$a_j = \frac{\left(\frac{1}{l}\sum_{j=1}^{l}\hat{F}^j\right)^2}{\frac{1}{l-1}\sum_{j=1}^{l}\left(\hat{F}^j - \frac{1}{l}\sum_{j=1}^{l}\hat{F}^j\right)^2}, \text{ and } b_j = \frac{\frac{1}{l}\sum_{j=1}^{l}\hat{F}^j}{\frac{1}{l-1}\sum_{j=1}^{l}\left(\hat{F}^j - \frac{1}{l}\sum_{j=1}^{l}\hat{F}^j\right)^2}.$$

Simulation results are shown in Tables 1−2 and Figures 2–11. The following conclusions can be constructed in considering the simulation results that were obtained as:

1. The MLE of ϑ generally has higher ABs and MSEs compared to BEs under SELF; this means that the performances of BEs under SELF of ϑ are better than those of MLEs (see Table 1).
2. Notably, estimates of ϑ exhibit the consistency property for both MLEs and BEs. This property implies that as the number of records (n, m) increases, the estimates of ϑ converge to the true parameter values.
3. For both MLEs and BEs, the ABs and MSEs of ϑ estimates tend to decrease as n and m increase (see Table 1 and Figures 2 and 3).

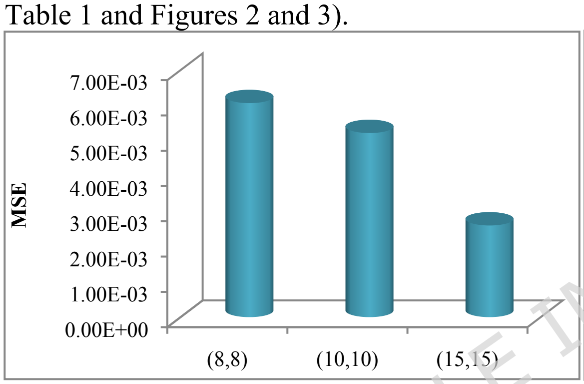

Figure 2: The MSEs of ϑ_{MLE} for different record numbers at true value ϑ=0.75.

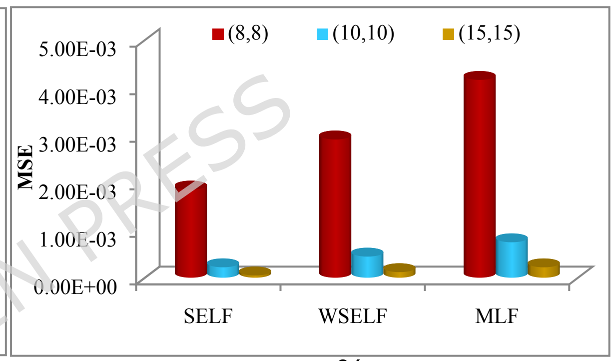

Figure 3: The MSEs of ϑ_{BE} in case of INP for different (n, m) at true value ϑ=0.90.

4. In most cases, the BEs of ϑ under SELF perform better than those under WSELF and MLF. While BEs of ϑ under MLF are the worst compared to others (see Table 1and Figures 3−5).
5. As seen, the MSEs and the ABs of ϑ_{BE} in the case of INP have the smallest value corresponding to the other N-INPs in all situations (see Table 1 and Figures 4 and 5).

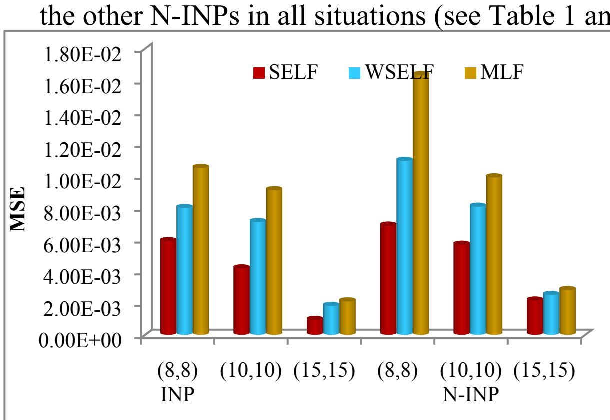

Figure 4: The MSEs of ϑ_{BE} for different record numbers at true value ϑ=0.50.

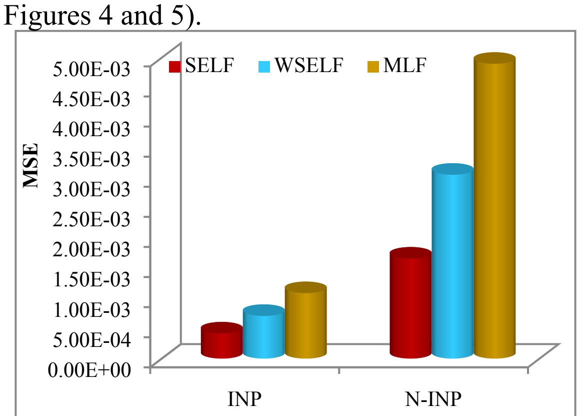

Figure 5: The MSEs of ϑ_{BE} for $(n,m)=(10,10)$ at true value ϑ=0.50.

6. In comparison to boot-t, we observed that boot-p has lower AILs, while the CP for boot-p is better than boot-t. Also, the length of BCIs reduces as record numbers rise (see Table 2 and Figures 6 and 7).

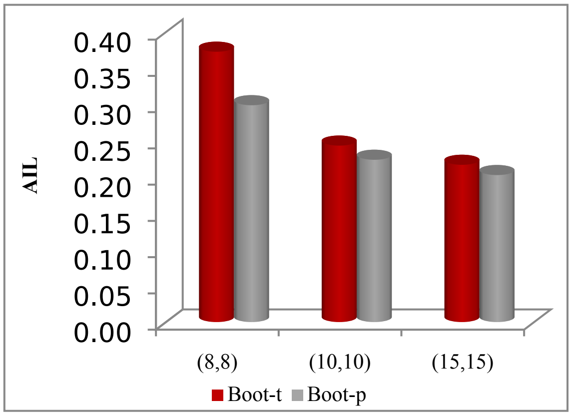

Figure 6: The AIL of boot-t and boot-p for different (n, m) at true value θ=0.75.

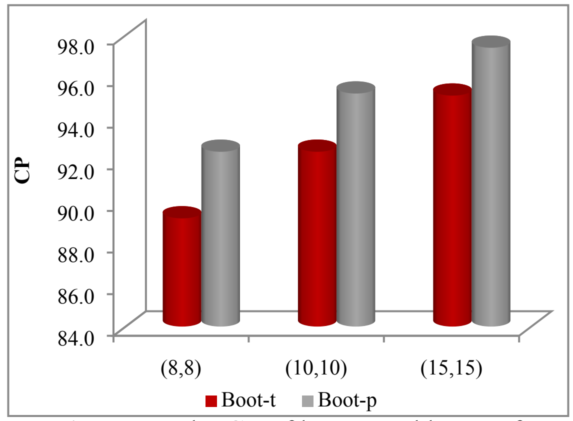

Figure 7: The CP of boot-t and boot-p for different (n, m) at true value θ=0.75.

7. The HPD credible intervals in INP outperform those in N-INP based on AIL and CP in all situations (see Figures 8 and 9).

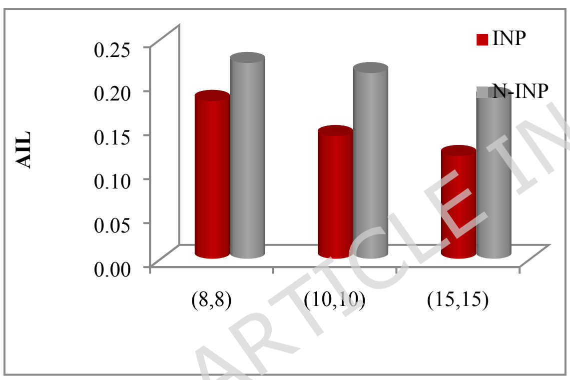

Figure 8: The AIL of INP and N-INP estimates for different (n, m) at true value

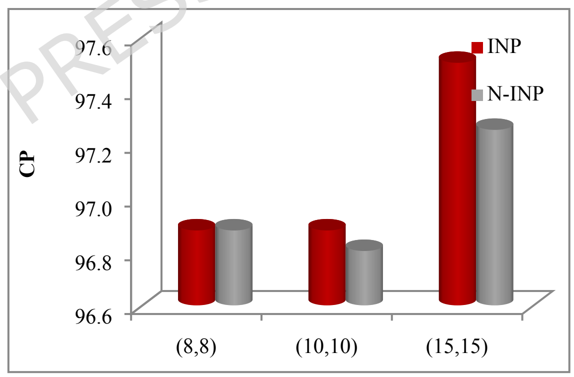

Figure 9: The CP of INP and N-INP for different record numbers at true value

8. It is also noted that Bayesian intervals for both INP and N-INP outperform boot-p and boot-t intervals, in approximately most of situations.
9. Finally, convergence graphs of MCMC for θ of the IEPD at θ=0.75 with record numbers $(n,m)=(8,8)$ and $(15,15)$ are displayed in Figures 10 and 11. The findings demonstrate a high degree of similarity between the θ estimates and the theoretical posterior density functions. As mentioned, it is quite likely that further MCMC technique repetitions would yield similar or even better results.

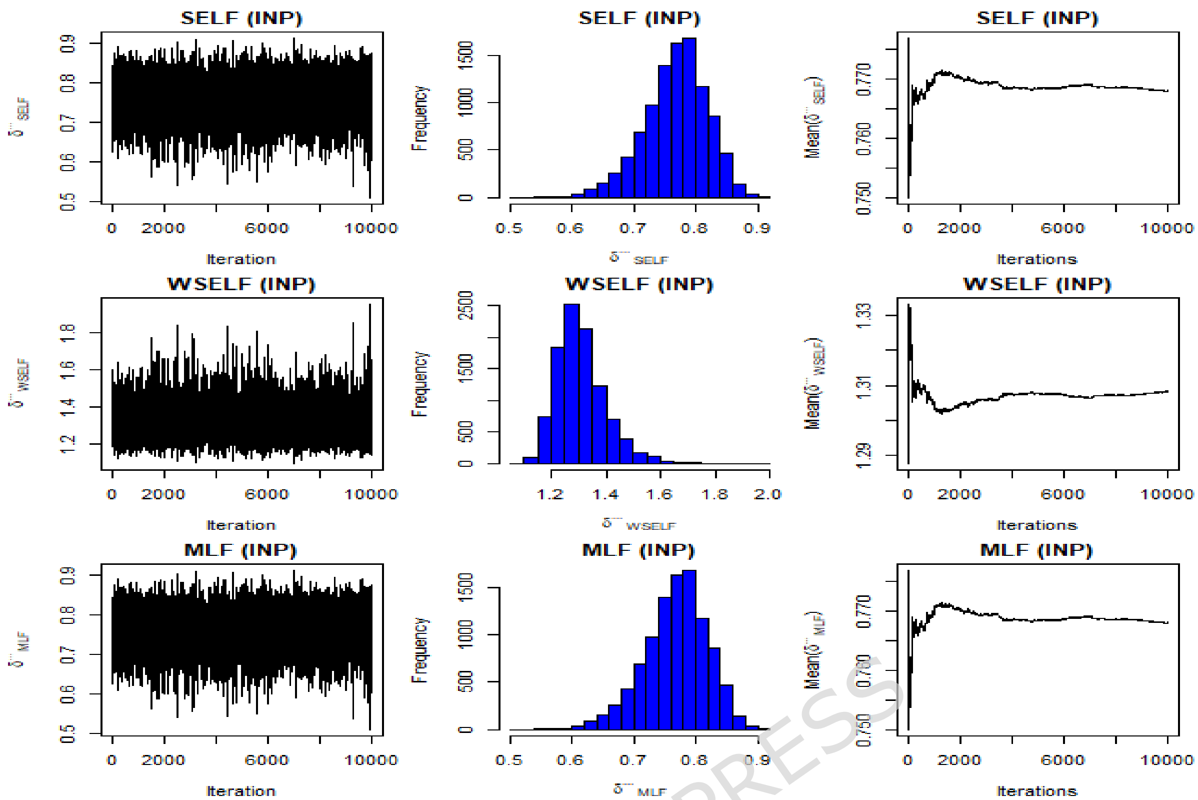

Figure 10: The MCMC plots of δ=0.75. for INP at (n, m) = (8,8)

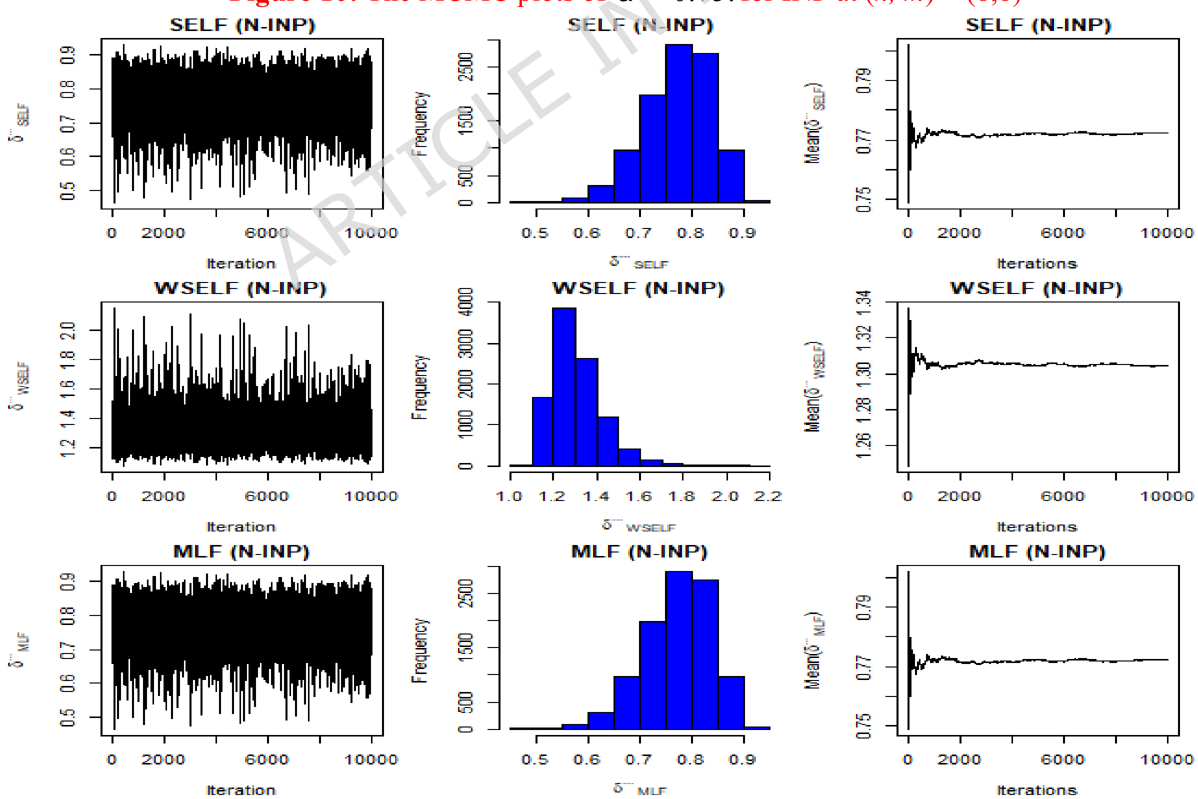

Figure 11: The MCMC plots of δ=0.75 for N-INP at (n, m) = (15,15)

Table 1: ABs and MSEs for IEPD for MLEs and both INP and N-INP

n	m	MLE		INP						N-INP					
				SELF		WSELF		MLF		SELF		WSELF		MLF	
		AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
[illegible]=0.25															
8	8	0.0714	0.83754	0.0430	0.18549*	0.0539	0.290861	0.0645	0.41614	0.0523	0.27370	0.0708	0.50230	0.0892	0.79683
	1	0.0653	0.68123	0.0137	0.01892*	0.0052	0.02803*	0.0029	0.05097	0.0060	0.03661	0.0110	0.03210	0.0272	0.07432
	1	0.1080	1.69024	0.0990	0.98026*	0.0914	0.83653*	0.0839	0.70459	0.0996	0.99351	0.0861	0.94125	0.0728	0.83106
10	8	0.0584	0.50663	0.0572	0.32730*	0.0644	0.41477*	0.0713	0.50941	0.0780	0.60900	0.0935	0.87592	0.1089	1.18761
	1	0.0493	0.40002	0.0148	0.021913	0.0210	0.04442*	0.0273	0.07453	0.0245	0.06013	0.0389	0.15131	0.0525	0.27634
	1	0.0762	0.91060	0.0638	0.40703*	0.0578	0.33436*	0.0519	0.26956	0.0638	0.40778	0.0518	0.46860	0.0400	0.16028
15	8	0.0872	0.92509	0.0549	0.30234*	0.0617	0.38157*	0.0683	0.46776	0.0180	0.03264	0.0311	0.96815	0.0442	0.19564
	1	0.0695	0.61808	0.0828	0.68635*	0.0882	0.77798*	0.0933	0.87149	0.1037	1.07660	0.1122	1.26101	0.1206	1.45509
	1	0.0410	0.26449	0.0068	0.00467*	0.0108	0.01172*	0.0147	0.02184	0.0094	0.00889	0.0188	0.03551	0.0283	0.08026
[illegible]=0.50															
8	8	0.0877	0.01197	0.0767	0.00588	0.0892	0.00797	0.1024	0.01049	0.0828	0.00686	0.1045	0.01093	0.1277	0.01633
	1	0.0800	0.00960	0.0188	0.00036	0.0119	0.00014	0.0049	0.00002	0.0000	0.00000	0.0152	0.00023	0.0309	0.00096
	1	0.1266	0.02127	0.1202	0.01446	0.1160	0.01347	0.1118	0.01251	0.1117	0.01249	0.1032	0.01065	0.0945	0.00894
10	8	0.0804	0.00941	0.0947	0.00899	0.1042	0.01087	0.1141	0.01303	0.1229	0.01511	0.1430	0.02046	0.1643	0.02701
	1	0.0680	0.00716	0.0204	0.00042	0.0266	0.00071	0.0329	0.00109	0.0407	0.00166	0.0552	0.00305	0.0699	0.00489
	1	0.0936	0.01233	0.0813	0.00661	0.0771	0.00596	0.0730	0.00533	0.0732	0.00537	0.0647	0.00419	0.0559	0.00313
15	8	0.1268	0.02041	0.0687	0.00473	0.0770	0.00594	0.0854	0.00730	0.0291	0.00085	0.0420	0.00177	0.0557	0.00311
	1	0.0970	0.01237	0.1288	0.01660	0.1359	0.01849	0.1430	0.02046	0.1634	0.02671	0.1755	0.03081	0.1878	0.03529
	1	0.0511	0.00405	0.0096	0.00931*	0.0134	0.00018	0.0173	0.00030	0.0165	0.02752	0.0255	0.00065	0.0348	0.00121
[illegible]=0.75															
8	8	0.0634	0.60634	0.0695	0.48423*	0.0760	0.57763*	0.0828	0.68668	0.0768	0.59120	0.0884	0.78205	0.1010	0.80216
	1	0.0577	0.49701	0.0139	0.01944*	0.0115	0.01334*	0.0091	0.08296	0.0059	0.03579	0.0120	0.01451	0.0183	0.03374
	1	0.0860	0.93809	0.0825	0.68113*	0.0815	0.66436*	0.0804	0.64764	0.0726	0.52802	0.0702	0.49374	0.0678	0.45989
10	8	0.0632	0.60564	0.0973	0.94784*	0.1037	1.07543*	0.1104	1.22078	0.1136	1.29171	0.1258	1.58467	0.1392	1.93830
	1	0.0569	0.52159	0.0176	0.03104*	0.0203	0.04121*	0.0230	0.05317	0.0377	0.14231	0.0441	0.19526	0.0509	0.25913
	1	0.0666	0.58765	0.0580	0.33727*	0.0569	0.32405*	0.0557	0.31097	0.0483	0.23337	0.0456	0.20809	0.0428	0.18366
1	8	0.1049	1.55541	0.0455	0.20784*	0.0497	0.24703*	0.0539	0.29086	0.0268	0.07187	0.0324	0.10515	0.0383	0.14721

5	1	0.0773	0.88727	0.1163	1.35250*	0.1205	1.45297*	0.1248	1.55847	0.1509	2.27895	0.1590	2.52871	0.1674	2.80457
	1	0.0409	0.25967	0.0084	0.00712*	0.0099	0.00991*	0.0114	0.01320	0.0157	0.02491	0.0194	0.03779	0.0232	0.05410
[illegible]=0.90															
8	8	0.0340	0.28033	0.0422	0.17828*	0.0439	0.19334*	0.0458	2.0978*	0.0433	1.88291	0.0463	0.21450	0.0494	0.24415
	1	0.0296	0.13237	0.0045	0.00211*	0.0040	0.00164*	0.0035	0.00123	0.0049	2.41369	0.0061	0.00378	0.0074	0.00549
	1	0.0395	0.19819	0.0368	0.13588*	0.0366	0.13442*	0.0364	0.13295	0.0325	1.05609	0.0321	0.10306	0.0317	0.10052
10	8	0.0364	0.20743	0.0645	0.41602*	0.0666	0.44468*	0.0689	0.47598	0.0646	4.17454	0.0680	0.46253	0.0716	0.51289
	1	0.0333	0.10338	0.0127	0.01619*	0.0134	0.01817*	0.0142	0.02030	0.0209	4.37058	0.0223	0.04983	0.0237	0.05653
	1	0.0320	0.13527	0.0263	0.06930*	0.0260	0.06807*	0.0258	0.06684	0.0216	4.68458	0.0211	0.04483	0.0207	0.04284
15	8	0.0594	0.52638	0.0167	0.02797*	0.0177	0.03136*	0.0187	0.03502	0.0139	1.94197	0.0151	0.02288	0.0163	0.02675
	1	0.0459	0.32041	0.0737	0.54411*	0.0753	0.56792*	0.0769	0.59276	0.0843	7.10754	0.0866	0.75136	0.0891	0.79464
	1	0.0244	0.09074	0.0061	0.00376*	0.0065	0.00425*	0.0069	0.00477	0.0084	7.16862	0.0092	0.00849	0.0099	0.00997

* Indicate that the value multiply 10^{-2}

Table 2: AIL and CP (in %) for IEPD for BCIs and BEs

n	*m*	Boot-t		Boot-p		INP		N-INP	
		AIL	**CP**	**AIL**	**CP**	**AIL**	**CP**	**AIL**	**CP**
a=0.25									
8	8	0.273202	84.8	0.225892	89.9	0.179467	96.9	0.222589	96.9
	10	0.371627	87.1	0.263231	92.4	0.175766	97.0	0.244192	96.3
	15	0.322065	88.6	0.359027	93.6	0.190206	96.8	0.257346	96.4
10	8	0.202156	86.8	0.173570	91.9	0.139949	97.9	0.191588	97.0
	10	0.216624	88.9	0.212906	93.9	0.139448	96.9	0.211106	96.8
	15	0.189308	90.3	0.298819	95.1	0.160690	97.0	0.230269	96.3
15	8	0.200874	88.4	0.216189	93.5	0.133700	96.6	0.200794	96.1
	10	0.210237	90.0	0.113843	94.9	0.111117	97.0	0.128770	96.5
	15	0.192862	91.5	0.178774	96.0	0.116771	97.5	0.183102	97.3
a=0.50									
8	8	0.383075	83.9	0.366868	88.5	0.259517	96.3	0.345016	96.0
	10	0.428601	86.2	0.334319	90.3	0.219871	96.5	0.326453	96.1
	15	0.336041	87.9	0.302631	91.8	0.188328	96.8	0.261214	96.5
10	8	0.257718	86.0	0.316517	90.1	0.225907	96.1	0.318918	95.5
	10	0.310262	88.0	0.228460	91.8	0.199403	96.6	0.303931	96.4
	15	0.176108	89.6	0.280610	93.3	0.179245	96.4	0.257857	96.1
15	8	0.334118	87.8	0.326235	91.6	0.222569	95.8	0.287444	95.6
	10	0.277835	89.4	0.214544	93.2	0.192251	95.8	0.235250	96.4
	15	0.265952	91.2	0.227644	94.7	0.162665	97.0	0.245772	95.9
a=0.75									
8	8	0.372829	89.2	0.299053	92.4	0.233964	96.3	0.321517	95.8
	10	0.298903	90.9	0.272175	93.9	0.156432	97.8	0.252405	96.8
	15	0.250587	92.3	0.145503	95.1	0.106473	97.5	0.158379	97.0
10	8	0.257936	90.6	0.328776	93.7	0.228603	96.3	0.319767	96.0
	10	0.243350	92.4	0.223830	95.2	0.161107	96.9	0.252027	96.0
	15	0.159393	93.8	0.184739	96.4	0.111775	97.5	0.167737	96.5
15	8	0.297065	92.2	0.312685	95.0	0.199845	96.0	0.231644	95.9
	10	0.239414	93.7	0.262603	96.3	0.197830	96.0	0.258500	95.6
	15	0.216883	95.1	0.202738	97.4	0.126247	97.3	0.189103	96.3
a=0.90									
8	8	0.285184	90.2	0.183809	93.3	0.138376	97.3	0.180382	96.5
	10	0.260293	92.1	0.113655	94.9	0.079013	98.1	0.125777	97.5
	15	0.352444	93.3	0.257920	96.0	0.048745	99.0	0.068026	98.1
10	8	0.214408	92.0	0.208499	94.8	0.150118	96.6	0.192028	96.4
	10	0.224425	93.7	0.178638	96.4	0.092471	97.5	0.132383	97.3
	15	0.479199	94.9	0.197352	97.4	0.053652	99.0	0.074991	98.0
15	8	0.229288	93.3	0.129092	96.0	0.106339	98.0	0.116874	96.5
	10	0.265914	94.8	0.265001	97.3	0.133337	97.0	0.161295	96.3
	15	0.130515	95.7	0.098188	98.2	0.071221	98.9	0.093433	97.8

6. Real Data Analysis

This section examines a two real dataset to demonstrate how the suggested methodology can be implemented.

6.1. First dataset: Champion's League data

We consider a dataset that records the time in minutes for the first goal scored during the final stage matches of the European Champions League over two consecutive years (2011–2012 and 2012–2013). This dataset is considered separately for the return matches (T) and the first matches (Z) of the Champions League, and the recorded unit is minutes. We have selected just the matches with at least one goal scored. This real dataset is analyzed by Condino et al. [15]. The observed data obtained are the following:

Dataset (I): T = (2.97, 9.99, 30.96, 19.98, 7.02, 55.98, 11.97, 56.97, 37.98, 0.99, 9, 25.02, 8.01, 17.01, 45, 73.98, 74.97, 44.01, 57.96, 41.04, 19.98, 15.03, 27.99, 27, 86.04),

Dataset (II): Z = (24.03, 54.99, 30.96, 47.97, 2.97, 43.02, 18, 5.04, 50.04, 63.99, 7.02, 47.97, 82.98, 6.03, 35.01, 26.01, 20.97, 12.96, 9, 25.02, 45, 7.02, 26.01, 27.99, 10.98). Table 3 essentially analyzes the data sets.

Table 3: Some summary statistics of the first data sets.

	Sample size	Mean	Median	Variance	Skewness	Kurtosis	Range	Minimum	Maximum
Dataset I (T)	n = 25	32.27	27.00	589.23	0.67	2.40	85.05	0.99	86.04
Dataset II (Z)	m =25	29.24	26.01	434.98	0.73	2.92	80.01	2.97	82.98

Figures 12 and 13 each display some graphical visualization of these datasets. These consist of histograms, kernel density estimates, violin plots, box plots, total time on test plots (TTT), and quantile-quantile (QQ) plots.

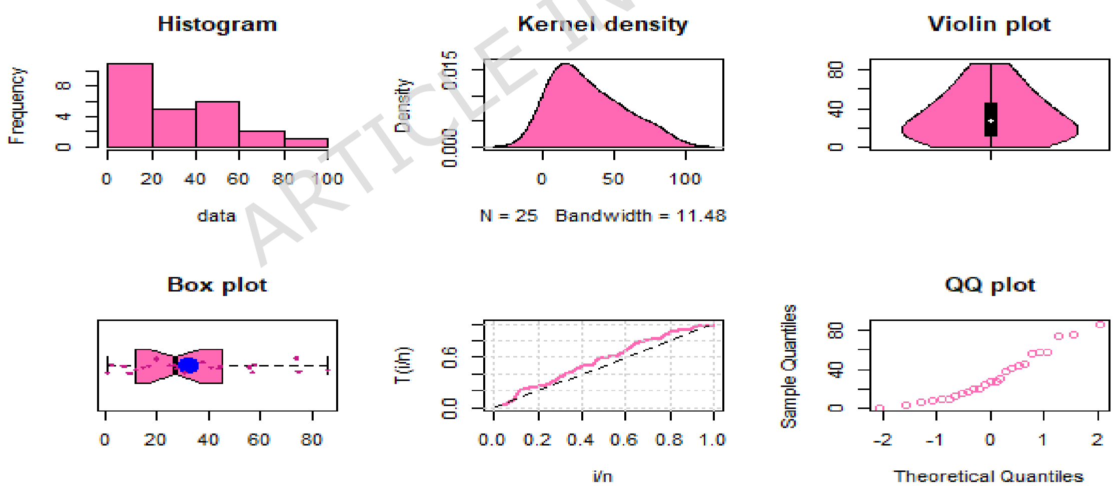

Figure 12. Some nonparametric plots of T for the first dataset

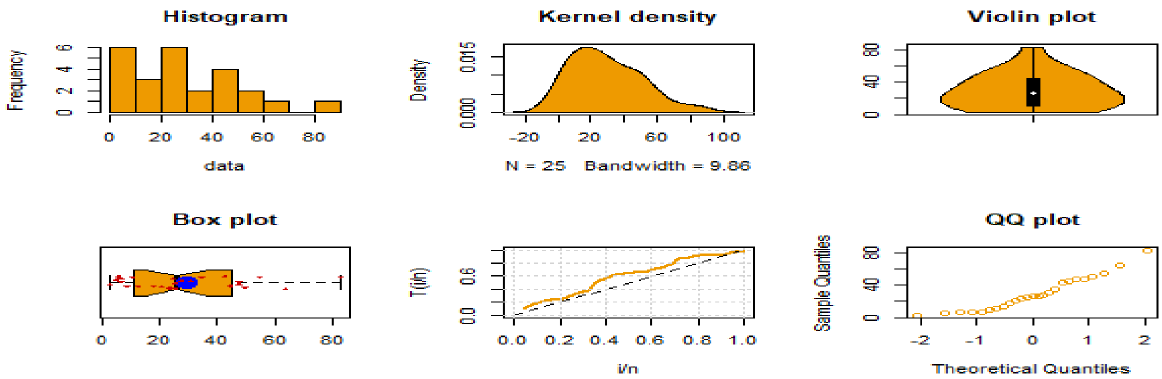

Figure 13. Some nonparametric plots of Z for the first dataset

The first and second datasets are right skewed and exhibit a moderate spread with some potential outliers on the higher end. Both datasets display an increasing hazard rate function. As the graphs demonstrate, the IEPD, as developed in the theoretical results, can handle these properties. The performance of the IEPD and other alternative models, such as the IERD, IEED, generalized inverse Weibull distribution (GIWD) [44] and extended inverse Lindley distribution (EILD) [45] are evaluated using goodness-of-fit measures. The Akaike information criteria (AIC), Bayesian information criteria (BIC), consistent AIC (CAIC), Hannan-Quinn information criterion (HQIC), Cramer-von Mises (W*), Anderson-Darling (A*), Kolmogorov-Smirnov (K-S) statistic, and its corresponding p-value (p-value) are some of these measures. Preliminary testing shows that our proposed model performs better than the others, which are well known for their capacity to apply the ML technique to the actual dataset in consideration. Model that demonstrates the lowest values for these statistics and measurements is deemed the best fit, while the highest p-value is also examined to determine the most suited distribution. All comparative measures for the two datasets are shown in Tables 4 and 5, respectively. All of the primary parameter estimates and standard errors (SEs) for the actual dataset are reported in Tables 4 and 5. According to Tables 4 and 5 and Figures 14 and 15, the IEPD is an appropriate model to fit to both datasets. The upper records are extracted from the datasets as follows:

Upper records based on Dataset (I): $\underline{r}$ = (2.97, 9.99, 30.96, 55.98, 56.97, 73.98, 74.97, 86.04).

Upper records based on Dataset (II): $\underline{s}$ = (24.03, 54.99, 63.99, 82.98). Now, we discuss two cases based on the above upper record sample combination (8, 4).

Table 4: Estimated parameters with their SEs for the first dataset

Data	Model	$\hat{a}_1$	$SE(\hat{a}_1)$	$\hat{b}$	$SE(\hat{b})$	$\hat{c}$	$SE(\hat{c})$
Dataset I (T)	IEPD	0.98083	0.10165	11.93156	0.24093		
	IERD	0.21934	0.10231	5.44568	0.24243		
	IEED	1.88844	0.10135	8.67476	0.24012		
	GIWD	3.79316	0.86895	0.78674	0.09579	2.52452	0.06029
	EILD	0.78663	0.02094	0.00776	0.79014	7.20946	0.60845
Dataset II (Z)	Model	$\hat{a}_2$	$SE(a_2)$	$\hat{b}$	$SE(\hat{b})$	$\hat{c}$	$SE(\hat{c})$
	IEPD	1.58287	0.10195	20.06664	0.24175		
	IERD	0.40477	0.10306	46.98934	0.24443		
	IEED	1.45053	0.10186	17.82961	0.24153		
	GIWD	4.96707	0.87372	1.09759	0.09669	2.99779	0.06031
	EILD	1.09799	0.02105	0.07321	0.78919	17.50412	0.61012

Table 5: Adequacy metrics for first dataset

Data	Model	AIC	BIC	CAIC	HQIC	A*	W*	K-S	P-value
Dataset I (T)	IEPD	241.6785	244.1162	248.1162	242.3546	1.5746	0.2715	0.1914	0.3189
	IERD	258.0944	260.5321	264.5321	258.7705	3.8878	0.7824	0.3105	0.0161
	IEED	245.7441	248.1819	252.1819	246.4202	2.0454	0.3664	0.2099	0.2207
	GIWD	244.5851	248.2417	254.2417	245.5993	1.7842	0.2915	0.1947	0.2994
	EILD	244.5742	248.2352	254.2352	245.5864	1.7953	0.2973	0.1943	0.2995
Dataset II(Z)	IEPD	225.5880	228.0258	232.0258	226.2642	1.0089	0.1891	0.2020	0.2595
	IERD	232.6706	235.1084	239.1084	233.3467	1.6962	0.3332	0.2441	0.1016
	IEED	226.3657	228.8034	232.8034	227.0418	1.0782	0.2036	0.2086	0.2269
	GIWD	229.5121	233.1178	239.1712	230.5127	1.0218	0.2718	0.2379	0.1409
	EILD	229.5254	233.1820	239.1820	230.5396	1.0227	0.2719	0.2379	0.1406

The MLEs as well as BEs of [illegible] are obtained following the procedure explained above. The BCIs of the 95% of boot-p and boot-t are obtained as (0.35119, 0.37910) and (0.36682, 0.43916). The BEs with their HPD interval are obtained under SELF, WSELF, and MLF. Since we do not have any prior information about the parameter, we assume that the values of hyper-parameters are $a_1 = a_2 = b_1 = b_2 = 0.0001$. The BEs have a lower posterior standard deviation (PSD) than the other MLEs. As a result, the BEs of the IEPD's parameters are the most accurate. Under SELF, the BEs of SS reliability outperform those under WSELF and MLF. Conversely, the BEs of SS reliability under MLF are the worst compared to others. The $P(Z < T)$ reliability of the Bayesian technique is greater than that of the ML method, which proves the displayed conclusion and confirms the results in the previous section. All results are presented in Table 6.

Table 6: MLEs and BEs of [illegible] and their SEs, PSD, and the HPD intervals

MLE		N-INP							
		SELF		WSELF		MLF		AIL	CP
[illegible]$_{MLE}$	[illegible]$_{SE}$	[illegible]$_{SELF}$	[illegible]$_{SELF\_PSD}$	[illegible]$_{WSELF}$	[illegible]$_{WSELF\_PSD}$	[illegible]$_{MLF}$	[illegible]$_{MLF\_PSD}$		
0.33741	0.17023	0.31996	0.00136	0.28940	0.00159	0.25565	0.00219	0.37559	95.4

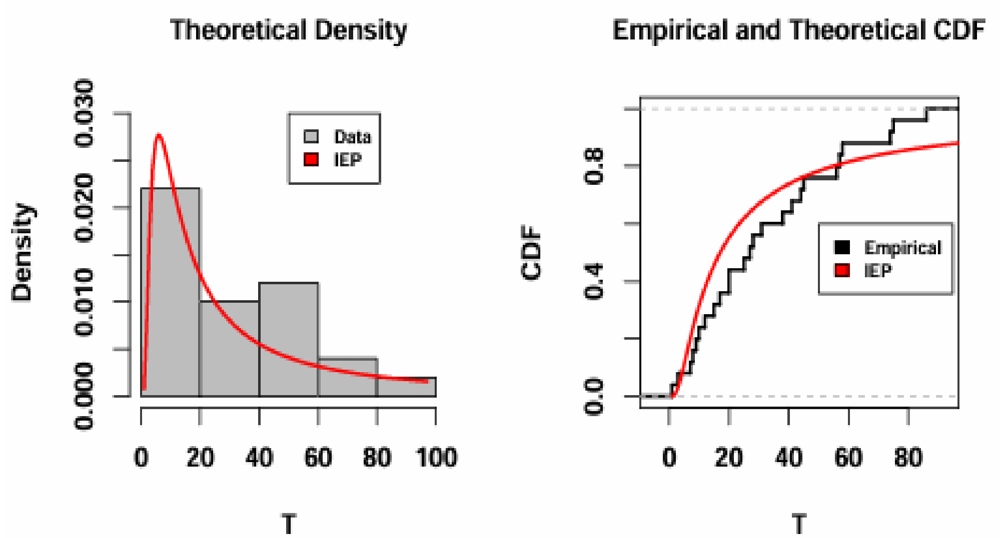

Figure 14: Estimated PDF and CDF of T for the first dataset

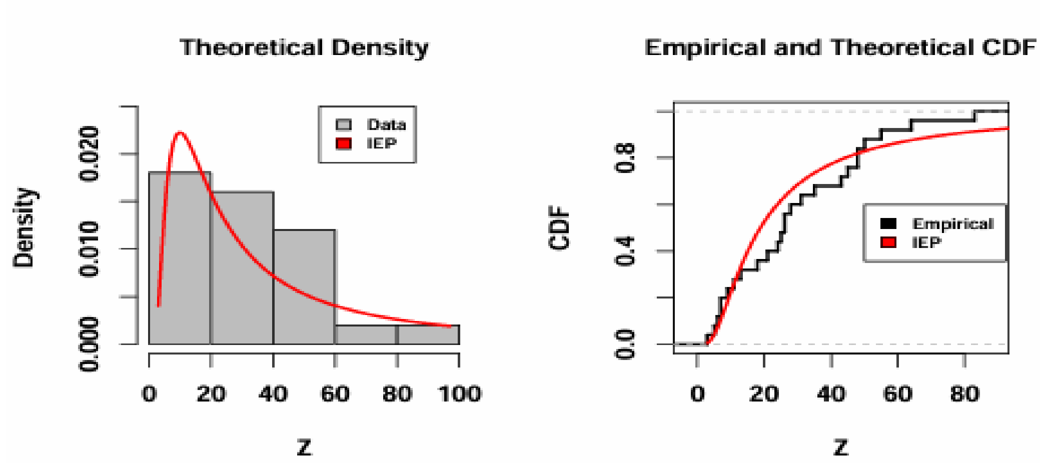

Figure 15: Estimated PDF and CDF of Z for the first dataset

6.2. Second dataset: Monthly levels of sulphur dioxide concentration data

In this real-world data, we analyze the monthly amounts of sulphur dioxide concentration in Long Beach, California, ranging from 1956 to 1974. This dataset was reported by Roberts [46] and investigated by Wang and Ye [47] to compare two alternative models. In this example, let t_1 through t_{20} represent the concentrations of sulphur dioxide in March, whereas z_1 through z_{20} represent those in August. The particular information is presented below:

Dataset (I): T = (21, 16, 20, 15, 9, 10, 10, 4, 25, 18, 18, 26, 25, 17, 40, 55, 19, 16, 9, 19.6),
Dataset (II): Z = (44, 20, 20, 20, 23, 20, 15, 27, 3, 9, 25, 32, 18, 55, 10, 20, 18, 8, 9, 20.8). This data set has been used by Pak et al. [48]. Table 7 essentially analyzes the data sets.

Table 7: Some summary statistics of the second dataset.

	Sample size	Mean	Median	Variance	Skewness	Kurtosis	Range	Minimum	Maximum
Dataset I (T)	$n = 20$	19.63	18.00	130.65	1.62	5.98	51.00	4.00	55.00
Dataset II (Z)	$m = 20$	20.84	20.00	148.55	1.23	4.64	52.00	3.00	55.00

A graphical representation of these datasets is shown in Figures 16 and 17. These consist of histograms, kernel density estimates, violin plots, box plots, TTT, and QQ plots.

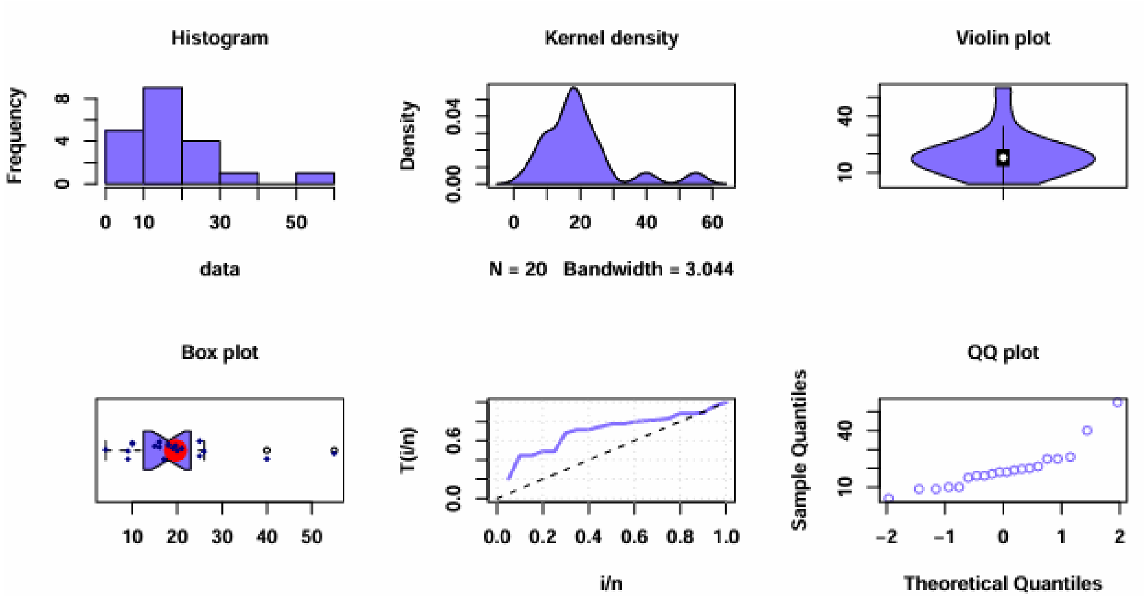

Figure 16. Some nonparametric plots of T for the second dataset

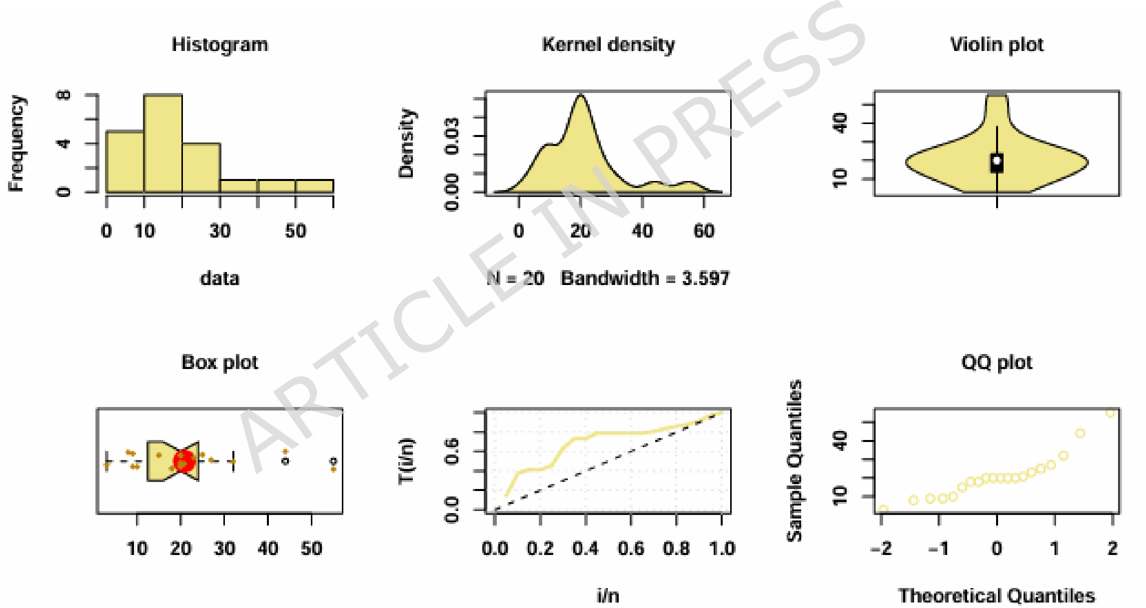

Figure 17. Some nonparametric plots of Z for the second dataset

The first and second datasets are right skewed, with a considerable dispersion and several potential outliers at the higher end. The hazard function in the first and second datasets is rising. These graphs show that the IEPD, as developed in the theoretical results, can manage these features. As in the previous dataset, we were evaluating the performance of the IEPD using the same previous alternative models (IERD, IEED, GIWD and EILD) and the same previous goodness-of-fit measures. All of the major parameter estimates and SEs for the actual dataset are given in Table 8. From Tables 8 and 9 and Figures 18 and 19 also indicate that the IEPD is an appropriate model to fit to both datasets.

Upper records based on Dataset (I): $\underline{r}$ = (21, 25, 26, 40, 55).

Upper records based on Dataset (II): $\underline{s}$ = (44, 55). Now, we discuss two cases based on the above upper record sample combination (5, 2).

Table 8: Estimated parameters with their SEs for the second dataset

Data	IEPD	$\hat{a}_1$	$SE(\hat{a}_1)$	$\hat{b}$	$SE(\hat{b})$	$\hat{c}$	$SE(\hat{c})$
Dataset I (T)	IERD	5.01247	0.11398	34.82113	0.27029		
	IEED	0.83842	0.11530	121.62699	0.27346		
	GIWD	4.52519	0.11391	31.91007	0.27012		
	EILD	6.55427	0.98088	1.62221	0.10872	2.92134	0.06746
	IEPD	1.62231	0.02357	0.11983	0.86352	61.82189	0.67462
Dataset II (Z)	IERD	$\hat{a}_2$	$SE(a_2)$	$\hat{b}$	$SE(\hat{b})$	$\hat{c}$	$SE(\hat{c})$
	IEED	3.52608	0.11394	29.59629	0.27020		
	GIWD	0.56694	0.11525	66.33116	0.27333		
	EILD	3.09720	0.11385	26.33568	0.26997		
	IEPD	5.46120	0.97975	1.33955	0.10852	3.03000	0.06745
	IERD	1.33959	0.02356	0.05073	0.86618	29.50028	0.67552

Table 9: Adequacy metrics for second dataset

Data	Model	AIC	BIC	CAIC	HQIC	A*	W*	K-S	P-value
Dataset I (T)	IEPD	151.2933	153.2848	157.2848	151.6821	0.5694	0.1006	0.1787	0.5454
	IERD	158.4466	160.4380	164.4380	158.8353	1.3870	0.2594	0.2691	0.1103
	IEED	151.8171	153.8086	157.8086	152.2059	0.6174	0.1094	0.1868	0.4878
	GIWD	158.7354	161.6223	167.6220	159.2357	1.0629	0.1824	0.2165	0.3051
	EILD	158.6303	161.6175	167.6175	159.2134	1.0628	0.1823	0.2164	0.3060
Dataset II(Z)	IEPD	159.3843	161.3758	165.3758	159.7731	0.9496	0.1851	0.2484	0.1693
	IERD	170.0210	172.0124	176.0124	170.4097	2.2033	0.4239	0.3157	0.0372
	IEED	160.4935	162.4849	166.4849	160.8822	1.0495	0.2027	0.2576	0.1406
	GIWD	167.0235	170.0221	176.0257	167.5241	1.3964	0.2468	0.2716	0.1633
	EILD	167.0486	170.0358	176.0358	167.6317	1.3965	0.2468	0.2617	0.1532

The MLEs as well as BEs of [illegible] are obtained following the procedure explained above. The BCIs of the 95% of boot-p and boot-t are obtained as (0.33154, 1.06216) and (0.33026, 0.51959). The BEs with their HPD interval are obtained under SELF, WSELF, and MLF. Since we do not have any prior information about the parameter, we assume that the values of hyper-parameters are $a_1 = a_2 = b_1 = b_2 = 0.0001$. The BEs have a lower PSD than the other MLEs. As a result, the BEs of the IEPD's parameters are the most accurate. Under SELF, the BEs of SS reliability outperform those under WSELF and MLF. Conversely, the BEs of SS reliability under MLF are the worst compared to others. The $P(Z < T)$ reliability of the Bayesian technique is greater than that of the ML method, which proves the displayed conclusion and confirms the results in the previous section. All results are presented in Table 10.

Table 10: MLEs and BEs of [illegible] and their SEs, PSD, and the HPD intervals

MLE		N-INP							
		SELF		WSELF		MLF		AIL	CP
[illegible]$_{MLE}$	[illegible]$_{SE}$	[illegible]$_{SELF}$	[illegible]$_{SELF\_PSD}$	[illegible]$_{WSELF}$	[illegible]$_{WSELF\_PSD}$	[illegible]$_{MLF}$	[illegible]$_{MLF\_PSD}$		
0.28575	0.21553	0.24363	0.00301	0.17518	0.00890	0.10441	0.00670	0.45182	95.26

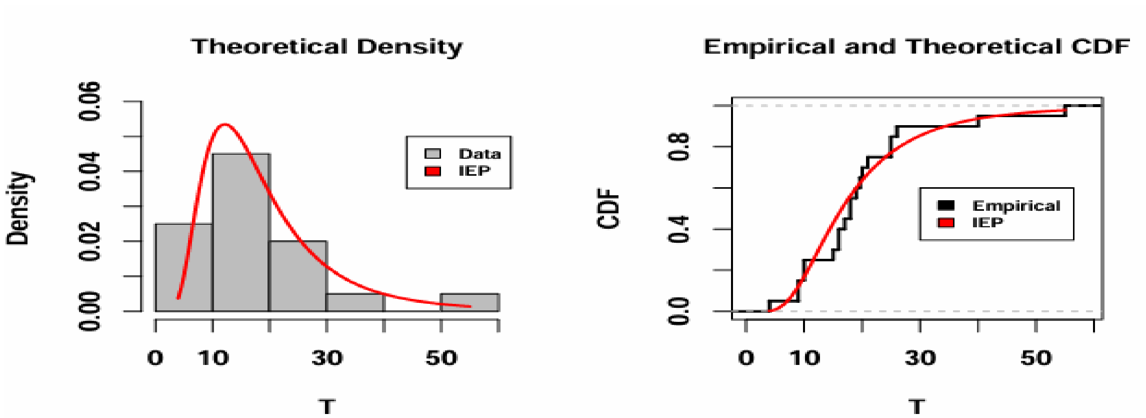

Figure 18: Estimated PDF and CDF of *T* for the second dataset

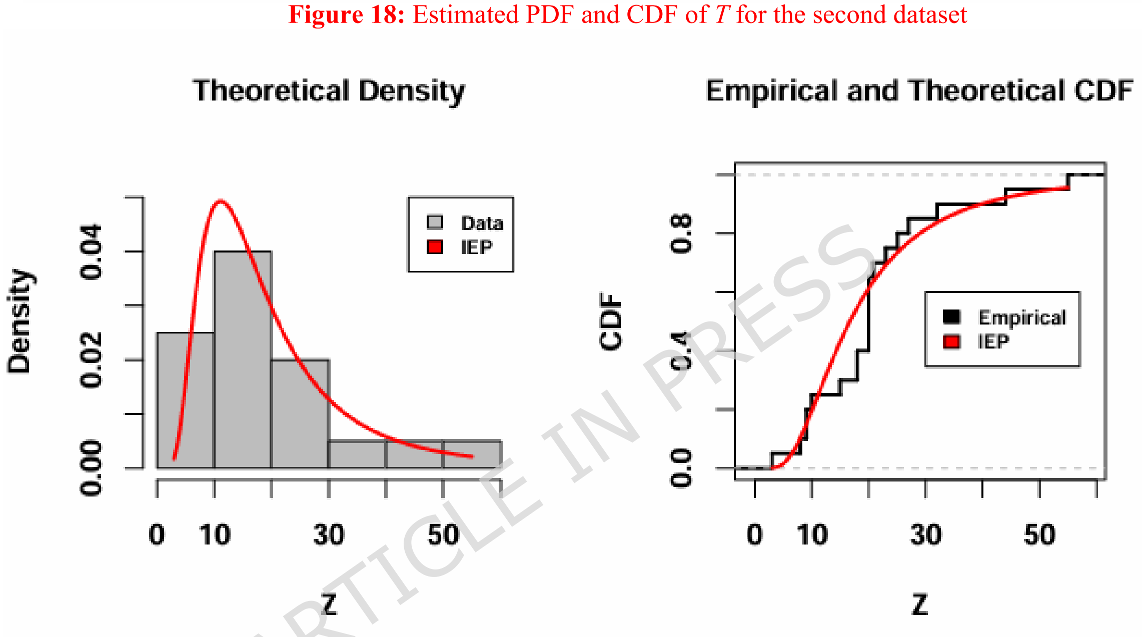

Figure 19: Estimated PDF and CDF of *Z* for the second dataset

7. Concluding Remarks

This study investigates the estimation issue of the SS reliability under the assumption that stress (*Z*) and strength (*T*) are independent random variables following the IEPD with a common second shape parameter. The Bayesian and maximum likelihood estimators of $\Re$ are acquired under URV. Bayesian estimators under INP (gamma) and N-INP (uniform) are investigated using three loss functions, including SELF, WSELF, and MLF. The results include the HPD credible intervals and two bootstrap-type confidence intervals. Furthermore, the HPD credible intervals, boot-t, and boot-p confidence intervals are obtained. Bayesian estimates of $\Re$ are obtained, based on the proposed loss functions, using Gibbs and Metropolis-Hasting samplers. Extensive simulation studies are conducted based on some measures, including mean squared errors, absolute biases, coverage probabilities, and average interval lengths, to examine the performance of suggested approaches.

Results from simulation experiments confirmed that the Bayesian estimates under SELF were more reliable than the others (WSELF and MLF) according to certain criteria measures. The performance of the Bayesian estimates under MLF was significantly worse than that of the estimates from other loss functions. Specifically, both the MLEs and BEs of SS reliability are consistent, meaning that they converge to the real value of [illegible] as the number of records increases. The Bayesian intervals for INP and N-INP provided significantly better results than the boot-p and boot-t intervals. As assessed by AIL and CP in every scenario, the HPD credible intervals obtained from INP performed better than those from N-INP. Actual data analysis proved that BEs perform better than MLE, validating the theoretical findings. The BEs of SS reliability perform better under SELF than under WSELF and MLF. On the other hand, the BEs of SS reliability under MLF are the worst. The results in the section of simulation study are confirmed, and the exhibited conclusion is supported by the fact that the Bayesian technique is more reliable than the ML method. Future studies can be extended to more another SS models using similar methods [49-52].

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