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A comparison of functions in Finnish, Singaporean, and Taiwanese middle-school mathematics textbooks

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This study adopted a content-analysis method to examine the differences in how the topic of functions was handled by Finnish, Singaporean, and Taiwanese middle-grade mathematics textbooks for children aged 13–15 (seventh-ninth grade). The results showed that the three countries' textbooks used different methods to introduce the concept of functions, with the Finnish ones using function machines with input and output tables; the Taiwanese ones, verbal and visual representations; and the Singaporean ones, verbal representations only. In all three countries, most function-related mathematics problems were provided in a purely mathematical form. However, the Finnish textbooks were more balanced in this regard, including some problems presented visually or verbally, whereas the Taiwanese and Singaporean textbooks were inclined to use purely mathematical presentations. The Singaporean textbooks featured more open-ended problems than the Finnish or Taiwanese textbooks did. And, as compared with the textbooks from the two Asian countries, the Finnish ones included both a greater overall number of problems and less difficult problems.

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Introduction

Mathematics textbooks have long been considered a crucial resource for teaching and learning mathematics (Lee et al. 2021). They can substantially influence not only what mathematics content is taught in the classroom but also how it is taught, as well as students' learning attitudes and, ultimately, what is learned (Bütüner, 2021; Wang and Lu, 2018). Unsurprisingly, strong linkages have been identified between the quality of textbooks and the effectiveness of classroom instruction (Bütüner, 2021). As such, research on different countries' textbooks, especially those of internationally high-performing countries, could provide meaningful information to the growth of student learning or educational reforms.

Functions, considered crucial to the learning of advanced mathematics, are a unifying theme of textbooks worldwide (Tesfamicael and Lundeby, 2019). The basic concept is usually introduced in early middle school, and its more advanced components, such as quadratic and trigonometric functions, are taught at the high school or college level; how well students learn functions influences their ability to master higher mathematical topics such as calculus in college (Leinhardt et al. 1990; Thompson and Milner, 2019). Nevertheless, functions are conceptually challenging for most students, and many teachers find it difficult to teach them (Krüger, 2019; Tesfamicael and Lundeby, 2019).

This study selected textbooks from Finland, Singapore, and Taiwan for comparison for several reasons: (1) These three countries have consistently had high rankings in the Program for International Student Assessment (PISA). Although Finland's performance has declined in recent years, its success in four consecutive PISA assessments continues to draw significant interest from educational scholars (Hemmi et al. 2020; Yang and Sianturi, 2022). (2) Despite overall high scores in functions, student performance in these countries varies significantly. Taiwanese students exhibit a wide variation in performance, while Finnish students tend to have a narrower distribution. Additionally, Trends in International Mathematics and Science Study (TIMSS) results indicate that algebra remains a challenging domain for Finnish students, whereas Singaporean and Taiwanese students perform better in algebra than in their overall mathematics scores across all four mathematics domains (Mullis et al. 2012; Mullis et al. 2020). (3) The textbooks from these three countries have been more frequently analyzed than those of other nations and are often considered a primary factor contributing to their students' success in international assessments (Yang and Sianturi, 2022). (4) It is likely to be instructive to compare both countries that have similar (Singapore and Taiwan) cultural backgrounds and those that have different (Singapore and Taiwan versus Finland) cultural backgrounds. While cultural differences are an important criterion for the selection of textbooks (Petersson et al. 2023), Hemmi et al. (2020) suggested that examining countries with similar cultures may be particularly valuable, as it can help researchers identify specific features or distinct perspectives within the textbooks. (5) Singapore's consistent top performance in international assessments has led to its textbooks being widely adopted around the world (Petersson et al. 2023). Singaporean textbooks are considered high-quality, and Vicente et al. (2019) recommended using them as a useful benchmark in research comparing textbooks.

Research purpose. The purpose of this study was to compare the different approaches to introducing the topic of functions and problem types among Finnish, Singaporean, and Taiwanese mathematics textbooks.

Literature review

Definition of functions. The generally accepted contemporary definition of functions, known as the Dirichlet-Bourbaki definition, is "a correspondence between two sets that assigns to each element from the first set exactly one element from the second set" (Ruhama, 1988, p. 305). Although many mathematics textbooks have adopted the Dirichlet-Bourbaki approach, not all function definitions in textbooks are the same. For example, some from England place more emphasis on the input-output relationship, illustrating it with flow diagrams, whereas several Chinese ones focus on the term "variables" and the dependent relationship between variables (Wang et al. 2015).

Over the past three decades, researchers have tended to divide definitions of functions into multiple categories (Viirman et al. 2010; Hohensee et al. 2022). For example, Sfard (1991) proposed two complementary approaches to functions: the operation approach and the structure approach. The operation approach stresses the operational process of how input values become output values. The use of input-output flow diagrams usually indicates this approach (Wang et al. 2015). On the other hand, the structured approach uses a more holistic and abstract view of functions, which includes knowing how to represent them in different ways (e.g., through tables, graphs, formulas, and verbal representations) and how to integrate different ideas of functions (e.g., being able to compare different functions or understand function families).

Sfard's (1991) different approaches to functions have heavily influenced later researchers' views of functions. By reviewing the ideas of Sfard and other researchers (e.g., Freudenthal, 1983), Doorman et al. (2012) summarized three interrelated approaches to functions: (1) the input-output assignment, (2) the dynamic process of co-variation and (3) the mathematical object. The input-output assignment approach is similar to Sfard's operation view, as both emphasize the calculation process of functions. The dynamic process of co-variation refers to how the dependent variable in the domain set causes the independent variable to change in the range set; that is, "the dependent variable co-varies with the independent" (p. 1246). Common representations used in this approach to functions include tables or graphs that trace the co-variation of variables' values. Finally, the mathematical object approach closely relates to Sfard's structure approach, as both refer to a holistic view of functions. The mathematical object approach stresses understanding various aspects of functions and connecting different representations of them (e.g., formulas and graphs).

Doorman et al. (2012)'s three approaches have served as a basis for later views of functions. For example, Wang et al. (2015) discussed operational and variable views of functions in their study. The former is similar to Doorman et al. (2012) input-output assignment approach, while the latter is similar to the dynamic process of co-variation. In addition, Ayalon and Wilkie's (2019) correspondence approach is similar to the input-output assignment approach, and their co-variation approach is similar to Doorman et al. (2012) dynamic process of co-variation. Furthermore, Hong and Choi (2018) summarized functional and structural approaches from previous research. The functional approach emphasizes using multiple representations of functions, while the structural approach emphasizes more symbolic representations. In this case, the functional approach is related to Doorman et al. (2012) mathematical object approach, but the structural approach is not emphasized in Doorman et al.'s model.

Based on the above literature, particularly Doorman et al. the present study also adopts a three-part typology that consists of input-output, co-variation, and multi-representational approaches. An input-output approach highlights the operational aspect of functions and implies that an input value will produce an output value. A co-variation approach, in contrast, focuses on the relationships between variables, including their dependency relationships, while a multi-representational approach presents the student with a range of different ways to represent functions, e.g., through arrow chains, tables, graphs, or phrases.

Introducing functions. The way a textbook introduces functions might relate to how it defines them or, more likely, to a viewpoint held by its writers (Doorman et al. 2012). Those who hold an input-output view will probably include an input-calculation-output chain (e.g., function machine) in their textbooks. Those who emphasize co-variation are likely to include context-based problems and ask students to create tables to explore the relationships between two variables (Wang et al. 2015); and those who hold a representation view may provide side-by-side representations of flow diagrams, tables, graphs, or algebraic expressions.

In spite of or perhaps because of their lack of consensus on the best way to introduce functions (Wang et al. 2015), researchers have emphasized some specific ways of doing so according to the various viewpoints explained above. The function-machine approach, for example, has been strongly endorsed by those who hold an input-output view (e.g., Kabael, 2009) and, in at least one case, asserted to be superior to a representation-based approach (Tall et al. 2000).

Reasons for selecting functions for comparison. We chose to make a comparative study of how textbooks in various countries teach functions for several reasons: (1) Functions serve as a foundation for advanced mathematics learning (Tall, 1996). (2) Many students at different school levels have difficulty understanding functions, making it a particularly challenging topic (Ayalon and Wilkie, 2019). (3) Different instructional approaches (e.g., function machine versus tables and values) are applied to teach functions; however, there remains no consensus on the most effective approach (Hohensee et al. 2022). (4) Research has indicated cultural differences in teaching and learning functions (Yang and Sianturi, 2022).

Functions in textbook studies. Several prior studies have compared the topic of functions between Western and Eastern countries' textbooks. For example, Hong and Choi's (2018) examination of Korean and American textbooks found that the American ones used more real-life problems and introduced the topic of functions by exploring rates of change, whereas their Korean counterparts emphasized a pure-mathematics context and presented the formal definition of a function in the first lesson. In addition, Korean textbooks were reported to have less content regarding different representations in functions. Nevertheless, both sets of textbooks provided similar opportunities for students to practice procedures and algorithms.

Similarly, Wang et al. (2015) compared the treatment of linear functions in various textbooks from England and China and found that the English ones strongly emphasized the input-output approach, thereby limiting students to a structural view of such functions; while the Chinese textbooks overwhelmingly reflected the co-variation approach, and couched it in highly abstract terms that students were unlikely to learn easily. The authors, therefore, recommended that both countries adopt a more balanced approach, combining these two perspectives. Fowler's (2015) head-to-head comparison of functions in Singaporean and U.S. textbooks, meanwhile, revealed that the latter contained more problems requiring multiple or open-ended responses, more connections between different types of representations, and more conceptual problems.

Yang and Sianturi (2022) analyzed algebraic problems in elementary textbooks from five countries – Finland, Indonesia, Malaysia, Singapore, and Taiwan – and found significant differences in their numbers of problems, forms of representation of problems, problems' cognitive-demand levels, and use of closed- vs open-ended types of problems. Specifically, the Finnish textbooks included significantly more problems ($n = 1450$) than those of the other four countries (SG = 245, ID = 296, MYS = 237, TW = 79), while the Taiwanese textbooks had the fewest problems ($n = 79$). Among the five countries' textbooks, a significantly higher percentage of problems in the Finnish and Singaporean textbooks emphasized the symbolic forms of functions (FI = 75.03%, SG = 52.24%), while the Indonesian textbooks were more likely to stress the visual form (42.57% of problems), and the Malaysian textbooks focused on the verbal form (41.35% of problems). Lastly, the Taiwanese textbooks included more problems in the combined forms (40.51% of problems featured two or more forms of representation). The Finnish, Singaporean, and Taiwanese textbooks included more problems that required lower levels of cognitive demand (FI = 74.83%, SG = 73.47%, TW = 72.15%), whereas the Indonesian and Malaysian textbooks included more challenging problems (ID = 58.11%, MYS = 62.03%). In addition, Malaysian and Taiwanese textbooks were more likely to emphasize context-based problems (MYS = 70.46%, TW = 80.01%), while Finnish, Indonesian, and Singaporean textbooks stressed non-contextual problems (FI = 91.72%, ID = 57.09%, SG = 73.47%). Finally, the majority of problems across all of the textbooks were closed-ended (96%–100%). The percentages of open-ended problems were quite low among all five countries' textbooks.

Though some discrepancies exist in the results of previous studies, they generally suggest that textbooks in East Asian countries take a structural approach that focuses on the symbolic, whereas those in the West adopt a functional approach that includes more multi-representational and real-life problems (Lee et al. 2021; Thompson and Milner, 2019). Additionally, textbooks in Western countries contain more problems, but less challenging ones, than their East Asian counterparts (Yang and Sianturi, 2022).

In sum, we identified six major categories of variation among different countries' textbooks from the studies discussed above: (1) content coverage and sequence in functions, (2) definition of functions, (3) major representational forms of problems, (4) number of context-based problems, (5) major response type of problems and (6) difficulty of problems. As many researchers have indicated, the designs and features of textbooks significantly influence students' opportunities to learn (Sayers et al. 2019; Zhu and Fan, 2006). As such, these major differences among textbooks were our primary focus in the current study.

Functions in the mathematics curricula of Finland, Singapore, and Taiwan. The national mathematics curricula of Finland, Singapore, and Taiwan share broad similarities but also exhibit some differences in terms of their core ideas and the specific content of functions. Below is a brief overview of each curriculum.

The Finnish national mathematics curriculum emphasizes developing students' mathematical thinking. As students advance, the curriculum increasingly focuses on modeling skills and enhancing their knowledge structure for future use in mathematical learning. The formal topic of functions appears in the 6th–9th grades (the third phase), where students engage with concepts traditionally associated with lower secondary education. These include observing correlation by using variables, understanding the concept of functions, and drawing and investigating function graphs (e.g., finding maximum and minimum values). In the first phase (1st–2nd grades) and the second phase (3rd–5th grades), the curriculum includes content in algebra that serves as foundational knowledge for learning functions later on. For example, students learn to identify patterns, ratios, and correlations in the first phase and to explore algebraic expressions, write, and interpret ratios and correlations in the second phase.

The Singaporean national mathematics curriculum centers on the idea of problem-solving, emphasizing heuristics, higher-order thinking, and self-reflection in solving mathematical problems (Fan and Zhu, 2007, p. 491). In the 7th grade, the national curriculum begins with graphing ordered pairs and exploring their algebraic forms (e.g., $y = ax + b$). Notably, the function notation "*f(x)*" is not used; instead, "y" is consistently employed in the entire curriculum in the 7th–10th grades. The curriculum also introduces the concept of slopes in linear graphs, a topic not covered in the Finnish or Taiwanese curricula. In the 8th grade, quadratic functions and their graphs are introduced. By the 9th–10th grades, the curriculum expands to include graphing quadratic functions in various forms (e.g., $y = (x - a)(x - b)$), power functions, exponential functions, and estimating the gradient of a curve by drawing a tangent. In comparison, the Taiwanese curriculum only includes graphing quadratic functions relating to the above four topics, while the Finnish curriculum does not address them in the middle school curriculum. This is perhaps because that content is usually taught at the 10^{th}-grade level (high school). The elementary curriculum in Singapore does not explicitly cover functions, but it does include content that prepares students for learning secondary-level functions, such as number patterns, line graphs, ratios, algebraic expressions, and simple linear equations (e.g., $2x + 10 = 50$).

The Taiwanese national mathematics curriculum is built on mathematics literacy, aligning with the PISA's definition by connecting mathematics to real-life applications. As such, the curriculum's core idea emphasizes mathematical reasoning and 21st-century skills, such as creativity in real-world contexts (Beccuti, 2024, p. 57). In Taiwan, functions are formally introduced in the 8th grade, including the definition of functions, linear functions, constant functions, and function graphs. In the 9th grade, students study quadratic functions, including their graphs, vertex identification, maximum or minimum values, and transformations. The elementary school curriculum introduces finding patterns, contingency tables with two variables (e.g., tables showing the number of boys and girls in a class), algebraic expressions, and simple equations, laying the groundwork for later function learning.

Methods

Selected textbooks. This study adopted a content-analysis method to analyze middle-grade textbooks aimed at 13 to 15-year-olds. Three Laskutaito mathematics textbooks from Finland (WSOY, 2009), three New Syllabus mathematics textbooks from Singapore (Teh and Loh, 2011), and three Kung-Hsuan mathematics textbooks from Taiwan (Kang Hsuan Educational Publishing Group, [KH], 2012) were identified as the most commonly used in each of their respective countries, and therefore as representative. The researchers coded all the problems in each textbook, including worked examples (with solutions), exercises (without solutions), and summary test problems provided at the end of chapters.

The Finnish textbooks (WSOY, 2009) analyzed consisted of three volumes (for seventh, eighth, and ninth grades) that were developed based on the Finnish Ministry of Education's 2004 mathematics curriculum guidelines (Yang et al. 2017). The seventh and eighth-grade textbooks cover three topics: numbers and operations, geometry, and algebra. Each of these textbooks comprises a total of 83 and 84 units, respectively. The ninth-grade textbook covers two topics, geometry and algebra, with a total of 78 units. The formal function topic appears in 27 units of the ninth-grade textbook, but the idea of the function machine is introduced earlier, in the variables and expressions units in the seventh and eighth-grade textbooks. A total of 526 problems were analyzed, comprising 31 units across seventh to ninth grades.

The Singaporean textbooks (Teh and Loh, 2011) analyzed also consisted of a series of three books (7th to 9th grade). They were developed under the Singaporean Ministry of Education's national mathematics curriculum guideline in 2001 (Yang et al. 2017). The first book has 16 chapters, and the second and third books contain 12 and 13 chapters, respectively. In general, the first book focuses on numbers and operations, while the second and third books emphasize algebra and geometry. However, the topic of functions is introduced in the first book (seventh grade), with seven units and a total of 121 problems. This topic is covered by one of the two algebra chapters in the first book (chapters are made up of units).

The Taiwanese textbooks (KH, 2012) analyzed consisted of a series of six books, with two volumes devoted to each grade (e.g., 7th grade has 7a and 7b two books). The Taiwanese textbooks were created based on the national mathematics curriculum guidelines of the Taiwanese Ministry of Education. The seventh-grade textbooks focus on numbers and operations and algebra, while the eighth and ninth-grade textbooks emphasize algebra and geometry. Functions are introduced in two units in the second volume for seventh grade and are also featured in three units of the second volume for ninth grade. A total of 171 problems cover the topic.

Analysis framework. This study selected vertical analysis as being able to yield an in-depth understanding of mathematical content. Following this approach, the problems in the sampled textbooks were analyzed according to the following six criteria:

(1) *Content Sequence*, which refers to how the concept of functions was introduced in terms of our three-part typology of approaches, as well as what related topics were developed throughout the textbook in question (see Hong and Choi, 2014).
(2) *Definition of Functions*, i.e., how the concept of functions as defined in each textbook related to (i) input-output, (ii) co-variation, and (iii) multi-representation.
(3) *Representation Forms*, divided by Zhu and Fan (2006) into four types, as follows:
   (i) *Purely Mathematical*, comprising mainly mathematical expressions;
   (ii) *Verbal*, comprising problems described mainly in words;
   (iii) *Visual*, problems to be solved using a graph, chart, table, figure, or other visual objects;
   (iv) *Combined*, problems containing two or three of the aforementioned forms, with no clear indication of which should be given the most prominence.
(4) *Context Type*, which Zhu and Fan (2006) divide into two: *application problems* that are posed in the context of a real-life situation and *non-application problems* that are unrelated to any practical or relatable background.
(5) *Response Type*, which Zhu and Fan (2006) again divide into two: *open-ended problems* with more than one correct

answer and *closed-ended problems* with only one correct answer (though many approaches may be used to arrive at it).

(6) The *Cognitive Demand of Problems*, as classified by Stein et al. (2000) into:

(i) *Memorization*, meaning that a problem requires students to recall information such as basic facts, rules, or formulas, and has no connection to concepts;
(ii) *Procedures without Connections*, usually step-by-step procedural calculations without connection to concepts and requiring no explanation;
(iii) *Procedures with Connections*, problems that require students not only to perform calculation procedures but also to make purposeful connections to meanings or to relevant mathematical concepts;
(iv) *Doing Mathematics*, problems that require complex thinking and exploration as well as a mere understanding of concepts, solved in a manner that is usually not predictable or rehearsed.

Data analysis. Data were analyzed according to the above framework. First, we analyzed which approaches the textbooks use to introduce functions as well as how they define functions. We also considered whether multiple approaches or definitions are used in each textbook. Finally, we examined four problem characteristics (representation form, context type, response type, and cognitive demand of problems) in the textbooks.

Sample coding results are provided in Appendix A. The first sample problem asks students to calculate the value of y ($y = 20–4x$) when $x = 3$ and 8. We coded this problem as the "purely mathematics" representation form because the mathematical expressions are the main focus of this problem. There is only one answer to each sub-question, so the problem is close-ended. In addition, there is no context for the problem, and students must use procedural knowledge to solve it. Therefore, it was coded as "non-application" and "procedures without connections." On the contrary, the second and third problems in Appendix A consist of graphs and a table, so they were coded as "visual form." The third problem asks students, "Do you notice any pattern formed by the points?" This question could have many different responses; therefore, it was coded as "open-ended." The fourth problem consists of a description of a situation and a corresponding picture. Therefore, it was coded as the "connection representation" form and as an "application" problem. Finally, the fifth problem was coded as "memorization" because students must remember the meaning of a and b in the form of $f(a) = b$.

Reliability. At the beginning of the study, the Finnish textbooks were translated into English (the Singaporean and Taiwanese textbooks were not translated because they were published in English and Chinese, respectively). Then, two coders who were both fluent in English and Chinese separately coded the same randomly chosen 20% of all the problems from each textbook. The interrater reliability (Cohen's kappa) ranged from 0.72 to 0.88. All differences in coding were then discussed until an agreement was reached.

Results

Introduction and definition of functions. The Finnish, Singaporean, and Taiwanese textbooks use different approaches to introducing and defining functions, though the Finnish and Singaporean textbooks seem to adopt a somewhat similar integrated approach involving both input-output and co-variation approaches. The Finnish textbooks use a function machine with an input and output table attached (coded as an input-output approach) to initially introduce the concept of functions and later use a mapping diagram to define functions (co-variation approach). On the other hand, when introducing functions, the Singaporean textbooks explicitly indicate the relationships between quantities in the beginning problem (the first problem for students to complete). As shown in Table 1, the Singaporean textbooks present an algebraic expression of the total cost and the number of tickets and identify these quantities' relationship as a function. When defining functions (Table 2), the Singaporean textbooks describe the relationship between x and $f(x)$ as input and output without using mathematical symbols (i.e., x and $f(x)$). We coded this definition as the input-output approach.

Unlike the Finnish and Singaporean textbooks, the Taiwanese textbooks use a more consistent mixed approach (consisting of both input-output and co-variation) in both the introduction and definition of functions. When introducing functions, the textbooks ask students to explore the relationships between quantities in a situational problem (Table 1). In contrast to the Singaporean textbooks, the Taiwanese ones do not explicitly indicate these relationships, but instead, ask students to use tables to explore the relationships before providing an explicit explanation. Therefore, the Singaporean situational problem seems to introduce the concept of functions in a more straightforward manner than the Taiwanese problem does, with the former seeming to be easier to solve. When defining functions, the Taiwanese textbooks use a traditional definition that specifies both the input-output and co-variation relationship between x and y. The Singaporean definition seems more straightforward, while the Taiwanese definition is more complex.

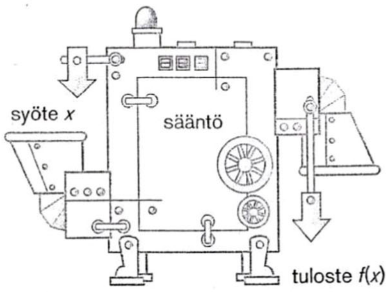
Topics and their placement. Before formally introducing different function-related topics, all three countries' textbooks covered algebraic expressions, linear equations, and basic algebraic operations. Tables 3 and 4 summarize the topics of lessons involving functions and their placement across the middle-school textbooks of each country. As noted above, the Finnish books highlighted the function-machine idea when initially introducing the concept, but they also deployed such machines repeatedly throughout each of the three studied grades. However, discussion of functions in the Finnish case remains fairly shallow until the end of the middle-school period, in contrast to the Taiwanese and Singaporean curricula, which treat them in greater depth much earlier.

The Taiwanese textbooks included the largest range of functions-related topics: including, for example, content about *identifying functions*, writing a function in the form $f(x) = a(x\text{-}h)^2 + k$ by completing the square and determining the number of x-intercepts *by using the discriminant* or *by completing the square*. These topics were not included in the Finnish or Singaporean textbooks. In their place, the Finnish textbooks focused on *function machines* and *finding the maximum, minimum, or intercepts by reading the graphs of functions*. In further contrast to the Taiwanese textbooks, the Finnish ones included content about reading values from the graphs of basic cubic functions. The Singaporean textbooks did not deal with quadratic functions at all, though they did introduce topics related to solving quadratic equations.

Representational forms. Table 5 shows the distribution of problems according to their representational forms. The most common mode of presenting problems in all three countries'

Table 1 Textbooks' introductions of the concept of functions.

Finnish textbooks (*input-output approach*): (WSOY, Grade 9, 2009, p. 62)



Syöte	Tuloste
1	11
2	21
3	31
4	

Tuloste riippuu syötteestä eli on syötteen *funktio*.

a) Päättele funktiokoneen sääntö.

Mikä on tuloste, kun syöte on

b) 4 c) 15 d) x?

▶ a) Funktiokone kertoo syötteen luvulla 10 ja lisää tuloon luvun 1.

b) $10 \cdot 4 + 1 = 41$

c) $10 \cdot 15 + 1 = 151$

d) $10 \cdot x + 1 = 10x + 1$

syöte *x=input x*; sääntö=*rule;* tuloste f(x)=*output f(x)*;

the output depends on the input, i.e., the input function (caption under the machine picture)

a) Find a rule for the function machine

What is the result when the input is:

b) 4 c) 15 d) x?

Singaporean textbooks (*co-variation approach*): (The and Loh, 2011, vol 1, p.273)

The total cost of 20 tickets to a particular musical at the Esplanade by the Bay, at \$75 per ticket is \$1500. In this example, the total cost is the first variable and the second variable is the number of tickets.

Let y be the first variable and x be the second variable. Then, we have $y = 75x$. We say that y is a function of x.

Taiwanese textbooks (input-output & *co-variation approach*): (KH, vol 2, 2012, p. 142)

明澤以每小時 60 公里的固定速率從甲地開車到乙地，若行駛時間為 x 小時、行駛距離為 y 公里，將 x、y 的變化情形以下表來呈現，則：

行駛時間 x (hr)	0.5	1	1.5	2	…	x
行駛距離 y (km)						

(1) 寫出變數 x、y 的關係式。

(2) 對給定的 x 值，在空格內填入對應的 y 值。

(3) 對於給定一個 x 值，是否恰有一個對應的 y 值？

Min drove a car from City A to City B at a fixed speed. If the time is x hours; the distance is y km; the table below shows the relationship between x and y.

Time x (hr)	*0.5*	*1*	*1.5*	*2*	*…*	*x*
Distance (km)						

(1) Write the relationship between x and y in algebraic form

(2) For any given x, please fill out the corresponding value of y in the table

(3) For any given x, is there only one corresponding value of y?

Table 2 Definitions of functions.

	Definition	Coding
FI	**Funktio** x f $f(x)$ 1 ▷ 11 2 ▷ 21 3 ▷ 31 $f(x) = 10x + 1$ Funktio f on sääntö, jonka mukaan jokaista muuttujan x arvoa vastaa täsmälleen yksi funktion arvo $f(x)$. Funktio määritellään usein antamalla funktion lauseke $f(x)$, esimerkiksi $f(x) = 10x + 1$, jonka avulla funktion arvot voidaan laskea. *Function f is a rule that every value of x corresponds to exactly one value of the function f(x).* *Function is often defined by an expression of the function f(x), for example, f(x)=10x+1, which allows function values to be calculated.* (WSOY, 2009, Grade 9, p. 62)	**Co-variation approach**
SG	For a function, each input produces only one output. For example, for *y=75x*, given *x=1*, we get only one value of *y*, that is 75. (The and Loh, 2011, vol 1, p.273)	**Input-output approach**
TW	前面的例子都牽涉到兩個變量，我們將兩個變量的數值用 x、y 表示後，當 x 的值確定時，y 的值也隨著唯一確定，也就是：對於給定的一個 x 值，都恰有一個 y 值與它對應，這時我們就說「y 是 x 的函數」。 *The preceding examples all involve two variables. We use x and y to represent the two variables. When the value of x is specified, the value of y is then decided. That is: For a given value of x, there is only one corresponding value of y. We say 'y is a function of x'.* (KH, vol 2, 2012, p. 144)	**Input-output and co-variation approach**

textbooks was in purely mathematical form (Finland: 37.1%, Singapore: 67.8%, Taiwan: 56.1%). However, the distribution of representational forms was more balanced in the Finnish textbooks than in the others, including a high proportion of visual-form (30.6%) and verbal-form problems (23.0%).

Context type. Table 6 shows the distribution of problems according to context type. The Taiwanese textbooks had the highest percentage of application problems, at 23.4%, as compared to less than 14% in the other two countries' textbooks. This result contrasts with an earlier study's finding that textbooks from East Asian countries usually have fewer application problems than those from elsewhere in the world (Zhu and Fan, 2006).

Response type. Table 7 shows the distribution of problems according to response type. It reveals that approximately 98% of the problems in the textbooks from Taiwan and Finland, but only 84% of those in the textbooks from Singapore, were closed-ended.

Levels of cognitive demand. Table 8 presents the distribution of problems according to cognitive demand and indicates that the Taiwanese and Singaporean textbooks mostly contained problems with high levels of cognitive demand, i.e., involving procedures with connections and doing mathematics (Taiwan: 63.9%, Singapore: 53.7%), whereas the Finnish textbooks featured mainly problems with low levels of cognitive demand, i.e., involving memorization and procedures without connections (62.8%).

Discussion

Introduction and definition of functions. This study sought to compare the different approaches to introduce and define the topic of functions in Finnish, Singaporean, and Taiwanese mathematics textbooks. The results of the analysis show that the Finnish and Singaporean textbooks use an integrated approach to introduce and define functions (e.g., input-output in the introduction, and then, co-variation in the definition), while the Taiwanese textbooks more consistently adopt both approaches (i.e., both input-output and co-variation in introducing and defining functions). The findings concerning the use of an integrated approach are against earlier findings that the viewpoint from which functions are *defined* is likely to influence how they are *introduced* (Doorman et al. 2012; Gok et al. 2019). This result also seems to imply that how a concept is defined in a textbook is not a necessary precursor to the way to introduce it in the same textbook. Perhaps the integrated approach of Finnish and Singaporean textbooks reflects

Table 3 Topics covered by Finnish, Taiwanese, and Singaporean mathematics textbooks.

	Finland	Taiwan	Singapore
Introducing Functions			
Defining functions	■	■	■
Function machines	■		
Linear Functions			
Evaluating functions	■	■	■
Identifying functions	■	■	■
Graphing functions	■	■	■
Finding function values by reading graphs	■		
Applying functions in real-life situations	■	■	■
Quadratic Functions			
Type $f(x)=ax^2+k$	■	■	
Type $f(x)=a(x-h)^2+k$		■	
Type $f(x)=ax^2+bx+c$		■	
Finding maximum, minimum, or intercepts by completing the square or by discriminant		■	
Finding maximum, minimum, or intercepts by reading graphs	■		
Applying functions in real-life situations	■	■	
Cubic Function			
Finding function values by reading graphs	■		

Shaded areas indicate that the textbook contained the topic.

Table 4 Functions-related topics according to their placement in the curriculum.

	Finland	Taiwan	Singapore
7th Grade	Function Machines	Introducing Functions Linear Functions	Introducing Functions Linear Functions
8th Grade	Function Machines		
9th Grade	Introducing Functions* Linear Functions Quadratic Functions	Quadratic Functions	

The section marked * included a few tasks in which students were asked to read cubic function graphs.

attempts to mitigate students' difficulties with learning functions. As past studies have indicated, functions have many different facets, and providing students with opportunities to understand and connect these different facets may help them understand functions more easily (Hong and Choi, 2018). In addition, earlier studies of textbooks usually found only one single approach (e.g., solely input-output) to functions used in a given country's textbook design (e.g., Lee et al. 2021). However, we may expect to see more integrated designs in the future, because researchers have recently suggested that a balanced design (i.e., using multiple approaches) in textbooks could best benefit students' learning (Wang et al. 2015; Yang et al. 2017; Yang and Sianturi, 2022).

Regarding defining functions, the Finnish textbooks were found to use a co-variation approach, while the Taiwanese textbooks used both input-output and co-variation approaches.

Table 5 Distribution of problems by representational form.

Representational	Finland		Singapore		Taiwan	
Form	*N* (526)	%	*N* (121)	%	*N* (171)	%
Purely math	195	37.1%	82	67.8%	96	56.1%
Visual	161	30.6%	25	20.7%	17	9.9%
Verbal	121	23.0%	13	10.7%	30	17.5%
Combined	49	9.3%	1	0.8%	28	16.4%

The total number of problems is shown in parentheses.

Table 6 Distribution of problems by context types.

Context	Finland		Singapore		Taiwan	
Types	N (526)	%	N (121)	%	N (171)	%
Application	70	13.3%	14	11.6%	40	23.4%
Non-Application	456	86.7%	107	88.4%	131	76.6%

The total number of problems is shown in parentheses.

Table 7 Distribution of problems by response type.

Response	Finland		Singapore		Taiwan	
Types	N (526)	%	N (121)	%	N (171)	%
Open-ended	9	1.7%	17	14.0%	4	2.3%
Close-ended	517	98.3%	104	86.0%	167	97.7%

The total number of problems is shown in parentheses.

Despite a co-variation approach being used in both textbooks, the Finnish textbooks used a mapping diagram combined with an easy-to-understand description, while the Taiwanese textbooks relied on a more abstract written description. The mapping diagram uses an arrow to show the relationship between input and output values, which is thought to be a more dynamic way to represent functions (Gosse and Karen Flanagan, 2003). As such, the Finnish definition seems to represent a more dynamic co-variation view, whereas the Taiwanese definition is more closely related to the static correspondence view. In fact, Finnish textbooks also place more emphasis on using function machines to introduce functions than Singaporean and Taiwanese textbooks. Moreover, emphasizing the function machine could also be seen as a more dynamic way to introduce functions than the use of tables in Taiwanese textbooks. Researchers have suggested that using dynamic approaches to learning mathematics can produce better learning outcomes (Latsi and Kynigos, 2021). From this point of view, the Finnish textbooks seem most aligned with the recommendations found in existing studies.

In addition, this study found that none of the textbooks examined use the multi-representation view of functions, which emphasizes using multiple representations to introduce or define functions. This result may derive from the fact that the analysis of the three views of functions was only applied to the sections of introduction and definition of functions in the textbooks, not to entire textbooks. In particular, a concept cannot be defined as its representations from the ontological assumption (Font et al. 2007). The multi-representation view also involves providing a structural (holistic) view of a concept. However, it does not mean that a new concept should be introduced in abstract terms or by presenting all of the information about the concept on a single page of a textbook (Sfard, 1992; Viirman et al. 2010). Therefore, we suggest future studies to explore this issue further.

The ideas of concept definition and concept image can be used to explain how one acquires a concept (Viirman et al. 2010). Concept definition refers to the formal definition of a concept, while concept image refers to one's mental construction of the concept. In the context of this theory, the definitions of functions provided by the textbooks can be characterized as concept definitions and students' conceptions of functions correlate to concept images. Although there is a gap between the concept definition and concept image, the way in which students encounter concept definitions may strongly influence their formation of concept images. In addition, Viirman et al. (2010) indicated that textbooks were the primary source for students' concept image of functions. As such, it would be meaningful for further study to investigate how three countries' students might form their concept images of functions from exposure to their textbooks. Would Students have different concept images from these three countries' different approaches to defining and introducing functions?

Function topics and placement in the textbooks. The current results showed that the topic of functions was introduced relatively early in Taiwanese and Singaporean textbooks, compared with Finnish ones. This result echoes those of earlier comparative studies of East Asian and Western countries' textbooks: e.g., Hong and Choi's (2018) finding that South Korean textbooks introduced functions two years earlier than their U.S. counterparts did. This is perhaps especially important, in light of prior research indicating that the earlier introduction of concepts is an important factor in East Asian countries' international test success (ÖZer and Sezer, 2014).

Although discovering the rationales behind different countries' placement of a given topic in different grade levels is not easy, the prior literature provides some clues. In the Finnish case, the textbook writers were likely to have been driven by considerations of the difficulty of learning functions and hence pitched this material to somewhat older students. Lue's (2013) study showed that, even though 10th graders had been taught during middle school to connect different representations of functions (e.g., algebraic expressions to graphs), they still had difficulties understanding functions and did not perform well on functions-related problems. In the Taiwanese and Singaporean cases, meanwhile, the priority seems to have been on keeping domain-related content together in one part of the curriculum.

Representation form of problems. Results of this research indicated that, in regard to the teaching of functions, middle-school mathematics textbooks in Singapore and Taiwan have more similarities with one another than they do with their

Table 8 Distributions of problems by cognitive demand.

Representation	Finland		Singapore		Taiwan	
Form	*N* (526)	%	*N* (121)	%	*N* (171)	%
Memorization	74	14.1%	0	0%	4	2.3%
Procedures without connections	256	48.7%	56	46.3%	59	34.5%
Procedures with connections	193	36.7%	48	39.7%	86	50.3%
Doing mathematics	3	0.6%	17	14.0%	22	12.9%

The total number of problems is shown in parentheses.

Finnish equivalents. Both the Asian countries' textbooks have more problems that are in the purely mathematical form, whereas the Finnish ones are more balanced in terms of the representational forms that their problems use: i.e., with nearly equal proportions of mathematical-, visual-, and verbal-form problems. Prior research has shown that connecting different representations is important to students' learning of functions (Senk et al. 2014), so a narrow range of representational forms of function problems could reasonably be expected to have a negative impact on students' mastery of this topic. In addition, Zhu and Fan (2006) indicated that limiting the quantity of visual-form problems in textbooks could hinder students' performance, not only by rendering them less able to solve those types of problems but also by placing constraints on their use of visual strategies for problem solving in a more general sense.

However, Zhu and Fan's (2006) findings regarding the negative impact of a lack of visual-form problems do not seem to be borne out in the Taiwanese and Singaporean cases. Although the textbooks from these two countries contain a majority of purely mathematical-form problems, their students regularly perform well in international assessments. Several reasons may contribute to this phenomenon. First, and perhaps most plausibly, textbooks are only one of many important factors that influence students' performance in international comparison tests (Zhu and Fan, 2006). There could be many other factors that contribute to the three countries' successes, including the ways that textbooks are used and cultural factors (Wei and Eisenhart, 2011). Second, the number of items in PISA that definitely require students to apply their knowledge of different representations of functions may be low. For example, a released PISA item on Drip Rate, designed to detect students' conceptions of the interrelationship between two variables, could be solved by a student who understood the problem description and algebraic representations, without necessarily applying any knowledge about visual representations of functions (e.g., connecting expressions to graphs). Finally, textbooks that strongly emphasize problems that are purely mathematical in form may not have a negative impact on student performance in the specific realm of function-related problems and could even help to boost it. For example, Koedinger et al. (2008) found that, in the case of more complex problems, students performed better when the problem's representation was abstract rather than situational; and termed this 'symbolic advantage' (p. 366). Their finding implies that learning initially from abstract-representation problems could help students' later reasoning when approaching more complex verbal-representation ones. Leung and Park (2002) likewise proposed that conceptual understanding does not always arise before procedural understanding does; that is, competence in algebraic procedures can lead to, as well as flow from, a clear conceptual understanding of algebra. This implies that, even when students are exposed mostly to problems in a purely mathematical form (usually related to their algebraic procedural skills), they are still able to develop a conceptual understanding of functions.

Cognitive demand of problems. As compared with the textbooks of the two East Asian countries, the Finnish ones included many more function problems, but these problems tended to place lower levels of cognitive demand on students than the function problems from the two East Asian countries did. This result echoes previous studies' findings that East Asian countries' textbooks usually have fewer, but more challenging, problems than those from Western countries (Hong and Choi, 2014; Yang and Sianturi, 2022). Prior studies have also shown that consistent use of problems with higher levels of cognitive demand yields greater learning gains (Zhu and Fan, 2006).

Although the sampled Finnish textbooks did not provide many high-cognitive-demand problems, Finnish students still performed well in the PISA, as noted above. Upon closer examination, the high-cognitive-demand problems in the Finnish textbooks were found to include a number of 'PISA-like' items, i.e., problems that used real data associated with day-to-day situations. For instance, one worked example showed a realistic 24-h temperature chart for a famous Finnish tourist region, Utsjoki, and introduced students to how to read the (function) graph: e.g., When was the temperature 0 degrees? As such, Finnish students' high PISA-test performance may be due in part to their familiarity with PISA-style problems.

Zhu and Fan (2006) and Yang and Sianturi (2022) have indicated that textbooks from Western countries have higher total numbers of problems than those from East Asian ones. Research has also shown that a student's frequency of exposure to problems can substantially influence their learning and, in turn, their problem-solving performance (Stigler et al. 1982). Nevertheless, it should be borne in mind that textbook data on the total number of problems may underrepresent the actual problem-exposure frequencies of Taiwanese and Singaporean students, who usually receive tutoring or attend additional classes after school (Wei and Eisenhart, 2011).

Comparing the results from a cultural perspective. Cultural perspectives may explain the differences observed in the representation forms and cognitive demands of problems in mathematics textbooks.

East Asian countries, such as Singapore and Taiwan, are strongly influenced by Chinese culture, especially the Confucian tradition (Leung et al. 2006). In Confucian cultural values, educational qualifications are highly valued and significantly impact an individual's reputation and career prospects (Jin et al. 2018). Consequently, teachers and parents place great emphasis on students achieving high grades in examinations at all school levels. This emphasis fosters a highly competitive examination environment. As such, it drives students to work hard and to memorize basic mathematical facts and problem-solving procedures to save time for learning advanced topics and for solving challenging problems. For the same reason, many students in Taiwan and Singapore attend cram schools or have private tuition after school (Leung et al. 2006; Wei and Eisenhart, 2011).

In contrast, while Finland also places a high value on education, it differs from Singapore and Taiwan in prioritizing equity and creativity over educational qualifications (Sahlberg, 2007). This is reflected in several policies (Ahonen, 2021; Hemmi et al. 2020): (1) The Finnish government provides free education at all levels, from kindergarten to higher education. (2) Teachers have much autonomy in lesson planning, and their lessons are not necessarily focused on annual tests or national examinations. (3) The teaching approach is learner-centered rather than teacher-centered, with special attention given to supporting students who are falling behind in mathematics. (4) Creative or innovative teaching methods are encouraged.

From a Confucian perspective, it is reasonable that Singaporean and Taiwanese textbooks included more challenging problems than Finnish textbooks did. Regarding how problems are represented, Leung et al. (2006) note that Chinese culture emphasizes abstract thinking, while Western culture tends to emphasize intuitive thinking. This distinction may be due to the emphasis in Chinese culture on problem-solving skills, such as calculation and algebraic manipulation, in contrast to the emphasis in Western culture on hands-on activities. As a result, Finnish textbooks included more problems in visual or combined forms, while Singaporean and Taiwanese textbooks favored problems in a purely mathematical form.

Finally, the results showed that Finnish textbooks contain more problems than those used in Singapore and Taiwan. This may be because Finnish teachers have greater flexibility in using textbooks and are not required to cover all the problems found in them. In contrast, teachers in Singapore and Taiwan must closely follow their textbooks, as these form the basis for examinations, making it necessary to include fewer problems.

Comparing the results from national mathematics curricula. The differences observed in the textbooks of Finland, Singapore, and Taiwan can also be partially attributed to their respective national mathematics curricula. In Finland, the curriculum placed slightly more emphasis on drawing and investigating function graphs, which is reflected in the Finnish textbooks' focus on graph-related content. In contrast, the curricula in Singapore and Taiwan emphasized the operations of functions, such as evaluating functions or converting between different forms of functions. This focus likely contributes to the prevalence of problems presented in purely mathematical form in the Singaporean and Taiwanese textbooks.

However, some discrepancies between the curricula and the textbooks were noted. For instance, although the Singaporean curriculum mentioned *quadratic functions*, the textbooks primarily focused on *quadratic equations* rather than functions. In Finland, while the curriculum did not explicitly mention the use of *function machines*, this concept was prominently featured in Finnish textbooks. Regarding Taiwanese textbooks, despite the curriculum's emphasis on mathematics literacy, which aligns with the framework of the PISA, there were actually fewer PISA-like problems in Taiwanese textbooks compared to the Finnish ones.

Conclusion

In summary, approaches to teaching functions differed markedly across typical middle-school mathematics textbooks from Finland, Singapore, and Taiwan. The Finnish textbooks placed greater emphasis on visual understanding, while the two East Asian countries' textbooks were more focused on abstract understanding. Here, visual understanding implies using more visual representations and emphasizing the dynamic input-output process (e.g., via function machines), while abstract understanding refers to using more verbal or symbolic representations and emphasizing the abstract level of thinking about functions as a concept (e.g., using more algebraic representations). Wang et al. (2015) proposed that the East and the West hold different perspectives on understanding: with the former associating it with abstract thinking, usually not facilitated by visual representations; whereas in the West, visual representations are an indispensable part of understanding. This dichotomy seems to be supported by the fact that the sampled Finnish textbooks emphasized various visual representations (e.g., function machines and graphs) and the connections between them and included many function-machine-related problems, whereas the Singaporean and Taiwanese textbooks used more algebraic expressions and more high-cognitive-demand problems. However, as all three countries are high performers in the sphere of middle-school mathematics, it should also be clear that the pathway to students' success at learning functions is not unitary or fixed.

Finally, despite prior research findings that technology such as graphing calculators and dynamic geometry software can positively affect students' learning of functions (Senk et al. 2014), we did not find any function-related content in any of the three countries' textbooks that were specifically designed for use with such technology. We, therefore, recommend that publishers consider the availability and affordances of such technology when determining what problems to include in their textbooks.

Implications for textbook design. The results of this study suggest that textbook design plays a significant role in helping students learn functions, as shown by these three countries' high performance in the PISA. We hypothesize that the Finnish textbook design may be more suitable for students with lower mathematics achievement or motivation, as the content is more accessible. The Finnish textbooks offered more visual representations and emphasized the use of function machines, making the material more approachable. Moreover, despite containing a greater number of problems, the Finnish textbooks had fewer high cognitive-demand problems.

On the other hand, the textbook designs used in Singapore and Taiwan may be particularly beneficial for high-achieving mathematics students, as they help develop abstract mathematical thinking. These textbooks presented more challenging problems and emphasized the abstract structure of functions, potentially providing a stronger foundation for advanced mathematics learning at the high school level[1].

Data availability

All data generated or analyzed during this study are included in this published article and its supplementary information file.

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Note

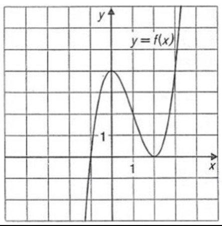
1 An earlier version of this paper was presented at the International Conference on Mathematics Textbook Research and Development (ICMT2014), University of Southampton, UK, 29–31 July, 2014

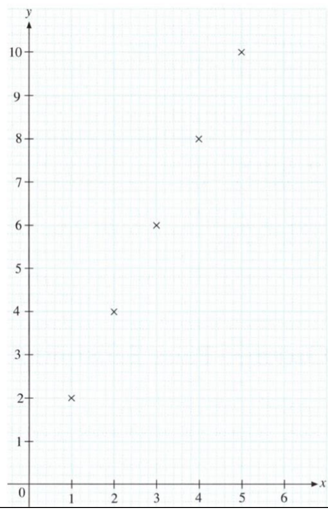
Appendix A

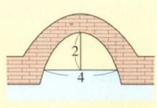
Table 9

Table 9 Sample coding results.

<table>
<tr><th></th><th>Problems</th><th>Coding Results</th></tr>
<tr><td>1</td><td>In the function $y = 20 - 4x$, calculate the value of y when
(i) $x = 3$, (ii) $x = 8$.

(The and Loh, 2011, Grade 9, p.274)</td><td>Purely mathematics
Close-ended
Non-application
Procedures without connections</td></tr>
<tr><td>2</td><td>

Read the function f by
(a) f(0),
(b) f(1),
(c) f(3),
(d) zeros of the function
(WSOY, Grade 9, 2009, p.67)</td><td>Visual
Close-ended
Non-application
Procedures with connections</td></tr>
<tr><td>3</td><td>The table below shows five pairs of values of x and y that satisfy the function y=2x
<table>
<tr><td>x</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td></tr>
<tr><td>y = 2x</td><td>2</td><td>4</td><td>6</td><td>8</td><td>10</td></tr>
</table>
The five pairs of numbers (x, y) in the table above are plotted as points in the Cartesian plane, as shown in Fig. 12.4 below. The scale for both the x- and y-axes is 1 cm to 1 unit.


Do you notice any pattern formed by the points? How does the value of y change as the value of x increases by 1?
Do all the values of y change accordingly as the values of x increase?
(The and Loh, vol 1, 2011, pp.274-275)</td><td>Visual
Open-ended
Non-application
Procedural with connection</td></tr>
<tr><td>4</td><td>有一個拋物線形的拱橋，當這座拱橋下的水面離拱頂 2 公尺時，水面寬 4 公尺，如右圖。當水位下降 1 公尺時，水面寬度為多少公尺？


There is a bridge. The arch of the bridge is a parabola. The width of the river is 4 meters, and the height is 2 meters. When the water levels drop 1 meter, what is the width of the river? (KH, 2012, vol 6, p.52)</td><td>Connection (visual and verbal)
Close-ended
Application
Doing mathematics</td></tr>
<tr><td>5</td><td>Mikä merkinnässä $f(4) = 17$ on
a) muuttujan arvo
b) funktion arvo? (WSOY, Grade 9, 2009, p.65)

f(4)=17, what is its a) input value b) output value?</td><td>Purely mathematics
Close-ended
Non-application
Memorization</td></tr>
</table>

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