

## Holographic cone of average entropies

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The holographic entropy cone identifies entanglement entropies of field theory regions, which are consistent with representing semiclassical spacetimes under gauge/gravity (holographic) duality. It is currently known up to five regions. Here we point out that average entropies of  $p$ -partite subsystems can be similarly analyzed for arbitrarily many regions. We conjecture that the holographic cone of average entropies is simplicial and specify all its bounding inequalities and extreme rays, which combine features of perfect tensor and bipartite entanglement. Heuristically, the conjecture posits that bipartite entanglement achieves the most efficient purification consistent with a holographic spacetime interpretation. We also explain that the extreme forms of entanglement allowed by our conjecture are realized by evaporating black holes.

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One of the main approaches to quantum gravity is gauge/gravity duality or AdS/CFT correspondence<sup>1</sup>. It posits that certain nongravitational systems (conformal field theories, CFT) in  $d$  dimensions provide a dual description of theories of gravity with anti-de Sitter (AdS) boundary conditions in  $d + 1$  dimensions. Perhaps the deepest insight into the fundamental nature of gravity ushered by the AdS/CFT correspondence is its intimate connection to quantum information theory. Succinctly put, gravitational spacetimes function like maps of quantum entanglement in the dual field theory<sup>2,3</sup>. On the other hand, most quantum systems do not have a dual gravitational description; only certain special systems are holographic in that sense. This motivates a question at the intersection of information theory and gravity<sup>4</sup>: What necessary conditions must a quantum state satisfy if its quantum entanglement is to be holographically represented as a gravitational spacetime?

A technical intermediary between information theory and gravity is the Ryu-Takayanagi proposal<sup>5–7</sup>, which is a holographic generalization of the Bekenstein-Hawking formula for black hole entropy<sup>8,9</sup>. The proposal asserts that the von Neumann entropy of the reduced state of a CFT subregion is represented in the bulk AdS geometry as the area of the smallest extremal codimension-2 surface, which is homologous to the said boundary region. A fact in geometry is that areas of minimal surfaces homologous to fixed boundary regions automatically satisfy certain inequalities, for example<sup>10</sup>:

$$S_{AB} + S_{BC} + S_{CA} - S_A - S_B - S_C - S_{ABC} \geq 0 \quad (1)$$

(Here,  $AB$  denotes the union of disjoint boundary regions  $A$  and  $B$ ). The Ryu-Takayanagi proposal and the existence of a semiclassical bulk dual demand that the same inequalities be satisfied by entanglement entropies of CFT subregions. To be sure, states violating inequality (1) and other holographic inequalities do exist in quantum mechanics, but they cannot be consistently represented as areas of surfaces stipulated in the Ryu-Takayanagi proposal.

Entropies of subregions, which are consistent with a semiclassical bulk interpretation, comprise the holographic entropy cone<sup>4</sup>. An explanation of this terminology is that linear inequalities such as (1) are saturated on hyperplanes in entropy space—the vector space of hypothetical assignments of numbers (entropies) to regions. As a collection of hyperplanes, holographic inequalities bound a convex cone. The holographic entropy cone is currently known for up to five regions<sup>11</sup>. We also know one infinite family of inequalities for arbitrarily many regions<sup>4</sup> and several properties, which all holographic inequalities (known and unknown) must possess<sup>12–14</sup>.

Yet progress in studies of the holographic entropy cone has been more quantitative than qualitative. With a few exceptions—inequality (1) as a count of perfect 4-tensor entanglement<sup>15</sup>, and the infinite series of inequalities from ref. <sup>4</sup> as discrete analogues of differential entropy<sup>16</sup>—much of what we know about the holographic entropy cone is still waiting to be deciphered in heuristic terms. The difficulty in interpretation is partly due to a lack of data. If we knew the inequalities for more than five regions, we would presumably discern some patterns, which could then be examined for qualitative insights.

This paper makes a step in this direction. We identify a sector of the holographic entropy cone, which is exactly solvable for arbitrarily many CFT regions. The identification is conjectural: we hypothesize the exact form of this sector, including all bounding inequalities and extreme rays, but do not give proof. As a qualitative insight, our conjecture characterizes the most efficient purification consistent with a semiclassical bulk dual as coming from bipartite entanglement; see inequality (16) for a

technical statement. Extreme rays achieve the most efficient purification on higher-partite subsystems but maximally violate it on lower-partite subsystems. This characterization of extreme rays implies that they describe stages of evaporation of old black holes.

## Results

**Holographic permutation invariants.** Our exactly solvable sector involves permutation invariants. The permutations in question simply relabel regions, for example,  $A \rightarrow B \rightarrow C \rightarrow A$ . When we have  $N$  named regions, the relevant permutation group is  $S_{N+1}$ . This is because every  $N$ -partite mixed state can be purified with the addition of an  $(N + 1)$ st system, whose label  $O$  can also be switched with  $A, B$ , etc. We remark that switching a named region  $A$  with the purifier  $O$  generally rewrites inequalities in a nontrivial way. For example, for subadditivity—an inequality which holds true even outside holography, by virtue of quantum mechanics alone<sup>17</sup>—we have in the  $(N = 3)$ -region context:

$$S_A + S_B - S_{AB} \geq 0 \xrightarrow{A \rightarrow O = \overline{ABC}} S_{ABC} + S_B - S_{AC} \geq 0 \quad (2)$$

On the other hand, the monogamy of mutual information—inequality (1)—is  $S_4$ -invariant<sup>15</sup>.

Entropies of  $N$ -partite states contain  $N/2$  or  $(N + 1)/2$   $S_{N+1}$ -invariants, whichever is an integer. (We denote this number  $\lfloor (N + 1)/2 \rfloor$ , where  $\lfloor \dots \rfloor$  is the floor function.) The claim becomes obvious when we write the invariants explicitly. They are average entropies of  $p$ -component regions, henceforth denoted  $S^p$ , with  $1 \leq p \leq \lfloor (N + 1)/2 \rfloor$ . For example, in the  $(N = 3)$ -region context, we have:

$$S^1 = \frac{1}{4}(S_A + S_B + S_C + S_O) \quad (3)$$

$$S^2 = \frac{1}{6}(S_{AB} + S_{AC} + S_{AO} + S_{BC} + S_{BO} + S_{CO}) \quad (4)$$

Note that, if we avoid using the purifier explicitly in formulas,  $S^p$  contains entropies of complementary  $(N + 1 - p)$ -partite regions. For example, (3)–(4) are equivalent to:

$$S^1 = \frac{1}{4}(S_A + S_B + S_C + S_{ABC}) \quad (5)$$

$$S^2 = \frac{1}{3}(S_{AB} + S_{AC} + S_{BC}) \quad (6)$$

The equivalence between  $p$ -partite and  $(N + 1 - p)$ -partite regions' entropies explains why the count of permutation invariants only goes up to  $\lfloor (N + 1)/2 \rfloor$ .

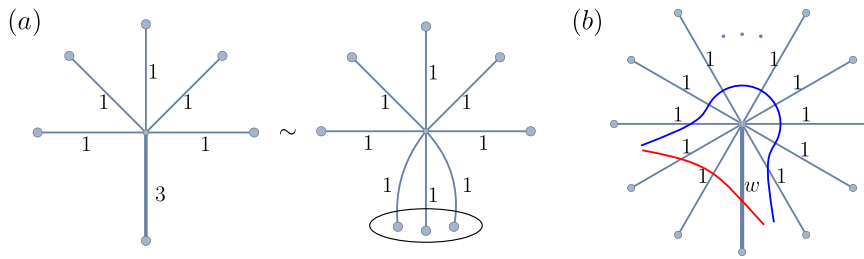
**Inequalities on average entropies:** A typical holographic inequality concerns more than just permutation invariants. But it is easy to convert it to a statement about averages: we simply replace every  $p$ - and  $(N + 1 - p)$ -component term in the inequality with  $S^p$ . Thus, for subadditivity (inequality (2)) and monogamy (inequality (1)) we have:

$$2S^1 - S^2 \geq 0 \quad (7)$$

$$-4S^1 + 3S^2 \geq 0 \quad (8)$$

To confirm the validity of this substitution, observe that if an entropy vector obeys an inequality, then so do all of its  $S_{N+1}$ -images. Replacing a  $p$ -component entropy with  $S^p$  applies the original inequality to the average of all permutation images.

**Holographic cone of average entropies.** While all holographic inequalities—known and unknown—carve out the full holographic entropy cone, the permutation-invariant inequalities



**Fig. 1 Flower graphs are conjectured extremal rays of the cone of average entropies.** **a** A flower graph with  $N$  ‘petals’ of weight 1 and one ‘stem’ of weight  $w$  (here  $w = 3$ ) can be obtained from a perfect tensor graph with  $N + w$  legs by bundling up  $w$  legs. **b** The entropy of each  $p$ -partite region in a flower graph is the smaller of two cuts (red and blue), which sever  $p$  or  $N + 1 - p$  legs.

carve out a smaller cone: the cone of average entropies. This cone is a projection of the full holographic entropy cone to the subspace of entropy space spanned by  $S^p$ . Its bounding inequalities, such as inequalities (7), (8), constrain the ratios of average  $p$ -partite entropies. For  $N$  named regions, the cone of averages lives in an  $\lfloor (N + 1)/2 \rfloor$ -dimensional linear space.

Conjecture: The holographic cone of average entropies, for any number  $N$  of regions, is as follows:

- The cone is simplicial. That is, it has  $\lfloor (N + 1)/2 \rfloor$  bounding facets (maximally tight inequalities) and  $\lfloor (N + 1)/2 \rfloor$  extreme rays—loci where  $\lfloor (N - 1)/2 \rfloor$  inequalities are saturated.
- The extreme rays are obtained by minimal cuts on weighted graphs shown in Fig. 1. We call them *flowers* with one *stem* of weight  $w$  and  $N$  *petals* of weight 1. The weights  $w$ , which realize extreme rays, are all odd (or even) positive integers up to and including  $N$ .
- The bounding facets of the cone are generated by subadditivity ( $F_1 \equiv S^1 - S^2/2 \geq 0$ ) and inequalities indexed by  $2 \leq p \leq \lfloor (N + 1)/2 \rfloor$ :

$$F_p \equiv \frac{2S^p}{p} - \frac{S^{p-1}}{p-1} - \frac{S^{p+1}}{p+1} \geq 0 \tag{9}$$

$F_2 \geq 0$  symmetrizes monogamy (inequality (1)) while  $F_3 \geq 0$  symmetrizes known inequalities from the  $N = 5$  holographic entropy cone<sup>11</sup>; the inequalities  $F_{>3} \geq 0$  are novel. In Supplementary Note 1, we verify that the conjectured extreme rays and facets are complete and consistent.

Inequality (9) requires a minor clarification. When  $p = \lfloor (N + 1)/2 \rfloor$ ,  $(p + 1)$ -composite regions are related by  $S_{N+1}$  permutations to  $(N - p)$ -composite regions. (We saw this in equations (5), (6)). We may state this fact as an equivalence between quantities  $S^p$ :

$$S^{(N+2)/2} \equiv S^{N/2} \quad (N \text{ even}) \tag{10}$$

$$S^{(N+3)/2} \equiv S^{(N-1)/2} \quad (N \text{ odd}) \tag{11}$$

Thus, in the special case of  $p = \lfloor (N + 1)/2 \rfloor$ , inequality (9) takes superficially modified forms:

$$(N + 3)(N - 1)S^{(N+1)/2} - (N + 1)^2 S^{(N-1)/2} \geq 0 \quad (N \text{ odd}) \tag{12}$$

$$(N + 4)(N - 2)S^{N/2} - N(N + 2)S^{(N-2)/2} \geq 0 \quad (N \text{ even}) \tag{13}$$

In Fig. 2 and Supplementary Note 2 we show the cones of average entropies up to  $N = 6$  named regions. They conform with our conjecture. We also display known inequalities and rays pertaining to the  $N = 7$  cone of averages. There is a small region

in the space of averages not excluded by previously known inequalities, for which we have been unable to find a consistent assignment of entropies (Following ref. 4, every entropy vector in the holographic entropy cone should be realizable by minimal cuts on a weighted graph). Inequality (12) with  $N = 7$  eliminates that questionable region. These remarks establish that our conjecture is consistent with known facts.

**Discussion**

To elucidate the meaning of our conjecture, we consider conditional entropy  $S(Y|X) = S_{YX} - S_X$ . It characterizes region  $Y$ 's contribution to purifying  $X$ . Conditional entropy equals  $S_Y$ —the maximum allowed by subadditivity—when  $Y$  does not purify  $X$  at all. To minimize  $S(Y|X)$  is to find a  $Y$ , which best purifies  $X$ .

The average conditional entropy of one region, conditioned on a  $p$ -partite system, is  $S^{p+1} - S^p$ . This quantity captures how much, on average, adjoining a single region purifies a  $p$ -partite system. Using strong subadditivity<sup>18</sup>, which holds generally in quantum mechanics, one can prove that it can never exceed  $S^p/p$ , or equivalently:

$$\frac{S^{p+1}}{S^p} \leq \frac{p + 1}{p} \tag{14}$$

In Supplementary Note 3 we prove inequality (14). We further show that it saturates if and only if all regions  $X$  and  $Y$ , which together cover  $p + 1$  or fewer basic constituents, have zero mutual information—that is, when they do not help to purify one another.

Now observe that inequalities (9) also imply inequality (14):

$$\frac{S^p}{p} - \frac{S^{p+1}}{p + 1} = \sum_{p'=1}^p F_{p'} \tag{15}$$

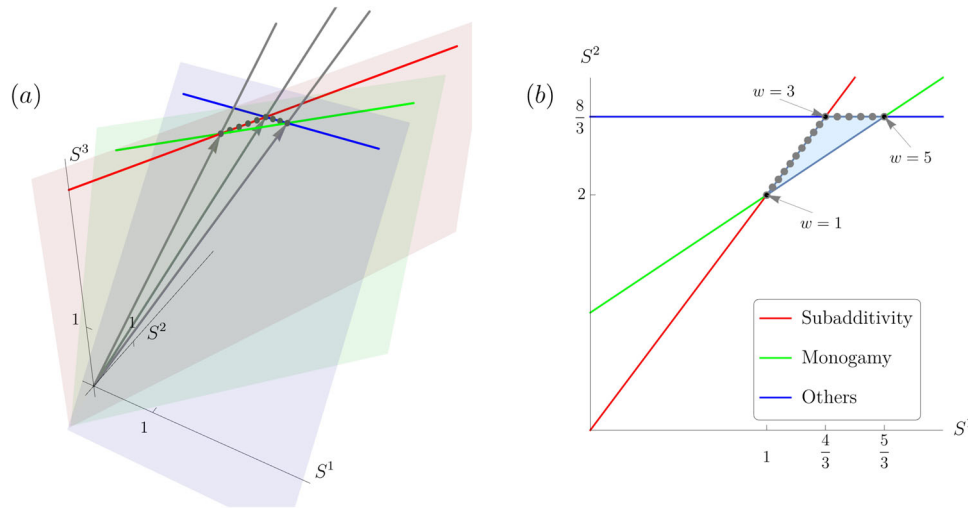
By itself, this fact does not characterize our conjecture because—we stress—inequality (14) follows from strong subadditivity alone. It does, however, set an illuminating counterpoint to the holographic lower bound on  $S^{p+1} - S^p$ :

$$\frac{S^{p+1}}{S^p} \geq \frac{p + 1}{p} \cdot \frac{N - p}{N - p + 1} \tag{16}$$

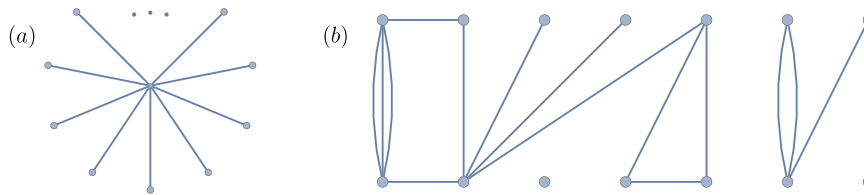
Bound (16) also follows from inequalities (9):

$$\frac{S^{p+1}}{(p + 1)(N - p)} - \frac{S^p}{p(N - p + 1)} \tag{N odd} \\ \propto \frac{N + 3}{4} F_{\lfloor (N+1)/2 \rfloor} + \sum_{p'=p+1}^{\lfloor (N-1)/2 \rfloor} (N + 1 - p') F_{p'} \tag{17}$$

(For even  $N$  the coefficient of  $F_{N/2}$  is  $(N + 2)/2$ ). But unlike bound (14), bound (16) cannot be derived from previously known entropic inequalities, general or holographic.



**Fig. 2 The  $N = 5$  holographic cone of average entropies.** **a** The cone is bound by three planes in three dimensions, which originate from subadditivity (red), monogamy (green), and other known inequalities (blue). **b** The  $S^3 = 3$  cross-section of the cone, with the same color scheme.



**Fig. 3 Graph models of perfect tensor and bipartite entanglement.** Minimal cuts through these graphs are identified with entropies of subregions. **a** Perfect tensor ‘PT’, equation (18). **b** Bipartite entanglement ‘EPR’, equation (19).

Owing to rewritings (15) and (17), bounds (14) and (16) enjoy many parallels. Whereas saturating bound (14) at  $p$  implies saturation of the same bound for all  $p' < p$ , saturating bound (16) at  $p$  implies saturation of the same bound for all  $p' > p$ . (The latter statement holds if and only if our conjecture does.) Whereas saturating bound (14) describes when adjoining one extra region is least helpful in purifying  $p$ -partite systems, saturating bound (16) describes when it is most helpful. In the form of bound (16), our conjecture describes the most efficient rate of purifying  $p$ -partite systems (by the addition of one constituent), which is consistent with a semiclassical bulk dual.

The fact that bounds (14) and (16) can only be saturated on entire ranges of  $p$  is significant. We shall see momentarily that it is responsible for the simplicial character of the holographic cone of average entropies. As a preliminary, we define two special patterns of entanglement, where (14) and (16) are saturated for all  $1 \leq p \leq \lfloor (N + 1)/2 \rfloor$ :

$$\text{PT} : S^p \propto p \quad \text{for } p \leq \lfloor (N + 1)/2 \rfloor \quad (18)$$

$$\text{EPR} : S^p \propto p(N + 1 - p) \quad \text{for all } p \quad (19)$$

(We explain the nomenclature ‘PT’ and ‘EPR’ momentarily.) Note that both patterns saturate inequalities (9). Our conjecture thus distinguishes (18), (19) as two extreme entanglement patterns, which are uniform over  $p$ .

The label ‘PT’ stands for perfect tensor entanglement. It is known to play an important role in holography; see for example<sup>15,19,20</sup>. In the holographic entropy cone, even-membered perfect tensor states are extreme rays and form a complete basis for entropy space<sup>13</sup>. We illustrate them as weighted graphs in Fig. 3a. PT is the pattern of least efficient purification.

The label ‘EPR’ stands for bipartite entanglement. The acronym follows common nomenclature, which refers to ref. <sup>21</sup>. Here is why  $S^p \propto p(N + 1 - p)$  describes EPR-like entanglement. A handy way to track bipartite entanglement is by drawing lines that connect pairs of regions; see Fig. 3b. Such lines have been studied extensively in the literature following ref. <sup>22</sup> and are known as bit threads. Suppose the average number of bit threads between any two distinct regions is  $K$ . The EPR-like assumption on the entanglement structure means that a  $p$ -partite entropy equals the number of threads, which connect the  $p$  constituents of the region with the  $N + 1 - p$  constituents of the complement. For the average, this gives  $S^p = Kp(N + 1 - p)$ .

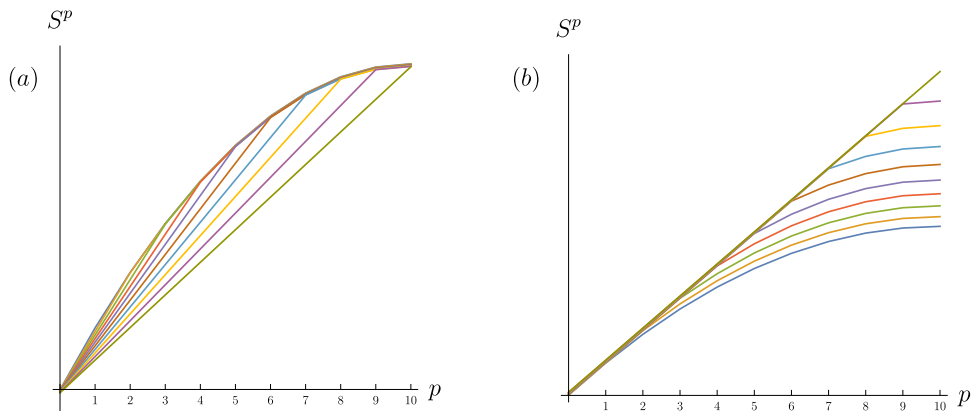
Armed with definition (19), we restate our conjecture in one final way, as advertised in the Introduction:

**Conjecture:** The most efficient rate of purifying mixed multipartite states, which is consistent with a semiclassical bulk dual, is set by EPR-like entanglement.

A transparent way of presenting the average entropies  $S^p$  is to plot them over the  $p$ -axis. We display such plots in Fig. 4. In a general state with a semiclassical bulk dual, the slope of this plot at any  $p$  is bounded above and below by inequalities (14) and (16). Extreme plots are those, which saturate either the lower or the upper bound at each  $p$ . Because extremizing (14) always happens over a range  $1 \leq p \leq \tilde{p}$  and extremizing (16) always happens over a range  $\tilde{p} \leq p \leq \lfloor (N + 1)/2 \rfloor$ , such extreme plots are uniquely specified by a single threshold parameter  $\tilde{p}$ :

$$S^p \propto \begin{cases} p(N + 1 - \tilde{p}) & p \leq \tilde{p} \\ p(N + 1 - p) & p \geq \tilde{p} \end{cases} \quad (20)$$

In other words, an extreme vector of the cone of average entropies must be PT-like for  $p \leq \tilde{p}$  and EPR-like for  $p \geq \tilde{p}$ . We verify in



**Fig. 4 Average entropies of flower graphs—example.** We take  $N = 19$  as an example. There are  $\lfloor (19 + 1)/2 \rfloor = 10$  extremal graphs; their plots of  $S^p$  versus  $p$  are shown in distinct colors. Each plot is characterized by a threshold value  $\tilde{p}$ , which separates the linear and quadratic dependence of  $S^p$  in equation (20). Because the overall normalization of  $S^p$  is tunable, we display  $S^p$  of flower graphs **a** normalized to fix  $S^{\lfloor (N+1)/2 \rfloor} = S^0$  and **b** normalized to fix  $S^1$ .

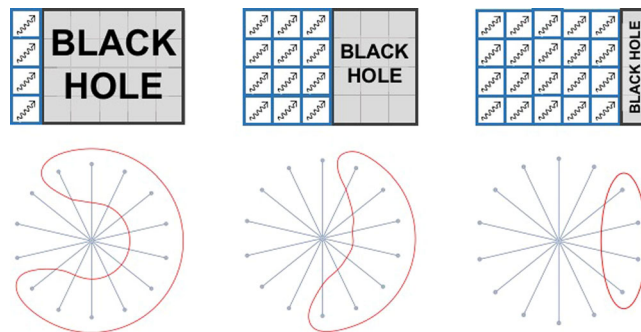
Supplementary Note 1 that this is exactly the dependence of  $S^p$  in the flower graphs of Fig. 1, with the threshold related to stem weight  $w$  via  $\tilde{p} = (N + 2 - w)/2$ . The holographic cone of average entropies is simplifical because there are precisely  $\lfloor (N + 1)/2 \rfloor$  possible cross-overs  $\tilde{p}$  to separate the PT- and EPR-like behaviors. The actual EPR pair (respectively, perfect tensor) is recovered by setting  $\tilde{p} = 1$  (respectively  $\tilde{p} = \lfloor (N + 1)/2 \rfloor$ ).

This description of flower graphs adds a heuristic justification to our conjecture. We claim that bound (16) identifies the most efficient way of purifying  $p$ -partite states. We know that bound (16) is saturated by EPR pairs. We can still saturate bound (16) yet depart from EPR-like entanglement (equation (19)) by using  $p$ -local operations to suppress any  $p$ -local mutual information. We posit that such  $p$ -local operations do not help in purifying  $p$ -partite systems.

We would like to situate the holographic cone of average entropies in a broader physical context. As the extreme rays are organized by  $p$ -dependence of mutual information, we may anticipate that the cone will be relevant to information scrambling<sup>23</sup> (dilution of mutual information) and to radial depth (because  $p$  captures a degree of nonlocality in the CFT). These heuristics suggest inspecting the cone from the viewpoint of black hole physics. And indeed, it turns out that the extreme rays of our cone correspond precisely to stages of unitary evaporation of old black holes, as described by Page<sup>24,25</sup> and more recently in the islands proposal<sup>26–28</sup>.

To explain this connection, we need one fact about flower graphs: that they can be obtained by grouping together  $w$  constituents of a perfect tensor state on  $N + w$  parties (We thank Xiaoliang Qi for this insightful observation.). This is illustrated in Fig. 1a. We argue that the same type of grouping is an essential aspect of black hole evaporation à la Page; see Fig. 5.

The argument in ref. <sup>24</sup> considers a globally pure state of the combined black hole (BH) plus radiation (R) system. By unitarity, the global state remains pure at all times. Because the black hole evaporates, BH makes up a decreasing proportion of the total Hilbert space. Assuming the global state is nearly random, Page’s theorem<sup>29</sup> tells us that the entropy of the BH system is the size of BH or the size of R, whichever is smaller. This yields the famous Page curve. Now, a random state in a large Hilbert space has nearly perfect tensor entanglement<sup>20</sup>. In isolating BH from a random state of the total system, Page effectively groups together  $A_{\text{BH}} \equiv w$  constituents of a perfect tensor, where  $A_{\text{BH}}$  is the horizon area. This ends up constructing flower graphs.



**Fig. 5 Extreme rays versus black hole evaporation.** In ref. <sup>24</sup> Page discusses black hole evaporation as a pure state in a box, divided into two parts: the black hole (whose size decreases in time) and radiation (which is complementary so its size increases in time). By Page’s theorem<sup>29</sup>, the entropy of radiation in a random such state is the smaller of the two sizes, which gives the Page curve. A sliding division of a random multi-partite state into a black hole component (circled red) and radiation components effectively constructs flower graphs. In this figure time flows from left to right.

The petals of the flower are small constituents of a global pure state—that is, particles of Hawking radiation. The idea of isolating Hawking radiation from the boundary CFT is what propelled the islands proposal of<sup>26</sup>; it is also what makes the connection with flower graphs explicit. A transparent way to compare flower graphs with stages of black hole evaporation is to consult the geometrized model of the latter formulated in ref. <sup>30</sup>. That model involves ‘octopus’ diagrams, which are identical to our flower graphs.

Both here and in refs. <sup>24,30</sup> the Hawking particles are assumed uncorrelated and identical. The ‘uncorrelated’ assumption is motivated by the equivalence principle—‘no drama’ in the language of ref. <sup>31</sup>. But the ‘identical’ assumption can be removed. In Fig. 5, the black hole arises from bundling together many constituents of a parent perfect tensor state. We may also consider bundling up Hawking particles into bins of distinct sizes. This process, which further breaks permutation symmetry, can yield extreme rays of the full holographic entropy cone.

Observe that flowers with  $w > N$  have the same entropies as the flower with  $w = N$ . The  $A_{\text{BH}} > N$  regime is when the horizon area exceeds the total Hawking radiation; this is a black hole before

Page time. Thus, flower graphs describe the stages of evaporation of an old black hole.

Finally, we ask: is it surprising that the holographic entropy cone ‘knows about’ black hole evaporation? We think not. Its extreme vectors are supposed to be marginally consistent with a semiclassical bulk interpretation. If not black holes, what other spacetimes might come closest to a breakdown of semiclassical gravity?

Reference<sup>4</sup> identified a mechanical technique for proving candidate inequalities, which involves constructing so-called contraction maps. We have used two implementations of a greedy algorithm that searches for contractions<sup>32,33</sup>. Both of them overloaded our standard Mac computers before proving or disproving the simplest inequality we conjecture: inequality (9) with  $p = 4$ . It may be more realistic to identify valid non-symmetric inequalities, which reduce inequality (9) under symmetrization and try to prove them by contraction. For example, the  $p = 4$  instance of inequality (9) predicts that some inequality

$$15 \text{ terms } S_{RRRR} \geq 10 \text{ terms } S_{RRR} + 6 \text{ terms } S_{RRRRR}$$

should be valid. The prediction that similar inequalities exist for every instance of formula (9) is a nontrivial corollary of our conjecture, which may aid ongoing and future efforts to describe the full holographic entropy cone.

A simplifying observation is that inequalities (9) need not be proven separately for each  $N$ . It suffices to prove them at the lowest  $N$  that makes sense,  $N = 2p - 1$ , where they take the simplified form of inequality (12). For all higher  $N'$ , inequality (9) follows from inequality (12) by symmetrization over  $S_{N'+1}$ .

After an earlier version of this paper was announced on the arXiv preprint server on 1 December 2021, paper<sup>34</sup> appeared. That paper overlaps with ours in conjecturing inequalities (9). In addition, it contrasts the holographic cone of average entropies with the quantum cone of average entropies. However, it does not exhibit bound (16), does not interpret the conjecture in terms of EPR-like entanglement, and does not relate the conjecture to black hole evaporation.

## Data availability

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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## References

- Maldacena, J. M. The large  $N$  limit of superconformal field theories and supergravity. *Adv. Theor. Math. Phys.* **2**, 231–252 (1998).
- Van Raamsdonk, M. Building up spacetime with quantum entanglement. *Gen. Rel. Grav.* **42**, 2323–2329 (2010).
- Maldacena, J. & Susskind, L. Cool horizons for entangled black holes. *Fortsch. Phys.* **61**, 781–811 (2013).
- Bao, N. et al. The holographic entropy cone. *JHEP* **09**, 130 (2015).
- Ryu, S. & Takayanagi, T. Holographic derivation of entanglement entropy from AdS/CFT. *Phys. Rev. Lett.* **96**, 181602 (2006).
- Ryu, S. & Takayanagi, T. Aspects of holographic entanglement entropy. *JHEP* **08**, 045 (2006).
- Hubeny, V. E., Rangamani, M. & Takayanagi, T. A Covariant holographic entanglement entropy proposal. *JHEP* **07**, 062 (2007).
- Bekenstein, J. D. Black holes and entropy. *Phys. Rev. D* **7**, 2333–2346 (1973).
- Hawking, S. W. Black hole explosions. *Nature* **248**, 30–31 (1974).
- Hayden, P., Headrick, M. & Maloney, A. Holographic mutual information is monogamous. *Phys. Rev. D* **87**, 046003 (2013).
- Hernández Cuenca, S. Holographic entropy cone for five regions. *Phys. Rev. D* **100**, 2 (2019).
- Hubeny, V. E., Rangamani, M. & Rota, M. The holographic entropy arrangement. *Fortsch. Phys.* **67**, 1900011 (2019).
- He, T., Headrick, M. & Hubeny, V. E. Holographic entropy relations repackaged. *JHEP* **10**, 118 (2019).
- He, T., Hubeny, V. E. & Rangamani, M. Superbalance of holographic entropy inequalities. *JHEP* **07**, 245 (2020).
- Cui, S. X. et al. Bit threads and holographic monogamy. *Commun. Math. Phys.* **376**, 609–648 (2019).
- Balasubramanian, V., Chowdhury, B. D., Czech, B., de Boer, J. & Heller, M. P. Bulk curves from boundary data in holography. *Phys. Rev. D* **89**, 086004 (2014).
- Araki, H. & Lieb, E. H. Entropy inequalities. *Commun. Math. Phys.* **18**, 160 (1970).
- Lieb, E. H. & Ruskai, M. B. Proof of the strong subadditivity of quantum-mechanical entropy. *J. Math. Phys.* **14**, 1938–1941 (1973).
- Pastawski, F., Yoshida, B., Harlow, D. & Preskill, J. Holographic quantum error-correcting codes: toy models for the bulk/boundary correspondence. *JHEP* **06**, 149 (2015).
- Hayden, P. et al. Holographic duality from random tensor networks. *JHEP* **11**, 009 (2016).
- Einstein, A., Podolsky, B. & Rosen, N. Can quantum-mechanical description of physical reality be considered complete? *Phys. Rev.* **47**, 777–780 (1935).
- Freedman, M. & Headrick, M. Bit threads and holographic entanglement. *Commun. Math. Phys.* **352**, 407–438 (2017).
- Sekino, Y. & Susskind, L. Fast scramblers. *JHEP* **10**, 065 (2008).
- Page, D. N. Information in black hole radiation. *Phys. Rev. Lett.* **71**, 3743–3746 (1993).
- Page, D. N. Time dependence of Hawking radiation entropy. *JCAP* **09**, 028 (2013).
- Penington, G. Entanglement wedge reconstruction and the information paradox. *JHEP* **09**, 002 (2020).
- Almheiri, A., Engelhardt, N., Marolf, D. & Maxfield, H. The entropy of bulk quantum fields and the entanglement wedge of an evaporating black hole. *JHEP* **12**, 063 (2019).
- Almheiri, A., Mahajan, R., Maldacena, J. & Zhao, Y. The Page curve of Hawking radiation from semiclassical geometry. *JHEP* **03**, 149 (2020).
- Page, D. N. Average entropy of a subsystem. *Phys. Rev. Lett.* **71**, 1291–1294 (1993).
- Akers, C., Engelhardt, N. & Harlow, D. Simple holographic models of black hole evaporation. *JHEP* **08**, 032 (2020).
- Almheiri, A., Marolf, D., Polchinski, J. & Sully, J. Black holes: complementarity or firewalls? *JHEP* **02**, 062 (2013).
- Stoica, B. An implementation of the greedy algorithm for finding contraction maps on Mathematica®.
- Walter, M. “The holographic contractor,” program in C++, Copyright 2015.
- Fadel, M. & Hernández-Cuenca, S. Symmetrized holographic entropy cone. *Phys. Rev. D* **105**, 08600 (2022).

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## Author contributions

B.C. identified inequality (9) and the extremal rays, analyzed the significance of the cone in terms of perfect tensor and bipartite entanglement as well as black hole evaporation, and wrote the manuscript. S.S. described the cones of average entropies for  $N \leq 7$ , contributed calculations and prepared the figures.

## Competing interests

The authors declare no competing interests.

## Additional information

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