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Experimental multistable states for small network of coupled pendula

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Chimera states are dynamical patterns emerging in populations of coupled identical oscillators where different groups of oscillators exhibit coexisting synchronous and incoherent behaviors despite homogeneous coupling. Although these states are typically observed in the large ensembles of oscillators, recently it has been shown that so-called weak chimera states may occur in the systems with small numbers of oscillators. Here, we show that similar multistable states demonstrating partial frequency synchronization, can be observed in simple experiments with identical mechanical oscillators, namely pendula. The mathematical model of our experiment shows that the observed multistable states are controlled by elementary dynamical equations, derived from Newton's laws that are ubiquitous in many physical and engineering systems. Our finding suggests that multistable chimera-like states are observable in small networks relevant to various real-world systems.

Chimera states correspond to the spatiotemporal patterns in which synchronized and phase locked oscillators coexist with desynchronized and incoherent ones^{1–25}. These patterns have been reported both in simulations^{1–18,26} and experiments^{19–25} of the large networks of coupled oscillators with a variety of topologies. Recently, Ashwin & Burylko²⁷ defined a *weak chimera state* as one referring to a trajectory in which two or more oscillators are frequency synchronized and one or more oscillators drift in phase and frequency with respect to the synchronized group. It has been found that these states can be observed in small networks as few as 4 phase oscillators (two groups of in-phase and antiphase oscillators)^{27–29}.

Up to now weak chimera states in small networks have been reported in simulation and theory of coupled phase oscillators. Here, we show that similar multistable chimera-like states can be observed experimentally in small networks of more general oscillators. As a proof of concept, we use the network of four coupled externally excited double pendula. Each pendulum is characterized by the coexistence of rotational or oscillatory periodic solutions of different frequencies. We argue that such multistability implies the occurrence of these states and present evidence that they can persist for a positive measure set of coupling strength.

We consider the system of 4 identical coupled double pendula arranged into a cross configuration, as shown in Fig. 1(a). The lower pendula's bobs (marked with symbols II_i , $i = 1, 2, 3, 4$) can rotate or oscillate around their horizontal axes at points D_1, D_2, D_3, D_4 . The displacements of these bobs are given by $\varphi_{iz}(t)$. Lower bobs are connected to the upper bobs by the rotational pivots at D_i . The upper bobs (I_i) can only oscillate around the horizontal axes marked by A_1, A_2, A_3 and A_4 and located on the base III . One of the bob's ends is connected to the base by the rotational pivot at A_i and the second ends are suspended on the springs characterized by the stiffness coefficient k_s . The displacements of upper bobs are given by $\varphi_{ii}(t)$. The upper bobs I_i of length η_1 have mass m_1 and moment of inertia J_1 , while the lower bobs II_i of length η_2 have mass m_2 and moment of inertia J_2 . The detailed geometry is shown in Fig. 1(b). The viscoelastic damping is assumed in the pivots at D_i (with damping coefficient c_c) and A_i (with damping coefficient k_c). The base, mounted on the shaker, is excited in the vertical direction by the kinematic displacement, $y = A \cos \omega t$. The upper pendula's bobs are coupled to the nearest neighbor by the plane springs (with stiffness coefficient α) shown in green. The similar system in which pendula have not been coupled i.e., the system without plane springs has been considered by Strzalko *et al.*³⁰.

The dynamics of the system of Fig. 1(a,b) can be analyzed using the equations of motion (see Methods).

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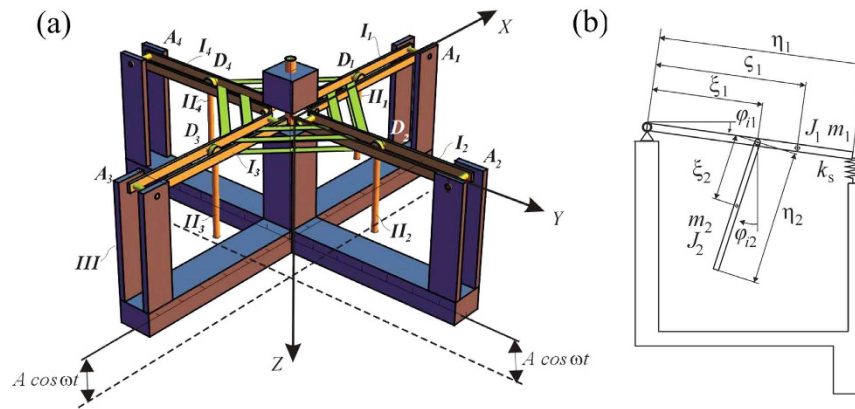


Figure 1. (a) Model of a set of ($N=4$) double pendula located at an oscillating platform, (b) geometry of i -th double pendulum.

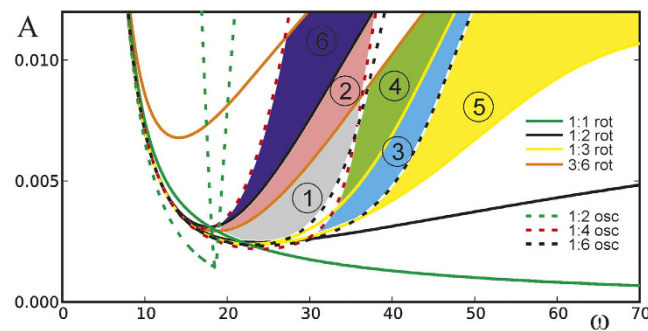


Figure 2. Regions of existence of different types of rotational or oscillatory responses of the uncoupled pendulum in the space of parameters A and ω . In regions 1–6 double pendulum is multi stable with co-existing solutions of different frequencies. In these regions for $\alpha > 0$ the multistable chimera-like states can be observed.

Results

In the absence of coupling (when one removes green planar springs and thus, coupling parameter $\alpha = 0$ in Equation(1) in Methods) it is possible to identify excitation parameters (A and ω) for which each double pendulum exhibits multistability. In Fig. 2 we present regions of existence of various $N:M$, where N is the number of rotation/oscillation of lower pendulum II_{1-4} and M is the number of periods of excitations, eg., 1:1 means that pendula II_{1-4} oscillate or rotate with the frequency of the excitation ω , 2:1 (pendula II_{1-4} oscillate or rotate with the frequency of the excitation $\frac{1}{2}\omega$), etc. One can identify six main regions, indicated from 1 to 6 in Fig. 2, in which the excited double pendulum is multistable. In region 1 three solutions exist: 1:1 rotations (above the green line), 1:4 oscillations (between the dashed red lines) and 1:2 rotations (between solid black lines). Region 2 is characterized by the co-existence of four solutions: 1:1 rotations (above the green line), 1:4 oscillations (between the dashed red lines), 1:2 rotations (between solid black lines) and 3:6 rotations (between solid orange lines). Four solutions are stable also in region 3: 1:1 rotations (above the green line), 1:6 oscillations (between the dashed black lines), 1:2 rotations (between solid black lines), 1:3 rotations (between solid yellow lines). Three solutions: 1:1 rotations (above the green line), 1:6 oscillation (between dashed black lines), 1:2 rotation (between solid black lines) can be observed in region 4. Region 5 is another example of the co-existence of three solutions: 1:1 rotations (above the green line), 1:2 rotations (between solid black lines), 1:3 rotation (between solid yellow lines). Finally in region 6 we observe four solutions: 1:1 rotations (above green line), 1:4 oscillations (between the dashed black lines), 3:6 rotations (between solid orange lines).

In regions 1–6 each of four uncoupled double pendula can exhibit M (equal to 3 or 4) various independent dynamical responses, i.e., 1:1, 2:1 or 3:1 rotational and oscillatory solutions. The set of 4 pendula is characterized by M^4 configurations. One can see that the number of configurations grows exponentially with the number of pendula (i.e., in the case of n pendula we have M^n configurations) so there is spatial chaos³¹ in an uncoupled system. For sufficiently small coupling one can observe multistable chimera-like states which persist over the wide range of system parameters and can be captured experimentally. These states coexist with various cases of complete, phase and cluster synchronous states.

Experimentally observed multistable chimera-like states are illustrated in Fig. 3(a–f). Upper images present general view of the pendula's configurations while lower plots show time series of the lower pendula bobs. The

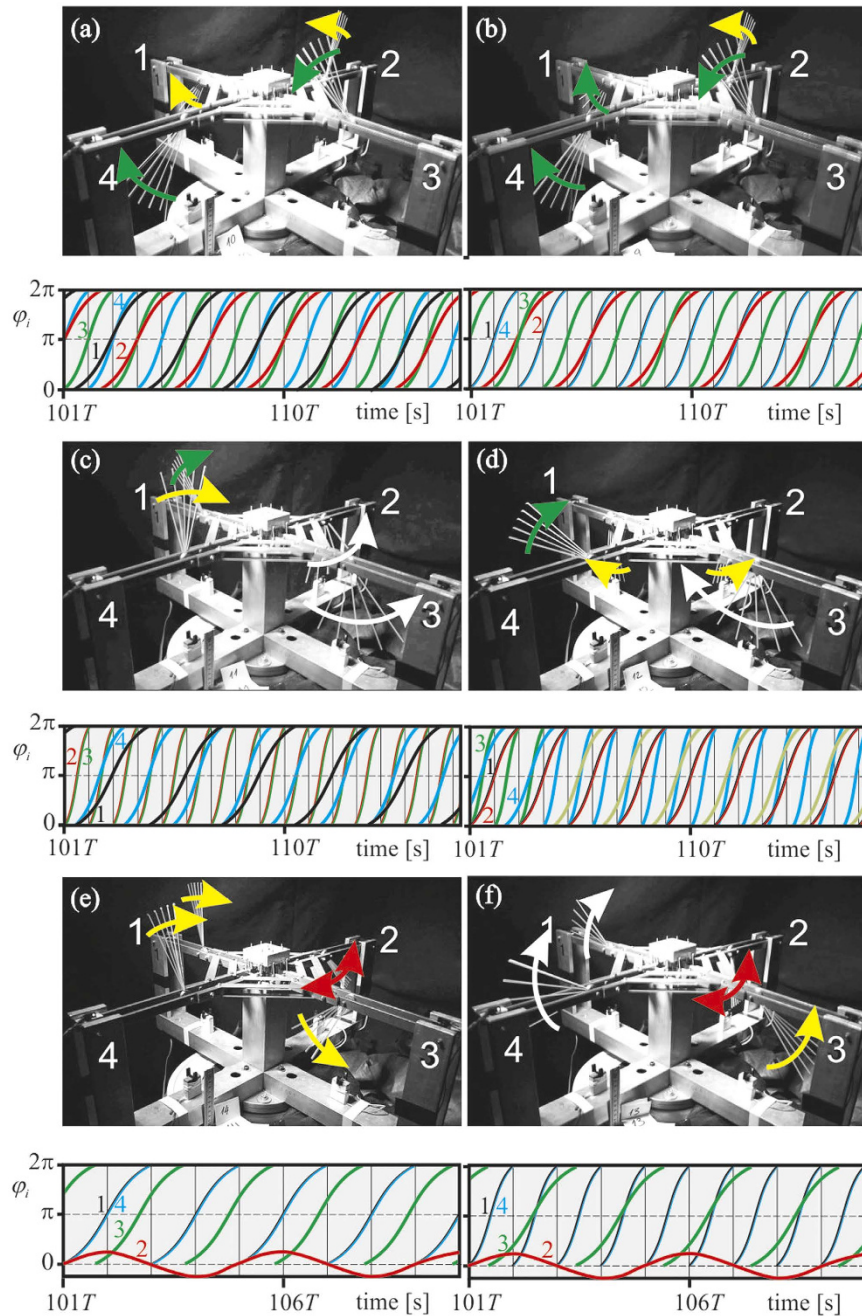


Figure 3. Experimentally observed multistable chimera-like states: **(a–d)** $A = 0.01$ [m], $\omega = 18\pi$ [rad/s] (region 5 of Fig. 2), **(e,f)** $A = 0.005$ [m], $\omega = 10\pi$ [rad/s] (region 1 of Fig. 2); **(a)** pendula 1 and 2 rotate with frequency $\frac{1}{3}\omega$, pendula 3 and 4 with frequency $\frac{1}{2}\omega$, **(b)** pendula 1, 3 and 4 rotate with frequency $\frac{1}{2}\omega$ and pendulum 2 with frequency $\frac{1}{3}\omega$, **(c)** pendulum 1 rotates with frequency $\frac{1}{3}\omega$, pendula 2 and 3 with frequency ω and pendulum 4 with frequency $\frac{1}{3}\omega$, **(d)** pendula 1, 2 and 4 rotate with frequency $\frac{1}{3}\omega$ and pendulum 3 with frequency ω , **(e)** pendula 1, 3 and 4 rotate with frequency $\frac{1}{2}\omega$, pendulum 2 oscillates with frequency $\frac{1}{4}\omega$, **(f)** pendula 1 and 4 rotate with frequency ω , pendulum 2 rotates with frequency $\frac{1}{2}\omega$ and pendulum 3 oscillates with the frequency $\frac{1}{4}\omega$.

figures present a kind of a stroboscope type images of the pendula motion in different cases. All experiments have been recorded using Vision Research Phantom v711 high speed camera. Typical recording speed was 1000 frames per second (fps) and for the purpose of a still photograph visualization a set of 5 of them every fifth frame: $5 \times 0.001 = 0.005$ seconds have been chosen. Then, the images were combined to a single image presenting all chosen images overlaid with the assumed transparency level. The wider area covered by the set of frozen images of each pendulum, the faster speed of its rotation or oscillation and vice versa. In Fig. 3(a–d) we show multistable states in which all the pendula rotate ($A = 0.01$ [m], $\omega = 18\pi$ [rad/s]—region 5 of Fig. 2). In Fig. 3(a) pendula 1 and 2 rotate with frequency $\frac{1}{3}\omega$ and pendula 3 and 4 with frequency $\frac{1}{2}\omega$. Pendula 3 and 4 are in antiphase to each

other (see movie W1). The case in which pendula 1, 3 and 4 rotate with frequency $\frac{1}{2}\omega$ and pendulum 2 with frequency $\frac{1}{3}\omega$ is shown in Fig. 3(b). Pendula 1 and 4 are synchronized in phase and pendulum 3 is in antiphase to pendula 1 and 4 (see movie W2). Configuration of Fig. 3(c) presents the case when pendulum 1 rotates with a frequency $\frac{1}{3}\omega$, pendula 2 and 3 with frequency ω and pendulum 4 with frequency $\frac{1}{2}\omega$. Pendula 2 and 3 are synchronized (see movie W3). Figure 3(d) shows the configuration in which pendula 1, 2 and 4 rotate with frequency $\frac{1}{3}\omega$ and pendulum 3 with frequency ω . Pendula 1 and 2 are synchronized in phase (see movie W4).

In Fig. 3(e,f) we observe multistable states in which the pendula show both rotational and oscillatory behavior ($A = 0.005[\text{m}]$, $\omega = 10\pi [\text{rad/s}]$ —region 1 of Fig. 2). Figure 3(e) shows the chimera-like state in which pendula 1, 3 and 4 rotate with frequency $\frac{1}{2}\omega$ while pendulum 2 oscillates with frequency $\frac{1}{4}\omega$. Pendula 1 and 4 are synchronized in phase (see movie W5). The chimera-like state shown in Fig. 3(f) is characterized by 3 rotating and one oscillating pendula. Pendula 1 and 4 rotate with the frequency ω and are synchronized in phase. Pendula 2 and 3 respectively rotate with frequency $\frac{1}{2}\omega$ and oscillate with frequency $\frac{1}{4}\omega$ (see movie W6).

The presented multistable states coexist with various synchronous states. Movies W7–W9 present the case of the complete synchronization of all pendula in rotational motion (W7), the case when all pendula oscillate with frequency ω and pendula 2, 3, 4 are synchronized in phase and pendulum 1 is in antiphase to them (W8) and the case when all pendula oscillate with the frequency ω and pendula 1, 3 and 2, 4 create two clusters of phase synchronized pendula respectively. These clusters are in antiphase to each other (W9).

In conclusion, we have constructed the simple experimental setup to explore the spatio-temporal dynamics of the small network of the locally coupled pendula. The nodes in the network are externally excited double pendula. Despite a small number of nodes, namely 4, we observe the formation of spatio-temporal patterns of multistable chimera-like states. This behavior is observed experimentally, confirmed in numerical simulations, persistent over a positive measure set of system parameters and seems to be characteristic for the small networks of coupled multistable general oscillators relevant to various real-world systems.

Methods

The dynamics of the system of coupled pendula shown in Fig. 1(a) is given by:

$$\begin{aligned} & (J_1 + m_1\zeta_1^2 + m_2\xi_1^2)\ddot{\varphi}_{i1} + k_c\dot{\varphi}_{i1} + \frac{1}{2}\eta_1^2k_s \sin 2\varphi_{i1} + m_2\xi_1\xi_2(\ddot{\varphi}_{i2} \sin(\varphi_{i1} - \varphi_{i2}) \\ & - \dot{\varphi}_{i2}^2 \cos(\varphi_{i1} - \varphi_{i2})) - (m_1\zeta_1 + m_2\xi_1)(A\omega^2 \cos \omega t + g)\cos \varphi_{i1} + \alpha\eta_1(\varphi_{i1} - \varphi_{(i-1)1}) \\ & + \alpha\eta_1(\varphi_{i1} - \varphi_{(i+2)i}) = 0, \\ & (J_2 + m_2\xi_2^2)\ddot{\varphi}_{i2} + m_2\xi_2(A\omega^2 \cos \omega t + g)\sin \varphi_{i2} \\ & + m_2\xi_2(\dot{\varphi}_{i1}\xi_1(\cos(\varphi_{i1} - \varphi_{i2}) + \ddot{\varphi}_{i1} \sin(\varphi_{i1} - \varphi_{i2}))) + c_c\dot{\varphi}_{i2} = 0, \end{aligned} \quad (1)$$

where $i = 1, 2, 3, 4$.

Numerical simulations. We used the following parameter values: $J_1 = 4.521 \times 10^{-3}[\text{kgm}^2]$, $J_2 = 2.908 \times 10^{-5}[\text{kgm}^2]$, $m_1 = 0.5562[\text{kg}]$, $m_2 = 0.0166[\text{kg}]$, $\xi_1 = 0.153[\text{m}]$, $\xi_2 = 0.096[\text{m}]$, $\zeta_1 = 0.180[\text{m}]$, $\eta_1 = 0.315[\text{m}]$, $\eta_2 = 0.145[\text{m}]$, $k_s = 6850[\text{N/m}]$, $c_c = 0.5 \times 10^{-4}[\text{Nms}]$ and $k_c = 0.050[\text{Nms}]$. The parameters values used in experiment have been independently measured.

Eqs (1) have been integrated by the 4th order Runge-Kutta method. Bifurcation curves in Fig. 2 have been calculated using path following method AUTO³².

Experimental observations. In our experiments, the rig with four coupled double pendula has been mounted on the shaker LDS V780 Low Force Shaker (basic data are as follows: sine force peak 5120[N], max random force (rms) 4230[N], max acceleration sine peak $g_n = 111 \text{ g}$ [m/s^2], system velocity sine peak 1.9[m/s], displacement pk-pk $g_n = 25.4[\text{mm}]$, moving element mass 4.7[kg]). The shaker introduces practically kinematic periodic excitation $A \cos \omega t$, where A and ω are the amplitude and the frequency of the excitation, respectively. All experiments were recorded at motion videos taken by Vision Research Phantom v711 high speed camera. Typical recording speed used was 1000 frames per second (fps). Different random initial conditions have been given to each pendulum.

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Author Contributions

Y.M. and T.K. initiated this work, P.P., J.G. and D.D. performed the modeling and simulations, J.W. and T.K. designed the experiment, J.W. build experimental set up and performed experiments. D.D., J.G., J.W., P.P., Y.M. and T.K. wrote the paper.

Additional Information

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